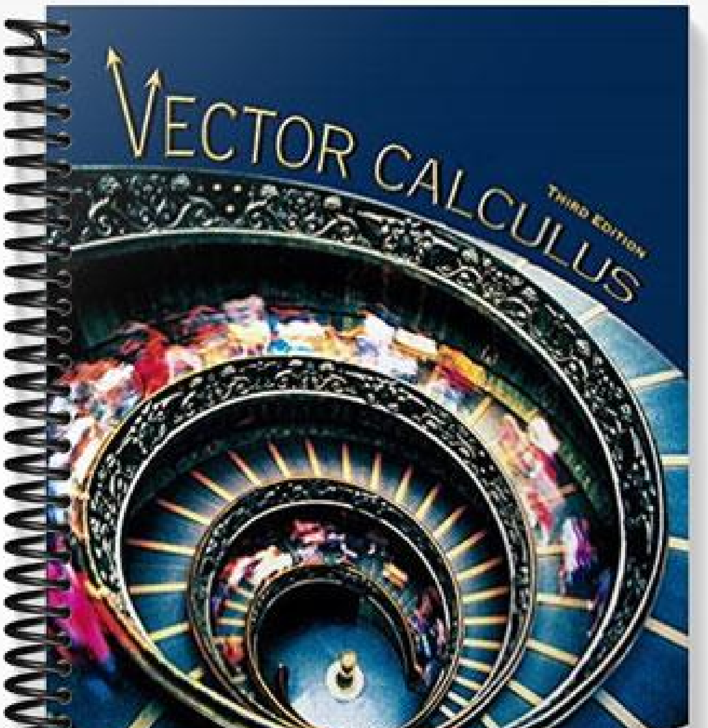
SOLUTIONS MANUAL



C H A P T E R 2

Differentiation in Several Variables

2.1 FUNCTIONS OF SEVERAL VARIABLES; GRAPHING SURFACES

1. $f: \mathbf{R} \to \mathbf{R}: x \mapsto 2x^2 + 1$

(a) Domain $f = \{x \in \mathbf{R}\}$, Range $f = \{y \in \mathbf{R} | y \geq 1\}$.

(b) No. For instance $f(1) = 3 = f(-1)$.

(c) No. For instance if $y = 0$, there is no x such that $f(x) = 0$.

2. $f: \mathbf{R}^2 \to \mathbf{R}: (x, y) \mapsto 2x^2 + 3y^2 - 7$

(a) Domain $g = \{(x, y) \in \mathbf{R}^2\}$, Range $g = \{z \in \mathbf{R} | z \geq -7\}$.

(b) Let Domain $g = \{(x, x) \in \mathbf{R}^2 | x \geq 0\}$.

(c) Let Codomain g = Range g.

3. Domain $f = \{(x, y) \in \mathbf{R}^2 | y \neq 0\}$, Range $f = \mathbf{R}$.

4. Domain $f = \{(x, y) \in \mathbf{R}^2 | x + y > 0\}$, Range $f = \mathbf{R}$.

5. Domain $g = \mathbf{R}^3$, Range $g = \{w \in \mathbf{R} | w \geq 0\}$.

6. Domain $g = \{\mathbf{x} \in \mathbf{R}^3 |\ \|\mathbf{x}\| < 2\}$, Range $g = \{y \in \mathbf{R} | y \geq 1/2\}$.

7. Domain $\mathbf{f} = \{(x, y) \in \mathbf{R}^2 | y \neq 1\}$, Range $\mathbf{f} = \{(x, y, z) \in \mathbf{R}^3 | y \neq 0, y^2 z = (xy - y - 1)^2 + (y + 1)^2\}$.

8. If $\mathbf{x} = (x_1, x_2, x_3)$, then $f_1(\mathbf{x}) = x_1$, $f_2(\mathbf{x}) = x_2 + 3$, and $f_3(\mathbf{x}) = x_3$.

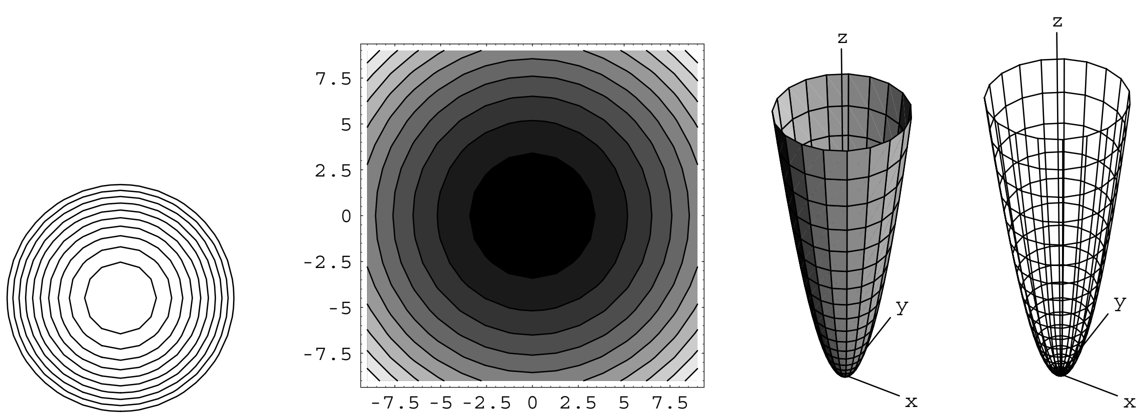
9. **(a)** $\mathbf{f}(\mathbf{x}) = -2\mathbf{x}/\|\mathbf{x}\|$.

(b) The component functions are

$$f_1(x, y, z) = \frac{-2x}{\sqrt{x^2 + y^2 + z^2}}, \quad f_2(x, y, z) = \frac{-2y}{\sqrt{x^2 + y^2 + z^2}}, \quad \text{and} \quad f_3(x, y, z) = \frac{-2z}{\sqrt{x^2 + y^2 + z^2}}.$$

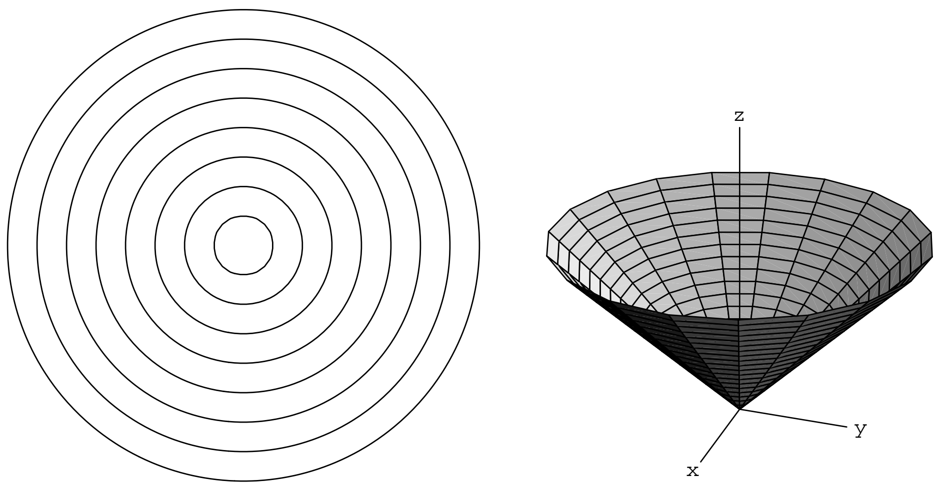
10. Here there is nothing to show. Everything is at level 3. This surface is a plane parallel to the xy-plane 3 units above it so the level set is the entire xy-plane if $c = 3$ and is the empty set if $c \neq 3$.

11. For $c > 0$ the level sets are circles centered at the origin of radius $\sqrt{c}$. For $c = 0$ the level set is just the origin. There are no values corresponding to $c < 0$. Note that the curves get closer together, indicating that we are climbing faster as we head out radially from the origin. The second figure below shows the plot of the level curves shaded to indicate the height of the level set (lighter is higher). The surface is therefore a paraboloid symmetric about the z-axis. We show it with and without the surface filled in.

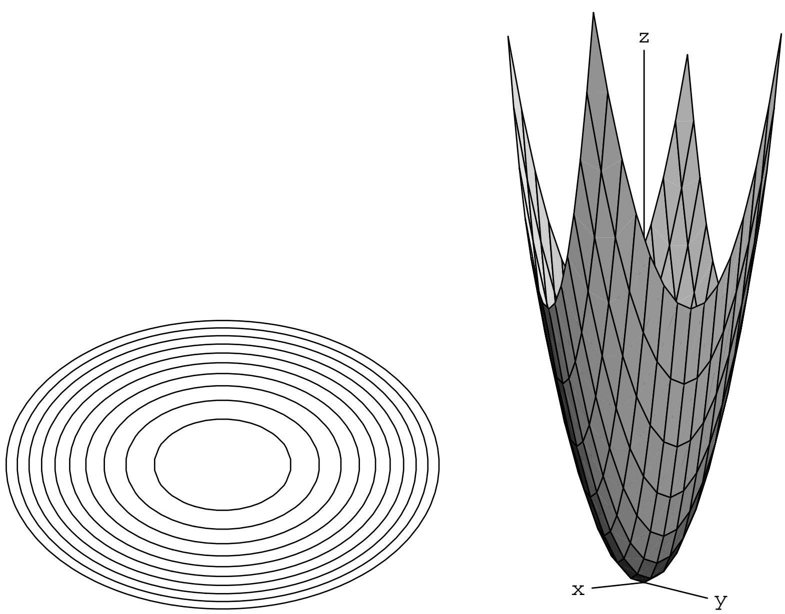


12. This is exactly the same as Exercise 11 except that the paraboloid has been shifted down 9 units so the level curves begin in the center at $c = -9$, not $c = 0$.

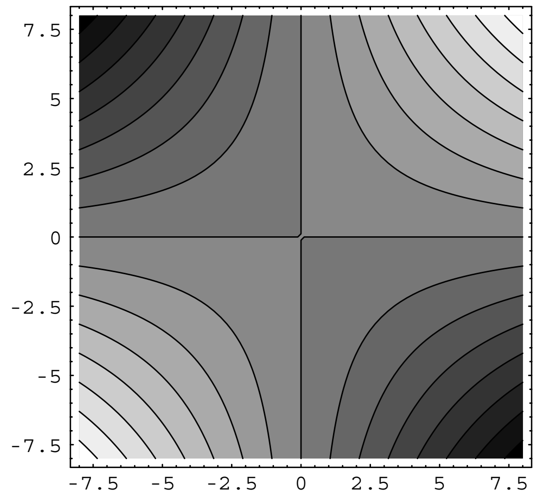
13. Again this time for $c > 0$ the level curves are circles. This time, however, the circles corresponding to the level sets at height c are of radius c. In other words, they are evenly spaced. We are climbing at a constant rate as we head out radially, so the surface is a cone.

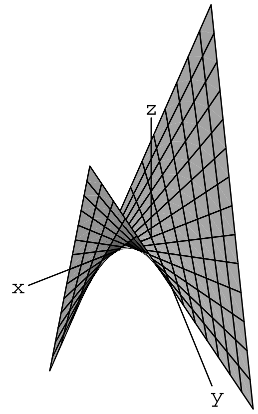


14. This time the level curves are ellipses. The sections as we cut in the direction x is constant or y is constant are still parabolas.



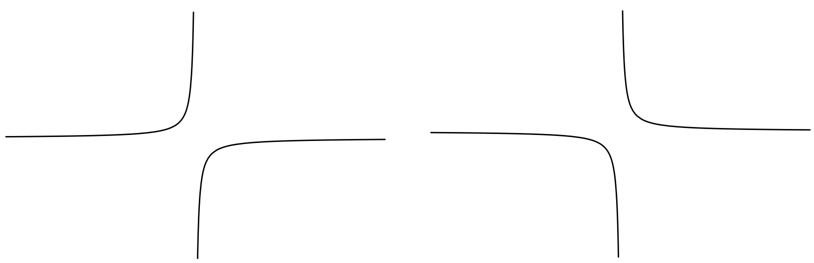
15. The graphs $xy = c$ are hyperbolas (unless $c = 0$ in which case it is the union of the two axes). When x and y are both positive the height of the level curves are positive and so the hyperboloid is increasing as we head away from the origin radially in either the first or third quadrant. When x and y are of different signs, the heights of the level curves are negative and so the hyperboloid is decreasing as we head out radially in either the second or fourth quadrant.



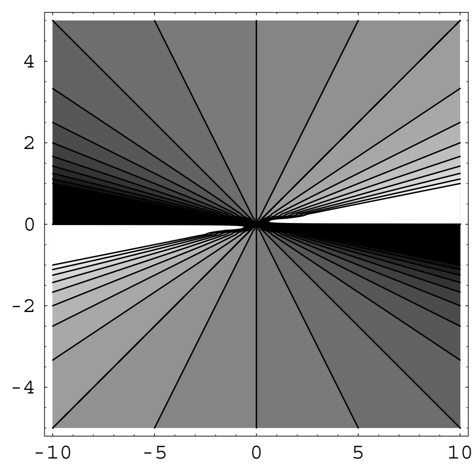


16. This is exactly the same as Exercise 17 except that the image has been reflected about the plane $y = x$.

17. We have a problem when $y = 0$. When $k < 0$, the section by $x = k$ looks like the hyperbola in the figure on the left, when $k > 0$, the section looks like the hyperbola in the figure on the right:

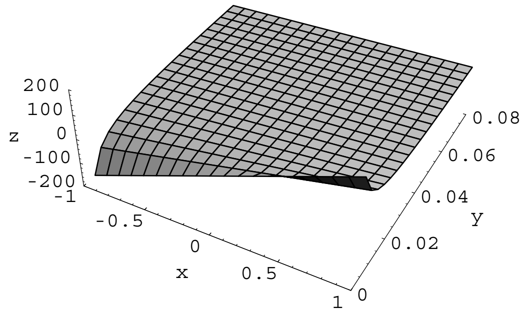


You can see that as $y \to 0$ from either side, along a line where x is constant and not 0, the z values won't match up. We are going to get a tear down the line $y = 0$. The level sets look like:

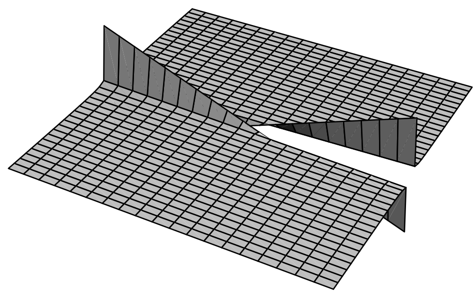


Notice that you can see that tear on the right center part of the above graph. The solid black and solid white areas which are on either side of the x-axis point to the behavior around the tear.

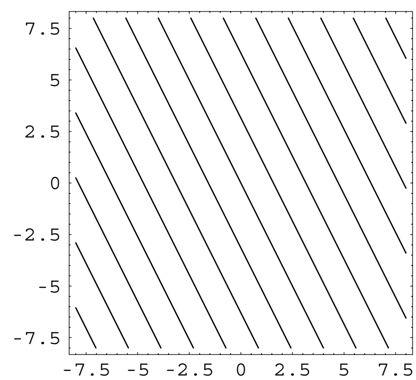
Graph each side of the x-axis and you will see the following piece of the surface:

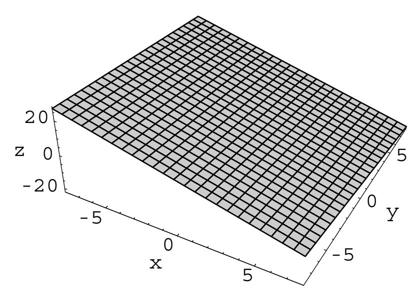


Our final surface is what you get when you try to glue two of those together:

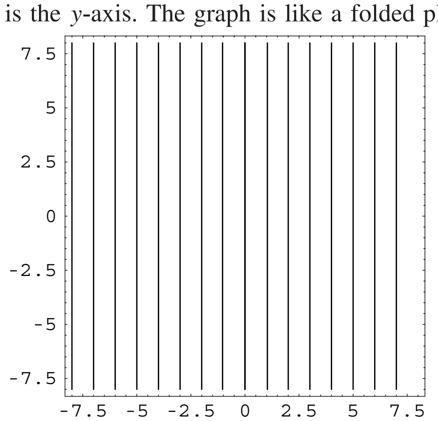


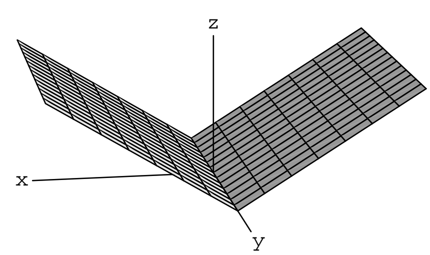
18. The surface is a plane. Level sets for which $f(x, y) = c$ are lines $c = 3 - 2x - y$ or $y = -2x + (3 - c)$. Level sets are pictured below on the left. The surface is pictured below on the right.





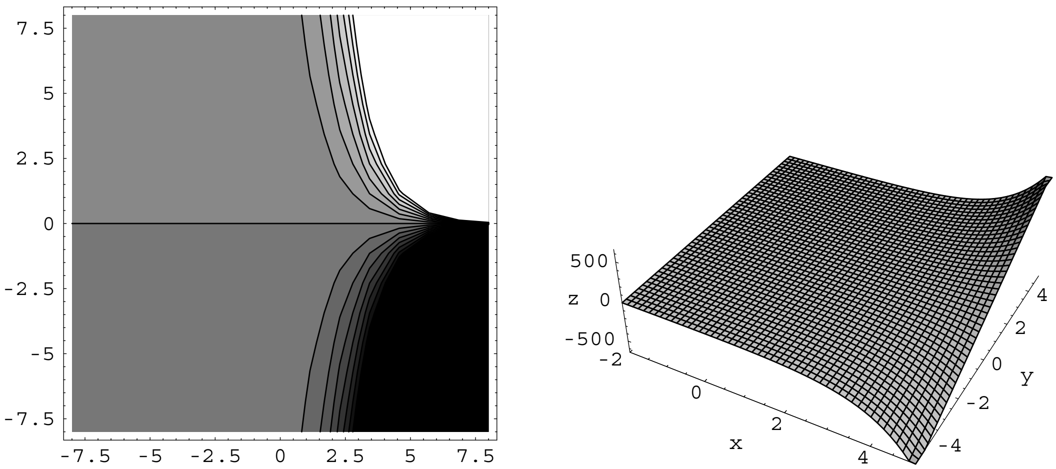
19. Here we are looking at the graph of $z = |x|$. For $c > 0$, level sets for $z = c$ will be the lines $x = \pm c$. For $c = 0$ the level set is the y-axis. The graph is like a folded plane.



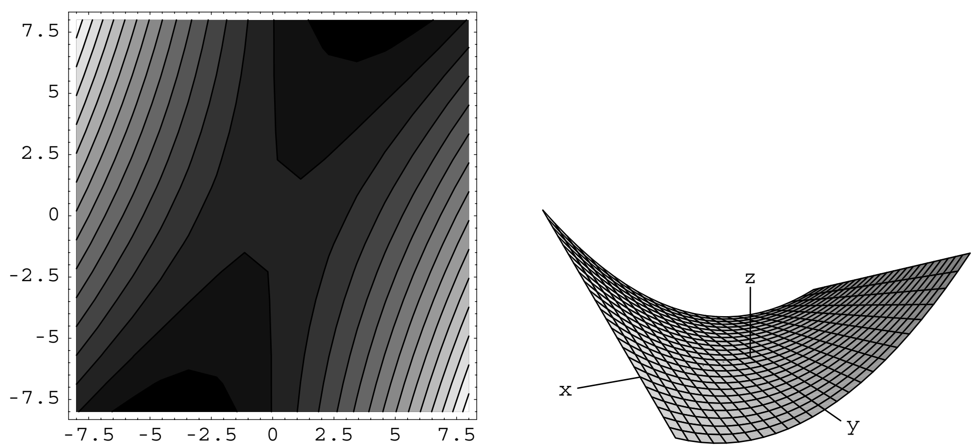


Note: In Problems 20–23 the level curves are shown along with the contour shading so you get an idea at what height to hang the curves. You should be able to figure out the orientation of the surface from the contour plot.

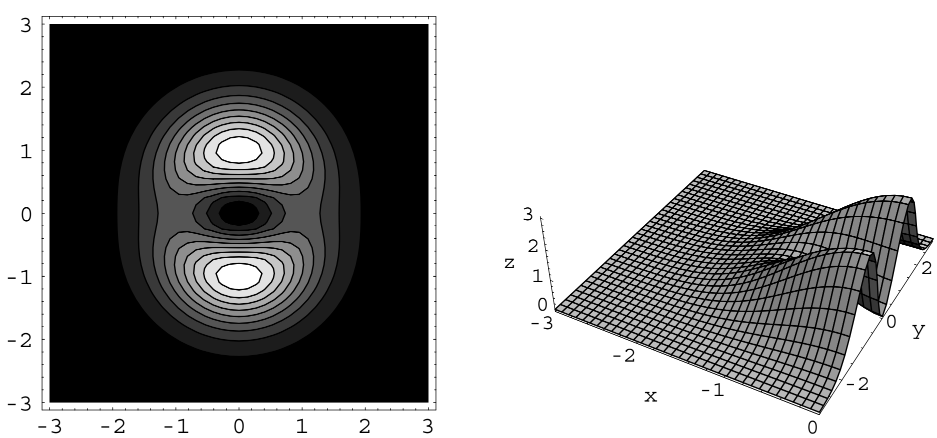
20. Figures below:



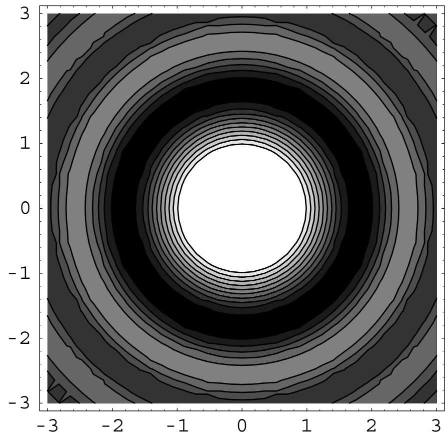
21. Figures below:

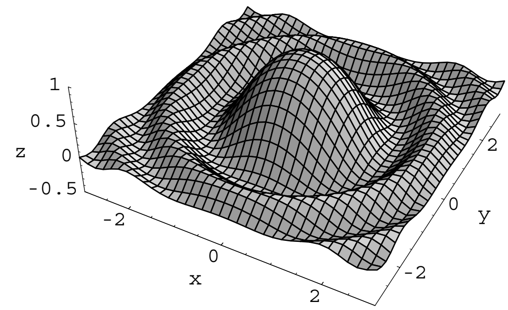


22. Figures below: (note only a portion of the surface has been sketched so that you get a better idea of what's going on)



23. Figures below:

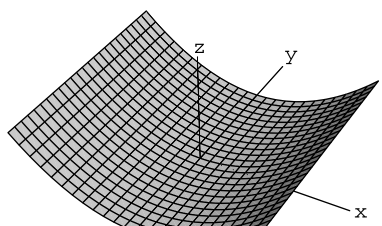


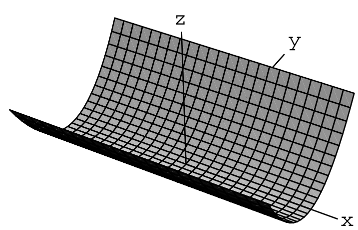


24. **(a)** We solve the equation $PV = kT$ for T, obtaining $T = f(P, V) = (1/k)PV$. This is the same as we considered in Exercise 15. See the figures for Exercise 15 for the general shape of the level curves.

(b) Here $V = g(P, T) = \frac{kT}{P}$. This is the same as the cases we considered in Exercises 16 and 17. We will get a "torn" surface similar to the one shown in Exercise 17. The level curve $V = c$ is the line through the origin: $P = (k/c)T$.

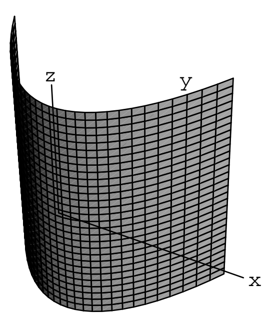
25. **(a)** The surface $z = x^2$ is graphed below left and $z = y^2$ below right.





(b) Consider first the surface $z = f(x)$ by considering the curve in the uv-plane given by $v = f(u)$. The intersection of the surface with planes of the form $y = c$ will look the same as the curve in the uv-plane for any value of y. This helps us see that if we "drag" this curve in each direction along the y-axis, the trail will trace out the surface. Similarly, but along the x-axis for surfaces of the form $z = f(y)$. The lack of dependence on x is our clue.

(c) The graph of the surface $y = x^2$ is shown below. It's what we would expect from parts (b) and (a).



26. See the solution to Exercise 17 and the note in Exercise 16.

27. They can't intersect—even though they may sometimes appear to. Say that two different level curves $f(x, y) = c_1$ and $f(x, y) = c_2$ where $c_1 \neq c_2$ intersect at some point (a, b). Then $f(a, b)$ would have assigned to it two non-equal values. This can't happen for a function (it's our vertical line test). On the other hand, if the limit as you approach (a, b) along different paths is different, those level curves may appear to intersect at (a, b) no matter how good the resolution on your contour plot.

28. The level surfaces are planes $x - 2y + 3z = c$.

29. The level surfaces at level $w = c$ are elliptic paraboloids.

30. The level surfaces at level $w = c$ are nested spheres of radius $\sqrt{c}$ centered at the origin.

31. The level surfaces at level $w = c$ are nested ellipsoids.

32. The level surfaces are of the form $y(x - z) = c$. If $c = 0$ we get the union of the xz-plane and the plane $x = z$. If $c \neq 0$ we get the hyperbola in the xy-plane $y = c/x$; this generates the solution surfaces when translated by $m(1, 0, -1)$.

33. **(a)** These are cylinders with the z-axis being the axis of the cylinder. For the surface at level $w = c$, the radius of the cylinder is $\sqrt{c}$.

(b) This is related to Exercise 25. A level surface at $w = c$ will be the surface generated by building a cylinder on the curve $h(x, y) = c$ in the $z = 0$ plane. You are dragging the curve both directions along the z-axis so that all cross sections for $z = c_1$ look identical.

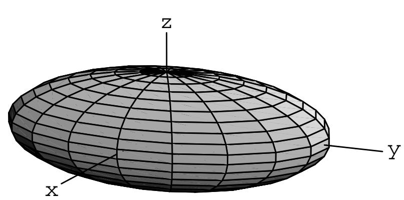
(c) Same thing in the y direction.

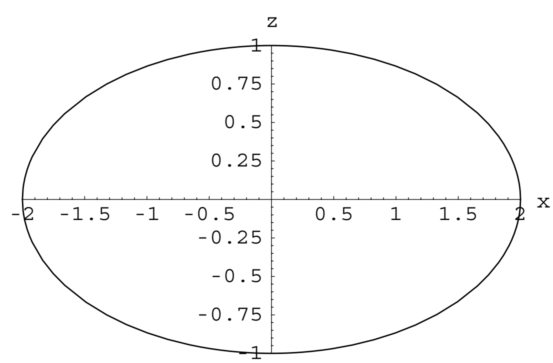
(d) If you said "same thing in the x direction," read the problem again. You are solving equations that look like $h(x) = c$. For each x_i that solves this equation, you have no dependency on y or z so the level set looks like a plane in $\mathbf{R}^3$ parallel to the yz-plane of the form $x = x_i$.

34. **(a)** F is, of course, not uniquely determined. But if we let $F(x, y, z) = x^2 + xy - xz - 2$, then the surface is the level set $F(x, y, z) = 0$.

(b) $x^2 + xy - xz = 2$ is equivalent to $z = \dfrac{x^2 + xy - 2}{x} = f(x, y)$.

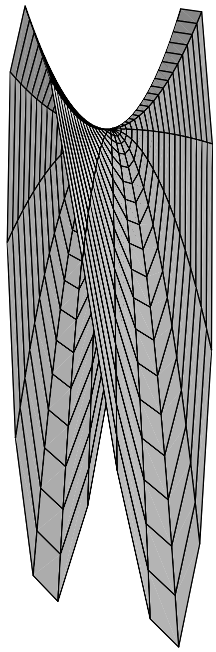
35. The ellipsoid is pictured below left. To see why you couldn't express the surface as one function $z = f(x, y)$, look for example at the intersection of the ellipsoid and the plane $y = 0$ pictured below on the right.

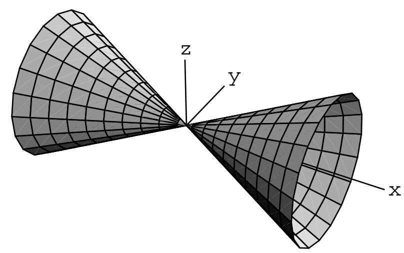




You can see that for $-2 < x < 2$ there correspond two values of z. We could express the top portion of the ellipsoid as $f(x, y) = \sqrt{1 - (x^2/4 + y^2/9)}$ and the bottom portion as $g(x, y) = -\sqrt{1 - (x^2/4 + y^2/9)}$.

