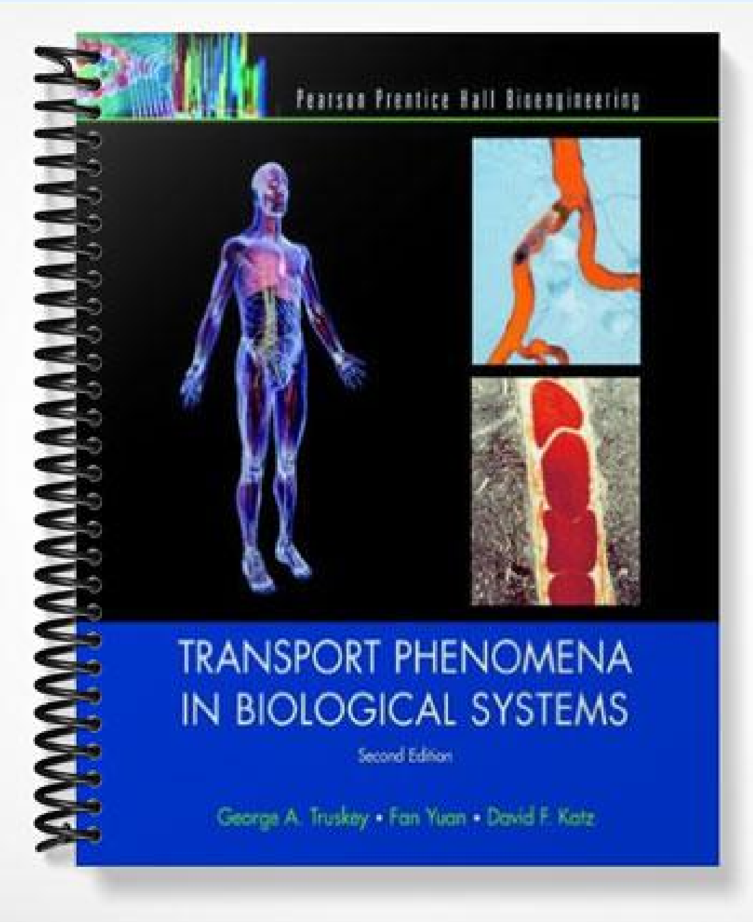
SOLUTIONS MANUAL



Solution Manual for
Transport Phenomena in Biological Systems

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Solution to Problems in Chapter 1, Section 1.10

1.1. The relative importance of convection and diffusion is evaluated by Peclet number,

$$Pe = \frac{vL}{D_{ij}} \qquad \text{(S1.1.1)}$$

(a) Solving for L, $L = PeD_{ij}/v$. When convection is the same as diffusion, Pe =1, L is 0.11cm.
(b) The distance between capillaries is 10^{-4} m, O_2 needs to travel half of this distance, and Pe = 0.0455. Therefore, convection is negligible compared with diffusion.

1.2. Since $H_{O_2} = H_{Hb}$, equation (1.6.4) is simplified to the following:

$$C_{O_2} = H_{O_2} P_{O_2} + 4C_{Hb}\overline{S}Hct \qquad \text{(S1.2.1)}$$

P_{O_2} and $\overline{S}$ are 95 mmHg and 95% for arterial blood and 38 mmHg 70% for venous blood. C_{Hb} is 0.0203 mol L^{-1} x 0.45 = 0.0091 M for men, and 0.0203 mol L^{-1} x 0.40 = 0.0081 M for women. Based on these data, the fraction of oxygen in plasma and bound to hemoglobin is 1.5% and 98.5% in arterial blood, and 0.83% and 99.17% in venous blood for men. Corresponding values for women are 1.7% and 98.3% in arterial blood, and 0.93% and 99.07% in venous blood. Most oxygen in blood is bound to hemoglobin.

1.3. For CO_2 70% is stored in plasma and 30% is in red blood cell. Therefore, the total change of $CO_2$is 2.27(0.70)+1.98(0.30) = 2.18 cm^3 per 100 cm^3. For O_2, P_{O_2} changes from 38 to 100 mmHg after blood passes through lung artery. Using data in problem (1.2), the total O_2 concentration in blood is 0.0088 M in arterial blood and 0.0063 M in venous blood. At standard temperature (273.15 K) and pressure (1 atm = 101,325 Pa), 1 mole of gas occupies 22,400 cm^3. Thus, the O_2 concentration difference of 0.0025 M corresponds to 5.58 cm^3 O_2 per 100 cm^3. While larger than the difference for CO_2, the pressure difference driving transport is much larger for O_2 than CO_2.

1.4. The diffusion time is $L^2/D_{ij} = (10^{-4}\ cm)^2/(2x10^{-5}\ cm^2\ s^{-1}) = 0.0005$ s. Therefore, diffusion is much faster than reaction and does not delay the oxygenation process.

1.5. $V = \pi R^2 L$ and the $S= 2\pi RL$ where R is the vessel radius and L is the length

Order	volume, cm^3	surface area, cm^2	cumulative volume, cm^3	cumulative surface area, cm^2
1	0.0158	26.27	0.0158	26.27
2	0.03885	35.32	0.05	61.59
3	0.05738	31.44	0.11	92.99
4	0.09219	30.23	0.20	123.21
5	0.12788	26.64	0.33	149.86
6	0.20487	23.28	0.54	173.14
7	0.20733	15.56	0.74	188.70
8	0.24132	11.03	0.99	199.73
9	0.31010	8.17	1.30	207.89
10	0.23046	3.71	1.53	211.60
11	0.50671	3.99	2.03	215.59

1.6.

Order	Volume (cm^3)	Surface Area (cm^2)	Cumulative Volume (cm^3)	Cumulative Surface Area (cm^2)
0	30.54	67.86	30.54	67.86
1	11.13	36.49	41.66	104.34
2	4.11	19.82	45.78	124.2
3	1.50	10.70	47.27	134.9
4	3.23	28.72	50.51	163.6
5	3.29	37.65	53.80	201.2
6	3.54	50.67	57.35	251.9
7	4.04	70.29	61.39	322.2
8	4.45	95.74	65.84	418.0
9	5.15	133.76	70.99	551.7
10	6.25	192.38	77.24	744.1
11	7.45	273.51	84.70	1018
12	9.58	403.41	94.27	1421
13	11.68	569.79	106.0	1991
14	16.21	876.05	122.2	2867
15	22.42	1358.86	144.6	4226
16	30.57	2038.28	175.2	6264
17	42.33	3135.25	217.5	9399
18	60.223	4817.76	277.7	14217
19	90.05	7663.95	367.8	21881
20	138.42	12303.82	506.2	34185
21	213.18	19831.06	719.4	54015
22	326.72	31874.64	1046.	85890
23	553.75	54024.81	1600	139915

1.7. (a) The water content is 55% and 60% of the whole blood for men and women, respectively. Then the water flow rate through kidney is 990 L day^{-1} for men and 1,080 L day^{-1} for women. Then the fraction of water filtered across the glomerulus is 18.2% for men and 16.67% for women.

(b) renal vein flow rate = renal artery flow rate – excretion rate = 1.19 L min^{-1}

renal vein flow rate = 1.25 L min^{-1} – (1.5 L day^{-1})/(1440 min day^{-1}) = 1.249 L min^{-1}

(c) Na^+ leaving glomerulus = 25,200 mmole day^{-1}/180 L day^{-1} = 140 mM.

Na^+ in renal vein = Na^+ in renal artery - Na^+ excreted

(1.25 L min^{-1} x 150mM – 150 mM day^{-1}/(1440 min day^{-1}))/1.249 L min^{-1}

= 150.037 mM

There is a slight increase in sodium concentration in the renal vein due to the volume reduction.

1.8. (a) $Bi = k_m L/D_{ij} = 5 \times 10^{-9}$ cm s^{-1} x 0.0150cm/(1 x 10^{-10} cm^2 s^{-1}) = 0.75.

(b) The results indicate that the resistance to LDL transport provided by the endothelium is similar to that provided by the arterial wall.

1.9 The oxygen consumption rate is $\dot{V}_{O_2} = Q\left(C_v - C_a\right)$ where Q is the pulmonary blood flow and C_v and C_a are the venous are arterial oxygen concentrations. The oxygen concentrations are obtained from Equation (1.6.4)

$$C_{O_2} = H_{O_2}P_{O_2}\left(1-\text{Hct}\right)+\left(4C_{Hb}\bar{S}+H_{Hb}P_{O_2}\right)\text{Hct}$$

The fractional saturation $\bar{S}$ is given by Equation (1.6.5). For the data given, the venous fraction saturation is 0.971. The arterial fractional saturation is 0.754 under resting conditions and 0.193 under exercise conditions.

	Men	Women
Rest	C_a = 0.0070 M	C_a = 0.0063 M
Exercise	C_a = 0.0019 M	C_a = 0.0017 M
	C_V = 0.0090 M	C_v = 0.0080 M

The oxygen consumption rates are

	Men	Women
Rest	0.0115 mole min^{-1}	0.0102 mole min^{-1}
Exercise	0.1776 mole min^{-1}	0.1579 mole min^{-1}

1.10. (a) To obtain the rate of oxygen removal from the lungs, we use the mass balance discussed in class that equates the oxygen removed from the inspired air with the oxygen uptake in the blood.

$$\dot{V}_I\left(C_I - C_{alv}\right) = Q\left(C_v - C_a\right) \qquad \text{(S1.10.1)}$$

We want to assess the left hand side of Equation (S1.10.1) which represents the rate of oxygen removal from the lungs. From the data provided and the ideal gas equation:

$$C_{alv} = \frac{p_{alv}}{RT} = \frac{(105 \text{ mm Hg})/(760 \text{ mm Hg/atm})}{(0.08206 \text{ L atm/(mol K)})(310 \text{ K})} = 0.00543 \text{ M}$$

$$C_I = \frac{p_{alv}}{RT} = \frac{0.21(1 \text{ atm})}{(0.08206 \text{ L atm/(mol K)})(310 \text{ K})} = 0.00826 \text{ M}$$

$$\dot{V}_I = \left(10 \text{ breaths/min}\right)\left(0.56 - 0.19 \text{ L}\right) = 3.7 \text{ L/min} \quad \text{males}$$

$$\dot{V}_I = \left(10 \text{ breaths/min}\right)\left(0.45 - 0.41 \text{ L}\right) = 3.1 \text{ L/min} \quad \text{females}$$