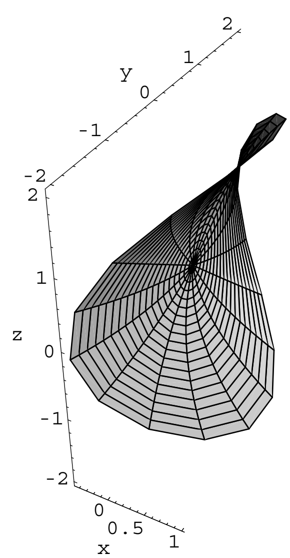
36. The figure is a hyperbolic paraboloid shown below left.

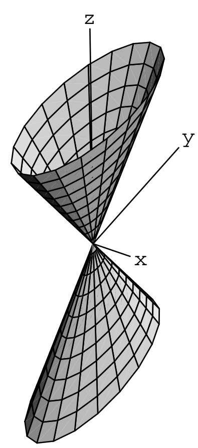




37. The only difference here is that z is squared. Here we get a cone with axis of symmetry the x-axis. The figure is shown above right.

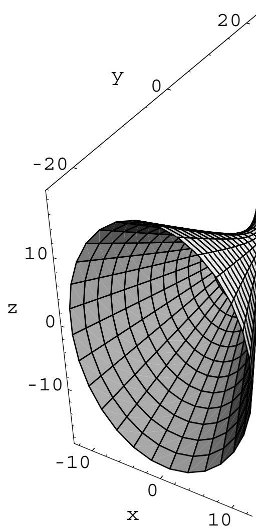
38. This is Exercise 36 with the roles of x, y and z permuted and a change in the constants. The figure is shown below left.

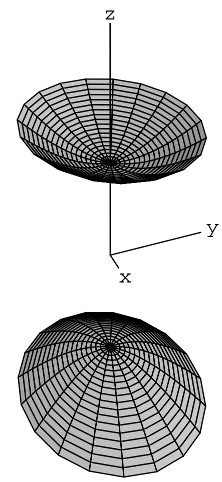




39. This is "cone" where the cross sections are ellipses, not circles. The figure is shown above right.

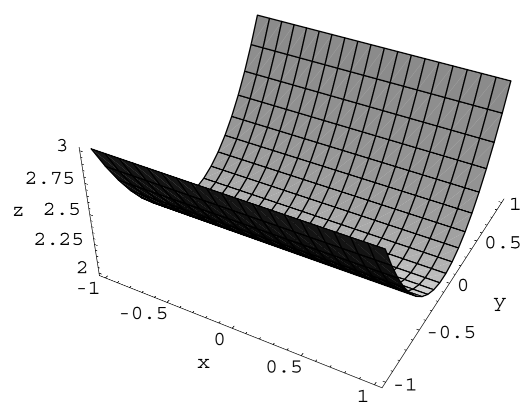
40. We see the figure is a hyperboloid. It is shown below left.





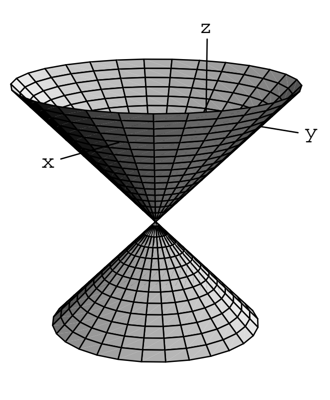
41. This is a hyperboloid of two sheets. It is shown above right.

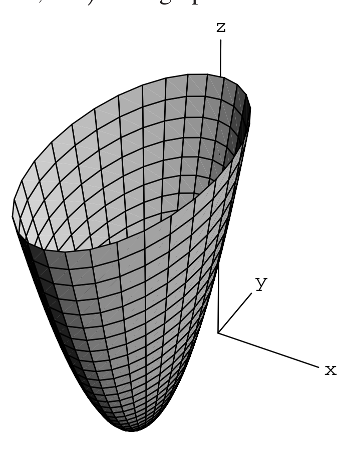
42. Here we have the parabola $z = y^2 + 2$ translated arbitrarily in the x direction. This is what we call a cylinder over the parabola $z = y^2 + 2$.



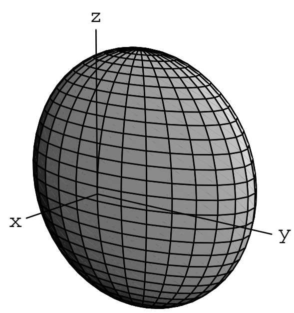
Note: Except that your students have to complete the square first, these are similar to Exercises 36–42 above. You may want them to be more explicit in reporting the translation as that's sometimes hard to pick up from a diagram.

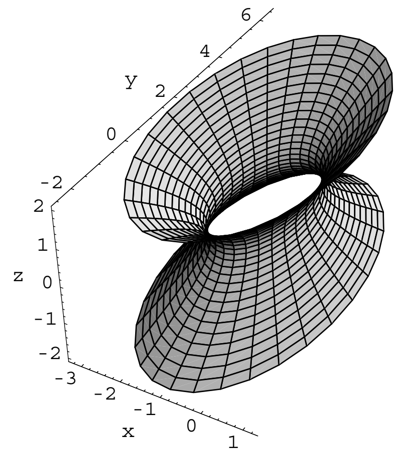
43. This is the equation of an elliptic cone with vertex at $(1, -1, -3)$. The graph is shown below left.





44. Here we have an elliptic paraboloid. The graph is shown above right.
45. This is the equation of an ellipsoid $4(x + 1)^2 + y^2 + z^2 = 4$. The graph is shown below left.





46. This is the equation of a hyperboloid of one sheet $4(x + 1)^2 + (y - 2)^2 - 4z^2 = 4$. The graph is shown above right.
47. This is similar to Exercise 44. The equation is equivalent to $z - 1 = (x - 3)^2 + 2y^2$.
48. Here we get $9x^2 + 4(y - 1)^2 - 36(z + 4)^2 = 684$ which is similar to Exercise 46.

2.2 LIMITS

Note: In Exercises 1–6, the rule of thumb is that a set is closed if it contains all of its boundary points.

1. This is an annulus which doesn't include its inner or outer boundary and so is **open**.
2. This is an annulus which includes all of its boundary points and so is **closed**.
3. This is an annulus which includes its inner boundary but not its outer boundary and so it is **neither open nor closed**.
4. This is a hollowed out sphere which includes its boundary points and so is **closed**.
5. This may be a bit harder to see. This is the union of an infinite open strip in the plane $(-1 < x < 1)$ and a closed line in the plane $(x = 2)$ and so is **neither open nor closed**.
6. This is the open infinite cylinder in $\mathbf{R}^3$ and so is **open**. *You could follow up on this by asking about* $\{(x, y, z) \in \mathbf{R}^3 | 1 \le x^2 + y^2 \le 4\}$.

Note: As pointed out in the text, the most common and convincing way to prove that a limit of a function with domain in $\mathbf{R}^2$ *doesn't exist is to show that you get two different answers when you follow two different paths. After doing Exercises 7–18 students may get in the habit of thinking that it is sufficient to check a few straight paths. Exercise 23 should make them think twice.*

7. There's no trick to taking this limit. Just let $(x, y, z) \to (0, 0, 0)$ and $x^2 + 2xy + yz + z^3 + 2 \to 2$.
8. We can see that $\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$ doesn't exist by looking at the limit along the paths $x = 0$ and $y = 0$. On the one hand

$$\lim_{(0,y)\to(0,0)} \frac{|y|}{\sqrt{x^2 + y^2}} = \frac{|y|}{\sqrt{y^2}} = 1 \quad \text{while} \quad \lim_{(x,0)\to(0,0)} \frac{|y|}{\sqrt{x^2 + y^2}} = \frac{0}{\sqrt{x^2}} = 0.$$

9. Again, the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x^2 + y^2} = 1 + \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}.$$

When $x = y$,

$$1 + \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = 1 + \lim_{x\to 0} \frac{2x^2}{x^2 + x^2} = 1 + 1 = 2.$$

When $x = 0$,

$$1 + \lim_{(0,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = 1 + \lim_{y\to 0} \frac{0}{y^2} = 1.$$

10. Here nothing goes wrong so we can evaluate the limit by substituting in the expression.

$$\lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x + y + 2} = \frac{e^0 e^0}{0 + 0 + 2} = \frac{1}{2}.$$

11. No limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + y^2}{x^2 + y^2} = 1 + \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}.$$

We reason, as above, that if $x = y$ then the limit is 3/2, but if $y = 0$ the limit is 2.

12. Here we can evaluate the function at the limit point and find that

$$\lim_{(x,y)\to(-1,2)} \frac{2x^2 + y^2}{x^2 + y^2} = \frac{6}{5}.$$

13. Just as with limits in first semester Calculus, this is begging to be simplified.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x + y} = \lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x + y} = \lim_{(x,y)\to(0,0)} (x + y) = 0.$$

14. This is the same as the limit in Exercise 9 (once we simplified it). The limit does not exist.

15. This, too, is begging to be simplified.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0.$$

16. This is the same as the limit in Exercise 11 (once we simplified it). The limit does not exist.

17. This is another standard trick from first year Calculus.

$$\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0),x\neq y} \frac{x(x - y)}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0),x\neq y} \frac{x(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})}{\sqrt{x} - \sqrt{y}}$$
$$= \lim_{(x,y)\to(0,0),x\neq y} x(\sqrt{x} + \sqrt{y}) = 0.$$

18. You can see that you would get different values depending on the path you took to $(x, y) = (2, 0)$. If you followed the path $(2, y) \to (2, 0)$ the limit would be -1. If you followed the path $(x, 0) \to (2, 0)$ the limit would be 1. So the limit doesn't exist.

19. The function is continuous so the limit is $f(0, \sqrt{\pi}, 1) = e^0 \cos\pi - 0 = -1$.

20. As in Exercise 18, you get different values depending on the path you choose. Look, for example, at paths along the three axes. Along $(x, 0, 0) \to (0, 0, 0)$ the limit is 2, along $(0, y, 0) \to (0, 0, 0)$ the limit is 3 and along $(0, 0, z) \to (0, 0, 0)$ the limit is 1. We can see that no limit can exist.

21. Again the limit doesn't exist because the value would differ on different paths. If you followed a path $(t, t, t) \to (0, 0, 0)$ the limit would be 1/3. If you followed the path $(x, 0, 0) \to (0, 0, 0)$ the limit would be 0.

22. **(a)** We know from single-variable calculus (either using l'Hôpital's rule or the direct geometric argument) that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Note: Exercise 23 is a classic and cool problem. You may wish to set it up in class before assigning it. Write the function on the board and ask the students to evaluate the limit or explain why the limit fails to exist. For those who get it right, this is wonderful. For those who get it wrong, they are now in a position to appreciate the subtlety of the problem.

23. Our goal is to evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ or explain why the limit fails to exist. We divide the answer into parts to make it easier to follow—there are no corresponding parts (a)–(d) in the text.

(a) If you evaluate the limit along the lines $x = 0$ and $y = 0$ the limit is 0. We might be tempted to guess that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ but as we saw in Exercise 14, we could get a limit of 0 along the paths $x = 0$ and $y = 0$ but perhaps not along $x = y$.

(b) So now let's follow the line $y = mx$ into the origin and see where f heads off to.

$$\begin{aligned}\lim_{(x,y)\to(0,0),\,y=mx} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{x\to 0} \frac{x^4(mx)^4}{(x^2+(mx)^4)^3}\\ &= \lim_{x\to 0}\frac{m^4x^8}{(x^2(1+m^4x^2))^3}\\ &= m^4\lim_{x\to 0}\frac{x^8}{(x^6)(1+m^4x^2)^3}\\ &= m^4\lim_{x\to 0}\frac{x^2}{(1+m^4x^2)^3} = 0.\end{aligned}$$

This means then if we head into the origin along any straight line the limit of f is 0. *Here is the point of this problem*: If we head into the origin in any constant direction, the limit of f is 0 and yet $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist!

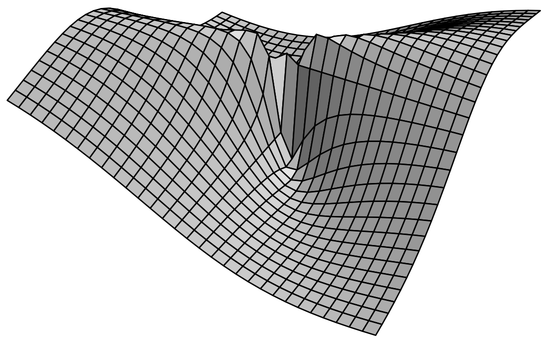
(c) For the limit to exist f must approach the same number no matter what path we choose to take to the origin. So let's approach along the parabola $x = y^2$.

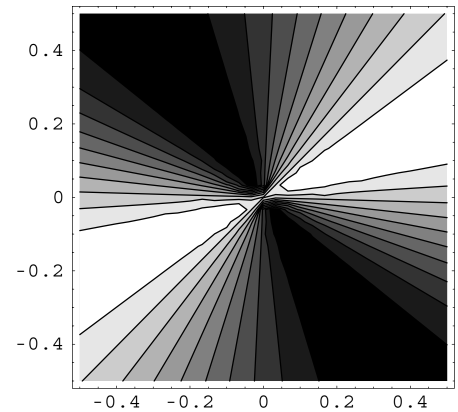
$$\begin{aligned}\lim_{(x,y)\to(0,0),\,x=y^2} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{y\to 0}\frac{(y^2)^4y^4}{((y^2)^2+y^4)^3}\\ &= \lim_{y\to 0}\frac{y^{12}}{(2y^4)^3}\\ &= \lim_{y\to 0}\frac{y^{12}}{8y^{12}} = \frac{1}{8}.\end{aligned}$$

(d) So we get different answers for $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ depending on what path we follow into the origin. So the limit does not exist.

Note—In Exercises 24–27 your students may find better visual information by using a contour plot than a three-dimensional plot.

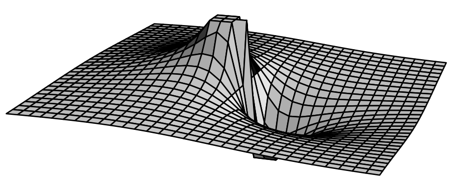
24. Below see two graphs of the function. The three dimensional plot makes it seem as if there are mountains and valleys quite close to the origin. The contour plot helps you see from the diagonal lines that meet at the origin that the limit doesn't exist.

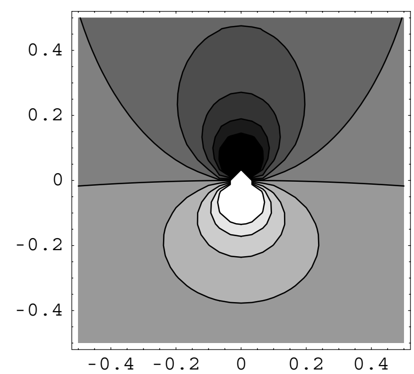




Analytically, f is equivalent to $1 + (x^2 + 2xy)/(3x^2 + 5y^2)$. Head in toward the origin on a path where $x = y$ and the limit is 13/8. Head in toward the origin on a path where $x = 0$ and the limit is 1. Head in toward the origin on a path where $y = 0$ and the limit is 11/3. So the limit doesn't exist.

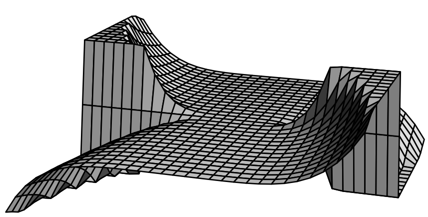
25. Below see two graphs of the function. You actually get most of the picture from the three dimensional graph—except that it looks as if things are joined smoothly. The contour plot shows the dramatic problems near the origin. Particularly if you look along the vertical line $x = 0$ you'll see that the limit does not exist at the origin.

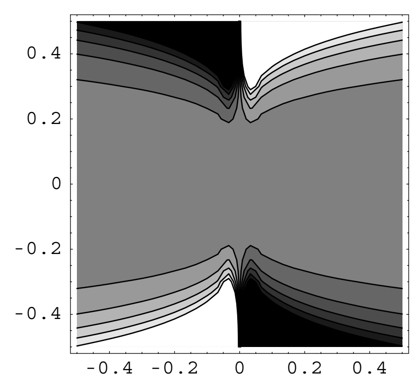




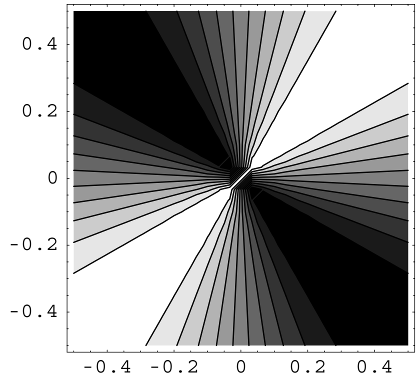
Analytically, look at the path $x = 0$. Here we're looking at the graph of $z = -1/y$. The limits as we approach from positive and negative y values is $\pm\infty$ so no limit exists.

26. In the three-dimensional graph below you can see that the extreme behavior calms down near the origin. This is confirmed in the contour plot. From the graphs it appears that the limit exists at the origin.



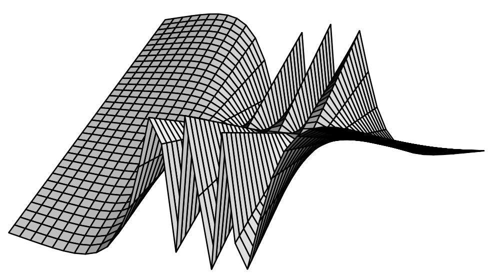


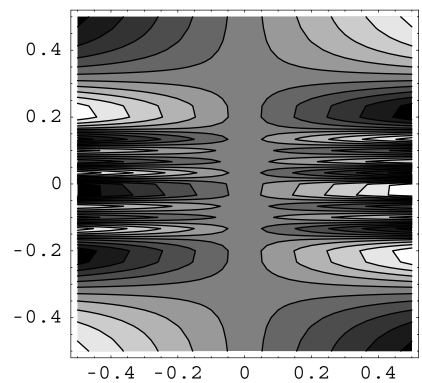
Before exploring this one analytically, consider the graph $g(x,t) = xt/(x^2 + t^2)$ Its contour plot is shown below.



So really the problem we are considering is the same with $t = y^5$. We're not looking along a path that shows us enough. Let's look at the limit for our original function f as we approach the origin. Along a path where $x = 0$ or $y = 0$ the limit is 0. Along a path where $x = y^5$ the limit is 1/2. This is a good place to encourage your students to be careful drawing conclusions from even very good graphs.

27. You'd think we would have learned our lesson from Exercise 26. On the other hand, it sure looks as if things are calming down near the origin. Sure $\sin 1/y$ oscillates madly between -1 and 1 but x seems to dampen it. We'll boldly assert that the limit exists at the origin.





Actually, the discussion above leads us to the truth. The product of a bounded function and one going to 0 goes to 0. The limit exists and is 0.

28.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2} = \lim_{r\to 0} r\sin\theta\cos^2\theta = 0$$

29.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \lim_{r\to 0}\cos^2\theta = \cos^2\theta$$

Limit does not exist as the result depends on θ.

30.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2 + (r\cos\theta) r\sin\theta}{r^2} = \lim_{r\to 0}(1 + \cos\theta\sin\theta) = 1 + \cos\theta\sin\theta.$$

Thus the limit does not exist.

31.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho\to 0} \frac{(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)}{\rho^2}$$

$$= \lim_{\rho\to 0} \rho\sin^2\varphi\cos\varphi\cos\theta\sin\theta = 0$$

32.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + z^2}} = \lim_{\rho\to 0} \frac{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho \sin^2\varphi = 0$$

33.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2 + y^2 + z^2} = \lim_{\rho\to 0} \frac{\rho^2 \sin\varphi \cos\varphi \cos\theta}{\rho^2} = \lim_{\rho\to 0} \sin\varphi\cos\varphi\cos\theta = \sin\varphi\cos\varphi\cos\theta$$

The Limit does not exist.

In Exercises 34–41: as the rules on continuity show, if the components are continuous and we put the functions together by adding, subtracting, multiplying, or composing, then the result is continuous. It should be clear to the students what points need checking.

34. This is a polynomial and is continuous everywhere.
35. This too is a polynomial and is continuous everywhere.

To make the point about composition, you may want to assign Exercises 36 and 37 together.

36. The only place we could get into trouble is where the denominator is 0, but $x^2 + 1 \neq 0$ so g is always continuous.
37. Here we are composing a continuous function (cos) with the continuous function g from Exercise 22, so the composition is continuous.
38. You can even rewrite the function as $(\cos x)^2 - 2(\sin xy)^2$ so that it is clear that this is just the composition of continuous functions.
39. The only place we need to check is the origin. We need to show that the limit of f as we approach (0, 0) is 0. If we add and subtract y^2 to the numerator we find that:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = 1 - 2 \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}.$$

In Exercise 16 we showed that this limit doesn't exist (in this case you get two different answers if you follow the paths $y = 0$ and $y = x$) and so f is not continuous at (0, 0).
40. As in Exercise 39, the only point we need to check is the origin.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + x^2 + xy^2 + y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x + 1)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x + 1) = 1.$$

The good news is that the limit exists, the bad news is that

$$\lim_{(x,y)\to(0,0)} g(x, y) = 1 \neq 2 = g(0, 0),$$

so g is not continuous at the origin.
41. A vector-valued function is continuous if each of its component functions is continuous. Each clearly is, so **F** is continuous.
42. Notice that when $(x, y) \neq 0$,

$$\frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \frac{(x^2 + y^2)(x + 2)}{x^2 + y^2} = x + 2.$$

So $c = 2$ and the function $g(x, y)$ is seen to be equivalent to $x + 2$.
43. Here you can view f as being a function $\mathbf{R}^3 \to \mathbf{R}$; then $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3$ which is linear in x_1, x_2, and x_3 and therefore continuous.
44. This is equivalent to $\mathbf{f}(x, y, z) = (-5y, 5x - 6z, 6y)$. Since each of the component functions from $\mathbf{R}^3 \to \mathbf{R}$ is continuous, so is $\mathbf{f}$.

We make students do at least a few of the following because "it's good for them." Exercise 45 is a review of how they looked at limits in first semester Calculus—it prepares them for Exercise 46. Exercise 47 is a generalization of Exercise 46.

45. Here $f(x) = 2x - 3$.
(a) If $|x - 5| < \delta$, then $|f(x) - 7| = |(2x - 3) - 7| = |2x - 10| = 2|x - 5| < 2\delta$.
(b) For any $\epsilon > 0$, if $0 < |x - 5| < \epsilon/2$, then $|f(x) - 7| < \epsilon$. This means that $\lim_{x\to 5} f(x) = 7$.

46. Now the function is $f(x, y) = 2x - 10y + 3$.

(a) Really we're just arguing that the hypotenuse of a right triangle is at least as long as either leg.

$$\delta > \|(x, y) - (5, 1)\| = \sqrt{(x-5)^2 + (y-1)^2} \geq \sqrt{(x-5)^2} = |x - 5|.$$

And

$$\delta > \|(x, y) - (5, 1)\| = \sqrt{(x-5)^2 + (y-1)^2} \geq \sqrt{(y-1)^2} = |y - 1|.$$

(b) First:

$$|f(x, y) - 3| = |2x - 10y + 3 - 3| = |2x - 10y| = |2(x-5) - 10(y-1)|.$$

(c) By the triangle inequality

$$|2(x-5) - 10(y-1)| \leq |2(x-5)| + |10(y-1)| = 2|x-5| + 10|y-1|.$$

But we are assuming that $\|(x, y) - (5, 1)\| < \delta$ and from part (a) we know that this implies that $|x - 5| < \delta$ and $|y - 1| < \delta$, so

$$2|x-5| + 10|y-1| < 2\delta + 10\delta = 12\delta.$$

(d) We put these together to obtain: For any $\varepsilon > 0$, if $0 < \|(x, y) - (5, 1)\| < \varepsilon/12$, then $|f(x, y) - 3| < \varepsilon$. In other words,

$$\lim_{(x,y)\to(5,1)} f(x, y) = 3.$$

47. This is just a generalization of Exercise 46. We can use the same steps outlined there:

(a)

$$\delta > \|(x, y) - (x_0, y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} \geq \sqrt{(x-x_0)^2} = |x - x_0|.$$

And

$$\delta > \|(x, y) - (x_0, y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} \geq \sqrt{(y-y_0)^2} = |y - y_0|.$$

(b) Assume that $\|(x, y) - (x_0, y_0)\| < \delta$, then follow the steps in part (b) of Exercise 46:

$$\begin{aligned} |f(x, y) - (Ax_0 + By_0 + C)| &= |Ax + By + C - (Ax_0 + By_0 + C)| \\ &= |A(x - x_0) + B(y - y_0)| \leq |A(x - x_0)| + |B(y - y_0)| \\ &= |A||x - x_0| + |B||y - y_0| < |A|\delta + |B|\delta = (|A| + |B|)\delta. \end{aligned}$$

(c) Now we're ready to put this together: For any $\varepsilon > 0$, if $0 < \|(x, y) - (x_0, y_0)\| < \varepsilon/(|A| + |B|)$, then $|f(x, y) - (Ax_0 + By_0 + C)| < \varepsilon$. In other words,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = Ax_0 + By_0 + C.$$

48. (a) This is really what we just showed in Exercise 47 with $x_0 = 0$ and $y_0 = 0$.

$$\|(x, y)\| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x|.$$

And

$$\|(x, y)\| = \sqrt{x^2 + y^2} \geq \sqrt{y^2} = |y|.$$

(b) We follow the hint given in the text: $|x^3 + y^3| \leq |x^3| + |y^3| = |x|^3 + |y|^3$. But by part (a), $|x| \leq \|(x, y)\| = \sqrt{x^2 + y^2}$, and $|y| \leq \|(x, y)\| = \sqrt{x^2 + y^2}$. Therefore,

$$|x^3 + y^3| \leq |x|^3 + |y|^3 \leq 2(\sqrt{x^2 + y^2})^3 = 2(x^2 + y^2)^{3/2}.$$

(c) If $0 < \|(x, y)\| < \delta$ then by part (b),

$$\left|\frac{x^3 + y^3}{x^2 + y^2}\right| \le \left|\frac{2(x^2 + y^2)^{3/2}}{x^2 + y^2}\right| = 2\sqrt{x^2 + y^2} = 2\|(x, y)\| < 2\delta.$$

(d) First we know by part (c) that $\frac{x^3 + y^3}{x^2 + y^2}$ can be made to be arbitrarily close to 0 by choosing (x, y) close enough to the origin. This means that the limit is 0.

Assemble the pieces: For any $\varepsilon > 0$, if $0 < \|(x, y)\| < \varepsilon/2$, then $|f(x, y)| < \varepsilon$. This shows that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0.$$

49. (a) $0 \le (a + b)^2 = a^2 + 2ab + b^2$, so $-2ab \le a^2 + b^2$. Also $0 \le (a - b)^2 = a^2 - 2ab + b^2$, so $2ab \le a^2 + b^2$. We combine these two results to get: $2|ab| \le a^2 + b^2$.

(b) If $\|(x, y)\| < \delta$, then we'll use part (a) to rewrite $|xy|$ in the following calculation:

$$\left|xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)\right| = \frac{|xy|(|x^2 - y^2|)}{x^2 + y^2} \le \frac{(1/2)(x^2 + y^2)|x^2 - y^2|}{x^2 + y^2} = \left(\frac{1}{2}\right)|x^2 - y^2|.$$

We can apply part (a) again with $a = x + y$ and $b = x - y$ so that

$$|(x + y)(x - y)| \le \frac{(x + y)^2 + (x - y)^2}{2} = x^2 + y^2.$$

Noting that $x^2 + y^2 = \|(x, y)\|^2 = \delta^2$, we have:

$$\left|xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)\right| \le \left(\frac{1}{2}\right)|x^2 - y^2| = \frac{\delta^2}{2}.$$

(c) As in Exercise 48, the limit has to be 0 because we can make f as small as we want by choosing (x, y) close enough to the origin.

We summarize the above as: For any $\varepsilon > 0$, if $0 < \|(x, y)\| < \sqrt{2\varepsilon}$, then $|f(x, y)| < \varepsilon$. This shows that

$$\lim_{(x,y)\to(0,0)} \left|xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)\right| = 0.$$

2.3 THE DERIVATIVE

The general strategy for Exercises 1–11 is to treat all variables except for the one with respect to which we are differentiating as constants.

1. $f(x, y) = xy^2 + x^2y$, so $\partial f/\partial x = y^2 + 2xy$, and $\partial f/\partial y = 2xy + x^2$.

2. $f(x, y) = e^{x^2+y^2}$, so $\partial f/\partial x = 2xe^{x^2+y^2}$, and $\partial f/\partial y = 2ye^{x^2+y^2}$.

3. $f(x, y) = \sin xy + \cos xy$, so $\partial f/\partial x = y\cos xy - y\sin xy$, and $\partial f/\partial y = x\cos xy - x\sin xy$.