Since we have all terms on the left hand side of Equation (1), the rate of oxygen removal from the lungs is:

$$\dot{V}_I\left(C_I - C_{alv}\right) = \left(3.7 \text{ L/min}\right)\left(0.00282 \text{ mole O}_2\text{/L}\right) = 0.0104 \text{ mole O}_2\text{/min} \quad \text{males}$$

$$\dot{V}_I\left(C_I - C_{alv}\right) = \left(3.1 \text{ L/min}\right)\left(0.00282 \text{ mole O}_2\text{/L}\right) = 0.00874 \text{ mole O}_2\text{/min} \quad \text{females}$$

To convert to mL O_2/L blood, multiply to oxygen removal rate by 22,400 L O_2 per mole of O_2.

For males the value is 233 mL O_2/min and for females the value is 196 mL O_2/min. These values are a bit low but within the range of physiological values under resting conditions.

(b) In this part of the problem, you are asked to find the volume inspired in each breadth or $\dot{V}_I$. Sufficient information is provided to determine the right hand side of Equation (1) which represents both the rate of oxygen delivery and oxygen consumption.

First, determine the oxygen concentrations in arteries and veins. The concentration in blood is:

$$C_{O_2} = H_{O_2} P_{O_2} (1 - \text{Hct}) + \left(4C_{Hb}\bar{S} + H_{Hb} P_{O_2}\right)\text{Hct}$$

Using the relation for the percent saturation to calculate the concentration in the pulmonary vein:

$$\bar{S} = \frac{\left(P_{O_2}/P_{50}\right)^{2.6}}{1 + \left(P_{O_2}/P_{50}\right)^{2.6}} = \frac{(100/26)^{2.6}}{1 + (100/26)^{2.6}} = 0.972$$

Likewise for the pulmonary artery:

$$\bar{S} = \frac{\left(P_{O_2}/P_{50}\right)^{2.6}}{1 + \left(P_{O_2}/P_{50}\right)^{2.6}} = \frac{(20/26)^{2.6}}{1 + (20/26)^{2.6}} = 0.3357$$

This is substantially less than the value in the pulmonary artery under resting conditions, S = 0.754.

The concentration in blood is:

$$C_{O_2} = H_{O_2} P_{O_2} (1 - \text{Hct}) + \left(4C_{Hb}\bar{S} + H_{Hb} P_{O_2}\right)\text{Hct}$$

For men

$$C_v = \left(1.33 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(20 \text{ mmHg})0.55 + \left((0.0203 \text{ M})(0.3357) + \left(1.50 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(20 \text{ mmHg})\right)0.45 = 0.0031\ M$$

$$C_a = \left(1.33 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(100 \text{ mmHg})0.55 + \left((0.0203 \text{ M})(0.972) + \left(1.50 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(100 \text{ mmHg})\right)0.45 = 0.0090\ M$$

For women

$$C_v = \left(1.33 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(20 \text{ mmHg})0.60 + \left((0.0203 \text{ M})(0.3357) + \left(1.50 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(20 \text{ mmHg})\right)0.40 = 0.00275\ M$$

$$C_a = \left(1.33 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(100 \text{ mmHg})0.60 + \left((0.0203 \text{ M})(0.972) + \left(1.50 \text{ x } 10^{-6} \text{ M mmHg}^{-1}\right)(100 \text{ mmHg})\right)0.40 = 0.0080\ M$$

Thus, the oxygen consumption rates are

$Q(C_v - C_a)$	0.148 mole O_2/min	men
	0.132 mole O_2/min	women

These values are about 14 times larger than the values under resting conditions.

From Equation (1)

$\dot{V}_I = Q\dfrac{(C_v - C_a)}{(C_I - C_{alv})}$ 52.5 L O_2/min men 46.8 L O_2/min women

For a respiration rate of 30 breaths per minutes, the net volume inspired in each breadth is: 1.75 L/min for men and 1.56 L/min for women. In terms of the total air inspired in each breadth, it is 1.94 L/min for men and 1.70 L/min for women.

1.11. CO = HR x SV where CO is the cardiac output (L min^{-1}), SV is the stroke volume (L) and HR is the hear rate in beat min^{-1}.

	Stroke Volume, L	
	Rest	Exercise
Athlete	0.0833	0.238
Sedentary person	0.0694	0.2

The peripheral resistance is $R = \overline{p}_a / CO$

	Peripheral resistance, mm Hg/(L/min)	
	Rest	Exercise
Athlete	20	5.2
Sedentary person	20	6

$W = \int \overline{p}_a dV = \overline{p}_a \Delta V$ since the mean arterial pressure is assumed constant. DV corresponds to the stroke volume.

Note 1 L = 1000 cm^3 *(1 m/100 cm)^3 = 0.001 m^3
100 mm Hg = 13,333 Pa

Sedentary person
W = (100 mm Hg)(133.3 Pa/mm Hg)(0.069 L)(1000 cm^3/L)(1 m^3/1x10^6 cm^3) =

	Work, J (N m)	
	Rest	Exercise
Athlete	1.11	4.12
Sedentary person	0.925	4.00

	Power, W (J/s)	
	Rest	Exercise
Athlete	1.11	7.22
Sedentary person	0.924	8.33

1.12. Although the pressure drops from 760 mm Hg to 485 mm Hg, the partial pressures are unchanged. The inspired air at 3,650 m is 101.85 mm Hg. For a 30 mm Hg drop, the alveolar air is at 71.85 mm Hg.

The oxygen consumption rate is $\dot{V}_{O_2} = \dot{V}_I\left(C_I - C_{alv}\right)$

Assuming that the inspired air is warmed to 37 C

$$C_I = \frac{p_I}{RT} = \frac{\left(101..85 \text{ mm Hg}\right)/\left(760 \text{ mm Hg/atm}\right)}{\left(0.08206 \text{ L atm/(mol K)}\right)\left(310 \text{ K}\right)} = 0.00527 \text{ M}$$

$$C_{alv} = \frac{p_{alv}}{RT} = \frac{\left(71.85 \text{ mm Hg}\right)/\left(760 \text{ mm Hg/atm}\right)}{\left(0.08206 \text{ L atm/(mol K)}\right)\left(310 \text{ K}\right)} = 0.00372 \text{ M}$$

Assuming that the inspired and dead volumes are the same as at sea level

$$\dot{V}_I = f\left(V_I - V_{dead}\right) = 20\left(0.56 \text{ L} - 0.19 \text{ L}\right) = 7.4 \text{ L min}^{-1}$$

The venous blood is at a partial pressure of 0.98(71.85) = 70.32 mm Hg

The corresponding saturation is 0.930.

1.13. (1650 kcal/day)*4.184 kJ/kcal*(1day/24 h)*(1 h/3600 s) = 79.9 J/s

	Rest
Athlete	0.014
Sedentary person	0.014

1.14. The concentrations are found as the ratio of the solute flow rate/fluid flow rate

	Urine, M	Plasma, M	Urine/Plasma
Sodium	0.1042	0.08444	1.233
Potassium	0.0694	0.004	17.36
Glucose	0.000347	0.00444	0.0781
Urea	0.32431	0.005183	62.57

The results indicate that urine concentrates sodium to a small extent, potassium to a higher level and urea to very high levels. Glucose is at a lower concentration in urine than plasma, suggesting that its transport across the glomerulus is restricted.

1.15. Assuming that inulin is not reabsorbed by the kidneys and returned to the blood, then the mass flow rate of inulin across the glomerulus must equal the mass flow rate in urine. The mass flow rate is the product of the mass concentration (mass/volume) multiplied by the flow rate (volume/time). Thus,

$$C_{inulin}^{plasma} GFR = C_{inulin}^{urine} Q_{urine}$$

Solving for the glomerular filtration rate:

$$GFR = \frac{C_{inulin}^{urine}}{C_{inulin}^{plasma}} Q_{urine} = \left(\frac{0.125}{0.001}\right)\left(1 \text{ mL min}^{-1}\right) = 125 \text{ mL min}^{-1}$$

Solution to Problems in Chapter 2, Section 2.10

2.1. $Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int_{y=0}^{3} \int_{x=0}^{2} \left(\frac{3}{\sqrt{2}} x + \frac{6}{\sqrt{2}} y \right) dx dy = \int_{y=0}^{3} \left(\frac{3}{2\sqrt{2}} x^2 + \frac{6}{2\sqrt{2}} yx \right) \Bigg|_{x=0}^{2} dy$

$Q = \int_{y=0}^{3} \left(\frac{6}{\sqrt{2}} + \frac{12}{\sqrt{2}} y \right) dy = \left(\frac{6}{\sqrt{2}} y + \frac{6}{\sqrt{2}} y^2 \right) \Bigg|_{y=0}^{3} = \frac{72}{\sqrt{2}}$

Q = 50.91 $cm^3 s^{-1}$

2.2. $|\boldsymbol{n}| = 1 = \sqrt{a^2 + a^2 + a^2} = \sqrt{3} a$

Rearranging, $a = 1/\sqrt{3}$

2.3.

$$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \bullet (\rho \mathbf{v}\mathbf{v}) = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \bullet \left(\rho \mathbf{e}_x \mathrm{v_x} \mathbf{v} + \rho \mathbf{e}_y \mathrm{v_y} \mathbf{v} + \rho \mathbf{e}_z \mathrm{v_z} \mathbf{v} \right)$$

$$= \frac{\partial}{\partial x}(\rho \mathrm{v_x} \mathbf{v}) + \frac{\partial}{\partial y}(\rho \mathrm{v_y} \mathbf{v}) + \frac{\partial}{\partial z}(\rho \mathrm{v_z} \mathbf{v})$$