4. $f(x, y) = \dfrac{x^3 - y^2}{1 + x^2 + 3y^4}$, so

$$\frac{\partial f}{\partial x} = \frac{(1 + x^2 + 3y^4)(3x^2) - (x^2 - y^2)(2x)}{(1 + x^2 + 3y^4)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{(1 + x^2 + 3y^4)(-2y) - (x^2 - y^2)(12y^3)}{(1 + x^2 + 3y^4)^2}.$$

5. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, so $\dfrac{\partial f}{\partial x} = \dfrac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} = \dfrac{4xy^2}{(x^2 + y^2)^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2} = \dfrac{-4x^2y}{(x^2 + y^2)^2}$.

6. $f(x,y) = \ln(x^2 + y^2)$, so $\dfrac{\partial f}{\partial x} = \dfrac{1}{x^2 + y^2}(2x) = \dfrac{2x}{x^2 + y^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2 + y^2}$.

7. $f(x,y) = \cos x^3 y$, so $\dfrac{\partial f}{\partial x} = (-\sin x^3 y)(3yx^2) = -3x^2 y \sin x^3 y$ and $\dfrac{\partial f}{\partial y} = -x^3 \sin x^3 y$.

8. $F(x,y,z) = xyz$, so $\partial F/\partial x = yz$, $\partial F/\partial y = xz$, and $\partial F/\partial z = xy$.

9. $F(x,y,z) = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$. The partials derivatives are:

$$\frac{\partial F}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}},$$
$$\frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \quad \text{and,}$$
$$\frac{\partial F}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

10. $F(x,y,z) = e^{ax}\cos by + e^{az}\sin bx$ so

$$\frac{\partial F}{\partial x} = ae^{ax}\cos by + be^{az}\cos bx,$$
$$\frac{\partial F}{\partial y} = -be^{ax}\sin by, \quad \text{and}$$
$$\frac{\partial F}{\partial z} = ae^{az}\sin bx.$$

11. $F(x,y,z) = \dfrac{x + y + z}{(1 + x^2 + y^2 + z^2)^{3/2}}$

$$F_x(x,y,z) = \frac{(1 + x^2 + y^2 + z^2)^{3/2} - (x + y + z)(3/2)(1 + x^2 + y^2 + z^2)^{1/2}(2x)}{(1 + x^2 + y^2 + z^2)^3}$$
$$= \frac{1 - 2x^2 + y^2 + z^2 - 3xy - 3xz}{(1 + x^2 + y^2 + z^2)^{5/2}}$$
$$F_y(x,y,z) = \frac{1 + x^2 - 2y^2 + z^2 - 3xy - 3yz}{(1 + x^2 + y^2 + z^2)^{5/2}}, \text{ and}$$
$$F_z(x,y,z) = \frac{1 + x^2 + y^2 - 2z^2 - 3xz - 3yz}{(1 + x^2 + y^2 + z^2)^{5/2}}.$$

12. $F(x,y,z) = \sin x^2 y^3 z^4$ so this is similar to Exercise 7 above. $F_x(x,y,z) = 2xy^3z^4\cos x^2y^3z^4$, $F_y(x,y,z) = 3x^2y^2z^4\cos x^2y^3z^4$ and $F_z(x,y,z) = 4x^2y^3z^3\cos x^2y^3z^4$.

13. $F(x,y,z) = \dfrac{x^3 + yz}{(x^2 + z^2 + 1)}$ We've seen this form a couple of times by now.

$$F_x(x,y,z) = \frac{(x^2 + z^2 + 1)(3x^2) - (x^3 + yz)(2x)}{(x^2 + z^2 + 1)^2} = \frac{x^4 + 3x^2z^2 + 3x^2 - 2xyz}{(x^2 + z^2 + 1)^2}$$
$$F_y(x,y,z) = \frac{(x^2 + z^2 + 1)(z) - (x^3 + yz)(0)}{(x^2 + z^2 + 1)^2} = \frac{z}{x^2 + z^2 + 1}$$
$$F_z(x,y,z) = \frac{(x^2 + z^2 + 1)(y) - (x^3 + yz)(2z)}{(x^2 + z^2 + 1)^2} = \frac{x^2y - yz^2 + y - 2x^3z}{(x^2 + z^2 + 1)^2}$$

The gradient of f is the function $(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$. *In Exercises 14–19 we are evaluating the gradient at a given point.*

14. $f(x,y) = x^2y + e^{y/x}$, so $\nabla f(x,y) = (2xy + (-y/x^2)e^{y/x}, x^2 + (1/x)e^{y/x})$. This means that $\nabla f(1,0) = (0,2)$.

15. $f(x,y) = \dfrac{x-y}{x^2+y^2+1}$, so

$$\begin{aligned}\nabla f(x,y) &= \left(\frac{(x^2+y^2+1)(1)-(x-y)(2x)}{(x^2+y^2+1)^2}, \frac{(x^2+y^2+1)(-1)-(x-y)(2y)}{(x^2+y^2+1)^2}\right)\\ &= \left(\frac{-x^2+y^2+1+2xy}{(x^2+y^2+1)^2}, \frac{-x^2+y^2-1-2xy}{(x^2+y^2+1)^2}\right).\end{aligned}$$

So

$$\nabla f(2,-1) = \left(-\frac{6}{36}, \frac{0}{36}\right) = \left(-\frac{1}{6}, 0\right).$$

16. $f(x,y,z) = \sin xyz$, so $\nabla f(x,y,z) = (\cos xyz)(yz, xz, xy)$. This means that

$$\nabla f(\pi, 0, \pi/2) = \cos 0(0, \pi^2/2, 0) = (0, \pi^2/2, 0).$$

17. $f(x,y,z) = xy + y\cos z - x\sin yz$, so $\nabla f(x,y,z) = (y - \sin yz, x + \cos z - xz\cos yz, -y\sin z - xy\cos yz)$. So,

$$\begin{aligned}\nabla f(2,-1,\pi) &= (-1-\sin(-\pi), 2+\cos(\pi) - 2(\pi)\cos(-\pi), \sin(\pi) + 2\cos(-\pi))\\ &= (-1, 1+2\pi, -2).\end{aligned}$$

18. $f(x,y) = e^{xy} + \ln(x-y)$, so $\nabla f(x,y) = (ye^{xy} + 1/(x-y), xe^{xy} - 1/(x-y))$. This means that $\nabla f(2,1) = (e^2+1, 2e^2-1)$.

19. $f(x,y,z) = (x+y)e^{-z}$, so $\nabla f(x,y,z) = (e^{-z}, e^{-z}, -(x+y)e^{-z})$. So, $\nabla f(3,-1,0) = (1,1,-2)$.

The nth row of the derivative matrix is the gradient of the nth component function.

20. $f(x,y) = \frac{x}{y}$, $Df(x,y) = \left[\frac{1}{y}, \frac{-x}{y^2}\right]$. So $Df(3,2) = [1/2, -3/4]$.

21. $\mathbf{f}(x,y,z) = (xyz, \sqrt{x^2+y^2+z^2})$, so

$$D\mathbf{f}(x,y,z) = \begin{bmatrix} yz & xz & xy \\ x/\sqrt{x^2+y^2+z^2} & y/\sqrt{x^2+y^2+z^2} & z/\sqrt{x^2+y^2+z^2}\end{bmatrix}.$$

This means,

$$D\mathbf{f}(1,0,-2) = \begin{bmatrix} 0 & -2 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5}\end{bmatrix}.$$

22. $\mathbf{f}(t) = (t, \cos 2t, \sin 5t)$, so

$$D\mathbf{f}(t) = \begin{bmatrix} 1 \\ -2\sin 2t \\ 5\cos 5t\end{bmatrix} \quad \text{and so} \quad D\mathbf{f}(0) = \begin{bmatrix} 1 \\ 0 \\ 5\end{bmatrix}.$$

23. $\mathbf{f}(x,y,z,w) = (3x - 7y + z, 5x + 2z - 8w, y - 17z + 3w)$ so

$$D\mathbf{f}(x,y,z,w) = \begin{bmatrix} 3 & -7 & 1 & 0 \\ 5 & 0 & 2 & -8 \\ 0 & 1 & -17 & 3\end{bmatrix}.$$

Since all of the entries are constant, the matrix doesn't depend on **a**.

24. $\mathbf{f}(x, y) = (x^2y, x + y^2, \cos \pi xy)$, so

$$D\mathbf{f}(x, y) = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -\pi y \sin \pi xy & -\pi x \sin \pi xy \end{bmatrix}.$$

This means,

$$D\mathbf{f}(2, -1) = \begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

25. $\mathbf{f}(s, t) = (s^2, st, t^2)$, so

$$D\mathbf{f}(s, t) = \begin{bmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{bmatrix}.$$

This means,

$$D\mathbf{f}(-1, 1) = \begin{bmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

We will appeal to Theorem 3.5 for Exercises 26–28.

26. $f(x, y) = xy - 7x^8y^2 + \cos x$ is differentiable because the two partials $f_x(x, y) = y - 56x^7y^2 - \sin x$ and $f_y(x, y) = x - 14x^8y$ are continuous.

27. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2}$ is differentiable because the three partials

$$f_x(x, y, z) = \frac{-x^2 + y^2 + z^2 - 2xy - 2xz}{(x^2 + y^2 + z^2)^2}$$

$$f_y(x, y, z) = \frac{x^2 - y^2 + z^2 - 2xy - 2yz}{(x^2 + y^2 + z^2)^2}$$

$$f_z(x, y, z) = \frac{x^2 + y^2 - z^2 - 2xz - 2yz}{(x^2 + y^2 + z^2)^2}$$

are all continuous.

28. $\mathbf{f}(x, y) = \left(\dfrac{xy^2}{x^2 + y^4}, \dfrac{x}{y} + \dfrac{y}{x}\right)$ is differentiable because the partials in the matrix

$$D\mathbf{f}(x, y) = \begin{bmatrix} \dfrac{y^6 - x^2y^2}{(x^2 + y^4)^2} & \dfrac{2x^3y - 2xy^5}{(x^2 + y^4)^2} \\ \dfrac{1}{y} - \dfrac{y}{x^2} & \dfrac{-x}{y^2} + \dfrac{1}{x} \end{bmatrix}$$

are continuous in the domain of $\mathbf{f}$.

29. (a) The graph of $z = x^3 - 7xy + e^y$ has continuous partial derivatives at $(-1, 0, 0)$.

(b) By Theorem 3.3, the equation for the tangent plane is: $z = f(-1, 0) + f_x(-1, 0)(x - (-1)) + f_y(-1, 0)(y - 0)$. In this case $f_x(x, y) = 3x^2 - 7y$ so $f_x(-1, 0) = 3$. Also $f_y(x, y) = -7x + e^y$ and so $f_y(-1, 0) = 8$. The equation of the plane is $z = 3(x + 1) + 8y$.

30. Again using Theorem 3.3, the equation for the tangent plane is: $z = f(\pi/3, 1) + f_x(\pi/3, 1)(x - \pi/3) + f_y(\pi/3, 1)(y - 1)$. Here $z = 4\cos xy$, so $f_x(x, y) = -4y \sin xy$ and $f_y(x, y) = -4x \sin xy$. Plugging in we get $z = 2 - 2\sqrt{3}(x - \pi/3) - (2\pi/\sqrt{3})(y - 1)$.

31. Again using Theorem 3.3, the equation for the tangent plane is: $z = f(0, 1) + f_x(0, 1)(x) + f_y(0, 1)(y - 1)$. Here $z = e^{x+y} \cos xy$, so $f_x(x, y) = e^{x+y}(\cos xy - y \sin xy)$ and $f_y(x, y) = e^{x+y}(\cos xy - x \sin xy)$. Plugging in we get $z = e - ex + e(y - 1)$ or $z = -ex + ey$.

32. First find the two partials $f_x(x,y) = 2x - 6$ and $f_y(x,y) = 3y^2$. Then putting the tangent plane equation into the same form as the plane $4x - 12y + z = 7$ gives us $z - (2a-6)(x-a) - (3b^2)(y-b) = a^2 - 6a + b^3$ or $z - (2a-6)x - 3b^2y = -a^2 - 2b^3$. So $2a - 6 = -4$ so $a = 1$ and $3b^2 = 12$ so $b = \pm 2$. This gives two tangent planes. The equation for one is $4x - 12y + z = -17$ and the equation for the other is $4x - 12y + z = 15$.

33. For $f(x_1, \ldots, x_4) = 10 - (x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2)$, we have

$$\nabla f = (-2x_1, -6x_2, -4x_3, -2x_4) \quad \text{so} \quad \nabla f(2,-1,1,3) = (-4, 12, -4, -6).$$

Formula (8) gives that the hyperplane has equation

$$\begin{aligned} x_5 &= -8 + (-4, 12, -4, -6)(x_1 - 2, x_2 + 1, x_3 - 1, x_4 - 3) \\ &= -8 - 4(x_1 - 2) + 12(x_2 + 1) - 4(x_3 - 1) - 6(x_4 - 3) \end{aligned}$$

or

$$x_5 = -4x_1 + 12x_2 - 4x_3 - 6x_4 + 34.$$

34. **(a)**

$$f_x(2,3) \approx \frac{f(1.98,3) - f(2,3)}{1.98 - 2} = \frac{12.1 - 12}{-.02} = \frac{.1}{-.02} = -5$$
$$f_y(2,3) \approx \frac{f(2,3.01) - f(2,3)}{3.01 - 3} = \frac{12.2 - 12}{0.01} = \frac{.2}{.01} = 20$$

Thus, formula (4) of §2.3 would give an approximate equation for the tangent plane as

$$z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) \approx 12 - 5(x-2) + 20(y-3)$$

or

$$z = -5x + 20y - 8.$$

(b)

$$\begin{aligned} f(1.98, 2.98) &\approx 12 - 5(1.98 - 2) + 20(2.98 - 3) = 12 - 5(-0.02) + 20(-0.02) \\ &= 11.7 \end{aligned}$$

Exercises 35–37 have the student investigate the linear approximation h of f near a given point a. We use the formula in Definition 3.8:

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

35. Here $f(x,y) = e^{x+y}$ so the partials are $f_x(x,y) = e^{x+y} = f_y(x,y)$.
(a) $h(.1,-.1) = f(0,0) + (e^0, e^0)\cdot(.1,-.1) = 1$.
(b) $f(.1,-.1) = e^0 = 1$. So the approximation is exact.

36. Here $f(x,y) = 3 + \cos \pi xy$ so the partials are $f_x(x,y) = -\pi y \sin \pi xy$ and $f_y(x,y) = -\pi x \sin \pi xy$.
(a) $h(.98,.51) = 3 + \cos \pi(1)(.5) - (\pi(.5)\sin[\pi(1)(.5)], \pi(1)\sin[\pi(1)(.5)])\cdot(-.02,.01) = 3 - \pi(.5,1)\cdot(-.02,.01) = 3$.
(b) $f(.98,.51) = 3 + \cos \pi(.98)(.51) \approx 3.00062832$.

37. $f(x,y,z) = x^2 + xyz + y^3z$, so the partials are $f_x(x,y,z) = 2x + yz$, $f_y(x,y,z) = xz + 3y^2z$, and $f_z(x,y,z) = xy + y^3$.
(a) $h(1.01, 1.95, 2.2) = f(1,2,2) + (f_x(1,2,2), f_y(1,2,2), f_z(1,2,2))\cdot(.01,-.05,.2) = 21 + (6,26,10)\cdot(.01,-.05,.2) = 21.76$.
(b) $f(1.01,1.95,2.2) = 21.665725$.

38.
$$f(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \text{ so}$$

$$f_{x_i}(x_1, x_2, \ldots, x_n) = \frac{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} - x_i(x_1 + x_2 + \cdots + x_n)(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}}{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$= \frac{x_1^2 + x_2^2 + \cdots + x_n^2 - x_i(x_1 + x_2 + \cdots + x_n)}{(x_1^2 + x_2^2 + \cdots + x_n^2)^{3/2}}$$

39. (a) For $(x, y) \neq (0, 0)$ we can find a neighborhood that misses the origin. In this neighborhood

$$f(x, y) = \frac{xy^2 - x^2y + 3x^3 - y^3}{x^2 + y^2} = x - y + \frac{2x^3}{x^2 + y^2}.$$

We can then easily compute the partials as

$$f_x(x, y) = 1 + \frac{2x^4 + 6x^2y^2}{(x^2 + y^2)^2} \quad \text{and} \quad f_y(x, y) = -1 - \frac{4x^3y}{(x^2 + y^2)^2}.$$

(b) Using Definition 3.2 of the partial derivative, if

$$f(x, y) = \begin{cases} x - y + \dfrac{2x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases},$$

then

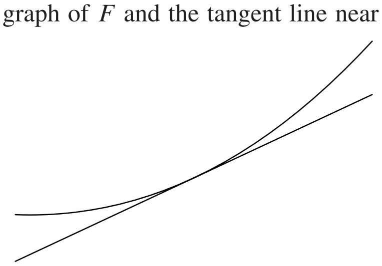
$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{3h}{h} = 3,$$

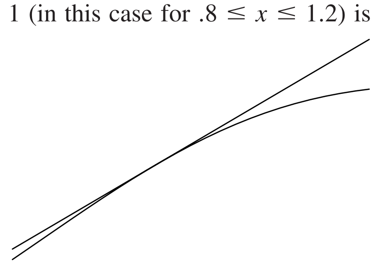
and

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

Note: Exercises 40–43 are review exercises for single-variable calculus. The idea is to see that near a point, the tangent line approximates the curve. This idea will then be extended to a tangent plane and a surface in Exercises 45–49. For Exercises 40–43 use either the point-slope equation $y - f(a) = f'(a)(x - a)$ or solve for y to get $y = f'(a)x + f(a) - f'(a)a$.

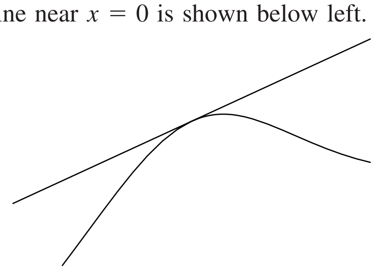
40. For the tangent line to $F(x) = x^3 - 2x + 3$ at $a = 1$ $F'(x) = 3x^2 - 2$ so $F'(1) = 1$. The tangent line is $y = x + 1$. The graph of F and the tangent line near $x = 1$ (in this case for $.8 \leq x \leq 1.2$) is shown below left.

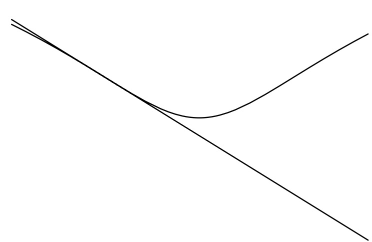




41. For the tangent line to $F(x) = x + \sin x$ at $a = \pi/4$ $F'(x) = 1 + \cos x$ so $F'(\pi/4) = 1 + \sqrt{2}/2$. The tangent line is $y = (1 + \sqrt{2}/2)x + (\pi/4 + \sqrt{2}/2 - (1 + \sqrt{2}/2)\pi/4)$. The graph of F and the tangent line near $x = \pi/4$ is shown above right.

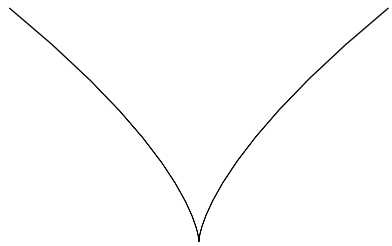
42. For the tangent line rewrite $F(x) = x - 3 + 3/(x^2 + 1)$. $F'(x) = 1 - 6x/(x^2 + 1)^2$ so $F'(0) = 1$ and $F(0) = 0$. The tangent line is $y = x$. We can see that by looking at our rewritten version of F. The graph of F and the tangent line near $x = 0$ is shown below left.





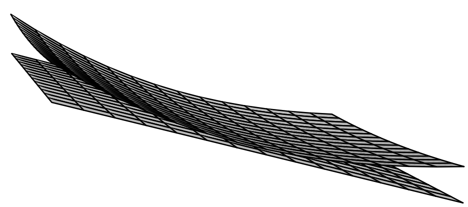
43. For the tangent line to $F(x) = \ln(x^2 + 1)$ at $a = -1 F'(x) = 2x/(x^2 + 1)$ so $F'(-1) = -1$. The tangent line is $y = -x + \ln 2 - 1$. The graph of F and the tangent line near $x = -1$ is shown above right.

44. Looking at the graph below, we can see that there is a cusp at $x = 2$ (trust me, that's where the cusp is). You can also see that the limit of the derivative using points to the left of 2 would not be the same as the derivative using points to the right of 2 as one set is negative and the other is positive. Finally, the tangent line looks to be a vertical line. This has no slope and so the derivative wouldn't exist.



45. **(a)** For the function $f(x, y) = x^3 - xy + y^2$, $f_x(x, y) = 3x^2 - y$ and $f_y(x, y) = -x + 2y$. So at the point (2, 1) these become $f(2, 1) = 7$, $f_x(2, 1) = 11$, and $f_y(2, 1) = 0$. The equation of the tangent plane is $z = 7 + 11(x - 2)$.

(b)



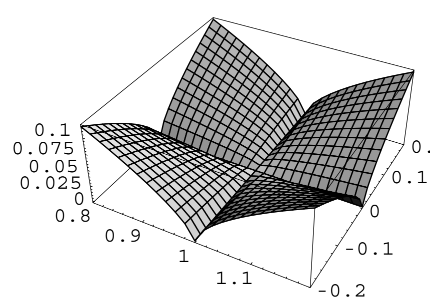
(c) The partials are continuous so by Theorem 3.5, f is differentiable.

46. **(a)** To find the partial derivatives $f_x(1, 0)$ and $f_y(1, 0)$, we must look at appropriate partial functions of $f(x, y) = ((x - 1)y)^{2/3}$:

$$f(x, 0) \equiv 0 \Rightarrow f_x(1, 0) = 0$$
$$f(1, y) \equiv 0 \Rightarrow f_y(1, 0) = 0$$

Since $f(1, 0) = 0$, the candidate tangent plane has equation $z = 0 + 0(x - 1) + 0(y - 0)$ or $z = 0$.

(b) A computer graph looks as follows.



Zooming in closer to the point (1, 0, 0) doesn't make things appear very different, so it's tempting to conclude that f must not be differentiable at (1, 0).

(c) From our calculations in part (a), the linear function $h(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = 0$. Thus, for $(x, y) \neq (1, 0)$ we have

$$0 \leq \frac{|f(x, y) - h(x, y)|}{\|(x, y) - (1, 0)\|} = \frac{|f(x, y)|}{\sqrt{(x - 1)^2 + y^2}}.$$

Now

$$|f(x,y)| = |x-1|^{2/3}|y|^{2/3} \le ((x-1)^2 + y^2)^{1/3}((x-1)^2 + y^2)^{1/3}$$
$$= ((x-1)^2 + y^2)^{2/3}.$$

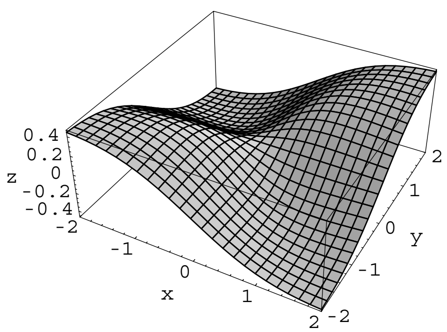
Thus

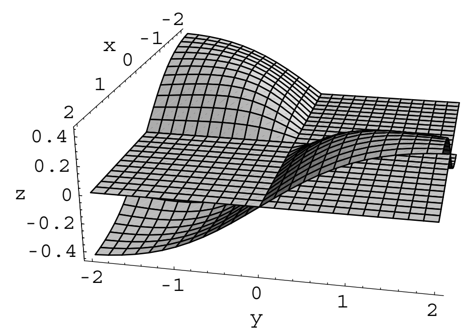
$$\frac{|f(x,y)|}{\sqrt{(x-1)^2 + y^2}} \le \frac{((x-1)^2 + y^2)^{2/3}}{((x-1)^2 + y^2)^{1/2}} = ((x-1)^2 + y^2)^{1/6}.$$

Since this last expression approaches zero as $(x, y) \to (1, 0)$, we see that f must be differentiable at (1, 0) by Definition 3.4.

47. (a) For the function $f(x,y) = \dfrac{xy}{x^2 + y^2 + 1}$, $f_x(x,y) = \dfrac{-x^2y + y^3 + y}{(x^2 + y^2 + 1)^2}$ and $f_y(x,y) = \dfrac{x^3 - xy^2 + y}{(x^2 + y^2 + 1)^2}$. So at the point (0, 0) these become $f(0,0) = 0$, $f_x(0,0) = 0$, and $f_y(0,0) = 0$. The equation of the tangent plane is $z = 0$.

(b) The surface is shown below left. It is shown with the tangent plane below right.

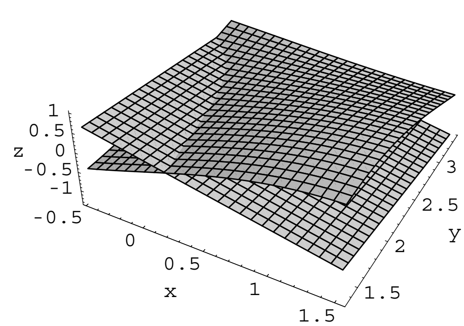




(c) This is the plane that best approximates the surface at that point. But we can see that it's not a very good approximation as you move way in any direction other than the two axes lines. Analytically, the reason is that the partials are continuous in a neighborhood of (0, 0).

48. (a) For the function $f(x,y) = \sin x \cos y$, $f_x(x,y) = \cos x \cos y$ and $f_y(x,y) = -\sin x \sin y$. So at the point $(\pi/6, 3\pi/4)$ these become $f(\pi/6, 3\pi/4) = -\sqrt{2}/4$, $f_x(\pi/6, 3\pi/4) = -\sqrt{6}/4$, and $f_y(\pi/6, 3\pi/4) = -\sqrt{2}/4$. The equation of the tangent plane is $z = -\sqrt{2}/4 - \sqrt{6}/4(x - \pi/6) - \sqrt{2}/4(y - 3\pi/4)$.

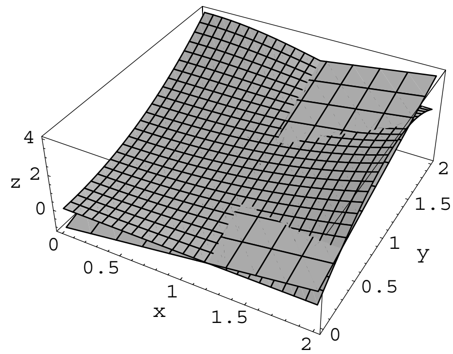
(b)



(c) Again the partials are continuous in a neighborhood of $(\pi/6, 3\pi/4)$ so by Theorem 3.5, f is differentiable at the point.

49. (a) For the function $f(x,y) = x^2 \sin y + y^2 \cos x$, $f_x(x,y) = 2x \sin y - y^2 \sin x$ and $f_y(x,y) = x^2 \cos y + 2y \cos x$. So at the point $(\pi/3, \pi/4)$ these become $f(\pi/3, \pi/4) = \pi^2\sqrt{2}/18 + \pi^2/32$, $f_x(\pi/3, \pi/4) = \pi\sqrt{2}/3 - \pi^2\sqrt{3}/32$, and $f_y(\pi/3, \pi/4) = \pi^2\sqrt{2}/18 + \pi/4$. The equation of the tangent plane is $z = (\pi^2\sqrt{2}/18 + \pi^2/32) + (\pi\sqrt{2}/3 - \pi^2\sqrt{3}/32)(x - \pi/3) + (\pi^2\sqrt{2}/18 + \pi/4)(y - \pi/4)$.

(b)



(c) The partials are continuous near $(\pi/3, \pi/4)$ so by Theorem 3.5, f is differentiable there.

50. **(a)** Yes $g(x, y) = (xy)^{1/3}$ is continuous at (0, 0).

(b) $\partial g/\partial x = (1/3)x^{-2/3}y^{1/3}$, and $\partial g/\partial y = (1/3)x^{1/3}y^{-2/3}$.

(c) Unfortunately we can't just substitute the point (0, 0) in our answers to (b), but using Definition 3.2 of partial derivatives, we see that the two partials must be 0. In other words we define $g_x(0, 0) = 0$, and $g_y(0, 0) = 0$.

(d) No (choose a path that crosses the x- and y-axes).

(e) You can see this answer if you look along the line $y = x$. There $g(x, x) = x^{2/3}$ which has a corner at (0, 0). So there can't be a tangent plane.

(f) No g isn't differentiable at (0, 0).

51. If $\mathbf{f}(\mathbf{x}) = A\mathbf{x} = \left(\sum_{k=1}^{n} a_{1k}x_k, \sum_{k=1}^{n} a_{2k}x_k, \ldots, \sum_{k=1}^{n} a_{mk}x_k\right)$. Let's look at the entry in row i column j of $D\mathbf{f}(\mathbf{x})$. This will be

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} a_{ik}x_k\right) = a_{ij}.$$

So $D\mathbf{f}(\mathbf{x}) = A$.

2.4 PROPERTIES; HIGHER-ORDER PARTIAL DERIVATIVES; NEWTON'S METHOD

In Exercises 1–4 there isn't much to show … the students just need to verify that the sum of the derivative is the derivative of the sum (Proposition 4.1).