Differentiating term by term,

$$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \mathbf{v} \left(\frac{\partial}{\partial x}(\rho \mathrm{v_x}) + \frac{\partial}{\partial y}(\rho \mathrm{v_y}) + \frac{\partial}{\partial z}(\rho \mathrm{v_z}) \right) + \rho \mathrm{v_x} \frac{\partial}{\partial x}(\mathbf{v}) + \rho \mathrm{v_y} \frac{\partial}{\partial y}(\mathbf{v}) + \rho \mathrm{v_z} \frac{\partial}{\partial z}(\mathbf{v})$$

$$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \mathbf{v} \nabla \bullet (\rho \mathbf{v}) + \rho \mathbf{v} \bullet \nabla \mathbf{v}$$

2.4. (a) For a two-dimensional steady flow, the acceleration is:

$$\mathbf{a} = \mathrm{v}_x \frac{\partial \mathbf{v}}{\partial \mathrm{x}} + \mathrm{v}_y \frac{\partial \mathbf{v}}{\partial \mathrm{y}}$$

For $\mathbf{v} = U_o(x^2 - y^2 + x)\mathbf{e}_x - U_o(2xy + y)\mathbf{e}_y$,

$$\frac{\partial \mathbf{v}}{\partial \mathrm{x}} = U_o(2x+1)\mathbf{e}_\mathrm{x} - U_o 2\mathrm{y}\mathbf{e}_\mathrm{y} \qquad \frac{\partial \mathbf{v}}{\partial \mathrm{y}} = U_o(-2y)\mathbf{e}_\mathrm{x} - U_o(2\mathrm{x}+1)\mathbf{e}_\mathrm{y}$$

$$\mathbf{a} = \mathrm{U}_o^{\,2}\left(x^2 - y^2 + x\right)\left((2x+1)\mathbf{e}_x - 2\mathrm{y}\mathbf{e}_\mathrm{y}\right) - \mathrm{U}_o^{\,2}\left(2xy + y\right)\left(-2y\mathbf{e}_\mathrm{x} - (2\mathrm{x}+1)\mathbf{e}_\mathrm{y}\right)$$

Collecting terms:

$$\mathbf{a} = \mathrm{U}_o^{\,2}\left[\left(x^2 - y^2 + x\right)(2x+1) + (2xy+y)2y\right]\mathbf{e}_x - \mathrm{U}_o^{\,2}\left[\left(x^2 - y^2 + x\right)2y - (2xy+y)(2x+1)\right]\mathbf{e}_\mathrm{y}$$

$$\mathbf{a} = \mathrm{U}_o^{\,2}\left[2x^3 + 3x^2 - 2xy^2 - y^2 + x + 4xy^2 + 2y^2\right]\mathbf{e}_x - \mathrm{U}_o^{\,2}\left[2yx^2 - 2y^3 + 2xy - 4x^2y + 2xy + 2xy + y\right]\mathbf{e}_\mathrm{y}$$

$$\mathbf{a} = \mathrm{U}_o^{\,2}\left[2x^3 + 3x^2 + x + 2xy^2 + y^2\right]\mathbf{e}_x - \mathrm{U}_o^{\,2}\left[-2yx^2 - 2y^3 + 6xy + y\right]\mathbf{e}_\mathrm{y}$$

$$\mathbf{a} = \mathrm{U}_o^{\,2}\left[\left(2x^2 + 2y^2 + 3x + 1\right)x + y^2\right]\mathbf{e}_x + \mathrm{U}_o^{\,2}\left(2x^2 + 2y^2 - 6x - 1\right)y\mathbf{e}_\mathrm{y}$$

At y = 1 and x = 0 $\quad \mathbf{a} = (2)^2\left(\mathbf{e}_x + \mathbf{e}_y\right) = 4\mathbf{e}_x + 4\mathbf{e}_y$

At y = 1 and x = 2

$$\mathbf{a} = (2)^2\left[(8+2+6+1)2+1\right]\mathbf{e}_x + (2)^2(8+2-12-1)\mathbf{e}_y = 140\mathbf{e}_x - 12\mathbf{e}_y$$

(b) From equation 2.2.6

$$Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int \mathrm{v}_x dydz$$

since $\mathbf{n} = \mathbf{e}_x$.

$$Q = \int_{y=0}^{5}\int_{z=0}^{3} U_o\left(x^2 - y^2 + x\right)\Big|_{x=5} dydz = 3U_o \int_{y=0}^{5}\left(30 - y^2\right)dy = 3U_o\left(30y - \frac{y^3}{3}\right) = 6\left(150 - \frac{125}{3}\right) = 650 \text{ m}^3\text{s}^{-1}$$

2.5. (a) $a_x = \boldsymbol{e}_x \boldsymbol{a} = \mathrm{v}_x \dfrac{\partial \mathrm{v}_x}{\partial x} = U_0\left(1 - \dfrac{x}{L}\right)^{-2} \dfrac{\partial}{\partial x}\left[U_0\left(1 - \dfrac{x}{L}\right)^{-2}\right]$

$$\frac{\partial}{\partial x}\left[\left(1 - \frac{x}{L}\right)^{-2}\right] = \frac{2}{L}\left(1 - \frac{x}{L}\right)^{-3}$$

$$a_x = \boldsymbol{e}_x \boldsymbol{a} = \mathrm{v}_x \frac{\partial \mathrm{v}_x}{\partial x} = U_0^{\,2}\left(1 - \frac{x}{L}\right)^{-2} \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{L}\right)^{-2}\right] = \frac{2U_0^{\,2}}{L}\left(1 - \frac{x}{L}\right)^{-5}$$

For values given:

$$a_x = \frac{50 \text{ m}^2/\text{s}^2}{2 \text{ m}}(1 - 0.5)^{-5} = \left(25 \text{ m/s}^2\right)/(1/32) = 800 \text{ m/s}^2$$

(b) (1) The "no slip" boundary condition is not satisfied.
(2) At x = L, the acceleration is undefined!

2.6. (a) Using the definition of the volumetric flow rate, Q

$$Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int_0^{2\pi}\int_0^{R_i} \mathrm{v}_z r dr d\theta$$

The cross-sectional area element in cylindrical coordinates is rdrdθ. Since the velocity does not vary with angular position, substitution for v_z and integration in the angular direction yields:

$$Q = \int_0^{2\pi}\int_0^{R_i} \mathrm{v}_{\max}\left(1 - \frac{r^2}{R^2}\right) r dr d\theta = 2\pi \mathrm{v}_{\max}\int_0^{R_i}\left(1 - \frac{r^2}{R_i^2}\right) r dr$$

R_i is used to denote the local radius within the stenosis. Integrating in the radial direction yields:

$$Q = 2\pi \mathrm{v}_{\max}\int_0^{R_i}\left(1 - \frac{r^2}{R_i^2}\right) r dr = 2\pi \mathrm{v}_{\max}\left(\frac{r^2}{2} - \frac{r^4}{4R_i^2}\right)_{r=0}^{R} = \frac{\pi R_i^2}{2}\mathrm{v}_{\max}$$

Solving for v_{max}: $$v_{max} = \frac{2Q}{\pi R_i^2} = \frac{2Q}{\pi R_0^2\left[1-0.5\left(1-4\left(\frac{z}{L}\right)^2\right)^{1/2}\right]^2}$$

Outside the stenosis, $R_i = R_0$ and:

$$v_{max} = \frac{2Q}{\pi R_0^2}$$

(b) At z = 0, the velocity in the stensosis is

$$v_{max} = \frac{2Q}{\pi R_i^2} = \frac{2Q}{\pi R_0^2[0.5]^2} = \frac{8Q}{\pi R_0^2}$$

$$R_i = R_0\left[1-0.5\left(1-4\left(\frac{z}{L}\right)^2\right)^{1/2}\right] = 0.5R_0$$

The shear stress in the stenosis is:

$$\tau_{rz}\big|_{stenosis} = \mu\frac{\partial v_r}{\partial z} = \mu\frac{\partial}{\partial r}\left[v_{max}\left(1-\frac{r^2}{R_i(z=0)^2}\right)\right]_{r=R_i} = -\frac{2\mu R_i(z=0)v_{max}}{R_i(z=0)^2} = -\frac{32\mu Q}{\pi R_0^3}$$

Outside the stenosis the shear stress is:

$$\tau_{rz} = \mu\frac{\partial}{\partial r}\left[v_{max}\left(1-\frac{r^2}{R_0^2}\right)\right]_{r=R_0} = -\frac{2\mu v_{max}}{R_0} = -\frac{4\mu Q}{\pi R_0^3}$$

2.7. Evaluating Equation (2.7.30) for y = -h/2 yields:

$$\tau_w = \tau_{yx}(y=-h/2) = \frac{\Delta p}{L}\frac{h}{2} \qquad \text{(S2.7.1)}$$

From Equations (2.7.23) and (2.7.26),

$$\frac{\Delta p}{L} = \frac{8\mu v_{max}}{h^2} = \frac{12\mu Q}{wh^3} \qquad \text{(S2.7.2)}$$

Replacing $\Delta p/L$ in Equation (S2.7.1) with the expression in Equation (S2.7.2) yields

$$\tau_w = \frac{6\mu Q}{wh^2}$$

Solving for h: $h = \sqrt{\frac{6\mu Q}{w\tau_w}}$

Inserting the values provided for Q, w, μ and τ_w yields h = 0.051 cm.

2.8. (a) $\Delta p = \rho gh = (1\ \text{g cm}^{-3})(980\ \text{cm s}^{-2})(2.5 \times 10^{-4}\ \text{cm}) = 0.245\ \text{dyne cm}^{-2}$

(b) Rearranging equation (2.4.16) we have

$$T_c = \frac{\Delta p}{2\left(\frac{1}{R_p} - \frac{1}{R_c}\right)}$$

T_c = 1.838 x 10^{-5} dyne cm^{-1}

2.9. (a) To find the radius use Equation (2.4.16) and treat the pipet radius as the capillary radius R_c = R_{cap}.

$$\Delta p = 2T_c\left(\frac{1}{R_{cap}} - \frac{1}{R_c}\right)$$

For R_c = 6.5 μm
T_c = 0.06 mN/m= 6 x 10^{-5} N/m(1 x 10^{-6} m/μm) = 6 x 10^{-11} N/μm