1. $f(x, y) = xy + \cos x$, and $g(x, y) = \sin(xy) + y^3$, so $Df = [y - \sin x, x]$, $Dg = [y\cos xy, x\cos xy + 3y^2]$, and $D(f + g) = [y - \sin x + y\cos xy, x + x\cos xy + 3y^2]$.

2. $\mathbf{f}(x, y) = (e^{x+y}, xe^y)$, and $\mathbf{g}(x, y) = (\ln(xy), ye^x)$, so

$$D\mathbf{f} = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^y & xe^y \end{bmatrix}, \quad D\mathbf{g} = \begin{bmatrix} \frac{y}{xy} & \frac{x}{xy} \\ ye^x & e^x \end{bmatrix}$$

and

$$D(\mathbf{f} + \mathbf{g}) = \begin{bmatrix} e^{x+y} + \frac{y}{xy} & e^{x+y} + \frac{x}{xy} \\ e^y + ye^x & xe^y + e^x \end{bmatrix}.$$

Note the use of the product rule in Exercise 3 when calculating $(g_1)_x$.

3. $\mathbf{f}(x, y, z) = (x\sin y + z, ye^x - 3x^2)$ and $\mathbf{g}(x, y, z) = (x^3\cos x, xyz)$, so

$$D\mathbf{f} = \begin{bmatrix} \sin y & x\cos y & 1 \\ -6x & e^z & ye^z \end{bmatrix}, \quad D\mathbf{g} = \begin{bmatrix} 3x^2\cos x - x^3\sin x & 0 & 0 \\ yz & xz & xy \end{bmatrix} \quad \text{and}$$

$$D(\mathbf{f} + \mathbf{g}) = \begin{bmatrix} \sin y + 3x^2\cos x - x^3\sin x & x\cos y & 1 \\ -6x + yz & e^z + xz & ye^z + xy \end{bmatrix}.$$

4. $\mathbf{f}(x, y, z) = (xyz^2, xe^{-y}, y \sin xz)$ and $\mathbf{g}(x, y, z) = (x - y, x^2 + y^2 + z^2, \ln(xz + 2))$, so

$$D\mathbf{f} = \begin{bmatrix} yz^2 & xz^2 & 2xyz \\ e^{-y} & -xe^{-y} & 0 \\ zy\cos xz & \sin xz & xy\cos xz \end{bmatrix}, D\mathbf{g} = \begin{bmatrix} 1 & -1 & 0 \\ 2x & 2y & 2z \\ z/(xz+2) & 0 & x/(xz+2) \end{bmatrix} \text{ and}$$

$$D(\mathbf{f}+\mathbf{g}) = \begin{bmatrix} 1 + yz^2 & -1 + xz^2 & 2xyz \\ e^{-y} + 2x & -xe^{-y} + 2y & 2z \\ zy\cos xz + z/(xz+2) & \sin xz & xy\cos xz + x/(xz+2) \end{bmatrix}.$$

Exercises 5–8 are again mainly calculations to convince the students of the formulas given in Proposition 4.2; we hope that they remember to apply them when confronted with a product or quotient. In Exercises 6 and 7 we notice that we just get the quotient rule in each component which factors into the quotient rule given in the proposition (and we drop the argument when convenient and clear).

5. $f(x, y) = x^2y + y^3$, $g(x, y) = x/y$, $f(x, y)g(x, y) = x^3 + xy^2$, and $\dfrac{f(x, y)}{g(x, y)} = xy^2 + y^4/x$.

$$\begin{aligned} \text{So} \quad Df &= [2xy, x^2 + 3y^2], \quad \text{and} \quad Dg = [1/y, -x/y^2], \\ D(fg) &= [3x^2 + y^2, 2xy] \\ &= (x^2y + y^3)[1/y, -x/y^2] + (x/y)[2xy, x^2 + 3y^2] \\ &= fD(g) + gD(f), \text{ and} \\ D\left(\frac{f}{g}\right) &= [y^2 - y^4/x^2, 2xy + 4y^3/x] \\ &= (y/x)[2xy, x^2 + 3y^2] - (y^2/x^2)(x^2y + y^3)[1/y, -x/y^2] \\ &= \frac{gDf - fDg}{g^2}. \end{aligned}$$

6. $f(x, y) = e^{xy}$, $g(x, y) = x \sin 2y$, $f(x, y)g(x, y) = xe^{xy}\sin 2y$, and $\dfrac{f(x, y)}{g(x, y)} = \dfrac{e^{xy}}{x \sin 2y}$.

$$\begin{aligned} \text{So} \quad Df &= [ye^{xy}, xe^{xy}], \quad \text{and} \quad Dg = [\sin 2y, 2x\cos 2y], \\ D(fg) &= [\sin 2y(e^{xy} + xy\,e^{xy}), x(xe^{xy}\sin 2y + 2e^{xy}\cos 2y)] \\ &= e^{xy}[\sin 2y, 2x\cos 2y] + x\sin 2y[ye^{xy}, xe^{xy}] \\ &= fD(g) + gD(f), \text{ and} \\ D\left(\frac{f}{g}\right) &= \left[\frac{xye^{xy}\sin 2y - e^{xy}\sin 2y}{x^2\sin^2 2y}, \frac{x^2e^{xy}\sin 2y - 2xe^{xy}\cos 2y}{x^2\sin^2 2y}\right] \\ &= \frac{x\sin 2y[ye^{xy}, xe^{xy}] - e^{xy}[\sin 2y, 2x\cos 2y]}{x^2\sin^2 2y} \\ &= \frac{gDf - fDg}{g^2}. \end{aligned}$$

7. $f(x, y) = 3xy + y^5$, $g(x, y) = x^3 - 2xy^2$, $f(x, y)g(x, y) = 3x^4y + x^3y^5 - 6x^2y^3 - 2xy^7$, and $\dfrac{f(x, y)}{g(x, y)} = \dfrac{3xy + y^5}{x^3 - 2xy^2}$. So

$$\begin{aligned} Df &= [3y, 3x + 5y^4], \quad \text{and} \quad Dg = [3x^2 - 2y^2, -4xy], \\ D(fg) &= [12x^3y + 3x^2y^5 - 12xy^3 - 2y^7, 3x^4 + 5x^3y^4 - 18x^2y^2 - 14xy^6] \end{aligned}$$

$$= (3xy + y^5)[3x^2 - 2y^2, -4xy] + (x^3 - 2xy^2)[3y, 3x + 5y^4]$$
$$= fD(g) + gD(f), \text{ and}$$
$$D\left(\frac{f}{g}\right) = \left[\frac{g(x,y)f_x(x,y) - f(x,y)g_x(x,y)}{[g(x,y)]^2}, \frac{g(x,y)f_y(x,y) - f(x,y)g_y(x,y)}{[g(x,y)]^2}\right]$$
$$= \frac{gDf - fDg}{g^2}.$$

8. $f(x, y, z) = x\cos(yz)$, $g(x, y, z) = x^2 + x^9y^2 + y^2z^3 + 2$, $f(x, y)g(x, y) = x^3\cos(yz) + x^{10}y^2\cos(yz) + xy^2z^3\cos(yz) + 2x\cos(yz)$, and $\dfrac{f(x,y)}{g(x,y)} = \dfrac{x\cos(yz)}{x^2 + x^9y^2 + y^2z^3 + 2}$.

So $Df = [\cos(yz), -xz\sin(yz), -xy\sin(yz)]$, and $Dg = [2x + 9x^8y^2, 2x^9y + 2yz^3, 3y^2z^2]$,

$$D(fg) = \begin{bmatrix} 3x^2\cos yz + 10x^9y^2\cos yz + y^2z^3\cos yz + 2\cos yz \\ -x^3z\sin yz + 2x^{10}y\cos yz - x^{10}y^2z\sin yz + 2xyz^3\cos yz - xy^2z^4\sin yz - 2xz\sin yz \\ -x^3y\sin yz - x^{10}y^3\sin yz + 3xy^2z^2\cos xy - xy^3z^3\sin yz - 2xy\sin yz \end{bmatrix}^T$$
$$= (x\cos yz)\begin{bmatrix} 2x + 9x^8y^2 \\ 2x^9y + 2yz^3 \\ 3y^2z^2 \end{bmatrix}^T + (x^2 + x^9y^2 + y^2z^3 + 2)\begin{bmatrix} \cos yz \\ -xz\sin yz \\ -xy\sin yz \end{bmatrix}^T$$
$$= fDg + gDf, \text{ and}$$
$$D\left(\frac{f}{g}\right) = \left[\frac{gf_x - fg_x}{g^2}, \frac{gf_y - fg_y}{g^2}, \frac{gf_z - fg_z}{g^2}\right]$$
$$= \frac{gDf - fDg}{g^2}.$$

In Exercises 9–17, students should verify that $f_{xy} = f_{yx}$. The fact that in these problems the derivative with respect to y of f_x is equal to the derivative with respect to x of f_y is not trivial. Problem 18 explicitly asks them to examine the mixed partials.

9. $f(x, y) = x^3y^7 + 3xy^2 - 7xy$ so $f_x(x, y) = 3x^2y^7 + 3y^2 - 7y$ and $f_y(x, y) = 7x^3y^6 + 6xy - 7x$. The second order partials are:

$$f_{xx}(x, y) = 6xy^7,$$
$$f_{xy}(x, y) = f_{yx}(x, y) = 21x^2y^6 + 6y - 7, \text{ and}$$
$$f_{yy}(x, y) = 42x^3y^5 + 6x.$$

10. $f(x, y) = \cos(xy)$ so $f_x(x, y) = -y\sin(xy)$ and $f_y(x, y) = -x\sin(xy)$. The second order partials are:

$$f_{xx}(x, y) = -y^2\cos xy,$$
$$f_{xy}(x, y) = f_{yx}(x, y) = -xy\cos xy - \sin xy, \text{ and}$$
$$f_{yy}(x, y) = -x^2\cos xy.$$

11. $f(x, y) = e^{y/x} - ye^{-x}$ so $f_x(x, y) = \dfrac{-y}{x^2}e^{y/x} + ye^{-x}$ and $f_y(x, y) = \dfrac{1}{x}e^{y/x} - e^{-x}$. The second order partials are:

$$f_{xx}(x, y) = \frac{2y}{x^3}e^{y/x} + \frac{y^2}{x^4}e^{y/x} - ye^{-x},$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{-1}{x^2}e^{y/x} - \frac{y}{x^3}e^{y/x} + e^{-x}, \text{ and}$$
$$f_{yy}(x,y) = \frac{1}{x^2}e^{y/x}.$$

12. $f(x,y) = \sin\sqrt{x^2+y^2}$ so

$$f_x(x,y) = \frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad\text{and}\quad f_y(x,y) = \frac{y\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}.$$

The second order partials are:

$$f_{xx}(x,y) = \frac{\sqrt{x^2+y^2}\left[\cos\sqrt{x^2+y^2} + (x)\frac{-x\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right] - (x\cos\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$
$$= \frac{y^2\cos\sqrt{x^2+y^2} - x^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \text{ and by symmetry}$$
$$f_{yy}(x,y) = \frac{x^2\cos\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \text{ and}$$
$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{-xy\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2} - xy\cos\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}.$$

13. $f(x,y) = \dfrac{1}{\sin^2 x + 2e^y}$ so

$$f_x(x,y) = \frac{-2\sin x\cos x}{(\sin^2 x + 2e^y)^2} = \frac{-\sin 2x}{(\sin^2 x + 2e^y)^2} \quad\text{and}\quad f_y(x,y) = \frac{-2e^y}{(\sin^2 x + 2e^y)^2}.$$

The second order partials are:

$$f_{xx}(x,y) = \frac{(\sin^2 x + 2e^y)^2(-2\cos 2x) + \sin 2x \cdot 2(\sin^2 x + 2e^y)\sin 2x}{(\sin^2 x + 2e^y)^4}$$
$$= \frac{(\sin^2 x + 2e^y)(-2\cos 2x) + 2\sin^2 2x}{(\sin^2 x + 2e^y)^3},$$
$$f_{xy}(x,y) = f_{yx}(x,y) = \frac{4e^y\sin 2x}{(\sin^2 x + 2e^y)^3}, \text{ and}$$
$$f_{yy}(x,y) = \frac{2e^y(2e^y - \sin^2 x)}{(\sin^2 x + 2e^y)^3}.$$

14. $f(x,y) = e^{x^2+y^2}$ so $f_x(x,y) = 2xe^{x^2+y^2}$ and $f_y(x,y) = 2ye^{x^2+y^2}$. The second order partials are:

$$f_{xx}(x,y) = 2e^{x^2+y^2} + 2x \cdot 2xe^{x^2+y^2}$$
$$= e^{x^2+y^2}(2+4x^2),$$
$$f_{xy}(x,y) = f_{yx}(x,y) = 4xye^{x^2+y^2}, \text{ and}$$
$$f_{yy}(x,y) = e^{x^2+y^2}(2+4y^2).$$

15. $f(x,y,z) = x^2yz + xy^2z + xyz^2$ so $f_x(x,y,z) = 2xyz + y^2z + yz^2$, $f_y(x,y,z) = x^2z + 2xyz + xz^2$, and $f_z(x,y,z) = x^2y + xy^2 + 2xyz$. The second order partials are:

$$f_{xx}(x,y,z) = 2yz$$

$$f_{yy}(x, y, z) = 2xz$$
$$f_{zz}(x, y, z) = 2xy$$
$$f_{xy}(x, y, z) = f_{yx}(x, y, z) = 2xz + 2yz + z^2$$
$$f_{xz}(x, y, z) = f_{zx}(x, y, z) = 2xy + y^2 + 2yz$$
$$f_{yz}(x, y, z) = f_{zy}(x, y, z) = x^2 + 2xy + 2xz$$

16. $f(x, y, z) = e^{xyz}$ so $f_x(x, y, z) = yze^{xyz}$, $f_y(x, y, z) = xze^{xyz}$, and $f_z(x, y, z) = xye^{xyz}$. The second order partials are:

$$f_{xx}(x, y, z) = y^2z^2e^{xyz}$$
$$f_{yy}(x, y, z) = x^2z^2e^{xyz}$$
$$f_{zz}(x, y, z) = x^2y^2e^{xyz}$$
$$f_{xy}(x, y, z) = f_{yx}(x, y, z) = ze^{xyz}(1 + xyz)$$
$$f_{xz}(x, y, z) = f_{zx}(x, y, z) = ye^{xyz}(1 + xyz)$$
$$f_{yz}(x, y, z) = f_{zy}(x, y, z) = xe^{xyz}(1 + xyz)$$

17. $f(x, y, z) = e^{ax}\sin y + e^{bx}\cos z$ so $f_x(x, y, z) = ae^{ax}\sin y + be^{bx}\cos z$, $f_y(x, y, z) = e^{ax}\cos y$, and $f_z(x, y, z) = -e^{bx}\sin z$. The second order partials are:

$$f_{xx}(x, y, z) = a^2e^{ax}\sin y + b^2e^{bx}\cos z$$
$$f_{yy}(x, y, z) = -e^{ax}\sin y$$
$$f_{zz}(x, y, z) = -e^{bx}\cos z$$
$$f_{xy}(x, y, z) = f_{yx}(x, y, z) = ae^{ax}\cos y$$
$$f_{xz}(x, y, z) = f_{zx}(x, y, z) = -be^{bx}\sin z$$
$$f_{yz}(x, y, z) = f_{zy}(x, y, z) = 0$$

18. $F(x, y, z) = 2x^3y + xz^2 + y^3z^5 - 7xyz$ so $F_x(x, y, z) = 6x^2y + z^2 - 7yz$, $F_y(x, y, z) = 2x^3 + 3y^2z^5 - 7xz$, and $F_z(x, y, z) = 2xz + 5y^3z^4 - 7xy$.

(a) $F_{xx}(x, y, z) = 12xy$, $F_{yy}(x, y, z) = 6yz^5$, and $F_{zz}(x, y, z) = 20y^3z^3 + 2x$.

(b) $F_{xy}(x, y, z) = 6x^2 - 7z = F_{yx}(x, y, z)$, $F_{xz}(x, y, z) = 2z - 7y = F_{zx}(x, y, z)$, and $F_{yz}(x, y, z) = 15y^2z^4 - 7x = F_{zy}(x, y, z)$.

(c) $F_{xyx}(x, y, z) = 12x = F_{xxy}(x, y, z)$. We knew that these would be equal because they are the mixed partials of F_x (i.e., $(F_x)_{yx} = (F_x)_{xy}$).

(d) $F_{xyz}(x, y, z) = -7 = F_{yzx}(x, y, z)$.

19. We will denote the degree of f by $deg(f)$ in this solution.

(a) $deg(p_x) = 16$, $deg(p_y) = 16$, $deg(p_{xx}) = 15$, $deg(p_{yy}) = 15$, and $deg(p_{yx}) = 15$.

(b) $deg(p_x) = 3$, $deg(p_y) = 3$, $deg(p_{xx}) = 2$, $deg(p_{yy})$ is undefined, and $deg(p_{yx}) = 2$.

(c) This is difficult because the term of highest degree can switch during the process of taking a derivative. For example consider $f(x, y) = xy^2 + x^3y$. Take the derivative with respect to y and the degree has decreased by one as we would expect: $f_y(x, y) = 2xy + x^3$ so $deg(f_y) = 3$. Now take another derivative with respect to y: $f_{yy}(x, y) = 2x$ and so the degree is now one.

For a polynomial $f(x_1, x_2, \ldots, x_n)$ which has degree $d = d_1 + d_2 + \cdots + d_n$ because of a term $cx_1^{d_1}x_x^{d_2}\ldots x_n^{d_n}$, $\partial^k f/\partial x_{i_1}\ldots\partial x_{i_k}$ has degree $d-k$ if x_j occurs at most d_j times in the partial derivative—otherwise we must look for the highest degree of any other surviving terms. If no terms survive, (i.e., $\partial^k f/\partial x_{i_1}\ldots\partial x_{i_k} = 0$) then the degree is undefined.

Exercises 20 and 21 have the students verify that certain functions are solutions to the given differential equations. When the students studied exponential equations in first semester calculus they may have seen that $f(x) = ce^{kx}$ solves the differential equation $y' = ky$. Here is a nice way to introduce the idea of a partial differential equation.

20. (a) For the first function, $f(x, y, z) = x^2 + y^2 - 2z^2$, $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$, and $f_x(x, y, z) = -4z$.

This means that $f_{xx}(x, y, z) = 2$, $f_{yy}(x, y, z) = 2$, and $f_{zz}(x, y, z) = -4$. We see that $f_{xx} + f_{yy} + f_{zz} = 0$ and conclude that f is harmonic.

For the second function, $f(x, y, z) = x^2 - y^2 + z^2$, $f_x(x, y, z) = 2x$, $f_y(x, y, z) = -2y$, and $f_z(x, y, z) = 2z$.

This means that $f_{xx}(x, y, z) = 2$, $f_{yy}(x, y, z) = -2$, and $f_{zz}(x, y, z) = 2$. We see that $f_{xx} + f_{yy} + f_{zz} \neq 0$ and conclude that f is not harmonic.

(b) One possible example is $f(x_1, x_2, \ldots, x_n) = x_1^2 - x_2^2 + 3x_3 + 4x_4 + 5x_5 + \cdots + nx_n$.

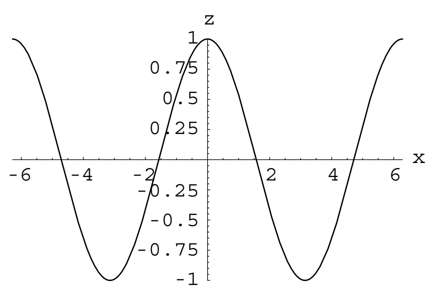
Here $f_{x_i x_i} = \begin{cases} 2 & \text{if } i = 1, \\ -2 & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$ and we see that $\sum_{i=1}^{n} f_{x_i x_i} = 0$ so f is harmonic.

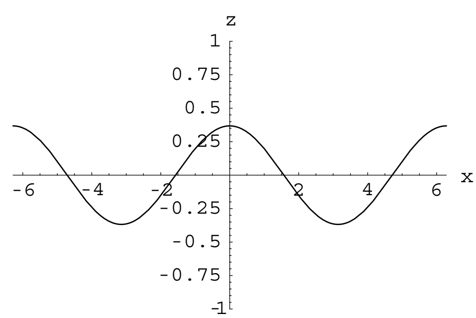
21. (a) To show that $T(x, t) = e^{-kt} \cos x$ satisfies the differential equation $kT_{xx} = T_t$ we calculate the derivatives:

$$T_x(x, t) = -e^{-kt} \sin x$$
$$T_{xx}(x, t) = -e^{-kt} \cos x$$
$$T_t(x, t) = -ke^{-kt} \cos x$$

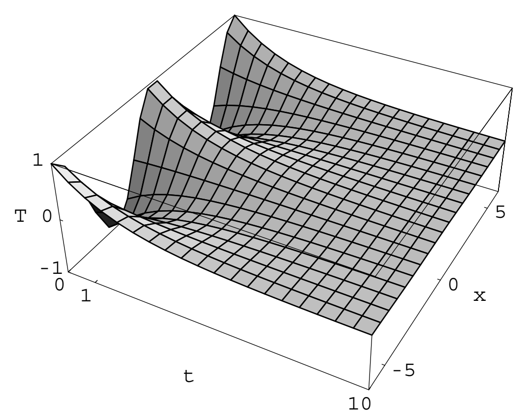
so $kT_{xx} = T_t$.

For $t_0 = 0$ and $t_0 = 1$ the graphs are:





For $t_0 = 10$ the graph is further damped. The graph of the surface $z = T(x, t)$ is:



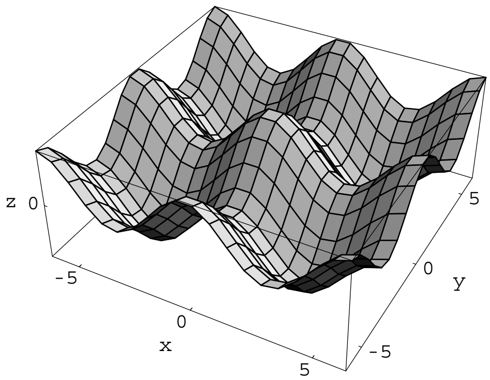
(b) To show that $T(x, y, t) = e^{-kt}(\cos x + \cos y)$ satisfies the differential equation $k(T_{xx} + T_{yy}) = T_t$ we calculate the derivatives:

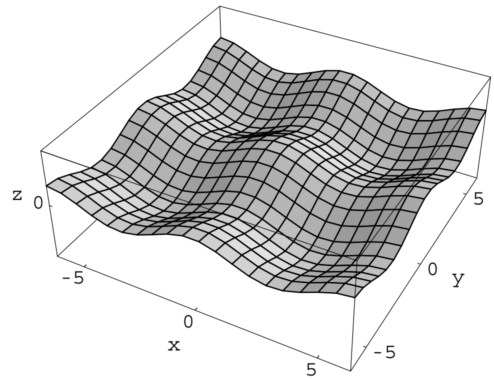
$$T_x(x, y, t) = -e^{-kt} \sin x$$
$$T_{xx}(x, y, t) = -e^{-kt} \cos x$$
$$T_y(x, y, t) = -e^{-kt} \sin y$$

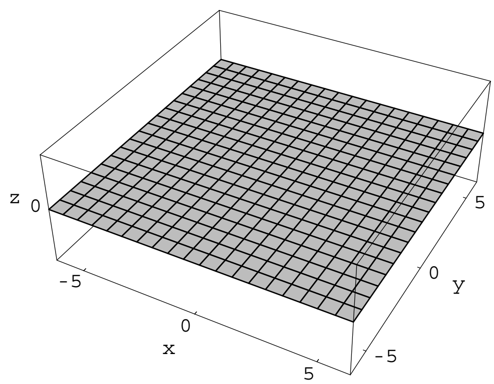
$$T_{yy}(x, y, t) = -e^{-kt}\cos y$$
$$T_t(x, y, t) = -ke^{-kt}(\cos x + \cos y)$$

so $k(T_{xx} + T_{yy}) = T_t$.

The graphs of the surfaces given by $z = T(x, y, t_0)$ for $t_0 = 0, 1,$ and 10 are:







(c) Finally, to show that $T(x, y, z, t) = e^{-kt}(\cos x + \cos y + \cos z)$ satisfies the differential equation $k(T_{xx} + T_{yy} + T_{xx}) = T_t$ we calculate the derivatives:

$$T_x(x, y, z, t) = -e^{-kt}\sin x$$
$$T_{xx}(x, y, z, t) = -e^{-kt}\cos x$$
$$T_y(x, y, z, t) = -e^{-kt}\sin y$$
$$T_{yy}(x, y, z, t) = -e^{-kt}\cos y$$
$$T_z(x, y, z, t) = -e^{-kt}\sin z$$
$$T_{zz}(x, y, z, t) = -e^{-kt}\cos z$$
$$T_t(x, y, z, t) = -ke^{-kt}(\cos x + \cos y + \cos z)$$

so $k(T_{xx} + T_{yy} + T_{zz}) = T_t$.

22. (a) For $(x, y) \neq (0, 0)$, compute the partial derivatives:

$$f_x(x, y) = y\left(\frac{x^2 - y^2}{x^2 + y^2}\right) + xy\left(\frac{[x^2 + y^2](2x) - [x^2 - y^2](2x)}{(x^2 + y^2)^2}\right)$$

$$= \frac{y(x^2 - y^2)(x^2 + y^2) + xy(4xy^2)}{(x^2 + y^2)^2}$$

$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2} \text{ and similarly}$$

$$f_y(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

(b) We use part (a):

$$f_x(0, y) = \frac{y(-y^4)}{(y^2)^2}$$

$$= -y \text{ for } y \neq 0, \text{ and}$$

$$f_y(x, 0) = x \text{ for } x \neq 0.$$

(c) From part (b), $f_{xy}(0, y) = -1$ while $f_{yx}(x, 0) = 1$ and $f_x(0, y)$ and $f_y(x, 0)$ are continuous at the origin so you can conclude that $f_{xy}(0, 0) = -1$ while $f_{yx}(0, 0) = 1$. Why aren't the mixed partials equal? The answer is that the second partials are not continuous at the origin. We can see this by calculating

$$f_{xy}(x, y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}.$$

$$\text{Therefore} \quad f_{xy}(x, 0) = 1 \text{ and}$$

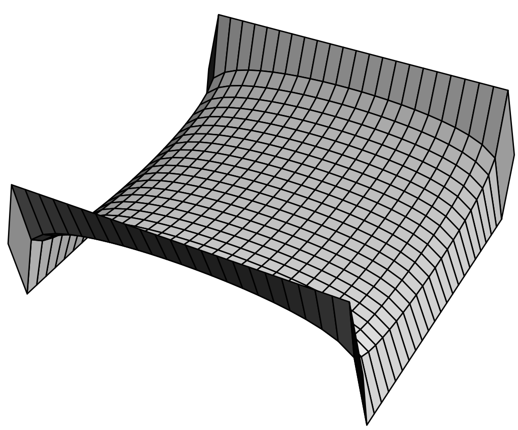
$$f_{xy}(0, y) = -1.$$

$$\text{Hence} \quad \lim_{(x,y)\to(0,0)} f_{xy}(x, y) \quad \text{does not exist.}$$

In other words, f_{xy} is not continuous at the origin.

23. An equation of a plane in the form $z = f(x, y)$ is $z = Ax + By + C$. Here $z_x = A$, $z_y = B$ and the second derivatives are all 0. The partial differential equation for minimal surfaces is therefore trivially satisfied and a plane is seen to be a minimal surface.

24. (a) Here's an image of Scherk's surface.



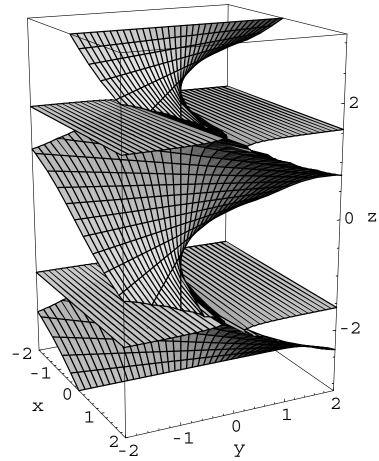
(b) In this case $z = \ln(\cos x / \cos y)$. So $z_x = -\tan x$, $z_y = \tan y$, $z_{xy} = 0$, $z_{xx} = -\sec^2 x$, and $z_{yy} = \sec^2 y$. So

$$(1 + z_y^2)z_{xx} + (1 + z_x^2)z_{yy} = (1 + \tan^2 y)(-\sec^2 x) + (1 + \tan^2 x)(\sec^2 y)$$

$$= -\sec^2 x \sec^2 y + \sec^2 x \sec^2 y = 0.$$

This agrees with the right side of the equation as $z_{xy} = 0$.