Δp = 0.2 mm Hg
Since 1.0133×10^5 N m^{-2} = 760 mm Hg
0.2 mm Hg = 26.7 N m^{-2}(1 m/10^6 μm)^2 = 2.67 x 10^{-11} N μm^{-2}

Solving for R_{cap}

$$\frac{1}{R_{pcap}} = \frac{1}{R_c} + \frac{\Delta p}{2T_c}$$

$$R_{cap} = \frac{1}{\frac{1}{R_c} + \frac{\Delta p}{2T_c}}$$

$$R_{cap} = \frac{1}{\frac{1}{R_c} + \frac{\Delta p}{2T_c}} = \frac{1}{\frac{1}{6.5} + \frac{2.67}{2(6)}} = 2.66 \ \mu m$$

While this result satisfies the law of Laplace, we need to assess whether the surface area is no greater than the maximum surface area of the cell, 1.4 times the surface area of a spherical cell, or 743.3 μm^2. The factor of 1.4 accounts for the excess surface area. Ideally, a larger cell entering a smaller capillary with look like a cylinder with hemispheres on each end. The cylinder will have length l and radius equal to the capillary. The hemispheres will have a radius equal to the capillary radius. The volume must remain constant, so

$$V = \frac{4}{3}\pi R_c^3 + \pi R_c^2 L$$

Solving for the length,

$$L=\frac{V-\frac{4}{3}\pi R_c^{\,3}}{\pi R_c^{\,2}}=\frac{\frac{4}{3}\pi\left(R^3-R_c^{\,3}\right)}{\pi R_c^{\,2}}=\frac{4}{3}\frac{\left(6.5^3-2.66^3\right)}{2.66^2}=48.2\ \mu\text{m}$$

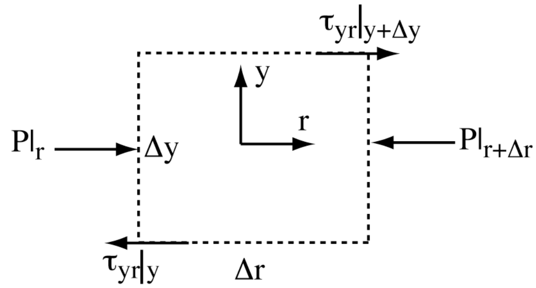
The resulting surface area is $SA=4\pi R_c^{\,2}+2\pi R_c L=\pi\left(4*2.66^2+2*48.2*2.66\right)=894.6\ \mu\text{m}^2$
This is larger than the surface area 530.9 μm^2 or 1.4 times the surface area 743.3 μm^2.

To find the radius and length, one could iteratively solve for L and surface area of use the fzero function in MATLAB. After several iterations, the result approaches a radius of 3.3 μm and L = 29.2 μm.

If the cell had no excess area, then the cell would have no capacity to enter a capillary smaller than itself!

(b) Whether or not excess area is not considered, a cell with a radius of 3.0 μm can enter the capillary.

2.10. A momentum balance is applied on a differential volume element, 2πrΔrΔy, as shown in the figure below.



$$p\big|_r 2\pi r\Delta y - p\big|_{r+\Delta r}2\pi(r+\Delta r)\Delta y+\tau_{yr}\big|_{y+\Delta y}2\pi r\Delta r-\tau_{yr}\big|_{y}2\pi r\Delta r=0 \qquad \text{(S2.10.1)}$$

Divide each term by 2πrΔrΔy and take the limit as Δr and Δy go to zero results in the following expression:

$$\frac{1}{r}\frac{d(rp)}{dr}=\frac{d\tau_{yr}}{dy} \qquad \text{(S2.10.2)}$$

Note that if the gap distance h is much smaller than the radial distance, then curvature is not significant. Each side is equal to a constant C_1. Solving for the shear stress, $\tau_{yr} = C_1 y + C_2$. Substituting Newton's law of viscosity and integrating yields:

$$\mathrm{v}_r=\frac{C_1 y^2}{2\mu}+\frac{C_2}{\mu}y+C_3 \qquad \text{(S2.10.3)}$$

Applying the boundary conditions that $\mathrm{v}_r = 0$ at $y=\pm h/2$,

$$0=\frac{C_1 h^2}{8\mu}+\frac{C_2}{\mu}\frac{h}{2}+C_3 \qquad \text{(S2.10.4a)}$$

$$0=\frac{C_1 h^2}{8}-\frac{C_2}{\mu}\frac{h}{2}+C_3 \qquad \text{(S2.10.4b)}$$

Adding Equations (S2.10.4a) and (S2.10.4b) and solving for C_3,

$$C_3 = -\frac{C_1 h^2}{8} \qquad \text{(S2.10.5)}$$

Inserting Equation (S2.10.5) into Equation (S2.10.4a) yields $C_2 = 0$. Thus the velocity is:

$$v_r = \frac{C_1}{\mu}\left(\frac{y^2}{2} - \frac{h^2}{8}\right) \qquad \text{(S2.10.6)}$$