25. **(a)** Here's an image of the helicoid:



(b) There's no reason not to think of this surface as $z = x \tan y$. Then $z_x = \tan y$, $z_y = x \sec^2 y$, $z_{xx} = 0$, $z_{xy} = \sec^2 y$, and $z_{yy} = 2 \tan y \sec^2 y$. So

$$\begin{aligned}(1 + z_y^2)z_{xx} + (1 + z_x^2)z_{yy} &= (1 + x^2 \sec^4 y)(0) + (1 + \tan^2 y)(2 \tan y \sec^2 y) \\ &= (\sec^2 y)(2 \tan y \sec^2 y) = 2(\tan y)(x \sec^2 y)(\sec^2 y) \\ &= 2z_x z_y z_{xy}\end{aligned}$$

26. Let $\mathbf{L}$ denote $\lim_{k\to\infty} \mathbf{x}_k$. Then $\lim_{k\to\infty} \mathbf{x}_{k-1} = \mathbf{L}$ and taking limits in (6), we have

$$\mathbf{L} = \mathbf{L} - [D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L}).$$

Hence $[D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L}) = \mathbf{0}$. Now multiply by $D\mathbf{f}(\mathbf{L})$ on the left to obtain $D\mathbf{f}(\mathbf{L})([D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L})) = D\mathbf{f}(\mathbf{L})\mathbf{0} = \mathbf{0} \Leftrightarrow I_n\mathbf{f}(\mathbf{L}) = \mathbf{0} \Leftrightarrow \mathbf{f}(\mathbf{L}) = \mathbf{0}$.

27. **(a)**

k	x_k	y_k
0	-1	1
1	-1.3	1.7
2	-1.2653846	1.55588235
3	-1.2649112	1.54920772
4	-1.2649111	1.54919334
5	-1.2649111	1.54919334

This table suggests that $\mathbf{x}_k \to (-1.2649111, 1.54919334) \approx (-\sqrt{8/5}, \sqrt{12/5})$.

(b)

k	x_k	y_k
0	1	-1
1	1.3	-1.7
2	1.26538462	-1.558824
3	1.2649115	-1.5492077
4	1.26491106	-1.5491933
5	1.26491106	-1.5491933

Here $\mathbf{x}_k \to (\sqrt{8/5}, -\sqrt{12/5})$ it seems.

k	x_k	y_k
0	-1	-1
1	-1.3	-1.7
2	-1.2653846	-1.555824
3	-1.2649112	-1.5492077
4	-1.2649111	-1.5491933
5	-1.2649111	-1.5491933

Here $\mathbf{x}_k \to (-\sqrt{8/5}, -\sqrt{12/5})$.

(c) The results don't seem too strange; each initial vector is in a different quadrant and the limit is an intersection point in the same quadrant.

28. (a)

k	x_k	y_k
0	1.4	10
1	54.7	−317.452
2	28.0832917	−75583.381
3	14.8412307	−9364.2812
4	8.35050251	−1128.3294
5	5.34861164	−119.96986
6	4.2264792	−4.8602841
7	4.00886454	4.73583325
8	4.0001468	4.99959722
9	4	5
10	4	5

(b)

k	x_k	y_k
0	1.3	10
1	−105.35	641.7815
2	−52.041661	606283.635
3	−25.420779	75622.9747
4	−12.17662	9372.7823
5	−5.6848239	1132.95037
6	−2.6823677	124.108919
7	−1.5599306	8.94078154
8	−1.3422068	−0.6614827
9	−1.333348	−0.9255223
10	−1.3333333	−0.9259259
11	−1.3333333	−0.9259259

(c) (1.3, 10) is a good deal closer to (4, 5) than it is to (−1.3333333, −0.9259259).

(d) It seems surprising that, beginning with $\mathbf{x}_0 = (1.3, 10)$, we found the limit we did, especially when $\mathbf{x}_0 = (1.4, 10)$ causes things to converge to (4, 5). This suggests that, when there are multiple solutions, it can be difficult to know to which solution the initial vector will converge.

29. Formula (6) says $\mathbf{x}_{k+1} = \mathbf{x}_k - [D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{f}(\mathbf{x}_k)$. But if $\mathbf{x}_k$ solves (2) exactly, then $\mathbf{f}(\mathbf{x}_k) = \mathbf{0}$. Thus $\mathbf{x}_{k+1} = \mathbf{x}_k - [D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{0} = \mathbf{x}_k$. By the same argument $\mathbf{x}_k = \mathbf{x}_{k+2} = \mathbf{x}_{k+3} = \cdots$

30. $D\mathbf{f}(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$. By Exercise 36 of §1.6, $[D\mathbf{f}(x, y)]^{-1} = \frac{1}{f_x g_y - f_y g_x}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}$. If we evaluate at (x_{k-1}, y_{k-1}) and calculate, we find that formula (6) tells us that

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \underbrace{\frac{1}{f_x g_y - f_y g_x}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix}}_{\text{all evaluated at } (x_{k-1}, y_{k-1})}.$$

Expanding and taking entries we obtain the desired formulas.

31. $DF(x, y) = [4y\cos(xy) + 3x^2, 4x\cos(xy) + 3y^2]$, so we want to solve $\begin{cases} 4y\cos xy + 3x^2 = 0 \\ 4x\cos xy + 3y^2 = 0 \end{cases}$. Using the result of Exercise 30, we have

$$x_k = \frac{6y_{k-1}^2\cos(x_{k-1}y_{k-1}) + x_{k-1}(6(x_{k-1}^3 + 3y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - x_{k-1}y_{k-1}(9 + 8\sin 2x_{k-1}y_{k-1}))}{2(2 - 9x_{k-1}y_{k-1} + 2\cos(2x_{k-1}y_{k-1}) + 6(x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - 4x_{k-1}y_{k-1}\sin(2x_{k-1}y_{k-1}))}$$

$$y_k = \frac{6x_{k-1}^2\cos(x_{k-1}y_{k-1}) + y_{k-1}(6(3x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - x_{k-1}y_{k-1}(9 + 8\sin(2x_{k-1}y_{k-1})))}{2(2 - 9x_{k-1}y_{k-1} + 2\cos(2x_{k-1}y_{k-1}) + 6(x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - 4x_{k-1}y_{k-1}\sin(2x_{k-1}y_{k-1}))}$$

(This was obtained using *Mathematica* to simplify.)

Using initial vector $(x_0, y_0) = (-1, -1)$ and iterating the formulas above we find

k	x_k	y_k	
0	-1	-1	
1	-0.9206484	-0.9206484	
2	-0.9073724	-0.9073724	
3	-0.9070156	-0.9070156	
4	-0.9070154	-0.9070154	
5	-0.9070154	-0.9070154	$\leftarrow$ Here's the approximate root.

32. (a) Here we're trying to solve the system $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 1 \\ 4y^2 + z^2 = 4. \end{cases}$ Hence we define $\mathbf{f}(x, y, z) = (x^2 + y^2 + z^2 - 4, x^2 + y^2 - 1, 4y^2 + z^2 - 4)$.

Thus $D\mathbf{f}(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 0 & 8y & 2z \end{bmatrix}$. It follows (see Exercise 37 of §1.6) that

$$[D\mathbf{f}(x, y, z)]^{-1} = \begin{bmatrix} \frac{1}{8x} & \frac{3}{8x} & -\frac{1}{8x} \\ -\frac{1}{8y} & \frac{1}{8y} & \frac{1}{8y} \\ \frac{1}{2z} & -\frac{1}{2z} & 0 \end{bmatrix}.$$

Thus $\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ z_{k-1} \end{bmatrix} - [D\mathbf{f}(x_{k-1}, y_{k-1}, z_{k-1})]^{-1} \begin{bmatrix} x_{k-1}^2 + y_{k-1}^2 + z_{k-1}^2 - 4 \\ x_{k-1}^2 + y_{k-1}^2 - 1 \\ 4y_{k-1}^2 + z_{k-1}^2 - 4 \end{bmatrix}$.

This simplifies to give

$$x_k = \frac{x_{k-1}}{2} + \frac{3}{8x_{k-1}}$$
$$y_k = \frac{y_{k-1}}{2} + \frac{1}{8y_{k-1}}$$
$$z_k = \frac{z_{k-1}}{2} + \frac{3}{2z_{k-1}}$$

Newton's method with $\mathbf{x}_0 = (1, 1, 1)$ gives the following set of results

k	x_k	y_k	z_k
0	1	1	1
1	0.875	0.625	2
2	0.86607143	0.5125	1.75
3	0.86602541	0.50015244	1.73214286
4	0.8660254	0.50000002	1.73205081
5	0.8660254	0.5	1.73205081
6	0.8660254	0.5	1.73205081

With $\mathbf{x}_0 = (1, -1, 1)$, we find

k	x_k	y_k	z_k
0	1	-1	1
1	0.875	-0.625	2
2	0.86607143	-0.5125	1.75
3	0.86602541	-0.5001524	1.73214286
4	0.8660254	-0.5	1.73205081
5	0.8660254	-0.5	1.73205081

(b) We solve $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 1 \\ 4y^2 + z^2 = 4 \end{cases}$ by hand. First insert the second equation into the first: $1 + z^2 = 4 \Leftrightarrow z = \pm\sqrt{3}$. Use this in the third equation $4y^2 + 3 = 4 \Leftrightarrow y = \pm\frac{1}{2}$.

Now use this in the second equation: $x^2 + \frac{1}{4} = 1 \Leftrightarrow x = \pm\frac{\sqrt{3}}{2}$.

So we have 8 solutions:

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right)$$

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}\right)$$

We found two of them above.

2.5 THE CHAIN RULE

In Exercises 1–3 students see that if you have a composite function you can take the derivative either by substituting or by using the chain rule.

1. $f(x, y, z) = x^2 - y^3 + xyz$, $x = 6t + 7$, $y = \sin 2t$, and $z = t^2$.

Substitution:

$$\begin{aligned} f(x(t), y(t), z(t)) &= (6t + 7)^2 - (\sin 2t)^3 + (6t + 7)(\sin 2t)(t^2) \\ &= (6t + 7)^2 - (\sin 2t)^3 + (6t^3 + 7t^2)(\sin 2t) \quad \text{and so} \\ \frac{df}{dt} &= 2(6t + 7)6 - 3(\sin 2t)^2(2\cos 2t) + (18t^2 + 14t)\sin 2t + (6t^3 + 7t^2)(2\cos 2t) \end{aligned}$$

Chain Rule:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (2x + yz)(6) + (-3y^2 + xz)(2\cos 2t) + (xy)(2t) \\ &= [2(6t + 7) + (\sin 2t)(t^2)](6) + [-3\sin^2 2t + (6t + 7)t^2](2\cos 2t) + [(6t + 7)\sin 2t](2t) \end{aligned}$$

2. $f(x, y) = \sin(xy)$, $x = s + t$, and $y = s^2 + t^2$.

(a) $f(x(t), y(t)) = \sin(x(t)y(t)) = \sin[(s + t)(s^2 + t^2)]$.

$$\begin{aligned} \frac{\partial f}{\partial s} &= \cos[(s + t)(s^2 + t^2)][(s^2 + t^2) + (s + t)(2s)] \\ \frac{\partial f}{\partial t} &= \cos[(s + t)(s^2 + t^2)][(s^2 + t^2) + (s + t)(2t)] \end{aligned}$$

(b)

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\cos(xy) + x\cos(xy)2s \\ &= \cos[(s + t)(s^2 + t^2)][(s^2 + t^2) + (s + t)(2s)] \text{ and} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{aligned}$$

$$= y\cos(xy) + x\cos(xy)2t$$
$$= \cos[(s+t)(s^2+t^2)][(s^2+t^2)+(s+t)(2t)]$$

3. (a) We want

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}\\ &= \frac{12xz}{y}(-2\sin t) - \frac{6x^2z}{y^2}(2\cos t) + \frac{6x^2}{y}(3)\\ &= \frac{12(2\cos t)(3t)}{2\sin t}(-2\sin t) - \frac{6(4\cos^2 t)3t}{4\sin^2 t}(2\cos t) + \frac{6(4\cos^2 t)}{2\sin t}(3)\\ &= -72t\cos t - 36t\frac{\cos^3 t}{\sin^2 t} + \frac{36\cos^2 t}{\sin t}.\end{aligned}$$

Therefore,

$$\left.\frac{dP}{dt}\right|_{t=\pi/4} = \frac{(36-27\pi)}{\sqrt{2}}.$$

(b) $P(x(t), y(t), z(t)) = \frac{6(2\cos t)^2(3t)}{2\sin t} = \frac{36t\cos^2 t}{\sin t}$, so

$$\frac{dP}{dt} = \frac{\sin t(36\cos^2 t - 36t \cdot 2\cos t\sin t) - 36t\cos^2 t(\cos t)}{\sin^2 t}.$$

Therefore,

$$\left.\frac{dP}{dt}\right|_{t=\pi/4} = \frac{(36-27\pi)}{\sqrt{2}}.$$

(c) Using differentials,

$$\Delta P \approx \left(\left.\frac{dP}{dt}\right|_{t=\pi/4}\right)(dt) = \left(\frac{36-27\pi}{\sqrt{2}}\right)(.01) \approx -.34523.$$

So (writing P as a function of t),

$$P(\pi/4 + .01) \approx P(\pi/4) + \Delta P \approx \frac{9\pi}{\sqrt{2}} - .34523 \approx 19.6477.$$

4. We are thinking of $z = z(s,t) = [x(s,t)]^2 + [y(s,t)]^3$. So

$$\frac{\partial z}{\partial t}(2,1) = \left.\frac{\partial z}{\partial x}\right|_{(2,1)} \cdot \left.\frac{\partial x}{\partial t}\right|_{(2,1)} + \left.\frac{\partial z}{\partial y}\right|_{(2,1)} \cdot \left.\frac{\partial y}{\partial t}\right|_{(2,1)} = 2x|_{(2,1)} \cdot s|_{(2,1)} + 0 = 8.$$

5. Here $V = LWH$, so

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial L}\frac{dL}{dt} + \frac{\partial V}{\partial W}\frac{dW}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt}\\ &= WH\left(\frac{dL}{dt}\right) + LH\left(\frac{dW}{dt}\right) + LW\left(\frac{dH}{dt}\right)\\ &= 5 \cdot 4(.75) + 7 \cdot 4(.5) + 7 \cdot 5(-1)\\ &= -6\ in^3/min.\end{aligned}$$

Since $\frac{dV}{dt} < 0$, the volume of the dough is decreasing at this instant.

6. Let the length of the butter be y and the length of an edge of the cross section be x. Then the volume $V = x^2 y$. The rate at which the volume is changing is

$$\frac{dV}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 2(1.5)(6)(-.125) + (1.5)^2(-.25) = -2.8125\ in^3/min.$$

7. Note that in 6 months:

$$x = 1 + .6 - \cos\pi = 2.6$$
$$y = 200 + 12\sin\pi = 200$$

The chain rule gives

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=6} &= \left.\frac{\partial P}{\partial x}\right|_{\substack{x=2.6\\y=200}} \left.\frac{dx}{dt}\right|_{t=6} + \left.\frac{\partial P}{\partial y}\right|_{\substack{x=2.6\\y=200}} \left.\frac{dy}{dt}\right|_{t=6}\\ &= 10(0.1x + 10)^{-\frac{1}{2}}(0.1)|_{x=2.6}\left.\left(0.1 - \frac{\pi}{6}\sin\frac{\pi t}{6}\right)\right|_{t=6}\\ &\quad - 4y^{-\frac{2}{3}}|_{y=200}\left.\left(2\sin\frac{\pi t}{6} + \frac{2\pi t}{6}\cos\frac{\pi t}{6}\right)\right|_{t=6}\\ &= (10.26)^{-\frac{1}{2}}(0.1) - 4(200^{-\frac{2}{3}})(-2\pi)\\ &= 0.031219527 + 0.734885812 = 0.766105339 \text{ units/month (demand in rising slightly).}\end{aligned}$$

8. (a) The chain rule gives

$$\begin{aligned}\frac{d}{dt}(\text{BMI}) &= \frac{\partial(\text{BMI})}{\partial w}\frac{dw}{dt} + \frac{\partial(\text{BMI})}{\partial h}\frac{dh}{dt}\\ &= \frac{10{,}000}{h^2}\frac{dw}{dt} - \frac{20{,}000\ w}{h^3}\frac{dh}{dt}\end{aligned}$$

On the child's 10th birthday: $w = 33$ kg, $h = 140$ cm,

$$\frac{dw}{dt} = 0.4, \quad \frac{dh}{dt} = 0.6.$$

So

$$\begin{aligned}\frac{d(\text{BMI})}{dt} &= \frac{10{,}000}{140^2}(0.4) - \frac{20{,}000 \cdot 33}{140^3}(0.6)\\ &\approx 0.0598 \text{ points/month.}\end{aligned}$$

(b) The rate we found in part (a) is greater than the typical rate by about 49%. I'd monitor the situation monthly so that it doesn't persist for too long, but I wouldn't be very concerned, since the current BMI is roughly 16.84, which is quite low.

9. Since $x = e^r\cos\theta$ and $y = e^r\sin\theta$ we can write

$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right) = \left(\frac{\partial z}{\partial x}\right)(e^r\cos\theta) + \left(\frac{\partial z}{\partial y}\right)(e^r\sin\theta).$$

Similarly,

$$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial \theta}\right) = \left(\frac{\partial z}{\partial x}\right)(-e^r\sin\theta) + \left(\frac{\partial z}{\partial y}\right)(e^r\cos\theta).$$

Therefore,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = e^{2r}\left[(\cos^2\theta + \sin^2\theta)\left(\frac{\partial z}{\partial x}\right)^2\right.$$

$$+(\cos^2\theta + \sin^2\theta)\left(\frac{\partial z}{\partial y}\right)^2 + (2\cos\theta\sin\theta - 2\cos\theta\sin\theta)\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\Bigg]$$

$$= e^{2r}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right].$$

The result follows.

Exercises 10–13 are fun exercises. You may want to stress that we are showing that the partial differential equations are true without even knowing the "outside" function.

10. We'll start by calculating the components on the left side:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ &= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)\\ &= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{and}\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\\ &= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1)\\ &= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad \text{so}\\ \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} &= \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\\ &= \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2.\end{aligned}$$

11. First calculate:

$$\frac{\partial u}{\partial x} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

Now

$$\begin{aligned}x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} &= x\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{y(y^2 - x^2)}{(x^2 + y^2)^2} + y\frac{x(x^2 - y^2)}{(x^2 + y^2)^2}\right)\\ &= 0.\end{aligned}$$

12. First calculate:

$$\frac{\partial u}{\partial x} = \frac{4xy^2}{(x^2 + y^2)^2} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = \frac{-4x^2y}{(x^2 + y^2)^2}$$

Now

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = x\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}$$
$$= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$
$$= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{4xy^2}{(x^2 + y^2)^2} + y\frac{-4x^2y}{(x^2 + y^2)^2}\right)$$
$$= 0.$$

13. $\frac{\partial u}{\partial x} = \frac{-1}{x^2}$, $\frac{\partial u}{\partial y} = \frac{1}{y^2}$, and $\frac{\partial u}{\partial z} = 0$. Also $\frac{\partial v}{\partial x} = \frac{-1}{x^2}$, $\frac{\partial v}{\partial y} = 0$, and $\frac{\partial v}{\partial z} = \frac{1}{z^2}$. Now it is just a matter of using the chain rule and plugging in:

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = x^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right] + y^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right] + z^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z}\right]$$
$$= \frac{\partial w}{\partial u}\left[x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}\right] + \frac{\partial w}{\partial v}\left[x^2\frac{\partial v}{\partial x} + y^2\frac{\partial v}{\partial y} + z^2\frac{\partial v}{\partial z}\right]$$
$$= \frac{\partial w}{\partial u}\left[x^2\left(\frac{-1}{x^2}\right) + y^2\left(\frac{1}{y^2}\right) + 0\right] + \frac{\partial w}{\partial v}\left[x^2\left(\frac{-1}{x^2}\right) + 0 + z^2\left(\frac{1}{z^2}\right)\right]$$
$$= 0.$$

14. $\frac{\partial u}{\partial x} = \frac{1}{y}$, $\frac{\partial u}{\partial y} = \frac{-x}{y^2}$, and $\frac{\partial u}{\partial z} = 0$. Also $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = \frac{-z}{y^2}$, and $\frac{\partial v}{\partial z} = \frac{1}{y}$. Again, it is just a matter of using the chain rule and plugging in:

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = x\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right] + y\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right] + z\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z}\right]$$
$$= \frac{\partial w}{\partial u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right] + \frac{\partial w}{\partial v}\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}\right]$$
$$= \frac{\partial w}{\partial u}\left[x\left(\frac{1}{y}\right) + y\left(\frac{-x}{y^2}\right) + 0\right] + \frac{\partial w}{\partial v}\left[0 + y\left(\frac{-z}{y^2}\right) + z\left(\frac{1}{y}\right)\right]$$
$$= 0.$$

15. (a) $\mathbf{f} \circ g = (3(s - 7t)^5, e^{2s-14t})$ so

$$D(\mathbf{f} \circ g) = \begin{bmatrix} 15(s - 7t)^4 & -105(s - 7t)^4 \\ 2e^{2s-14t} & -14e^{2x-14t} \end{bmatrix}$$

(b)

$$D\mathbf{f} = \begin{bmatrix} 15x^4 \\ 2e^{2x} \end{bmatrix} = \begin{bmatrix} 15(s - 7t)^4 \\ 2e^{2s-14t} \end{bmatrix} \quad \text{and} \quad Dg = \begin{bmatrix} 1 & -7 \end{bmatrix}$$

We can easily see that $D\mathbf{f}\, Dg = D(\mathbf{f} \circ g)$.

16. (a) $f \circ \mathbf{g} = (st)^2 - 3(s+t^2)^2 = s^2t^2 - 3s^2 - 6st^2 - 3t^4$, so

$$D(f \circ \mathbf{g}) = [2st^2 - 6s - 6t^2 \quad 2s^2t - 12st - 12t^3].$$

(b) $Df = \begin{bmatrix} 2x & -6y \end{bmatrix} = \begin{bmatrix} 2st & -6s - 6t^2 \end{bmatrix}$, and $D\mathbf{g} = \begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix}$, so

$$Df\, D\mathbf{g} = \begin{bmatrix} 2st & -6s - 6t^2 \end{bmatrix}\begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix} = [2st^2 - 6s - 6t^2 \quad 2s^2t - 12st - 12t^3].$$

17. (a) $\mathbf{f} \circ \mathbf{g} = \left((s/t)s^2t - \frac{s^2t}{s/t}, \frac{s/t}{s^2t} + s^6t^3\right) = \left(s^3 - st^2, \frac{1}{st^2} + s^6t^3\right)$, so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 3s^2 - t^2 & -2st \\ -1/(s^2t^2) + 6s^5t^3 & -2/(st^3) + 3s^6t^2 \end{bmatrix}$$

(b)

$$D\mathbf{f} = \begin{bmatrix} y + \frac{y}{x^2} & x - \frac{1}{x} \\ \frac{1}{y} & \frac{-x}{y^2} + 3y^2 \end{bmatrix} = \begin{bmatrix} s^2t + \frac{s^2t}{s^2/t^2} & \frac{s}{t} - \frac{t}{s} \\ \frac{1}{s^2t} & \frac{-s/t}{s^4t^2} + 3s^4t^2 \end{bmatrix} = \begin{bmatrix} s^2t + t^3 & \frac{s^2-t^2}{st} \\ \frac{1}{s^2t} & -\frac{1}{s^3t^3} + 3s^4t^2 \end{bmatrix}$$

and $D\mathbf{g} = \begin{bmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ 2st & s^2 \end{bmatrix}$ so

$$D\mathbf{f}\, D\mathbf{g} = \begin{bmatrix} s^2t + t^3 & \frac{s^2-t^2}{st} \\ \frac{1}{s^2t} & -\frac{1}{s^3t^3} + 3s^4t^2 \end{bmatrix}\begin{bmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ 2st & s^2 \end{bmatrix} = \begin{bmatrix} 3s^2 - t^2 & -2st \\ \frac{-1}{s^2t^2} + 6s^5t^3 & \frac{-2}{st^3} + 3s^6t^2 \end{bmatrix}.$$

18. (a) $\mathbf{f} \circ \mathbf{g} = ((t-2)^2(3t+7) + (3t+7)^2t^3, (t-2)(3t+7)t^3, e^{t^3})$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 45t^4 + 168t^3 + 156t^2 - 10t - 16 \\ 15t^4 + 4t^3 - 42t^2 \\ 3t^2e^{t^3} \end{bmatrix}.$$

(b)

$$D(\mathbf{f}) = \begin{bmatrix} 2xy & x^2 + 2yz & y^2 \\ yz & xz & xy \\ 0 & 0 & e^z \end{bmatrix}$$

$$= \begin{bmatrix} 2(t-2)(3t+7) & (t-2)^2 + 2(3t+7)t^3 & (3t+7)^2 \\ (3t+7)t^3 & (t-2)t^3 & (t-2)(3t+7) \\ 0 & 0 & e^{t^3} \end{bmatrix}$$

and $D(\mathbf{g}) = \begin{bmatrix} 1 \\ 3 \\ 3t^2 \end{bmatrix}$ so $D(\mathbf{f})D(\mathbf{g}) = \begin{bmatrix} 45t^4 + 168t^3 + 156t^2 - 10t - 16 \\ 15t^4 + 4t^3 - 42t^2 \\ 3t^2e^{t^3} \end{bmatrix}.$

19. (a) $\mathbf{f} \circ \mathbf{g} = (st + tu + su, s^3t^3 - e^{stu^2})$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} t + u & s + u & s + t \\ 3s^2t^3 - tu^2e^{stu^2} & 3s^3t^2 - su^2e^{stu^2} & -2stue^{stu^2} \end{bmatrix}.$$

(b)

$$D\mathbf{f} = \begin{bmatrix} 1 & 1 & 1 \\ 3x^2 & -ze^{yz} & -ye^{yz} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3s^2t^2 & -sue^{stu^2} & -tue^{stu^2} \end{bmatrix}$$

$$\text{and} \quad D\mathbf{g} = \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \quad \text{so} \quad D\mathbf{f}D\mathbf{g} = \begin{bmatrix} t+u & s+u & s+t \\ 3s^2t^3 - tu^2e^{stu^2} & 3s^3t^2 - su^2e^{stu^2} & -2stue^{stu^2} \end{bmatrix}.$$

20. This is a matter of seeing what we have to determine and which formula to use. We calculate $D(\mathbf{f} \circ \mathbf{g})(1, -1, 3)$ as $D\mathbf{f}(\mathbf{g}(1, -1, 3))D(\mathbf{g}(1, -1, 3))$. The second piece is given in the exercise. For the first we calculate

$$D\mathbf{f}(\mathbf{g}(1, -1, 3)) = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}\Bigg|_{\mathbf{g}(1,-1,3)} = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}\Bigg|_{(2,5)} = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}.$$

Then we can multiply the matrices to get the result

$$D(\mathbf{f} \circ \mathbf{g})(1, -1, 3) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}.$$

21. (a) This is similar to Exercise 20.

$$\begin{aligned} D(\mathbf{f} \circ \mathbf{g})(1,2) &= D\mathbf{f}(\mathbf{g}(1,2))D\mathbf{g}(1,2) = D\mathbf{f}(3,5)D\mathbf{g}(1,2) \\ &= \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 31 & 44 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} D(\mathbf{g} \circ \mathbf{f})(4,1) &= D\mathbf{g}(\mathbf{f}(4,1))D\mathbf{f}(4,1) = D\mathbf{g}(1,2)D\mathbf{f}(4,1) \\ &= \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ 2 & 31 \end{bmatrix} \end{aligned}$$

22. We'll start with the right hand side of the equation because we can easily calculate the partials of x and y with respect to r and θ.

$$\begin{aligned} \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2\left[\left(\frac{\partial x}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial x}{\partial \theta}\right)^2\right] + \left(\frac{\partial z}{\partial y}\right)^2\left[\left(\frac{\partial y}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial y}{\partial \theta}\right)^2\right] \\ &\quad + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\left[\frac{\partial y}{\partial r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta}\right] \\ &= \left(\frac{\partial z}{\partial x}\right)^2\left[\cos^2\theta + \frac{1}{r^2}(r^2\sin^2\theta)\right] + \left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta + \frac{1}{r^2}(r^2\cos^2\theta)\right] \\ &\quad + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\left[\sin\theta\cos\theta + \frac{1}{r^2}(-r\sin\theta)(r\cos\theta)\right] = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \end{aligned}$$

23. (a) From formula (10) in Section 2.5, we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \quad \text{and} \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}$$

Hence if $z = f(x, y)$, then

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right)$$
$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right)$$

Now use the product rule:

$$\frac{\partial^2 z}{\partial x^2} = \cos\theta\left(\cos\theta\frac{\partial^2 z}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial z}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta}\right)$$
$$- \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial z}{\partial r} + \cos\theta\frac{\partial^2 z}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial\theta^2}\right)$$
$$= \cos^2\theta\frac{\partial^2 z}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial z}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial\theta^2}.$$

Follow the same steps to calculate

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)$$
$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta}\right)$$
$$= \sin^2\theta\frac{\partial^2 z}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial\theta\partial r} + \frac{\cos^2\theta}{r}\frac{\partial z}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 z}{\partial\theta^2}.$$

(b) Adding the two equations above we easily see that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

24. Given Exercise 23, this is easy: We know $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{r^2}\frac{\partial^2}{\partial\theta^2}$. Since the z-coordinate means the same thing in both Cartesian and cylindrical coordinates, the result follows.

25. (a) The chain rule gives $\frac{\partial w}{\partial\rho} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial\rho} + \frac{\partial w}{\partial\theta}\frac{\partial\theta}{\partial\rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial\rho}$ for any appropriately differentiable function w. Now (6) of §1.7 gives $z = \rho\cos\varphi$, $r = \rho\sin\varphi$. Hence

$$\frac{\partial w}{\partial\rho} = \sin\varphi\frac{\partial w}{\partial r} + 0 + \cos\varphi\frac{\partial w}{\partial z} = \sin\varphi\frac{\partial w}{\partial r} + \cos\varphi\frac{\partial w}{\partial z}.$$

Also

$$\frac{\partial w}{\partial\varphi} = \rho\cos\varphi\frac{\partial w}{\partial r} - \rho\sin\varphi\frac{\partial w}{\partial z} \quad \text{from a similar chain rule computation.}$$

From this, we have

$$\rho\sin\varphi\frac{\partial w}{\partial\rho} + \cos\varphi\frac{\partial w}{\partial\varphi} = \left(\rho\sin^2\varphi\frac{\partial w}{\partial r} + \rho\sin\varphi\cos\varphi\frac{\partial w}{\partial z}\right) + \left(\rho\cos^2\varphi\frac{\partial w}{\partial r} - \rho\cos\varphi\sin\varphi\frac{\partial w}{\partial z}\right)$$
$$= \rho\frac{\partial w}{\partial r}.$$

Thus

$$\frac{\partial w}{\partial r} = \sin\varphi\frac{\partial w}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial w}{\partial\varphi} \quad \text{or} \quad \frac{\partial}{\partial r} = \sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}.$$

(Alternatively, consider formula (10) in this section with $x = z$, $y = r$, θ replaced by φ, and r replaced by ρ.)

(b) The cylindrical Laplacian is $\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$. From $z = \rho\cos\varphi$, $r = \rho\sin\varphi$, we may treat z and r as if they are Cartesian coordinates, so that

$$\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} \quad (\text{Cartesian}/\textit{cylindrical})$$

Now we know $\frac{\partial}{\partial r}$ from part (a). So, with $r = \rho\sin\varphi$, we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)$$
$$+ \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho\sin\varphi}\left(\sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)$$
$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\varphi}{\rho^2}\frac{\partial}{\partial \varphi} \quad \text{as desired.}$$

Exercises 26–28 puts the implicit differentiation techniques which the students learned in a previous course in the context of the current discussion. This is one of those problems where it would be immediately clear if we were able to talk to each other. The problem is explaining to you which derivative with respect to x is being considered. One solution is to introduce another variable. You might want to use this as an example of why the author introduces the notation she does for Exercises 31–35. One other note is that the results hold also for $F(x, y)$ or $F(x, y, z)$ being constant (not necessarily 0).

26. (a) View x and y as functions of t, where $x = x(t) = t$ and $y = y(t)$. Since $F(x, y) = 0$ we know that $F_t(x, y) = 0$. This means that we know:

$$0 = \frac{dF}{dt} = F_x(x, y)\frac{dx}{dt} + F_y(x, y)\frac{dy}{dt}.$$

But $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$ so $\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$.

(b-i) If $F(x, y) = x^3 - y^2$ then $F_x(x, y) = 3x^2$ and $F_y(x, y) = -2y$ so $\frac{dy}{dx} = -\frac{3x^2}{-2y} = \frac{3x^2}{2y}$.

(b-ii) $y^2 = x^3$ so $y = x^{3/2}$ so $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$. Multiply numerator and denominator by $x^{3/2}$ to get the answer in (b-i).

27. Here we'll just use the formula from Exercise 26(a) where here $F(x, y) = \sin(xy) - x^2y^7 + e^y$.

$$\frac{dy}{dx} = -\frac{y\cos(xy) - 2xy^7}{x\cos(xy) - 7x^2y^6 + e^y}.$$

The results of Exercise 28 are used in Exercises 33 and 35 in a nice way. None of them is very time consuming—it is worth assigning all three.

28. (a) We have the same problem here with ambiguity about what is meant by the derivative with respect to x and y. Let $x = x(s, t) = s$, $y = y(s, t) = t$, and $z = z(s, t)$. Then

$$0 = \frac{\partial F}{\partial s} = F_x(x, y, z)\frac{\partial x}{\partial s} + F_y(x, y, z)\frac{\partial y}{\partial s} + F_z(x, y, z)\frac{\partial z}{\partial s} = F_x(x, y, z) + F_z(x, y, z)\frac{\partial z}{\partial x}.$$

Solving we get

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}.$$

An analogous calculation gives

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}.$$

(b-i) $F(x, y, z) = xyz - 2$ so by part (a):

$$\frac{\partial z}{\partial x} = -\frac{yz}{xy} = -\frac{z}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{xz}{xy} = -\frac{z}{y}.$$

(b-ii) $z = 2/xy$ so

$$\frac{\partial z}{\partial x} = \frac{-2}{x^2 y} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-2}{xy^2}.$$

29. Use the equations from Exercise 28(a) for $F(x, y, z) = x^3 z + y\cos z + (\sin y)/z = 0$:

$$\frac{\partial z}{\partial x} = \frac{-3x^2 z}{x^3 - y\sin z - (\sin y)/z^2} = \frac{-3x^2 z^3}{x^3 z^2 - yz^2 \sin z - \sin y} \quad \text{and}$$
$$\frac{\partial z}{\partial y} = \frac{-\cos z - (\cos y)/z}{x^3 - y\sin z - (\sin y)/z^2} = \frac{-z^2\cos z - z\cos y}{x^3 z^2 - yz^2\sin z - \sin y}.$$

Exercise 30 is a good example of why you can not just blindly apply formulas such as the chain rule without first checking that all of the hypotheses are met.

30. (a) By definition

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0, \quad \text{and}$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

(b)

$$f \circ \mathbf{x} = \begin{cases} \dfrac{at}{1 + a^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

therefore $f \circ \mathbf{x} = \dfrac{at}{1 + a^2}$ and so $D(f \circ \mathbf{x})(0) = \dfrac{a}{1 + a^2}$.

(c) By definition, $D(f)(0,0) = [f_x(0,0), f_y(0,0)]$. We calculated these in part (a) to be 0 so

$$Df(0,0)D\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = 0.$$

The function f is not differentiable at the origin and so not all of the assumptions of the chain rule are met.

31. (a) $\left(\frac{\partial w}{\partial x}\right)_{y,z} = 1$, $\left(\frac{\partial w}{\partial y}\right)_{x,z} = 7$, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -10$, $\left(\frac{\partial w}{\partial x}\right)_{y} = 1 - 10(2x) = 1 - 20x$, and $\left(\frac{\partial w}{\partial y}\right)_{x} = 7 - 10(2y) = 7 - 20y$.

(b) $\left(\frac{\partial w}{\partial x}\right)_{y} = \left(\frac{\partial w}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\right)_{x,z}\left(\frac{\partial y}{\partial x}\right) + \left(\frac{\partial w}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)$. But $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$ so $\left(\frac{\partial w}{\partial x}\right)_{y} = \left(\frac{\partial w}{\partial x}\right)_{y,z} + \left(\frac{\partial w}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)$.

32. $\left(\frac{\partial w}{\partial x}\right)_{y,z} = 3x^2$, $\left(\frac{\partial w}{\partial y}\right)_{x,z} = 3y^2$, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = 3z^2$, $\left(\frac{\partial w}{\partial x}\right)_{y} = 3x^2 + 3z^2(2) = 3x^2 + 6(2x - 3y)^2$, and $\left(\frac{\partial w}{\partial y}\right)_{x} = 3y^2 + 3z^2(-3) = 3y^2 - 9(2x - 3y)^2$.

33. $\left(\frac{\partial s}{\partial z}\right)_{x,y,w} = xw - 2z$, so $\left(\frac{\partial s}{\partial z}\right)_{x,w} = \left(\frac{\partial s}{\partial z}\right)_{x,y,w} + \left(\frac{\partial s}{\partial y}\right)_{x,z,w}\left(\frac{\partial y}{\partial z}\right)_{x,w}$.

To calculate $\left(\dfrac{\partial y}{\partial z}\right)_{x,w}$ we can use the results of Exercise 28 with $F(x, y, z, w) = xyw - y^3 z + xz$:

$$\left(\frac{\partial y}{\partial z}\right)_{x,w} = -\frac{F_z(x, y, z, w)}{F_y(x, y, z, w)} = -\frac{-y^3 + x}{xw - 3y^2 z}.$$

So $\left(\dfrac{\partial s}{\partial z}\right)_{x,w} = xw - 2z + (x^2)\left(\dfrac{y^3 - x}{xw - 3y^2}\right).$

34. $U = F(P, V, T)$ and $PV = kT$.

(a) $\left(\dfrac{\partial U}{\partial T}\right)_P = \left(\dfrac{\partial U}{\partial T}\right)_{P,V} + \left(\dfrac{\partial U}{\partial V}\right)_{P,T}\left(\dfrac{\partial V}{\partial T}\right) = F_T(P, V, T) + F_V(P, V, T)\left(\dfrac{k}{P}\right).$

(b) $\left(\dfrac{\partial U}{\partial T}\right)_V = \left(\dfrac{\partial U}{\partial T}\right)_{P,V} + \left(\dfrac{\partial U}{\partial P}\right)_{V,T}\left(\dfrac{\partial P}{\partial T}\right) = F_T(P, V, T) + F_P(P, V, T)\left(\dfrac{k}{V}\right).$

(c) $\left(\dfrac{\partial U}{\partial P}\right)_V = \left(\dfrac{\partial U}{\partial P}\right)_{V,T} + \left(\dfrac{\partial U}{\partial T}\right)_{P,V}\left(\dfrac{\partial T}{\partial P}\right) = F_P(P, V, T) + F_T(P, V, T)\left(\dfrac{V}{k}\right).$

35. $\left(\dfrac{\partial x}{\partial y}\right)_z\left(\dfrac{\partial y}{\partial z}\right)_x\left(\dfrac{\partial z}{\partial x}\right)_y = \left(-\dfrac{F_y(x, y, z)}{F_x(x, y, z)}\right)\left(-\dfrac{F_z(x, y, z)}{F_y(x, y, z)}\right)\left(-\dfrac{F_x(x, y, z)}{F_z(x, y, z)}\right) = -1.$

36. In this case $P = kT/V$ so $(\partial P/\partial T)_V = k/V$. Similarly, $V = kT/P$ so $(\partial V/\partial P)_T = -kT/P^2$ and $T = PV/k$ so $(\partial T/\partial V)_P = P/k$. So

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{k}{V}\right)\left(\frac{-kt}{P^2}\right)\left(\frac{P}{k}\right) = \frac{-kTP}{VP^2} = \frac{-kT}{VP} = -1.$$

The last equality holds since $PV = kT$.

37. It is easiest to use implicit differentiation and solve. For example, for the equation $ax^2 + by^2 + cz^2 - d = 0$, hold z constant and take the derivative with respect to y. You get $2ax(\partial x/\partial y)_z + 2by = 0$. Solve this and get $(\partial x/\partial y)_z = -by/ax$. Similarly we get that $(\partial y/\partial z)_x = -cz/by$ and $(\partial z/\partial x)_y = -ax/cz$. So

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{-by}{ax}\right)\left(\frac{-cz}{by}\right)\left(\frac{-ax}{cz}\right) = -1.$$

2.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT

1. (a) $\nabla f(x, y, z) \cdot (-\mathbf{k})$ is the directional derivative of $f(x, y, z)$ in the direction $-\mathbf{k}$ (i.e., the negative z direction).

(b) $\nabla f(x, y, z) \cdot (-\mathbf{k}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (0, 0, -1) = -\frac{\partial f}{\partial z}.$

In Exercises 2–8, the students should notice that the given vector $\mathbf{u}$ *is not always a unit vector and that they may have to normalize it first.*

2. $\nabla f(x, y) = (e^y \cos x, e^y \sin x)$ so $\nabla f(\pi/3, 0) = (1/2, \sqrt{3}/2)$.

$$D_{\mathbf{u}} f(\pi/3, 0) = \nabla f(\pi/3, 0) \cdot (3, -1)/\sqrt{10} = \frac{3 - \sqrt{3}}{2\sqrt{10}}.$$

3. $\nabla f(x, y) = (2x - 6x^2 y, -2x^3 + 6y^2)$, so $\nabla f(2, -1) = (28, -10)$ and

$$D_{\mathbf{u}} f(2, -1) = (28, -10) \cdot \frac{(1, 2)}{\sqrt{5}} = \frac{8}{\sqrt{5}}.$$

4. As noted above, here we have to normalize **u** so $D_{\mathbf{u}}(f(\mathbf{a})) = \nabla f(\mathbf{a}) \cdot \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$.

$$\nabla f(x, y) = \left(\frac{-2x}{(x^2 + y^2)^2}, \frac{-2y}{(x^2 + y^2)^2}\right) \text{ so } \nabla f(3, -2) = (1/169)(-6, 4) \text{ and}$$

$$D_{\mathbf{u}} f(\mathbf{a}) = \left(\frac{-6}{169}, \frac{4}{169}\right) \cdot \frac{(1, -1)}{\sqrt{2}} = \frac{-10}{169\sqrt{2}}.$$

5. $\nabla f(x, y) = (e^x - 2x, 0)$ so $\nabla f(1, 2) = (e - 2, 0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (e - 2, 0) \cdot \frac{(2, 1)}{\sqrt{5}} = \frac{2e - 4}{\sqrt{5}}.$$

6. $\nabla f(x, y, z) = (yz, xz, xy)$ so $\nabla f(-1, 0, 2) = (0, -2, 0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (0, -2, 0) \cdot \frac{(-1, 0, 2)}{\sqrt{5}} = 0.$$

7. $\nabla f(x, y, z) = -e^{-(x^2+y^2+z^2)}(2x, 2y, 2z)$ so $\nabla f(1, 2, 3) = -e^{-14}(2, 4, 6)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = -e^{-14}(2, 4, 6) \cdot \frac{(1, 1, 1)}{\sqrt{3}} = -4\sqrt{3}e^{-14}.$$

8. $\nabla f(x, y, z) = \left(\frac{e^y}{3z^2+1}, \frac{xe^y}{3z^2+1}, \frac{-6xe^y z}{(3z^2+1)^2}\right)$ so $\nabla f(2, -1, 0) = (e^{-1}, 2e^{-1}, 0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (e^{-1}, 2e^{-1}, 0) \cdot \frac{(1, -2, 3)}{\sqrt{14}} = \frac{-3}{e\sqrt{14}}.$$

9. **(a)**

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

(b)

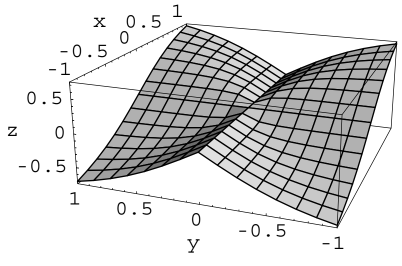
$$D_{(u,v)} f(0, 0) = \lim_{h\to 0} \frac{f(hu, hv) - 0}{h} = \lim_{h\to 0} \frac{hu|hv|}{h\sqrt{h^2u^2 + h^2v^2}}$$

But (u, v) is a unit vector so this

$$= \lim_{h\to 0} \frac{hu|h||v|}{h|h|(1)} = u|v|$$

for all unit vectors (u, v).

(c) The graph is shown below.



10. **(a)**

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

(b)

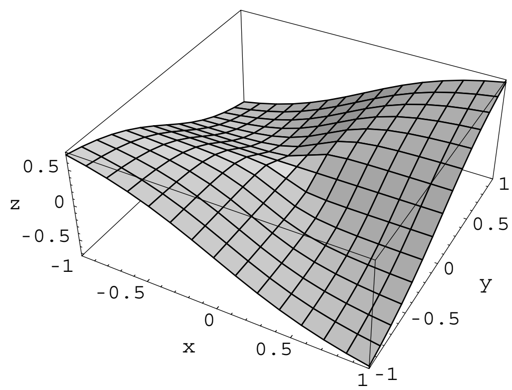
$$D_{(u,v)} f(0,0) = \lim_{h\to 0} \frac{f(hu, hv) - 0}{h} = \lim_{h\to 0} \frac{(hu)(hv)}{h\sqrt{h^2u^2 + h^2v^2}}$$

But (u, v) is a unit vector so this

$$= \lim_{h\to 0} \frac{h^2uv}{h|h|} = uv(\operatorname{sgn}(h))$$

for all unit vectors (u, v) where sgn(h) is 1 for $h \geq 0$ and -1 for $h < 0$. Unless u or v are zero, this limit doesn't exist.

(c) The graph is shown below.



11. The gradient direction for the function h is $\nabla h = (-6xy^2, -6x^2y)$.

(a) Head in the direction $\nabla h(1, -2) = (-24, 12)$. If you prefer your directions given by a unit vector, we normalize to obtain:

$$\frac{\nabla h(1,-2)}{\|\nabla h(1,-2)\|} = \frac{(-24, 12)}{\sqrt{24^2 + 12^2}} = \frac{(-2, 1)}{\sqrt{5}}.$$

(b) Head in a direction orthogonal to your answer for part (a): $\pm\frac{(1,2)}{\sqrt{5}}$.

12. $f_x(3,7) = 3$ and $f_y(3,7) = -2$ so the gradient is $\nabla f(3,7) = (3, -2)$.

(a) To warm up we head in the direction of the gradient; this is the unit vector $(3, -2)/\sqrt{13}$.

(b) To cool off we head in the opposite direction; this is the unit vector $(-3, 2)/\sqrt{13}$.

(c) To maintain temperature we head in a direction orthogonal to the gradient, namely $\pm(2, 3)/\sqrt{13}$.

13. We begin by heading east and keep heading towards lower levels while intersecting each level curve orthogonally. See the solution given in the text.

14. We're looking at the top half of this ellipsoid. The equation is $f(x, y) = z = \sqrt{4 - x^2 - y^2/4}$. For the path of steepest descent, we look at the negative gradient

$$-\nabla f(x, y) = (1/2)(4 - x^2 - y^2/4)^{-1/2}(2x, y/2).$$

This means that

$$\frac{dy}{dx} = \frac{y/2}{2x} = \frac{y}{4x}.$$

This is the separable differential equation $(4/y)\,dy = (1/x)\,dx$ or $4\ln y = \ln x + c$. Work the usual magic and get $y^4 = kx$. So the raindrops will follow curves of that form where z is constrained by the surface of the ellipsoid.

15. We want to head in the direction of the negative gradient. Since $M(x, y) = 3x^2 + y^2 + 5000$, the negative gradient is $-\nabla M(x, y) = (-6x, -2y)$. This means that

$$\frac{dy}{dx} = \frac{-2y}{-6x} = \frac{y}{3x}.$$

This is the separable differential equation $(3/y)\,dy = (1/x)\,dx$ or $3 \ln y = \ln x + c$. Work the usual magic and get $y^3 = kx$. Substitute in the point (8, 6) to solve for k to end up with the path $y^3 = 27x$.

For Exercises 16–22 we can use equations (5) and (6) from Section 2.6 in the text.

16. $f(x, y, z) = x^3 + y^3 + z^3 = 7$ so $\nabla f(x, y, z) = (3x^2, 3y^2, 3z^2)$ and $\nabla f(0, -1, 2) = (0, 3, 12)$. So the equation of the tangent plane is:

$$0 = (0, 3, 12) \cdot (x - 0, y + 1, z - 2) \quad \text{or} \quad y + 4z = 7.$$

17. $f(x, y, z) = ze^y \cos x = 1$ so $\nabla f(x, y, z) = (-ze^y \sin x, ze^y \cos x, e^y \cos x)$ and $\nabla f(\pi, 0, -1) = (0, 1, -1)$. So the equation of the tangent plane is:

$$0 = (0, 1, -1) \cdot (x - \pi, y, z + 1) \quad \text{or} \quad y - z = 1.$$

18. $f(x, y, z) = 2xz + yz - x^2y + 10 = 0$ so $\nabla f(x, y, z) = (2x - 2xy, z - x^2, 2x + y)$ and $\nabla f(1, -5, 5) = (20, 4, -3)$. So the equation of the tangent plane is:

$$0 = (20, 4, -3) \cdot (x - 1, y + 5, z - 5) \quad \text{or} \quad 20x + 4y - 3z = -15.$$

19. $f(x, y, z) = 2xy^2 - 2z^2 + xyz$ so $\nabla f(x, y, z) = (2y^2 + yz, 4xy + xz, xy - 4z)$ and $\nabla f(2, -3, 3) = (9, -18, -18)$. So the equation of the tangent plane is:

$$0 = (9, -18, -18) \cdot (x - 2, y + 3, z - 3) \quad \text{or} \quad x - 2y - 2z = 2.$$

20. (a) First we use the formula (4) from Section 2.3 in the text: $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. If $x^2 - 2y^2 + 5xz = 7$ then $z = \frac{7+2y^2-x^2}{5x} = f(x, y)$. Calculate the two partial derivatives:

$$f_x(x, y) = \frac{-7 - 2y^2 - x^2}{5x^2} \quad \text{so} \quad f_x(-1, 0) = \frac{-8}{5}$$

$$\text{and} \quad f_y(x, y) = \frac{4y}{5x} \quad \text{so} \quad f_y(-1, 0) = 0.$$

At $(-1, 0, -6/5)$ formula (4) gives the equation of the tangent plane as

$$z = \frac{-6}{5} + \frac{-8}{5}(x + 1).$$

(b) Now we'll use formula (6) from this section and calculate the gradient of $f(x, y, z) = x^2 - 2y^2 + 5xz$ as $\nabla f(x, y, z) = (2x + 5z, -4y, 5x)$ so $\nabla f(-1, 0, -6/5) = (-8, 0, -5)$ and so the equation for the plane is

$$0 = (-8, 0, -5) \cdot (x + 1, y, z + 6/5) \quad \text{or} \quad -8x - 5z = 14.$$

This agrees with the answer we found in part (a).

21. (a) First we use the formula (4) from Section 2.3 in the text: $x = f(a, b) + f_y(a, b)(y - a) + f_z(a, b)(z - b)$. If $x \sin y + xz^2 = 2e^{yz}$ then $x = \frac{2e^{yz}}{\sin y + z^2} = f(y, z)$. Calculate the two partial derivatives:

$$f_y(y, z) = 2e^{yz}\frac{z \sin y + z^3 - \cos z}{(\sin y + z^2)^2} \quad \text{so} \quad f_y(\pi/2, 0) = 0$$

$$\text{and} \quad f_z(y, z) = 2e^{yz}\frac{y \sin y + yz^2 - 2z}{(\sin y + z^2)^2} \quad \text{so} \quad f_z(\pi/2, 0) = \pi.$$

At $(2, \pi/2, 0)$ formula (4) gives the equation of the tangent plane as

$$x = 2 + \pi z.$$

(b) Now we'll use formula (6) from this section and calculate the gradient of $f(x, y, z) = x \sin y + xz^2 - 2e^{yz}$ as $\nabla f(x, y, z) = (\sin y + z^2, -x \cos y - 2ze^{yz}, 2xz - 2ye^{yz})$ so $\nabla f(2, \pi/2, 0) = (1, 0, -\pi)$ and so the equation for the plane is

$$0 = (1, 0, -\pi) \cdot (x - 2, y - \pi/2, z) \quad \text{or} \quad x - 2 - \pi z = 0.$$

This agrees with the answer we found in part (a).

22. Using formula (6) we get that the gradient of $f(x, y, z) = x^3 - 2y^2 + z^2$ at (x_0, y_0, z_0) is $\nabla f(x_0, y_0, z_0) = (3x_0^2, -4y_0, 2z_0)$. For this to be perpendicular to the given line, $(3x_0^2, -4y_0, 2z_0) = k(3, 2, -\sqrt{2})$. This means that $x_0^2 = -2y_0$ and $z_0 = -(\sqrt{2}/2)x_0^2$. Substituting this back into the equation of the surface, we get that $x_0^3 - 2x_0^4/4 + x_0^4/2 = 27$ or $x_0 = 3$. Our point is, therefore $(3, -9/2, -9\sqrt{2}/2)$.

23. The tangent plane to the surface at a point (x_0, y_0, z_0) is

$$0 = 18x_0(x - x_0) - 90y_0(y - y_0) + 10z_0(z - z_0).$$

For this to be parallel to $x + 5y - 2z = 7$, the vector

$$(18x_0, -90y_0, 10z_0) = k(1, 5, -2).$$

This means that $y_0 = -x_0$ and $z_0 = (-18/5)x_0$. Substitute these back into the equation of the hyperboloid: $9x^2 - 45y^2 + 5z^2 = 45$ to get:

$$45 = 9x_0^2 - 45x_0^2 + 5(18^2/5^2)x_0^2 \quad \text{therefore} \quad x_0 = \pm 5/4.$$

This means that the points are $(5/4, -5/4, -9/2)$ and $(-5/4, 5/4, 9/2)$.

24. First note that $(2, 1, -1)$ lies on both surfaces: $7 \cdot 2^2 - 12 \cdot 2 - 5 \cdot 1 = -1$, $2 \cdot 1(-1)^2 = 2$. The normal to the first surface at $(2, 1, -1)$ is given by $(f_x(2, 1), f_y(2, 1), -1)$ where $f(x, y) = 7x^2 - 12x - 5y^2$. This is $((14x - 12)|_{(2,1)}, -10y|_{(2,1)}, -1) = (16, -10, -1)$. The normal to the second surface at $(2, 1, -1)$ is $\nabla F(2, 1, -1)$ where $F(x, y, z) = xyz^2$. This is $(yz^2, xz^2, 2xyz)|_{(2,1,-1)} = (1, 2, -4)$. We have

$$(16, -10, -1) \cdot (1, 2, -4) = 16 - 20 + 4 = 0.$$

Since the normals are orthogonal, the tangent planes must be so as well.

25. The two surfaces are tangent at $(x_0, y_0, z_0) \Leftrightarrow$ the tangent planes at (x_0, y_0, z_0) are the same $\Leftrightarrow$ normal vectors at (x_0, y_0, z_0) are parallel (since the surfaces intersect at (x_0, y_0, z_0)) $\Leftrightarrow \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) = \mathbf{0}$.

26. (a) S is the level set at height 0 of $f(x, y, z) = x^2 + 4y^2 - z^2$ so $\nabla f = (2x, 8y, -2z) \Rightarrow$ $\nabla f(3, -2, -5) = (6, -16, 10)$. Thus formula (6) gives the equation of the tangent plane as $6(x - 3) - 16(y + 2) + 10(z + 5) = 0$ or $3x - 8y + 5z = 0$.

(b) $\nabla f(0, 0, 0) = (0, 0, 0)$ so formula (6) cannot be used. Note that there's no tangent plane at the origin, which is the vertex of the cone (i.e., the surface is not "locally flat" there).

27. (a) For $f(x, y, z) = x^3 - x^2y^2 + z^2, \nabla f(x, y, z) = (3x^2 - 2xy^2, -2x^2y, 2z)$ so $\nabla f(2, -3/2, 1) = (3, 12, 2)$. Thus the equation of the tangent plane is

$$3(x - 2) + 12(y + 3/2) + 2(z - 1) = 0 \quad \text{or} \quad 3x + 12y + 2z + 10 = 0.$$

(b) $\nabla f(0, 0, 0) = (0, 0, 0)$ so the gradient cannot be used as a normal vector. If we solve $z = \pm\sqrt{y^2x^2 - x^3} = \pm x\sqrt{y^2 - x}$, we see that $g(x, y) = x\sqrt{y^2 - x}$ fails to be differentiable at (0, 0)—so there is no tangent plane there.

28. **(a)** $2x + 2y\dfrac{dy}{dx} = 0$ so

$$\left.\frac{dy}{dx}\right|_{(-\sqrt{2},\sqrt{2})} = \left.\frac{-x}{y}\right|_{(-\sqrt{2},\sqrt{2})} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$$

The equation of the line is $y - \sqrt{2} = x + \sqrt{2}$.

(b) The equation of the tangent line is $0 = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$. Here $f(x, y) = x^2 + y^2 = 4$ so $\nabla f(x, y) = (2x, 2y)$ or $\nabla f(-\sqrt{2}, \sqrt{2}) = (-2\sqrt{2}, 2\sqrt{2})$. The equation of the tangent line is

$$0 = (-2\sqrt{2}, 2\sqrt{2}) \cdot (x + \sqrt{2}, y - \sqrt{2}) \quad \text{or} \quad x - y = -2\sqrt{2}.$$

29. **(a)** $3y^2\dfrac{dy}{dx} = 2x + 3x^2$ so $\left.\dfrac{dy}{dx}\right|_{(1,\sqrt[3]{2})} = \dfrac{5}{(3)2^{2/3}}$. The equation of the tangent line is

$$y - \sqrt[3]{2} = \frac{5}{(3)2^{2/3}}(x - 1).$$

(b) $f(x, y) = y^3 - x^2 - x^3$ so $\nabla f(x, y) = (-2x - 3x^2, 3y^2)$ so $\nabla f(1, \sqrt[3]{2}) = (-5, (3)2^{2/3})$. The equation of the tangent line is

$$0 = (-5, (3)2^{2/3}) \cdot (x - 1, y - \sqrt[3]{2}) \quad \text{or} \quad -5x + (3)2^{2/3}y = 1.$$

30. **(a)** $5x^4 + 2y + 2x\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$ so $\left.\dfrac{dy}{dx}\right|_{(2,-2)} = \dfrac{-76}{16} = \dfrac{-19}{4}$. The equation of the tangent line is

$$y + 2 = \frac{-19}{4}(x - 2).$$

(b) $f(x, y) = y^3 - x^2 - x^3$ so $\nabla f(x, y) = (5x^4 + 2y, 2x + 3y^2)$ so $\nabla f(2, -2) = (76, 16)$. The equation of the tangent line is

$$0 = (76, 16) \cdot (x - 2, y + 2) \quad \text{or} \quad 19x + 4y = 30.$$

31. If $f(x, y) = x^2 - y^2$ then $\nabla f(5, -4) = (10, 8)$ so the equations of the normal line are

$$x(t) = 10t + 5 \quad \text{and} \quad y(t) = 8t - 4 \quad \text{or} \quad 8x - 10y = 80.$$

32. If $f(x, y) = x^2 - x^3 - y^2$ then $\nabla f(-1, \sqrt{2}) = (5, 2\sqrt{2})$ so the equations of the normal line are

$$x(t) = 5t - 1 \quad \text{and} \quad y(t) = 2\sqrt{2}t - \sqrt{2} \quad \text{or} \quad 2\sqrt{2}x - 5y = -7\sqrt{2}.$$

33. If $f(x, y) = x^3 - 2xy + y^5$ then $\nabla f(2, -1) = (14, 1)$ so the equations of the normal line are

$$x(t) = 14t + 2 \quad \text{and} \quad y(t) = t - 1 \quad \text{or} \quad x - 14y = 16.$$

34. If $f(x, y, z) = x^3 z + x^2 y^2 + \sin(yz)$ then

$$\nabla f(x, y, z) = (3x^2 + 2xy^2, 2x^2 y + z\cos(yz), x^3 + y\cos(yz)).$$

(a) The plane is given by $0 = \nabla f(-1, 0, 3) \cdot (x + 1, y, z - 3) = 9(x + 1) + 3y - (z - 3)$ or $9x + 3y - z = -12$.

(b) The normal line to the surface at $(-1, 0, 3)$ is given by

$$\begin{cases} x = 9t - 1 \\ y = 3t \\ z = -t + 3. \end{cases}$$

35. Using the method above for $f(x, y, z) = e^{xy} + e^{zx} - 2e^{yz}$, we find that $\nabla f = (ye^{xy} + ze^{xz}, xe^{xy} - 2ze^{yz}, xe^{xz} - 2ye^{yz})$ so $\nabla f(-1, -1, -1) = e(-2, 1, 1)$. So

$$\begin{cases} x = -2et - 1 \\ y = et - 1 \\ z = et - 1 \end{cases} \quad \text{or, factoring out } e, \quad \begin{cases} x = -2t - 1 \\ y = t - 1 \\ z = t - 1. \end{cases}$$

36. Remember in the equation of a plane $0 = \mathbf{v} \cdot (x - x_0, y - y_0, z - z_0)$ that $\mathbf{v}$ is a vector orthogonal to the plane. We saw in this section that we can use $\nabla f(x_0, y_0, z_0)$ for $\mathbf{v}$. This means that the equation of the line normal to a surface given by the equation $F(x, y, z) = 0$ at a given point (x_0, y_0, z_0) is

$$(x, y, z) = \nabla F(x_0, y_0, z_0)t + (x_0, y_0, z_0).$$

37. The hyper surface is the level set at height -1 of the function $f(x_1, \ldots, x_5) = \sin x_1 + \cos x_2 + \sin x_3 + \cos x_4 + \sin x_5$. We find $\nabla f\left(\pi, \pi, \frac{3\pi}{2}, 2\pi, 2\pi\right) = (-1, 0, 0, 0, 1)$. Hence the tangent hyperplane has equation

$$-1(x_1 - \pi) + 1(x_5 - 2\pi) = 0 \quad \text{or} \quad x_5 - x_1 = \pi.$$

38. The surface is the level set at height $\frac{n(n+1)}{2}$ of the function $f(x_1, \ldots, x_n) = x_1^2 + 2x_2^2 + \cdots + nx_n^2$. We have $\nabla f = (2x_1, 4x_2, 6x_3, \ldots, 2nx_n) \Rightarrow \nabla f(-1, \ldots, -1) = -2(1, 2, 3, \ldots, n)$. An equation for the tangent hyperplane is thus

$$1(x_1 + 1) + 2(x_2 + 1) + 3(x_3 + 1) + \cdots + n(x_n + 1) = 0$$

or

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n + \frac{n(n+1)}{2} = 0$$

39. Here $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$ so $\nabla f(x_1, x_2, \ldots, x_n) = (2x_1, 2x_2, \ldots, 2x_n)$. Using the techniques of this section, the tangent hyperplane to the $(n-1)$-dimensional sphere $f(x_1, x_2, \ldots, x_n) = 1$ at $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n})$ is

$$\begin{aligned} 0 &= \nabla f(1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n}) \cdot (x_1 - 1/\sqrt{n}, x_2 - 1/\sqrt{n}, \ldots, x_{n-1} - 1/\sqrt{n}, x_n + 1/\sqrt{n}) \\ &= \frac{2}{\sqrt{n}}\left(x_1 - \frac{1}{\sqrt{n}}\right) + \frac{2}{\sqrt{n}}\left(x_2 - \frac{1}{\sqrt{n}}\right) + \cdots + \frac{2}{\sqrt{n}}\left(x_{n-1} - \frac{1}{\sqrt{n}}\right) + \frac{-2}{\sqrt{n}}\left(x_n + \frac{1}{\sqrt{n}}\right) \quad \text{or} \end{aligned}$$

$$0 = (x_1 - 1/\sqrt{n}) + (x_2 - 1/\sqrt{n}) + \cdots + (x_{n-1} - 1/\sqrt{n}) - (x_n + 1/\sqrt{n}) \quad \text{so}$$

$$\sqrt{n} = x_1 + x_2 + \cdots + x_{n-1} - x_n.$$

40. $F(x, y, z) = z^2y^3 + x^2y = 2$.

(a) We can write $z = f(x, y)$ when $F_z \neq 0$. $F_z(x, y, z) = 2zy^3$ is not 0 when both $z \neq 0$ and $y \neq 0$.

(b) We can write $x = f(y, z)$ when $F_x \neq 0$. $F_x(x, y, z) = 2xy$ is not 0 when both $x \neq 0$ and $y \neq 0$.

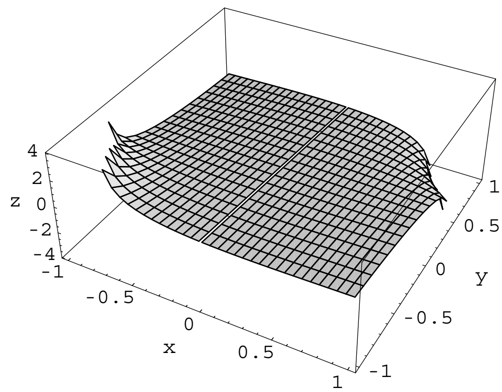
(c) We can write $y = f(x, z)$ when $F_y \neq 0$. $F_y(x, y, z) = 3z^2y^2 + x^2$ is not 0 everywhere but on the y- or z-axis (i.e., except when $x = 0$ at the same time that either $y = 0$ or $z = 0$).

41. (a) $\frac{\partial F}{\partial z} = xe^{xz}$. This is non-zero whenever $x \neq 0$. There we can solve for z to get

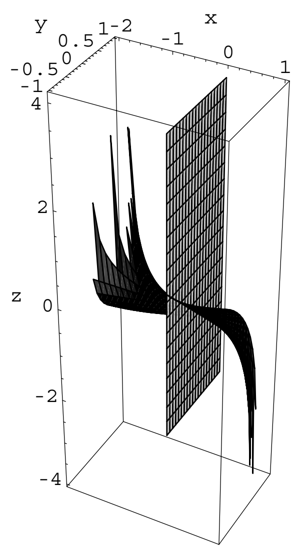
$$z = \frac{\ln(1 - \sin xy - x^3y)}{x}.$$

(b) Looking only at points in S we only need to stay away from points in yz-plane (i.e., where $x = 0$).

(c) You shouldn't then make the leap from your answer to part (b) that you can graph $z = \frac{\ln(1 - \sin xy - x^3y)}{x}$ for any values of x and y just so $x \neq 0$. Your other restriction is that $1 - \sin xy - x^3y > 0$ as it is the argument of the natural logarithm. A sketch that gives you an idea of the surface is:



Now the actual surface S includes the plane $x = 0$ since $x = 0$ satisfies the original equation: $\sin xy + e^{xz} + x^3 y = 1$. S will actually look a bit like:

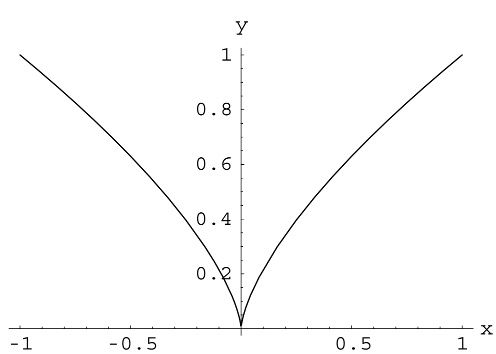


42. The point of this problem is that since $F(x, y) = c$ defines a curve C in $\mathbf{R}^2$ such that either $f_x(x_0, y_0) \neq 0$ or $f_y(x_0, y_0) \neq 0$ then by the implicit function theorem we can represent the curve near (x_0, y_0) as either the graph of a function $x = g(y)$ or a function $y = g(x)$.

Exercise 43 poses a bit of a puzzle. Here we can write the equation of C as $y = f(x)$ even though F_y is zero at the origin. Why doesn't this contradict the implicit function theorem? What "goes bad" in Exercise 43 is that we have a corner at the origin. You may also want to assign the students the same problem for the function $F(x, y) = x - y^3$.

43. (a) $F(0, 0) = 0$ so the origin lies on the curve C. $F_y(x, y) = 3y^2$ and so $F_y(0, 0) = 0$.

(b) We can write C as the graph of $y = x^{2/3}$. The graph of C is



(c) So here we are with $F_y(0,0) = 0$ but we can express the graph of C everywhere as $y = x^{2/3}$. On second look we see that C is not a C^1 function—it has a corner at the origin—and so the implicit function theorem doesn't apply.

44. (a) $F(x,y) = xy + 1$ so $F_y(x,y) = x$ and so we cannot solve $F(x,y) = c$ for y when $x = 0$ or when $c = 0(y) + 1 = 1$. In other words, level sets are unions of smooth curves in $\mathbf{R}^2$ except for $c = 1$.

(b) Here the function is $F(x,y,z) = xyz + 1$. Using a similar argument to that in part (a), $F_z(x,y,z) = xy$ and this is only 0 when $xy = 0$. This means that we cannot solve $F(x,y,z) = c$ for z when $xy = 0$ or when $c = z(0) + 1 = 1$. So level sets of this family are unions of smooth surfaces in $\mathbf{R}^3$ except for level $c = 1$.

45. (a) $G(-1,1,1) = F(-1-2+1, -1-1+3) = F(-2,1) = 0$.

(b) To invoke the implicit function theorem, we need to show that $G_z(-1,1,1) \neq 0$.

$$G_z(-1,1,1) = F_u(-2,1)\frac{\partial(x^3 - 2y^2 + z^5)}{\partial z}\bigg|_{(-1,1,1)} + F_v(-2,1)\frac{\partial(xy - x^2z + 3)}{\partial z}\bigg|_{(-1,1,1)}$$
$$= (7)(5) + (5)(1) = 40 \neq 0.$$

46. Let $F_1 = x_2y_2 - x_1\cos y_1 = 5$ and $F_2 = x_2\sin y_1 + x_1y_2 = 2$. Solving for y in terms of x means that we have to look at the determinant

$$\det\begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \end{bmatrix} = \det\begin{bmatrix} x_1\sin y_1 & x_2 \\ x_2\cos y_1 & x_1 \end{bmatrix} = x_1^2\sin y_1 + x_2^2\cos y_1.$$

To see that you can solve for y_1 and y_2 in terms of x_1 and x_2 near $(x_1, x_2, y_1, y_2) = (2, 3, \pi, 1)$, evaluate the determinant at that point. We get -9. This is not 0 so you can, at least in theory, solve for the y's in terms of the x's.

To see that you can solve for y_1 and y_2 as functions of x_1 and x_2 near $(x_1, x_2, y_1, y_2) = (0, 2, \pi/2, 5/2)$, evaluate the determinant at that point. We get 0. We can not solve for the y's in terms of the x's.

47. (a) Let $F_1 = x_1^2y_2^2 - 2x_2y_3 = 1$, $F_2 = x_1y_1^5 + x_2y_2 - 4y_2y_3 = -9$, and $F_3 = x_2y_1 + 3x_1y_3^2 = 12$. Solving for y's in terms of x's means that we have to look at the determinant

$$\det\begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} & \dfrac{\partial F_1}{\partial y_3} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} & \dfrac{\partial F_2}{\partial y_3} \\ \dfrac{\partial F_3}{\partial y_1} & \dfrac{\partial F_3}{\partial y_2} & \dfrac{\partial F_3}{\partial y_3} \end{bmatrix} = \det\begin{bmatrix} 0 & 2x_1^2y_2 & -2x_2 \\ 5x_1y_1^4 & x_2 - 4y_3 & -4y_2 \\ x_2 & 0 & 6x_1y_3 \end{bmatrix}$$
$$= -60x_1^4y_1^4y_2y_3 - 8x_1^2x_2y_2^2 + 2x_2^3 - 8x_2^2y_3.$$

Evaluating this at the point $(x_1, x_2, y_1, y_2, y_3) = (1, 0, -1, 1, 2)$ results in $-120 \neq 0$. This means that we can solve for y_1, y_2, and y_3 in terms of x_1 and x_2.

(b) Take the partials of the three equations with respect to x_1 to get

$$\begin{cases} y_2^2 + 2x_1y_2\dfrac{\partial y_2}{\partial x_1} - 2x_2\dfrac{\partial y_3}{\partial x_1} = 0 \\ y_1^5 + 5x_1y_1^4\dfrac{\partial y_1}{\partial x_1} + x_2\dfrac{\partial y_2}{\partial x_1} - 4y_3\dfrac{\partial y_2}{\partial x_1} - 4y_2\dfrac{\partial y_3}{\partial x_1} = 0 \\ x_2\dfrac{\partial y_1}{\partial x_1} + 3y_3^2 + 6x_1y_3\dfrac{\partial y_3}{\partial x_1} = 0. \end{cases}$$

At the point $(1, 0, -1, 1, 2)$ this system of equations becomes:

$$\begin{cases} 1 + 2\dfrac{\partial y_2}{\partial x_1} = 0 \\ -1 + 5\dfrac{\partial y_1}{\partial x_1} - 8\dfrac{\partial y_2}{\partial x_1} - 4\dfrac{\partial y_3}{\partial x_1} = 0 \\ 12 + 12\dfrac{\partial y_3}{\partial x_1} = 0. \end{cases}$$

Solving, we find that

$$\frac{\partial y_1}{\partial x_1} = \frac{-7}{5}, \frac{\partial y_2}{\partial x_1} = -\frac{1}{2}, \quad \text{and} \quad \frac{\partial y_3}{\partial x_1} = -1.$$

48. (a) We need to consider where the following determinant is non-zero.

$$\begin{vmatrix} \partial F_1/\partial r & \partial F_1/\partial \theta & \partial F_1/\partial z \\ \partial F_2/\partial r & \partial F_2/\partial \theta & \partial F_2/\partial z \\ \partial F_3/\partial r & \partial F_3/\partial \theta & \partial F_3/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

In other words, for any points for which $r \neq 0$.

(b) This makes complete sense. When the radius is 0 then r and z completely determine the point. You get no extra information from the θ component. Without the z coordinate, this is the standard problem when using polar coordinates in the plane.

49. (a) As with Exercise 48, we need to consider where the same determinant is non-zero. In this case the determinant is

$$\begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi\cos^2\varphi + \rho^2\sin^3\varphi = \rho^2\sin\varphi.$$

In other words, for any points for which $\rho \neq 0$ and for which $\sin\varphi \neq 0$.

(b) Again, this makes complete sense. When the radius is 0, then ρ completely determines the point as being the origin. When $\sin\varphi = 0$ you are on the z-axis so θ no longer contributes any information.

2.7 TRUE/FALSE EXERCISES FOR CHAPTER 2

1. False.
2. True.
3. False. (The range also requires $v \neq 0$.)
4. False. (Note that $\mathbf{f}(\mathbf{i}) = \mathbf{f}(\mathbf{j})$.)
5. True.
6. False. (It's a paraboloid.)
7. False. (The graph of $x^2 + y^2 + z^2 = 0$ is a single point.)
8. True.
9. False.
10. False. (The limit does not exist.)
11. False. ($\lim_{(x,y)\to(0,0)} f(x, y) = 0 \neq 2.$)
12. False.
13. False.
14. True.
15. False. ($\nabla f(x, y, z) = (0, \cos y, 0).$)
16. False. (It's a 4×3 matrix.)
17. True.

18. False.
19. False. (The partial derivatives must be continuous.)
20. True.
21. False. ($f_{xy} \neq f_{yx}$.)
22. False. (f must be of class C^2.)
23. True. (Write the chain rule for this situation.)
24. True.
25. False. (The correct equation is $x + y + 2z = 2$.)
26. False. (The plane is *normal* to the given vector.)
27. True.
28. False. (The directional derivative equals $-\partial f/\partial z$.)
29. False.
30. True.

2.8 MISCELLANEOUS EXERCISES FOR CHAPTER 2

1. (a) Calculate the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = (-x_2, x_1 - x_3, x_2).$$

More explicitly, the component functions are $f_1(x_1, x_2, x_3) = -x_2$, $f_2(x_1, x_2, x_3) = x_1 - x_3$, and $f_3(x_1, x_2, x_3) = x_2$.

(b) The domain is all of $\mathbf{R}^3$ while the range restricts the first component to be the opposite of the last component. In other words the range is the set of all vectors $(a, b, -a)$.

2. (a) It might help to see **f** explicitly first as

$$\left(\frac{(3, -2, 1) \cdot (x, y, z)}{(3, -2, 1) \cdot (3, -2, -1)}\right)(3, -2, 1) = \frac{3x - 2y + z}{14}(3, -2, 1).$$

(b) The domain is all of $\mathbf{R}^3$ and the range are vectors of the form $(3a, -2a, a)$.

3. (a) The domain of f is $\{(x, y) | x \geq 0 \text{ and } y \geq 0\} \cup \{(x, y) | x \leq 0 \text{ and } y \leq 0\}$. The range is all real numbers greater than or equal to 0.

(b) The domain is closed. The quarter planes are closed on two sides because they include the axes.

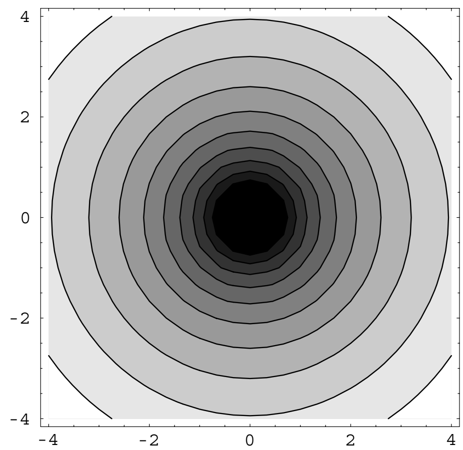
4. (a) The domain of f is $\{(x, y) | x \geq 0 \text{ and } y > 0\} \cup \{(x, y) | x \leq 0 \text{ and } y < 0\}$. The range is all real numbers greater than or equal to 0.

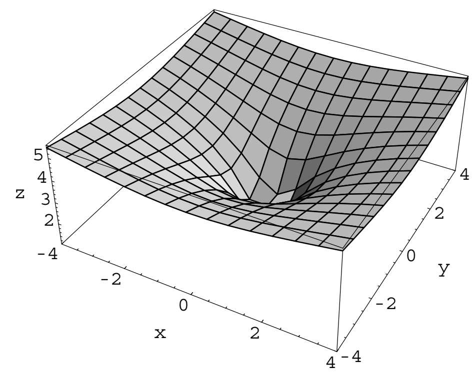
(b) The domain is neither open nor closed. The quarter planes are closed on one side because they include the y-axis but they don't include the x-axis and so aren't closed.

5.

$f(x, y)$	Graph	Level curves
$1/(x^2 + y^2 + 1)$	D	d
$\sin\sqrt{x^2 + y^2}$	B	e
$(3y^2 - 2x^2)e^{-x^2-2y^2}$	A	b
$y^3 - 3x^2y$	E	c
$x^2y^2e^{-x^2-y^2}$	F	a
$ye^{-x^2-y^2}$	C	f

6. (a) See below left.





(b) See above right.

7. First we'll substitute $x = r\cos\theta$ and $y = r\sin\theta$ while noting that $(x, y) \longrightarrow (0, 0)$ is equivalent to $r \longrightarrow 0$.

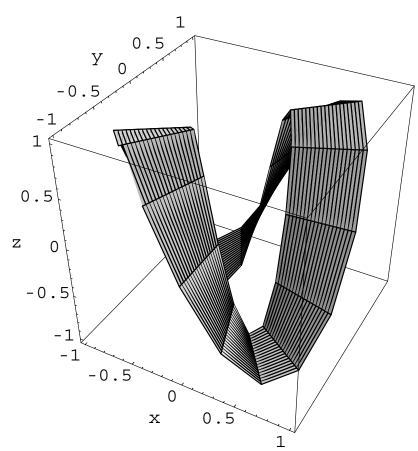
$$\lim_{(x,y)\to(0,0)} \frac{yx^2 - y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{(r\sin\theta)(r^2\cos^2\theta) - (r^3\sin^3\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \lim_{r\to 0} \frac{r^3(\cos^2\theta - \sin^2\theta)\sin\theta}{r^2}$$
$$= \lim_{r\to 0} r\cos 2\theta \sin\theta = 0$$

8. (a) $\dfrac{2xy}{x^2 + y^2} = \dfrac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta = \sin 2\theta$. So

$$f(x, y) = \begin{cases} \sin 2\theta & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

(b) We're looking for (x, y) such that $f(x, y) = c$. For $-1 < c < 1$ the level sets are pairs of radial lines symmetric about $\theta = \pi/4$.
For example, if $c = 1/2$ then we are looking for θ such that $\sin 2\theta = 1/2$. In this case, $\theta = \pi/12$, $5\pi/12$, $13\pi/12$, and $17\pi/12$. So the level sets are the lines $\theta = \pi/12$ and $\theta = 5\pi/12$. These could also be written as $\theta = \pi/4 \pm \pi/6$.
For $c = 1$ the level set is the line $\theta = \pi/4$, for $c = -1$ the level set is the line $\theta = 3\pi/4$ and for $|c| > 1$ the level set is the empty set.

(c) f is constant along radial lines, so the figure below just shows a ribbon corresponding to $.4 < r < 1$.



(d) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \sin 2\theta$ which doesn't exist.

(e) Since the limit doesn't exist at the origin, f couldn't be continuous there. Also, f takes on every value between 1 and -1 in every open neighborhood of the origin.

Before assigning Exercise 9 you may want to ask the students if it is true that if a function F(x, y) is continuous in each variable separately it is continuous. The calculations in Exercise 9 are fairly routine but the conclusion is very important.

9. $g(x) = F(x,0) \equiv 0$ and so is continuous at $x = 0$ and $h(y) = F(0,y) \equiv 0$ and so is continuous at $y = 0$. Consider $p(x) = F(x,x) = 1$ when $x \neq 0$ and $F(0,0) = 0$. Clearly, $p(x)$ is not continuous at 0 so $F(x, y)$ is not continuous at (0,0).

10. **(a)** You can see as x gets closer and closer to 0 that $1/x^2$ gets larger and larger. More formally, for any $N > 0$, if $0 < |x| < 1/\sqrt{N}$ then $1/x^2 > N$.

(b) Here $\|(x,y) - (1,3)\| = \sqrt{(x-1)^2 + (y-3)^2}$ so for any $N > 0$, if $0 < \|(x,y) - (1,3)\| < \sqrt{(2/N)}$, then

$$\frac{2}{(x-1)^2 + (y-3)^2} = \frac{2}{\|(x,y) - (1,3)\|^2} > \frac{2}{2/N} = N.$$

(c) The definition is analogous to that for above: $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = -\infty$ means that given any $N < 0$ there is some $\delta > 0$ such that if $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ then $f(\mathbf{a}) < N$.

(d) We are considering $\lim_{(x,y)\to(0,0)}$ so let's restrict our attention to $|x| < 1$ and $|y| < 1$. For $|x| < 1$ we have

$$\frac{1-x}{xy^4 - y^4 + x^3 - x^2} = \frac{-1}{y^4 + x^2}.$$

For $|y| < 1$ we have $y^4 < y^2$ so

$$\frac{-1}{y^4 + x^2} < \frac{-1}{y^2 + x^2} = \frac{-1}{\|(x,y)\|^2}.$$

So for any $N < 0$ if $0 < \|(x,y)\| < \min\{1, 1/\sqrt{-N}\}$ then $\dfrac{1-x}{xy^4 - y^4 + x^3 - x^2} < N.$

11. We read right from the table in the text:

(a) 15°F.

(b) 5°F.

12. **(a)** If the temperature of the air is 10°F we read off the chart that when the windspeed is 10 mph the windchill is -4; when the windspeed is 15 mph the windchill is -7. Since we are looking to estimate when the windchill is -5 you might be tempted to stop here and just conclude that the answer is between 10 mph and 15 mph (and you'd be correct) but we want to say more. Our first estimate will just use linear interpolation (similar triangles) to get $\frac{x}{2} = \frac{5}{3}$ or the distance from 15 is $x = 10/3$. We would then conclude that, to the nearest degree, the windspeed is $15 - 10/3 \approx 12$ mph.

(b) Before you feel too good about your answer to part (a) you should notice further that when the windspeed is 20 the windchill is -9 and when the windspeed is 25 the windchill is -11. In other words, the rate at which the windchill is dropping is slowing slightly. In calculus terms, for the function $f(s) = W(s \text{ mph}, 30^\circ)$, $f''(s)$ seems to be positive so the curve is concave up. The line used to estimate in part (a) then probably lies above the curve and our guess of 12 mph is, most likely, too high.

13. For the function $W(s \text{ mph}, t^\circ)$, we want to estimate

$$\left.\frac{\partial W}{\partial t}\right|_{(30 \text{ mph}, 35^\circ)} = \lim_{h\to 0} \frac{W(30, 35+h) - W(30,35)}{h}.$$

We will use the slopes of the two secant lines:

$$\frac{W(30,40) - W(30,35)}{5} = \frac{28-22}{5} = 1.2$$

$$\frac{W(30,30) - W(30,35)}{-5} = \frac{15-22}{-5} = 1.4$$

We average them to get an estimate of 1.3.

14. We will use the same technique as in Exercise 13 and estimate the derivative with respect to windspeed by averaging the slopes of the two secant lines.

$$\frac{W(20,25) - W(15,25)}{5} = \frac{11 - 13}{5} = -0.4$$
$$\frac{W(10,25) - W(15,25)}{-5} = \frac{15 - 13}{-5} = -0.4$$

so we average them to get an estimate of -0.4.

15. (a) Comparison with Exercise 11: With an air temperature of 25°F, windspeed of 10 mph,

$$W(10,25) = 91.4 + (25 - 91.4)(0.474 + 0.304\sqrt{10} - 0.203)$$
$$\approx 9.573 \quad \text{or} \quad 10°\text{F}$$

(as compared to 15°F in 11(a)).
If $s = 20$ mph, then $W = -15°$F if

$$91.4 + (t - 91.4)(0.474 + 0.304\sqrt{20} - 0.406) = -15.$$

Hence $t = 91.4 - \dfrac{15 + 91.4}{(0.474 + 0.304\sqrt{20} - 0.406)} \approx 16.866$ or 17°F (as compared to 5°F in 11(b)).

Comparison with Exercise 12: With $W(s,t) = 91.4 + (t - 91.4)(.474 + .304\sqrt{s} - .0203s)$, we must solve

$$20 = 91.4 + (30 - 91.4)(.474 + .304\sqrt{s} - .0203s)$$

or

$$\frac{20 - 91.4}{30 - 91.4} = .474 + .304\sqrt{s} - .0203s \quad \text{so that}$$
$$1.16287 \approx .474 + .304\sqrt{s} - .0203s \quad \text{or}$$
$$0 \approx .0203s - .304\sqrt{s} + .688866$$

Now solve the quadratic: $\sqrt{s} \approx \dfrac{.304 \pm \sqrt{.0364801}}{.0406}$. The two solutions are 7.74682 and 148.646.

(b) The windchill effect of windspeed appears to be greater in the Siple formula than that which may be inferred from the table.

(c) For temperatures greater than 91.4 the model has the wind actually making the apparent temperature warmer than air temperature. Physically, the model probably falls apart because between 91.4 and 106 you are too close to body temperature for the wind to have much effect and if you are in temperatures much greater than 106 a breeze won't replace a frosty beverage. For winds below 4 mph, the effect is negligible and won't be reflected in the model.

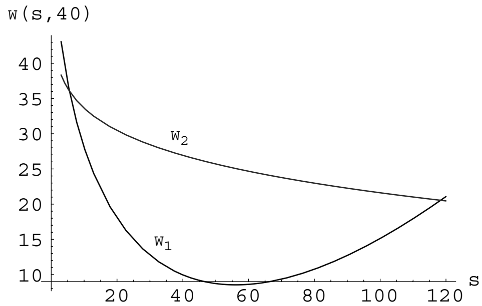
16. Comparison with Exercise 13: We want to calculate $W_t(30,35)$. $\partial W/\partial t = 0.621 + 0.4275s^{0.16}$, so $W_t(30,35) = 0.621 + (0.4275)30^{0.16} \approx 1.358$ (this is close).

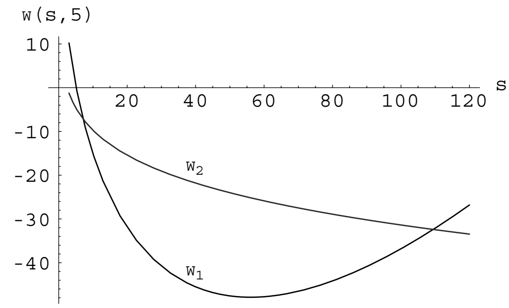
Comparison with Exercise 14: We want $W_s(15,25)$.

$$\partial W/\partial s = -35.75(0.16)s^{-0.84} + 0.4275(0.16)ts^{-0.84}$$
$$= (-5.72 + 0.0684t)s^{-0.84}$$

$W_s(15,25) = (-5.72 + 0.0684 \cdot 25)15^{-0.84} \approx -0.412$ (again close).

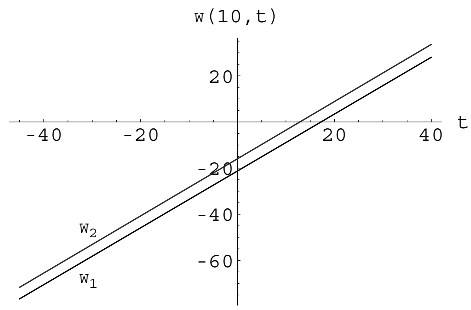
17. (a) Pictured (left) are the pairs $W_1(s,40)$ and $W_2(s,40)$ and, on the right, the pairs $W_1(s,5)$ and $W_2(s,5)$.

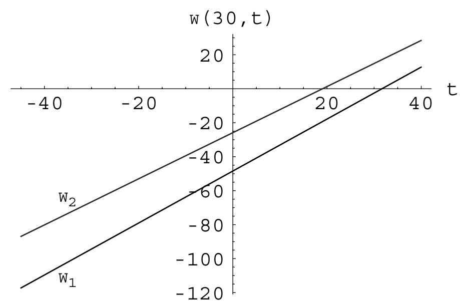




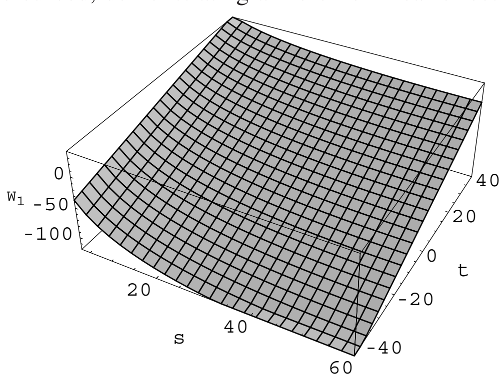
From these graphs, we see that windspeed depresses apparent temperature in the Siple formula much more than in the National Weather Service Formula.

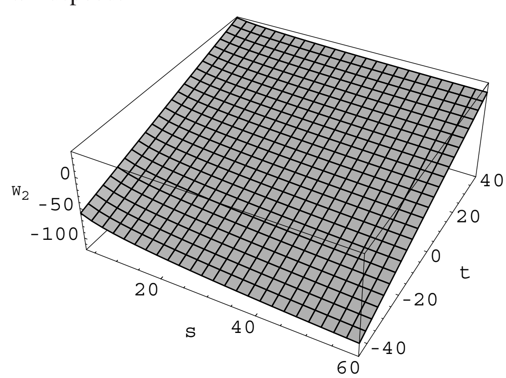
(b) Pictured (left) are the pairs $W_1(10, t)$ and $W_2(10, t)$ and, on the right, the pairs $W_1(30, t)$; and $W_2(30, t)$. Again we see that the Siple formula results in lower apparent temperatures predicted, only the effect appears to be more of a constant difference.





(c) The surfaces $z = W_1(s, t)$ and $z = W_2(s, t)$ are pictured. Note that the Siple surface determined by W_1 is more curved, demonstrating a more nonlinear effect of windspeed.





18. The equation of the sphere is $F(x, y, z) = x^2 + y^2 + z^2 = 9$ so $\nabla F = (2x, 2y, 2z)$ and the plane tangent to the sphere at (1, 2, 2) is $0 = (2, 4, 4) \cdot (x - 1, y - 2, z - 2)$ or $x + 2y + 2z = 9$. This intersects the x-axis when $y = 0$ and $z = 0$ so $x = 9$.

19. Without loss of generality we can locate the center of the sphere at the origin and so the equation of the sphere is $F(x, y, z) = x^2 + y^2 + z^2 = r^2$ so $\nabla F = (2x, 2y, 2z)$ and the equation of the plane tangent to the sphere at $P = (x_0, y_0, z_0)$ is $0 = (2x_0, 2y_0, 2z_0) \cdot (x - x_0, y - y_0, z - z_0)$ or $x_0 x + y_0 y + z_0 z = x_0^2 + y_0^2 + z_0^2$. This is orthogonal to the vector (x_0, y_0, z_0) which is the vector from the center of the sphere to P.

20. Because we're looking at a curve in the plane $2x - y = 1$ we know the x and y components of the parametric equations. What is left to determine is z. Substitute in $2x - 1$ for y in the equation of the surface to get $z = 3x^2 + x^3/6 - x^4/8 - 4(2x - 1)^2 = -5x^2 + x^3/6 - x^4/8 + 4$. We can now calculate the derivative $\partial z/\partial x = -10x + x^2/2 - x^3/2$ and evaluate it at the point $(1, 1, -23/24)$ to get -10. Because the value of z is $-23/24$ when $x = 1$, this component of the tangent line is derived by looking at $z + 23/24 = -10(x - 1)$. So the parametric equations for the tangent line are $(t, 2t - 1, -10t + 192/24)$.

21. **(a)** For the function $f(x, y, z) = x^2 + y^2 - z^2 = 0$ we consider $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. Here we get $(2(3), 2(-4), -2(5)) \cdot (x - 3, y + 4, z - 5) = 0$ or the equation is $6(x - 3) - 8(y + 4) - 10(z - 5) = 0$.

(b) In general we get $(2(a), 2(b), -2(c)) \cdot (x - a, y - b, z - c) = 0$. This amounts to $2a(x - a) + 2b(y - b) - 2c(z - c) = 0$.

(c) Note that (0, 0, 0) is a solution so the plane passes through the origin.

22. Show that the two surfaces

$$S_1 : z = xy \text{ and } S_2 : z = \frac{3}{4}x^2 - y^2$$

intersect perpendicularly at the point (2, 1, 2). First we see that $2 = 1(2)$ and $2 = (3/4)(4) - 1$ so (2, 1, 2) is a point on both surfaces. Rewrite the surfaces so that they are level sets of functions:

$$F_1(x, y, z) = xy - z \text{ and } F_2(x, y, z) = z + y^2 - \frac{3}{4}x^2.$$

The gradients are normal to the tangent planes (see Section 2.6, Exercise 36), so we calculate the two gradients at the given point: $\nabla F_1(2, 1, 2) = (1, 2, -1)$ and $\nabla F_2(2, 1, 2) = (-3, 2, 1)$ so

$$\nabla F_1(2, 1, 2) \cdot \nabla F_2(2, 1, 2) = 0.$$

So the two surfaces intersect perpendicularly at (2, 1, 2).

23. **(a)** As we have done before we find the plane tangent to the surface given by $F(x, y, z) = z - x^2 - 4y^2 = 0$ by formula (6):

$$0 = \nabla F(1, -1, 5) \cdot (x - 1, y + 1, z - 5) = (-2, 8, 1) \cdot (x - 1, y + 1, z - 5)$$
$$\text{or } -2x + 8y + z = -5.$$

(b) The line is parallel to a vector which is orthogonal to $\nabla F(1, -1, 5) = (-2, 8, 1)$ and with no component in the x direction. So it is of the form (0, a, b) with $(0, a, b) \cdot (-2, 8, 1) = 0$ so the line has the direction $(0, 1, -8)$ and passes through $(1, -1, 5)$. The equations are $\begin{cases} x = 1 \\ y = t - 1 \\ z = -8t + 5. \end{cases}$

24. We are assuming that the collar is fairly rigid so that it is maintaining a cylindrical shape throughout this process. We want $\dfrac{\partial V}{\partial t}$ at $t = t_0$. Since $V = \pi r^2 h$, $\dfrac{\partial V}{\partial t} = 2\pi r h \dfrac{dr}{dt} + \pi r^2 \dfrac{dh}{dt}$. We are given that the rate of change of the circumference at $t = t_0$ is $-.2$ *in/min*. This means

$$-.2 = \left.\frac{\partial C}{\partial t}\right|_{t_0} = \left.\frac{\partial(2\pi r)}{\partial t}\right|_{t_0} = 2\pi \left.\frac{dr}{dt}\right|_{t_0}.$$

We also know that at $t = t_0$, $2\pi r = 18$, $h = 3$, and $\dfrac{dh}{dt} = .1$. Substituting into the equation above, we get:

$$\left.\frac{\partial V}{\partial t}\right|_{t_0} = (18)(3)\left(\frac{-.2}{2\pi}\right) + \pi\left(\frac{18}{2\pi}\right)^2(.1) = \frac{-5.4}{\pi} + \frac{8.1}{\pi} = \frac{2.7}{\pi}.$$

So the volume is increasing at $t = t_0$.

25. First note that $0.2°\text{C/day} = 0.2 \cdot 24 = 4.8°\text{C/month}$. Then, with time measured in months, the chain rule tells us

$$\frac{dP}{dt} = \frac{\partial P}{\partial S}\frac{dS}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt}.$$

Here $\frac{dS}{dt} = -2$, $\frac{dT}{dt} = 4.8$. With $P(S, T) = 330S^{2/3}T^{4/5}$, we have

$$\left.\frac{dP}{dt}\right|_{(S=75,T=15)} = (220S^{-1/3}T^{4/5})|_{(75,15)}(-2) + 264S^{2/3}T^{-1/5}|_{(75,15)}(4.8)$$
$$= 220(75)^{-1/3}(15)^{4/5}(-2) + 264(75)^{2/3}(15)^{-1/5}(4.8)$$
$$= 12{,}201.4 \text{ units/month}$$
$$(\text{or } 508.392 \text{ units/day})$$

26. We want to know $\frac{du}{dt}$ (t in weeks) when $x = 80$, $y = 240$, given that $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = -15$. The chain rule tells us

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad = (0.002xe^{-0.001x^2-0.00005y^2})\frac{dx}{dt} + (0.0001ye^{-0.001x^2-0.00005y^2})\frac{dy}{dt}$$

Thus

$$\left.\frac{du}{dt}\right|_{x=80,y=240} = e^{(-0.001)80^2-0.00005(240)^2}[(0.002)80 \cdot 5 - (0.0001)240 \cdot 15]$$
$$\approx 0.000041.$$

So the utility function is increasing ever so slightly

27.

$$w = x^2 + y^2 + z^2,$$
$$x = \rho\cos\theta\sin\varphi,$$
$$y = \rho\sin\theta\sin\varphi \quad \text{and}$$
$$z = \rho\cos\varphi$$

(a)

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$
$$= 2x\cos\theta\sin\varphi + 2y\sin\theta\sin\varphi + 2z\cos\varphi$$
$$= 2\rho\cos^2\theta\sin^2\varphi + 2\rho\sin^2\theta\sin^2\varphi + 2\rho\cos^2\varphi$$
$$= 2\rho,$$
$$\frac{\partial w}{\partial \varphi} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \varphi}$$
$$= 2x\rho\cos\theta\cos\varphi + 2y\rho\sin\theta\cos\varphi - 2z\rho\sin\varphi$$
$$= 2\rho^2\cos^2\theta\cos\varphi\sin\varphi + 2\rho^2\sin^2\theta\cos\varphi\sin\varphi - 2\rho^2\cos\varphi\sin\varphi$$
$$= 0, \quad \text{and}$$
$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$
$$= -2x\rho\sin\theta\sin\varphi + 2y\rho\cos\theta\sin\varphi$$
$$= -2\rho^2\cos\theta\sin\theta\sin^2\varphi + 2\rho^2\cos\theta\sin\theta\sin^2\varphi$$
$$= 0.$$

(b) First substitute: $w = x^2 + y^2 + z^2 = (\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2 + (\rho\cos\varphi)^2 = \rho^2$. Now taking the derivatives from part (a) is trivial: $w_\rho = 2\rho$, $w_\varphi = 0$, and $w_\theta = 0$.

28. If $w = f\left(\frac{x+y}{xy}\right)$, let $u = \frac{x+y}{xy}$. So

$$\begin{aligned} x^2\frac{\partial w}{\partial x} - y^2\frac{\partial w}{\partial y} &= x^2\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} - y^2\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} \\ &= x^2\frac{\partial w}{\partial u}\left(\frac{-y^2}{x^2y^2}\right) - y^2\frac{\partial w}{\partial u}\left(\frac{-x^2}{x^2y^2}\right) \\ &= 0. \end{aligned}$$

29. (a) First use the chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$:

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}(e^r\cos\theta) + \frac{\partial z}{\partial y}(e^r\sin\theta), \quad \text{and} \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial z}{\partial x}(-e^r\sin\theta) + \frac{\partial z}{\partial y}(e^r\cos\theta). \end{aligned}$$

Now solve for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}, \quad \text{and} \\ \frac{\partial z}{\partial y} &= e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial \theta}. \end{aligned}$$

(b) Given the results for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in part (a), we compute:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = e^{-r}\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) - e^{-r}\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) \\ &= e^{-r}\cos\theta\frac{\partial}{\partial r}\left(e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}\right) - e^{-r}\sin\theta\frac{\partial}{\partial \theta}\left(e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}\right) \\ &= e^{-r}\cos\theta\left(-e^{-r}\cos\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial^2 z}{\partial r^2} + e^{-r}\sin\theta\frac{\partial z}{\partial \theta} - e^{-r}\sin\theta\frac{\partial^2 z}{\partial r\partial \theta}\right) \\ &\quad - e^{-r}\sin\theta\left(-e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial^2 z}{\partial \theta\partial r} - e^{-r}\cos\theta\frac{\partial z}{\partial \theta} - e^{-r}\sin\theta\frac{\partial^2 z}{\partial \theta^2}\right) \\ &= e^{-2r}\left[(\sin^2\theta - \cos^2\theta)\frac{\partial z}{\partial r} + \cos^2\theta\frac{\partial^2 z}{\partial r^2} + 2\sin\theta\cos\theta\frac{\partial z}{\partial \theta} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial r\partial \theta} + \sin^2\theta\frac{\partial^2 z}{\partial \theta^2}\right] \end{aligned}$$

A similar calculation gives:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = e^{-r}\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) + e^{-r}\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)$$

$$= e^{-r}\sin\theta\frac{\partial}{\partial r}\left(e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial\theta}\right) + e^{-r}\cos\theta\frac{\partial}{\partial\theta}\left(e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial\theta}\right)$$

$$= e^{-2r}\left[(\cos^2\theta - \sin^2\theta)\frac{\partial z}{\partial r} + \sin^2\theta\frac{\partial^2 z}{\partial r^2} - 2\sin\theta\cos\theta\frac{\partial z}{\partial\theta} + 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial r\partial\theta} + \cos^2\theta\frac{\partial^2 z}{\partial\theta^2}\right].$$

Now add these to get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}[(\cos^2\theta + \sin^2\theta)z_{\theta\theta} + (\cos^2\theta + \sin^2\theta)z_{rr}] = e^{-2r}[z_{\theta\theta} + z_{rr}].$$

30. (a) Consider $w = f(x, y) = x^y = e^{y\ln x}$. Then $\frac{d}{du}(u^u)$ can be calculated by taking the derivative and evaluating at the point (u, u).

$$\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = yx^{y-1} + \ln x e^{y\ln x} = u\cdot u^{u-1} + (\ln u)u^u = u^u(1 + \ln u).$$

(b) Here $x = \sin t$ and $y = \cos t$. So

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = yx^{y-1}\cos t + (\ln x)e^{y\ln x}(-\sin t) = \cos^2 t(\sin t)^{\cos t-1} - \sin t\ln(\sin t)\sin t^{\cos t}.$$

31. This is an extension of the preceding exercise. This time $w = f(x, y, z) = x^{y^z}$. If $x = u$, $y = u$, and $z = u$ we again calculate

$$\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du} = y^z x^{y^z-1} + e^{y^z\ln x}(z\ln x)y^{z-1} + e^{e^{z\ln y}\ln x}(\ln x)e^{z\ln y}\ln y$$

$$= u^u u^{(u^u-1)} + u^{u^u}(u\ln u)u^{u-1} + u^{u^u}(\ln u)^2 u^u = u^u u^{u^u}(1/u + \ln u + (\ln u)^2).$$

32. With

$$r = \|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + \cdots + x_n^2}} = \frac{x_i}{r}.$$

The chain rule gives $\frac{\partial f}{\partial x_i} = \frac{dg}{dr}\frac{\partial r}{\partial x_i} = g'(r)\frac{x_i}{r}$
By the product and chain rules:

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(g'(r)\frac{x_i}{r}\right) = \frac{g'(r)}{r} + x_i\frac{d}{dr}\left(\frac{g'(r)}{r}\right)\frac{\partial r}{\partial x_i}$$

$$= \frac{1}{r}g'(r) + x_i\left(\frac{rg''(r) - g'(r)}{r^2}\right)\frac{x_i}{r}$$

$$= \frac{1}{r}g'(r) + x_i^2\left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3}\right).$$

Add these to find

$$\nabla^2 f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} = \frac{n}{r}g'(r) + \left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3}\right)\underbrace{(x_1^2 + \cdots + x_n^2)}_{=r^2}$$

$$= \frac{n}{r}g'(r) + g''(r) - \frac{g'(r)}{r}$$

$$= \frac{1}{r}(n - 1)g'(r) + g''(r).$$

33. (a)

$$\begin{aligned}\nabla^2(\nabla^2 f(x,y)) &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\\ &= \frac{\partial^4 f}{\partial x^4} + \underbrace{\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^2 \partial x^2}}_{\text{these are equal} - f \text{ is of class } C^4} + \frac{\partial^4 f}{\partial y^4}\\ &= \text{desired expression.}\end{aligned}$$

(b) Similar:

$$\begin{aligned}\nabla^2(\nabla^2 f) &= \frac{\partial^2}{\partial x_1^2}\left(\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_j^2}\right) + \cdots + \frac{\partial^2}{\partial x_n^2}\left(\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_j}\right)\\ &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i{}^2}\left(\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_j^2}\right) = \sum_{i,j=1}^{n}\frac{\partial^4 f}{\partial x_i^2 \partial x_j^2}.\end{aligned}$$

34. Livinia is at (0, 0, 1) and $T(x,y,z) = 10(xe^{-y^2} + ze^{-x^2})$

(a) The unit vector in the direction from (0, 0, 1) to (2, 3, 1) is $\mathbf{u} = (2,3,0)/\sqrt{13}$.

$$D_{\mathbf{u}}T = \nabla T(0,0,1)\cdot \mathbf{u} = 10(1,0,1)\cdot(2,3,0)/\sqrt{13} = 20/\sqrt{13}\, deg/cm.$$

(b) She should head in the direction of the negative gradient: $(-1,0,-1)/\sqrt{2}$.

(c) $(3)10(1,0,1)\cdot(-1,0,-1)/\sqrt{2} = -30\sqrt{2}\, deg/sec.$

35. $z = r\cos 3\theta$

(a) $z = r[\cos\theta\cos 2\theta - \sin\theta\sin 2\theta] = r[\cos\theta(\cos^2\theta - \sin^2\theta) - \sin\theta(2\sin\theta\cos\theta)]$ so

$$z = \frac{r^3[\cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta]}{r^2} = \frac{x^3 - 3xy^2}{x^2+y^2}.$$

(b) Note that $\lim_{r\to 0} r\cos 3\theta = 0$ which is the value of the function at the origin. So yes, $f(x,y) = z$ is continuous at the origin.

(c) **(i)** $f_x = \dfrac{(x^2+y^2)(3x^2-3y^2) - (x^3-3xy^2)2x}{(x^2+y^2)^2} = \dfrac{x^4 - 3y^4 + 6x^2y^2}{(x^2+y^2)^2}.$

(ii) $f_y = \dfrac{(x^2+y^2)(-6xy) - (x^3-3xy^2)2y}{(x^2+y^2)^2} = \dfrac{-8x^3y}{(x^2+y^2)^2}.$

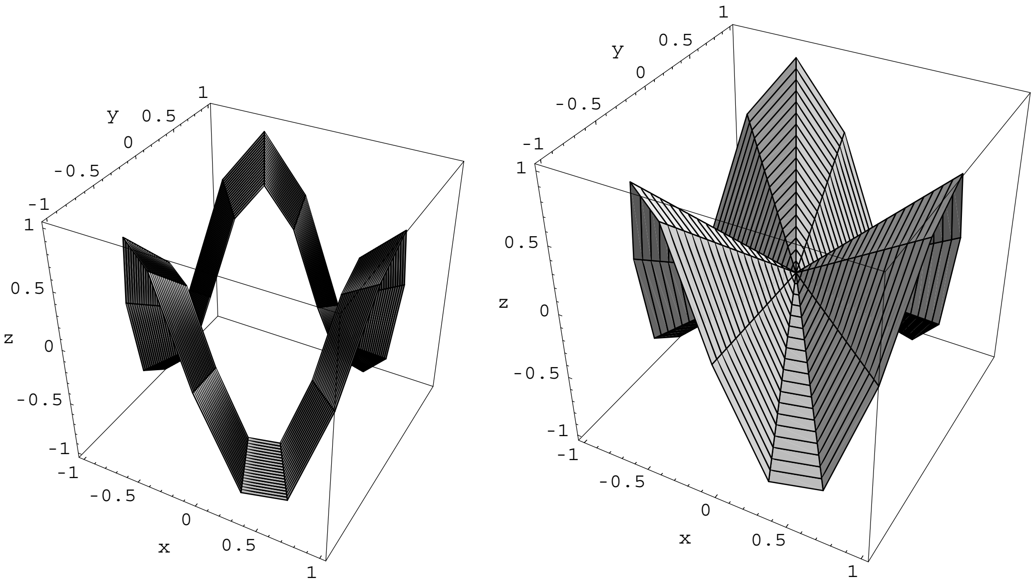
(iii) $f_x(0,0) = \lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{h-0}{h} = 1.$

(iv) $f_y(0,0) = \lim_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-0}{h} = 0.$

(d) $g(r,\theta) = r\cos 3\theta$ so $g_r(r,\theta) = \cos 3\theta$. This is the directional derivative $D_{\mathbf{u}}f$.

(e) When $(x,y)\neq(0,0)$, $f_y(x,y) = \dfrac{-8x^3y}{x^2+y^2}$. In particular, when $y = x$, $f_y = -2$. From part (c) $f_y(0,0) = 0$ so f_y is not continuous at the origin.

(f) Below are two sketches; the one on the left just shows a ribbon of the surface:



36. (a) $u = \cos(x - t) + \sin(x + t) - 2e^{z+t} - (y - t)^3$ so

- $u_x = -\sin(x - t) + \cos(x + t)$ and $u_{xx} = -\cos(x - t) - \sin(x + t)$.
- $u_y = -3(y - t)^2$ and $u_{yy} = -6(y - t)$.
- $u_z = -2e^{z+t}$ and $u_{zz} = -2e^{z+t}$.
- $u_t = \sin(x - t) + \cos(x + t) - 2e^{z+t} + 3(y - t)^2$ and $u_{tt} = -\cos(x - t) - \sin(x + t) - 2e^{z+t} - 6(y - t)$.

We have, therefore, the result: $u_{xx} + u_{yy} + u_{zz} = u_{tt}$.

(b) $u(x, y, z, t) = f_1(x - t) + f_2(x + t) + g_1(y - t) + g_2(y + t) + h_1(z - t) + h_2(z + t)$ so

- $u_x = (f_1)_{x-t}\dfrac{\partial(x - t)}{\partial x} + (f_2)_{x+t}\dfrac{\partial(x + t)}{\partial x} = (f_1)_{x-t} + (f_2)_{x+t}$ so
- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 f_1}{\partial(x - t)^2} + \dfrac{\partial^2 f_2}{\partial(x + t)^2}$
- $u_y = (g_1)_{y-t}\dfrac{\partial(y - t)}{\partial y} + (g_2)_{y+t}\dfrac{\partial(y + t)}{\partial y} = (g_1)_{y-t} + (g_2)_{y+t}$ so
- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 g_1}{\partial(y - t)^2} + \dfrac{\partial^2 g_2}{\partial(y + t)^2}$.
- $u_z = (h_1)_{z-t}\dfrac{\partial(z - t)}{\partial z} + (h_2)_{z+t}\dfrac{\partial(z + t)}{\partial z} = (h_1)_{z-t} + (h_2)_{z+t}$ so
- $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial^2 h_1}{\partial(z - t)^2} + \dfrac{\partial^2 h_2}{\partial(z + t)^2}$.
- $u_t = (f_1)_{x-t}\dfrac{\partial(x - t)}{\partial t} + (f_2)_{x+t}\dfrac{\partial(x + t)}{\partial t} + (g_1)_{y-t}\dfrac{\partial(y - t)}{\partial t} + (g_2)_{y+t}\dfrac{\partial(y + t)}{\partial t} + (h_1)_{z-t}\dfrac{\partial(z - t)}{\partial t} + (h_2)_{z+t}\dfrac{\partial(z + t)}{\partial t}$ so $u_t = -(f_1)_{x-t} + (f_2)_{x+t} - (g_1)_{y-t} + (g_2)_{y+t} - (h_1)_{z-t}\dfrac{\partial(z - t)}{\partial t} + (h_2)_{z+t}\dfrac{\partial(z + t)}{\partial t}$ and
- $u_{tt} = u_{xx} + u_{yy} + u_{zz}$.

37. $F(tx, ty) = t^3x^3 + t^3xy^2 - 6t^3y^3 = t^3F(x, y)$ so F is homogeneous of degree 3.

38. $F(tx, ty, tz) = t^3x^3y - t^4x^2z^2 + t^8z^8$ so, no, F is not homogeneous.

39. $F(tx, ty, tz) = t^3zy^2 - t^3x^3 + t^3x^2z = t^3F(x, y, z)$ so yes F is homogeneous of degree 3.

40. $F(tx, ty) = e^{ty/tx} = e^{y/x} = F(x, y)$ so F is homogeneous of degree 0.

41. $F(tx, ty, tz) = \dfrac{t^3x^3 + t^3x^2y - t^3yz^2}{t^3xyz + 7t^3xz^2} = F(x, y, z)$ so F is homogeneous of degree 0.

42. Make sure that the students realize (as in Exercises 40 and 41) that a function can be homogeneous and not be a polynomial. In the special case that F is a polynomial, F is homogeneous when all of the terms are of the same degree.

43. $F(tx_1, tx_2, \ldots, tx_n) = t^d F(x_1, x_2, \ldots, x_n)$ so that, by differentiating both sides with respect to t:

$$x_1 \frac{\partial F}{\partial x_1}(tx_1, \ldots, tx_n) + \cdots + x_n \frac{\partial F}{\partial x_n}(tx_1, \ldots, tx_n) = dt^{d-1} F(x_1, \ldots, x_n).$$

Now let $t = 1$ and we get the result:

$$x_1 \frac{\partial F}{\partial x_1} + \cdots + x_n \frac{\partial F}{\partial x_n} = dF.$$

44. The conjecture is:

$$\sum_{i_1,\ldots,i_k=1}^{n} = x_{i_1} x_{i_2} \cdots x_{i_k} F_{x_{i_1} x_{i_2} \cdots x_{i_k}} = \frac{d!}{(d-k)!} F.$$

Although not asked in the text, a good exercise is to ask the students to establish the formula given in this exercise. Show that

$$\frac{\partial}{\partial x_i}[dF] = \sum_{j=1}^{n} x_j \frac{\partial^2 F}{\partial x_i \partial x_j} + \frac{\partial F}{\partial x_i}.$$

Then you can show

$$d^2 F = d \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} + dF.$$

You can finish from there.