The volumetric flow rate is:

$$Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int_{y=-h/2}^{h/2} 2\pi r v_r dy = \frac{2\pi r C_1}{\mu}\int_{y=-h/2}^{h/2}\left(\frac{y^2}{2} - \frac{h^2}{8}\right)dy \qquad \text{(S2.10.7)}$$

$$Q = \frac{2\pi r C_1}{\mu}\left(\frac{y^3}{6} - \frac{h^2 y}{8}\right)\Bigg|_{y=-h/2}^{h/2} = \frac{2\pi r C_1 h^3}{\mu}\left(\frac{1}{24} - \frac{1}{8}\right) = -\frac{\pi r C_1 h^3}{6\mu} \qquad \text{(S2.10.8)}$$

Solving for C_1 and inserting into equation (S2.10.6)

$$v_r = -\frac{6Q}{\pi r h^3}\left(\frac{y^2}{2} - \frac{h^2}{8}\right) \qquad \text{(S2.10.9)}$$

The shear stress can thus be written as;

$$\tau_w = \tau_{yr}\Big|_{y=-h/2} = \mu\frac{dv_r}{dr}\Big|_{y=-h/2} = -\frac{6Q}{\pi r h^3}y\Big|_{y=-h/2} = \frac{3\mu Q}{\pi r h^2} \qquad \text{(S2.10.10)}$$

2.11. Flow rate per fiber, Q_f = Q/250 = 0.8 mL/60 s = 0.01333 mL/s

Average velocity per fiber: $\langle v_f \rangle = Q_f/\pi R_f^2$ = (0.01333 mL/s)/(3.14159*(0.01 cm)^2)
$\langle v_f \rangle$ = 42 cm/s

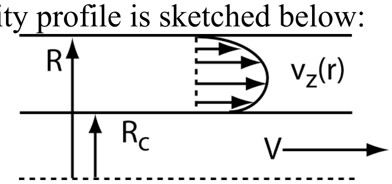
Re = $\rho\langle v_f\rangle D_f/\mu$ = 1.05*42*0.02/0.03 = 29.7.

L_e = 0.058DRe = 0.058*(0.02 cm)(29.7) = 0.034 cm << L = 30 cm.

2.12. (a) The momentum balance is the same as that used for the case of pressure-driven flow in a cylindrical tube in Section 2.7.3.

$$\frac{dp}{dz} = \frac{1}{r}\frac{d(r\tau_{rz})}{dr} \qquad \text{(S2.12.1)}$$

(b) The velocity profile is sketched below:



Integrating the momentum balance and substituting Newton's law of viscosity,

$$\tau_{rz} = -\frac{\Delta p}{2L}r + \frac{C_1}{r} = \mu\frac{dv_z}{dr} \qquad \text{(S2.12.2)}$$

Note that the shear stress and shear rate are a maximum at $r=\sqrt{\frac{2C_1}{\Delta p/L}}$. Assuming that C_1 is greater than zero, then r will have a maximum in the fluid.

(c) Integrating Equation (S2.12.2) yields:

$$v_z = -\frac{\Delta p}{4\mu L} r^2 + \frac{C_1}{\mu} ln(r) + C_2 \qquad \text{(S2.12.3)}$$

Applying the boundary conditions

$$V = -\frac{\Delta p}{4\mu L} R_C{}^2 + \frac{C_1}{\mu} ln(R_C) + C_2 \qquad \text{(S2.12.4a)}$$

$$0 = -\frac{\Delta p}{4\mu L} R^2 + \frac{C_1}{\mu} ln(R) + C_2 \qquad \text{(S2.12.4b)}$$

Subtracting

$$V = -\frac{\Delta p}{4\mu L}\left(R_C{}^2 - R^2\right) + \frac{C_1}{\mu} ln\left(\frac{R_C}{R}\right) \qquad \text{(S2.12.5)}$$

Solving for C_1:

$$C_1 = \frac{\mu V}{\ln\left(\frac{R_C}{R}\right)} + \frac{\Delta p}{4L}\frac{\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R_C}{R}\right)} \qquad \text{(S2.12.6a)}$$

Using this result to find C_2

$$C_2 = \frac{\Delta p}{4\mu L}R^2 - \left(\frac{V}{\ln\left(\frac{R}{R_C}\right)} + \frac{\Delta p}{4\mu L}\frac{\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R}{R_C}\right)}\right)\ln(R) \qquad \text{(S2.12.6b)}$$

The resulting expression for the velocity profile is

$$v_z = \frac{\Delta p R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right) + \left(V + \frac{\Delta p}{4\mu L}\left(R_C{}^2 - R^2\right)\right)\frac{\ln\left(\frac{r}{R}\right)}{\ln\left(\frac{R}{R_C}\right)} \qquad \text{(S2.12.7)}$$

(d) The shear stress is:

$$\tau_{zr} = \mu\frac{dv_z}{dr} = -\frac{r\Delta p}{2L} + \left(\frac{\mu V + \frac{\Delta p}{4L}\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R}{R_C}\right)}\right)\left(\frac{1}{r}\right) \qquad \text{(S2.12.8)}$$

(e) At r = R, the shear stress is: