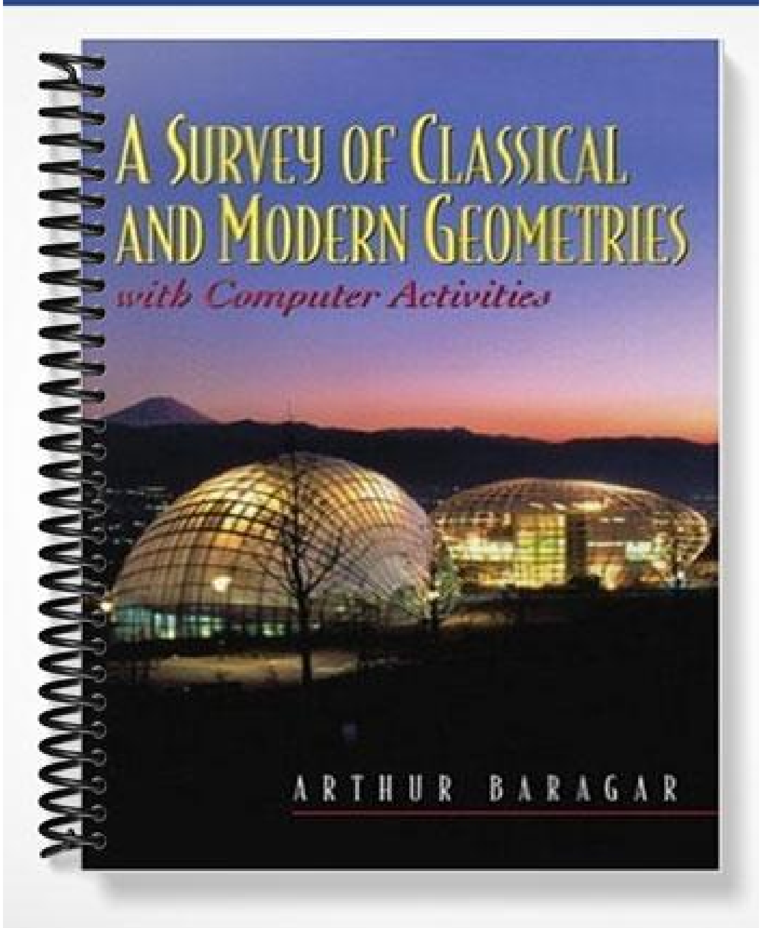
SOLUTIONS MANUAL



A Survey of Classical and Modern Geometries

Instructors' Solution Manual

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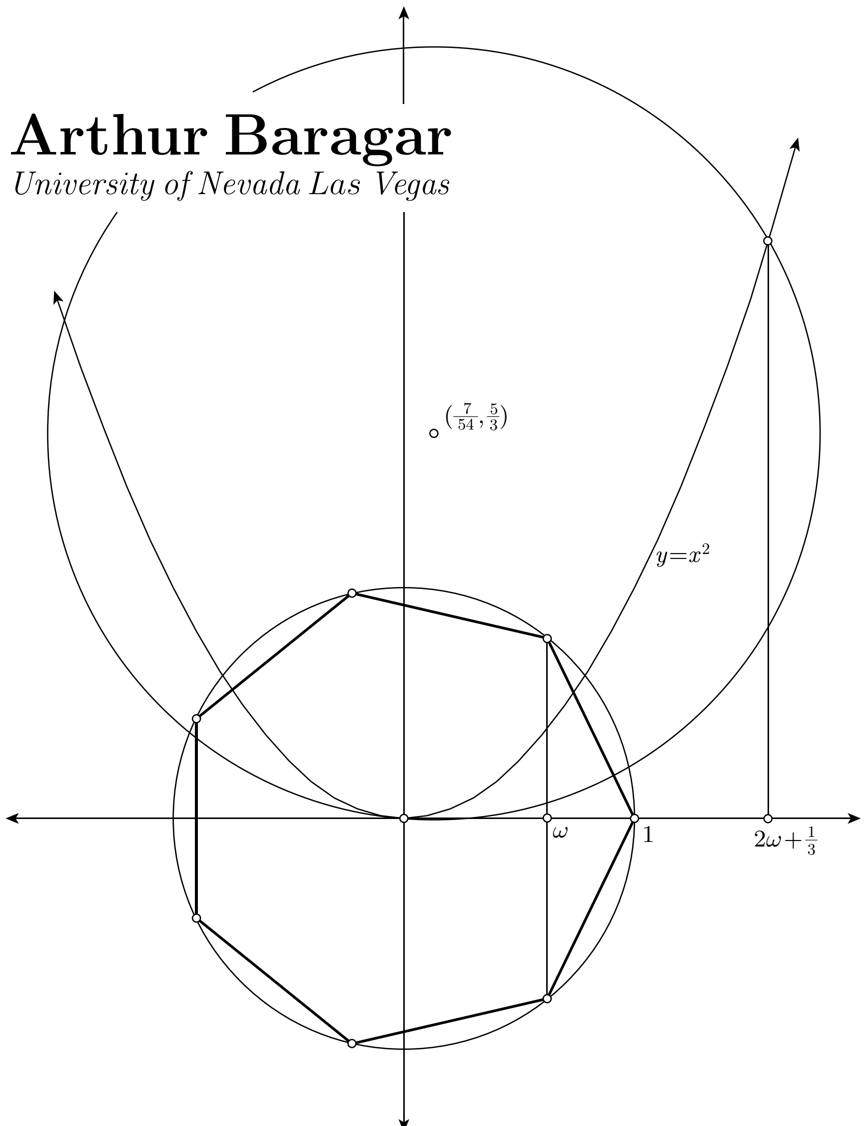

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Dear Instructor,

I was under a severe time crunch when I wrote this solution manual. Not all exercises are solved. Of the 600+ exercises in the text, about 500 are solved here. There are no solutions to exercises from Chapters 13 − 15. I believe all my solutions are correct, but there are many subtleties that are easy to miss. I am sure there are quite a few errors, probably in exposition or typing. I hope they are not too severe and that the reader will be able to sort them out. Nevertheless, reports of errors are appreciated and can be sent to the email address below. Despite the rush, there are some solutions that I am proud of. Some of the exercises in this text make good projects. For example, Exercises 1.101 (Feuerbach's Theorem), 3.7 (Mohr constructions), 3.8 (Rusty compass problem), 4.40 (Construction of the 13-gon using parabola paper in sketchpad), and 5.20 (classifying convex polyhedra with only equilateral faces). The intended use of this manual is two fold. An instructor can consult it to gauge the level of difficulty of an exercise, or copy portions of it to distribute to their class when they deem fit, presumably after homework has been handed in. I have tried to keep that latter intent in mind while writing this and hope that the exposition is clear enough for such an audience.

There is an electronic file that goes with this manual. The file contains scripts that solve most of the exercises in Chapter 4, as well as some in Chapters 7 and 8. These files are compressed and password protected (the password is Pi∼22/7) and will be available on the website

www.prenhall.com/baragar

As I am writing this, the site has not yet been set up, but I hope it will be up and running in the next week or so. Currently, the scripts are for Windows based PC's. I hope to make these available in Mac format too. I don't think there's a unix version of Sketchpad.

I hope you find this manual useful and enjoy teaching geometry as much as I have.

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Chapter 1

Euclidean Geometry

Exercise 1.1. Suppose that at the spring solstice the angle between a star, the Earth, and the sun measures 79.1°, and at the fall solstice the angle measures 100.8°. How far away is the star from the Earth? (The Earth is 93 million miles from the sun.)

The solution is in the text.

Exercise 1.2. The numbers in Exercise 1.1 were cooked up. From your general knowledge, are these numbers reasonable? Explain.

Solution. In Exercise 1.1, we found that the star is 100,000,000,000 miles away. Let us convert this figure to light years. A light year is how far light travels in a year. Since light travels at a speed of 186,000 miles per second, one light year is the same as

$$\begin{aligned} 1 \text{ lightyear} &= 186,000 \frac{\text{miles}}{\text{sec}} \text{ yr} \frac{3600\,\text{sec}}{\text{hr}} \frac{24 \text{ hrs}}{\text{day}} \frac{365 \text{ day}}{\text{yr}} \\ &= 5.87 \times 10^{12} \text{ miles.} \end{aligned}$$

Thus, the distance to the star in Exercise 1.1 is

$$10^{11} \text{ miles} \frac{1 \text{ lightyear}}{5.87 \times 10^{12} \text{ miles}} = .017 \text{ lightyear.}$$

Since this is much smaller than 4.3 lightyears, the distance to the nearest star (other than the Sun), we conclude that the numbers are not reasonable. □

Exercise 1.3. What order of magnitude is the difference between 180° and the sum of the angles for the summer and winter observations of Alpha Centauri? (See Figure 1.1.)

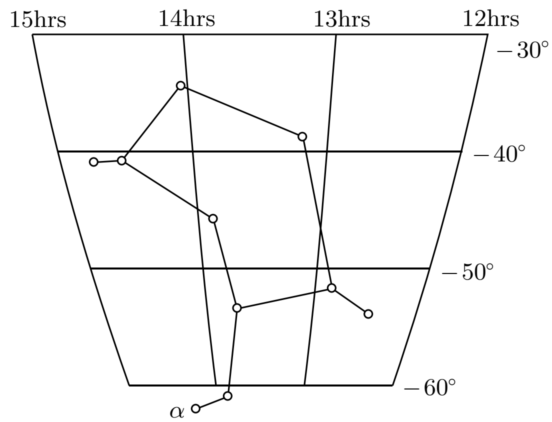

Figure 1.1. The nearest star (other than the sun) is Alpha Centauri, also known as Rigil Kentaurus, which is 4.3 light years away. This star is the α or brightest star in the Centaurus constellation, which is just south of Libra. It is visible from June to December in the nighttime sky of the southern hemisphere. It briefly appears above the southern horizon in parts of Florida and southern Texas.

Solution. Let Alpha Centauri be the vertex A, let the Earth during the summer be the vertex B, and let the Earth during the winter be the vertex C, forming a long narrow triangle ΔABC. By the Law of Sines, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

We are interested in the angle A. We know $a = 4.3$ lightyears and $b = 186,000,000$ miles. We do not know the exact measurement of B, but expect it to be close to $90°$. Thus,

$$\begin{aligned} \sin A &= \frac{b}{a \sin B} \approx \frac{4.3(5.87 \times 10^{12} \text{ miles})}{186,000,000 \text{ miles}} \\ &\approx 1.357 \times 10^{-5} \\ A &\approx .0008°. \end{aligned}$$

So, the order of magnitude of the angle at A is 10^{-3} degrees. □

Exercise 1.4*. In Figure 1.1, how much does Alpha Centauri move (use units of length) between the summer and winter observations?

Solution. By Exercise 1.3, the angle at the star is about $0.0008°$. We assume that the background stars are so far away that our lines of sight to them for the summer and winter observations are parallel. Thus, our line of sight to Alpha Centauri changes by 0.0008°. Let us now inspect the markings in Figure 1.1. The markings on the side are in degrees. Along the top, they are in hours, and with a little thought, we should come to the conclusion that a full circle is 24 hrs, so each hour represents 15°. However, the horizontal markings (those in hours) follow lines of latitudes, so vary, while the vertical markings follow lines of longitude, which are always great circles. Using a ruler, we find that 10° is approximately 16 mm, so 0.0008° is approximately 0.001 mm in this figure. Since this is so tiny, we conclude that to actually calculate this distance, we must take large pictures of a much smaller portion of the night time sky. □

1.1 The Pythagorean Theorem

Exercise 1.5. The diagram in Figure 1.2(a) suggests a different proof of the Pythagorean theorem. Fill in the details.

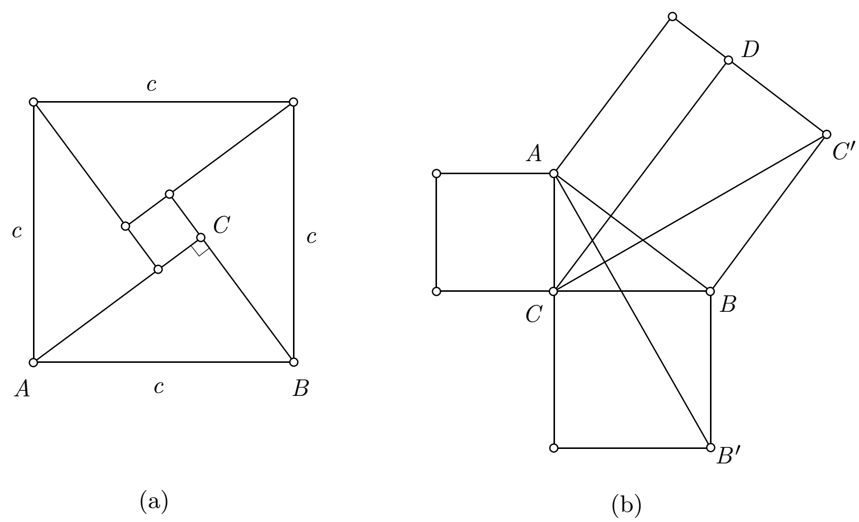

Figure 1.2. See Exercises 1.5 and 1.6.

Solution. Consider a square of side c. On each edge, place a copy of our triangle ΔABC with the hypotenuse on each edge, and the vertex C inside the square. Since the angles A and B are complimentary, we know that the edges of these triangles match up, as in Figure 1.2(a), leaving a small square in the center whose edges have length $|a - b|$. Calculating the area in two different ways, we find

$$\begin{aligned} c^2 &= 4(\frac{1}{2}ab) + (a-b)^2 \\ &= 2ab + a^2 - 2ab + b^2 \\ c^2 &= a^2 + b^2. \end{aligned}$$ □

Exercise 1.6. The diagram in Figure 1.2(b) suggests another proof of the Pythagorean theorem. Fill in the details. [H]

Solution. Let us first look at $\Delta ABB'$. Since the area of a triangle is half of its base times height, we know

$$|\Delta ABB'| = \frac{a^2}{2}.$$

To see this, think of BB' as being the base. Then the altitude at A is congruent to BC.

Now, note that $|BB'| = |BC|$, $|BA| = |BC'|$, and $\angle ABB' = \angle C'BC$, so by SAS, the two triangles $\Delta ABB'$ and $\Delta C'BC$ are congruent. In particular, their areas are equal.

We calculate the area of $\Delta C'BC$ by thinking of the base as BC', so the altitude is congruent to CD, and hence

$$|\Delta C'BC| = \frac{1}{2}c|C'D|.$$

Equating the areas of these two congruent triangles, we get

$$c|C'D| = a^2.$$

In a similar fashion, we can show that the area of the remaining rectangle in the square with side C (the rectangle with diagonal AD) has an area of b^2. Thus,

$$c^2 = a^2 + b^2,$$

as desired. □

Exercise 1.7 (Pappus' Variation on the Pythagorean Theorem). Let ΔABC be a triangle (not necessarily right). Let $ACDE$ and $BCFG$ be parallelograms whose sides DE and FG intersect at H (see Figure 1.3). Let $ABIJ$ be a parallelogram with sides AJ and BI parallel to and with the same length as CH. Prove that the area of $ABIJ$ is equal to the sum of the areas of the other two parallelograms.

Solution. Extend the line HC so that it intersects the parallelogram $ABIJ$. Label these points K and L. Also, draw a line segment MB parallel to HC with M on FG. Then, the area of $FGBC$ is the same as the area of $HMBC$, since they have the same base BC and the same altitude.

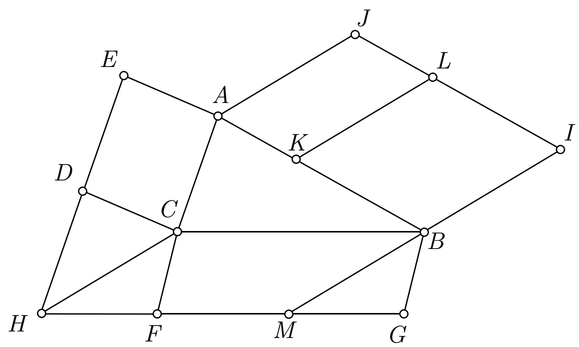

Figure 1.3. See Exercise 1.7

Now let us consider the parallelograms $HMBC$ and $KLIB$. Note that $|KL| = |BI| = |HC|$, by construction, so they have equal length bases. With respect to these bases, their altitudes are also equal, so these two parallelograms have equal areas. Thus, the area of $CBGF$ and $KLIB$ are equal. Similarly, the areas of $AECD$ and $AJLK$ are equal. Thus, the area of $AJIB$ is equal to the sum of the areas of $AECD$ and $CBGF$. $\square$

1.2 The Axioms of Euclidean Geometry

Exercise 1.8. How should we measure the length of a path?

Solution. This question is meant to be more of a source of discussion. One should realize that we measure paths 'one step at a time.' That is, consider a sequence of points along the path. Then the length of the path is approximated by the sum of the distances between these points. As we add more points, the approximation should get better. So, define the length of a path to be the limit of these sums as the length of the steps approach zero, if this limit exists. What we have described is essentially a path integral. $\square$

Exercise 1.9. The Cartesian plane $\mathbb{R}^2$ is a model of Euclidean geometry. In this model, explicitly describe an isometry which has no fixed points and is not a translation.

Solution. Such isometries are called *glide reflections*, a term which might ring a bell. We create such an isometry by translating along a line and then reflecting through that line. For example, the horizontal translation by one in $\mathbb{R}^2$ is given by $T(x, y) = (x + 1, y)$. Reflection through the x-axis is given by $R(x, y) = (x, -y)$. Thus, the composition

$$(R \circ T)(x, y) = R(x + 1, y) = (x + 1, -y)$$

is a glide reflection. It is clearly an isometry, since it is the composition of two isometries. It is not a translation, since it does not preserve orientation. Finally, it has no fixed points, since if it did, then we would have

$$(x, y) = (x + 1, -y),$$

which is not satisfied by any value of x.

Footprints are common examples of objects with glide reflection symmetry. □

Exercise 1.10. Which of Axioms 1 – 8 are local in nature, and which are global? It may help to ask yourself if any of these properties are true on a sphere.

This exercise is meant to be a source of discussion.

Exercise 1.11. The triangle inequality states

$$|PQ| + |QR| \geq |PR|.$$

Show that we can have equality if and only if Q is a point on the line segment PR.

Solution. We cannot yet answer this question with any rigor – our definition of a line segment is not yet sufficient. This exercise (as well as a few others in this section) should be thought of as motivation for defining a stronger foundation. This will be rectified in Chapter 9. At the heart of our deficiency is the question of how we should define line segments and what is meant by Axiom 1. For the moment, we are thinking of a line segment as the shortest path between two points. Thus, the existence part of Axiom 1 merely asserts the existence of a path between any two points. What we further need to know is that the length of a line segment between two points P and R is the same as the distance $|PR|$ between the two points. If we accept this (as a refinement of our definition of a line segment), then we can answer the question.

If Q is on the line segment PR, then the length of the segment PR is the same as the sum of the lengths of PQ and QR. That is,

$$|PR| = |PQ| + |QR|.$$

Conversely, if

$$|PR| = |PQ| + |QR|,$$

then the path which consists of the union of the line segments PQ and QR joins P to R and has length $|PR|$, which is the same length as the line segment joining P and R. Thus, if Q is not on this line segment, then we have found a path of equal length, contradicting the uniqueness part of Axiom 1. Thus, Q must be on PR. □

Exercise 1.12. It is sometimes possible to 'define away' an axiom by choosing a suitable definition. For example, with our definition of a circle, Euclid's third axiom is vacuous – all circles exist by definition, though some may be empty sets. Euclid's first axiom can also be defined away by making the following definition for a line segment:

$$PQ = \{R : |PQ| = |PR| + |RQ|\}.$$

With this definition, between any two points P and Q there exists a unique line segment PQ. Find an example of a familiar geometry where this definition for a line segment does not correspond with our intuitive notion of what a line segment should be.

Solution. Consider a sphere with North pole P and South pole Q. Then for any point R on the sphere, we have

$$|PQ| = |PR| + |RQ|.$$

Thus, the line segment PQ, as defined in this exercise, is the entire sphere. This does not correspond with our intuitive notion of a line segment, since we usually think of line segments as being one dimensional. □

1.3 SSS, SAS, and ASA

The exercises in this section are very tough. I have never assigned them as homework. On the surface, they are indicative of the types of questions which should naturally occur to us, and that we should be able to answer. Deeper analysis might best be done in guided discussions, rather than homework, and should demonstrate a need for a better set of axioms.

Exercise 1.13. Prove SAS.

Solution. Let ΔABC and $\Delta A'B'C'$ satisfy $\angle A = \angle A'$, $|AB| = |A'B'|$, and $|AC| = |A'C'|$. Since $\angle A = \angle A'$, there exists an isometry f which sends $\angle A$ to $\angle A'$ (by the definition of congruence of angles). That is, $f(A) = A'$, and the ray AB is sent to either the ray $A'B'$ or $A'C'$, while the ray AC is sent to the other.

If AB is sent to $A'B'$, then $f(B) = B'$, since $f(B)$ is on the ray AB and $|AB| = |A'B'|$. Similarly $f(C) = C'$, and hence f sends ΔABC to $\Delta A'B'C'$, so they are congruent.

If $f(B)$ lies on the ray $A'C'$, then let us call $B'' = f(B)$ and consider the triangle $\Delta A'B'B''$. Since $|A'B'| = |A'B''|$, we have the congruence

$$\Delta A'B'B'' \equiv \Delta A'B''B'$$

by SSS, and hence there exists an isometry g such that $g(A') = A'$, $g(B') = B''$, and $g(B'') = B'$. Furthermore, g sends $\angle A'$ to $\angle A'$. Thus, there exists an isometry $h = g \circ f$ which sends $\angle A$ to $\angle A'$ which further satisfies $h(B) = B'$. But then, as before, $h(C) = C'$, as desired, so ΔABC is congruent to $\Delta A'B'C'$. □

Exercise 1.14. Prove ASA.

Solution. Suppose there exist ΔABC and $\Delta A'B'C'$ with $|BC| = |B'C'|$, $\angle B = \angle B'$, and $\angle C = \angle C'$. If we also have $|BA| = |B'A'|$, then by SAS, the two triangles are congruent. So let us assume, without loss of generality, that $|BA| < |B'A'|$.

By the definition of congruence of angles, there exists an isometry which sends B to B' and the rays BA and BC to the rays $B'A'$ and $B'C'$. If C is sent to a point C'' on $B'A'$, then by SSS, $\Delta C'B'C'' \equiv \Delta C''B'C'$, so there exists an isometry g which fixes B', sends C'' to C', and sends $\angle C'B'C''$ to $\angle C''B'C'$ (that is, it fixes the angle at B'). Thus, there exists an isometry $h = g \circ f$ such that $h(B) = B'$, $h(C) = C'$, and h sends the angle at B to the angle at B'. If, on the other hand, $f(C)$ is on the ray $B'C'$, then $f(C) = C'$ already, so choose $h = f$.

Now, $A'' = h(A)$ is on the ray $B'A'$, and since $|BA| < |B'A'|$, the point A'' is between B' and A'. On the ray $C'A''$, find the point A''' such that $|C'A'''| = |C'A'|$. Since $\Delta B'C'A''$ is congruent to ΔABC (since it is the image of the latter under an isometry), we know $\angle A''C'B' = \angle ACB$, so $\angle A'''C'B' = \angle ACB = \angle A'C'B'$. Thus, by SAS, $\Delta A'''C'B' \equiv \Delta ACB$. In particular, $|B'A'''| = |B'A'|$. Let D be the midpoint of $A'A'''$. Note that the line $C'A'''$ enters the triangle $\Delta DA'''B'$ at A''', so it must intersect the segment $B'D$. Let us call that point of intersection E. It is clear that $E \neq C'$. Now, consider $\Delta B'A'D$ and $\Delta B'A'''D$. They are congruent by SSS. Thus, $\angle A'DB' = \angle A'''DB'$, and since these two angles sum to a straight line, they must both be right angles. Similarly, $\angle C'DA'$ is a right angle, so the angle $\angle C'DB'$ is a straight line. But this line intersects $C'A''$ at C' and E, a contradiction of Axiom 1. Thus, we could not have $|BA| < |B'A'|$, as assumed, so the two triangles are congruent, as desired. □

Exercise 1.15. Suppose f is an isometry and suppose there exist two distinct points P and Q such that $f(P) = P$ and $f(Q) = Q$. Show that f is either the identity or a reflection.

Solution. Suppose f is not the identity. Then there exists a point A such that $A' = f(A) \neq A$. Let D be the midpoint of AA'. Since P and Q are distinct, D is not one of them, so we may assume, without loss of generality, that $D \neq P$.

Suppose $B \neq P$ and $f(B) = B$. Then, since f is an isometry, $|PA| = |PA'|$, so by SSS, $\Delta PAD \equiv \Delta PA'D$. Hence, $\angle ADP \equiv \angle A'DP$, and since they sum to a straight line, they must both be right angles. Similarly, using B in place of P, we get that $\angle ADB$ is a right angle. Thus, P, D and B are collinear. Since this is true for any B such that $f(B) = B$, it is in particular true for $B = Q$. Thus, $f(B) = B$ only if B is on the line PQ.

Now suppose B is on the line PQ. Let $B' = f(B)$. Then we consider three cases. Either B is between P and Q, P is between B and Q, or Q is between P and B. Let us assume B is between P and Q. Then $|PB| + |BQ| = |PQ|$. Since f is an isometry, we therefore also have $|PB'| + |B'Q| =$

$|PQ|$, which implies B' is on the segment PQ (by Exercise 1.11). Finally, since both B and B' are on the segment PQ, and since $|PB'| = |PB|$, the points B and B' must be equal. The other two cases are similar, so $f(B) = B$ if B is on the line PQ.

Thus, we have shown that f is an isometry which fixes every point on the line PQ, and fixes no other points. That is, f is a reflection through the line PQ. □

Exercise 1.16. Suppose f is a reflection. Prove that f is not a direct isometry.

Solution. Let f be the reflection through the line PQ, and let A be a point not on PQ. Then $A' = f(A) \neq A$. Let D be the midpoint of AA'. Then D lies on PQ (see the solution to the previous exercise). But then ΔPQD and $\Delta PQD'$ have opposite orientations, so f is not a direct isometry. □

Exercise 1.17. Prove that if a line $l_1 \neq l$ is sent to itself under a reflection through l, then l_1 and l intersect at right angles.

Solution. Let A be a point on l_1 such that A is not on l. Then $A' = f(A) \neq A$. Let D be the midpoint of AA'. Note that D is on l_1, since both A and A' are on l_1. Let $P \neq D$ be a point on l. Then $\Delta ADP \equiv \Delta A'DP$ by SSS, so in particular, $\angle ADP = \angle A'DP$. Since they sum to a straight line, they are right angles. That is, $l_1 = AD$ is perpendicular to $l = PD$. □

Exercise 1.18. Suppose that f is an isometry for which there exists exactly one point P such that $f(P) = P$. Prove that f is a rotation. That is, prove that f is a direct isometry.

Solution. If an isometry preserves the orientation of one nondegenerate triangle, then it preserves the orientation of all nondegenerate triangles. This is shown in Exercise 9.12 and I do not think we can show this with any rigor without first coming up with a better set of axioms. So let us accept it as fact. As a corollary, if f reverses the orientation of any triangle, then it reverses the orientation of all (nondegenerate) triangles.

Suppose now that f is not a direct isometry. Then it reverses the orientation of all triangles. Consider a point $Q \neq P$ and its image $Q' = f(Q)$. If $\Delta PQQ'$ is not degenerate, then consider the image of $\Delta PQQ'$ under f. Note that $f(P) = P$, $f(Q) = Q'$. Let $f(Q') = Q''$. Note that Q'' lies on the circle centered at P with radius $|PQ|$, and also on the circle centered at Q' with center $|Q'Q|$. There are two points of intersection, one of which is Q. If it is the other point, then $\Delta PQQ'$ has the same orientation as $\Delta PQ'Q''$. Since we assumed f reverses orientation, we get $f(Q') = Q$. Finally, let M be the midpoint of QQ'. Then $f(M)$ is a distance $|MQ|$ from Q', and a distance $|MQ'|$ from Q. Since $|MQ| = |MQ'|$, these two circles intersect at exactly one point, M. Thus, $f(M) = M$. But this means $M = P$, so $\Delta PQQ'$ is degenerate, a contradiction. Thus, f is orientation preserving.

Now, suppose there does not exist a Q such that $\Delta PQQ'$ is nondegenerate. Then P, Q, and Q' are collinear, and since $Q \neq Q'$, P must be the midpoint of QQ'. Then it is clear that f is rotation by $180°$. □

Exercise 1.19†. Suppose f and g are two isometries such that $f(A) = g(A)$, $f(B) = g(B)$, and $f(C) = g(C)$ for some nondegenerate triangle ΔABC. Show that $f = g$. That is, show that $f(P) = g(P)$ for any point P.

Solution. Consider the isometry $h = g^{-1}f$. Note that $h(A) = A$, $h(B) = B$, and $h(C) = C$. By Exercise 1.15, h is either the identity, or a reflection which fixes the line AB, since $h(A) = A$ and $h(B) = B$. But since $h(A) = A$ and $h(C) = C$, it is also either the identity or a reflection which fixes the line AC. Since ΔABC is nondegenerate, the lines AB and AC are distinct, so h cannot fix both of them, so it must be the identity. Thus, $f = g$. □

Exercise 1.20*. Suppose P and Q are two distinct points. Prove that there exists exactly one translation which sends P to Q. [H]

Solution. There are two questions here – existence and uniqueness. I will ignore the question of existence. The only way I see of doing it is constructive, and in passing shows that Axioms 6 and 7 follow from Axiom 8.

Let f be a translation which sends P to Q. Let $Q' = f(Q)$, and let us now show that Q' is on the line PQ. Suppose it is not. Let M and M' be the midpoints of the segments PQ and QQ', respectively. Consider the perpendicular l to PQ through M, and the perpendicular l' to QQ' through M'. These lines intersect (and this is where we use the parallel postulate).

To see this, consider the line l'' which is perpendicular to PQ at Q. If l and l'' intersect on one side of PQ, then the must intersect on the other side too, contradicting Axiom 1 (actually, we have just assumed that there are two sides to a line, a concept not covered by our current set of axioms). Thus, l and l'' cannot intersect. Similarly, consider the line l''' perpendicular to QQ' and through Q. This line is parallel to l'. Since PQ and QQ' are distinct lines, the lines l'' and l''' are distinct. Thus, l'' must intersect l', since by Axiom 5, there is only one line (namely l''') through Q which is parallel to l'. Let this point of intersection be R. Then l'' is the unique line, through R, which is parallel to l, so since $l' \neq l''$ and l' goes through R, it cannot be parallel to l. Thus, l and l' intersect.

Let the point of intersection of l and l' be S. Let us now show that $|MS| = |M'S|$. Note that $|MQ| = |MP| = |M'Q|$ (the first equality follows since M is the midpoint of PQ, and the second equality follows since M' and Q are the images of M and P under the isometry f). Though we know the triangles ΔMQS and $\Delta M'QS$ are congruent by the Pythagorean theorem, we really shouldn't use that result, since we haven't proved it (recall that the proof we gave used results from Euclidean geometry which we had not yet proved, and motivated our discussion of axioms). So, let us instead reflect ΔMQS through l. Since the angle at M is a right angle, the point Q is sent to P, and the angle $\angle QMP$ is a straight line, giving us the triangle ΔPQS with M on the side PQ. Further, $|PS| = |QS|$. Do the same for $\Delta M'Q'S$ to get $\Delta QQ'S$, which is congruent to ΔPQS and ΔQPS by SSS. Now we know $\angle MQS = \angle PQS = \angle QPS = \angle Q'QS = \angle M'QS$, and we get $\Delta MQS \equiv \Delta M'QS$ by SAS. Thus, $|MS| = |M'S|$.

We will now show that $f(S) = S$. Since S is on l, and l' is the image of l under f, we know $f(S)$ lies on l'. Since $f(M) = M'$, there are only two possibilities for $f(S)$, one of which is S, the other is the reflection of S through the line QQ'. Since f is a translation, it preserves orientation, so the image of S must be S. Thus, we have shown that f has a fixed point, a contradiction. Thus, the assumption that $f(Q)$ is not on PQ is false.

Now, suppose that there exists another translation g such that $g(P) = Q$. Let $h = g^{-1}f$. Then $h(P) = P$ and $h(Q) = Q$. By Exercise 1.15, h is either the identity or a reflection. Since h is the composition of two translations, it preserves orientations, so it cannot be a reflection. Thus, it is the identity. Hence $g = f$, and the translation which sends P to Q is therefore unique. $\square$

1.4 Parallel Lines

Exercise 1.21*. Show that Euclid's version of Axiom 5 implies our version of Axiom 5.

Solution. Let l be a line and P a point not on l. We wish to show that there exists a unique line l' through P which does not intersect l. Let us first show that l' exists. By Lemma 1.4.2, there exists a point Q on l such that l and PQ intersect at right angles. Let l' be the line through P which is perpendicular to PQ. If l and l' intersect, say at R, then by Axiom 8, there exists an isometry which fixes the line PQ and sends R to a point $R' \neq R$. But since the angles at P and Q in ΔPQR are both right angles, the angles at P and Q in $\Delta PQR'$ are also right angles, so the angles $\angle RPR'$ and $\angle RQR'$ are both straight lines. That is, l and l' intersect at two points R and R', a contradiction of Axiom 1. Thus, R cannot exist, so l and l' are parallel.

Now, suppose l'' is any other line through P. Then the angles l'' makes with PQ are not right angles, so one of those angles is less than a right angle. The line PQ therefore meets l and l'' so that the sum of the angles on one side is less than a straight line, so l and l'' intersect. Thus, the line l' is unique. $\square$

Exercise 1.22. Prove that the angles in a quadrilateral sum up to 360°. Generalize this result to an n-sided polygon.

Solution. For a quadrilateral $ABCD$, draw the diagonal AC, creating two triangles ΔABC and ΔBCD. The sum of the angles in these two triangles are each 180°, giving a total of 360°. The sum of these angles is also equal to the sum of the angles in the quadrilateral.

In general, if we label the vertices of an n-gon P_1, P_2, ... P_n, then we can 'triangulate' it by drawing the segments P_1P_3, P_1P_4,...,P_1P_{n-1}. By doing so, we have drawn $n-3$ segments, and created $n-2$ triangles. Thus, the sum of the angles in an n-gon is $(n-2)180°$. □

Exercise 1.23. What is the sum of the exterior angles of a triangle? What is the sum of the exterior angles of a quadrilateral? What is the sum of the exterior angles of an n-gon?

Solution. The exterior angle for any angle is the difference of 180° less the interior angle. Thus, using the result in Exercise 1.22, the sum of the exterior angles of an n-gon is

$$n(180°) - (n-2)180° = 360°.$$ □

1.5 Pons Asinorum

Exercise 1.24†. Prove the converse of *pons asinorum.* That is, show that if in ΔABC we have $\angle ABC = \angle ACB$, then $|AB| = |AC|$.

Solution. By ASA, $\Delta ABC \equiv \Delta ACB$, so $|AB| = |AC|$. □

Exercise 1.25†. Prove that if a diameter of a circle bisects a chord which itself is not a diameter, then the diameter is perpendicular to the chord. Also, prove that the perpendicular bisector of a chord goes through the center of the circle. And finally, prove that if a diameter is perpendicular to a chord, then the diameter bisects the chord. [S]

1.6 The Star Trek Lemma

Exercise 1.26. Prove the Star Trek lemma for an acute angle for which the center O is outside the angle.

Solution. The proof is almost identical to the one in the book, so let us repeat it here, and point out how it differs. Note that OA, OB, and OC are radii, so we have several isosceles triangles. We have continued the segment OA to intersect the circle at D. Since ΔAOB is isosceles, $\angle BAO = \angle OBA$. Since the sum of angles in a triangle is 180°,

$$\angle BOD = \angle OBA + \angle BAO = 2\angle BAO.$$

Similarly,

$$\angle DOC = 2\angle OAC.$$

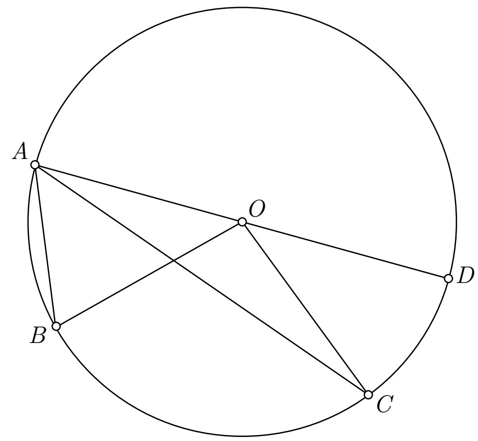

Figure 1.4

And now for the difference. Rather than adding these two angles, it is clear we want to taker the difference, which gives

$$\angle BOC = \angle BOD - \angle DOC = 2(\angle BAO - \angle OAC) = 2\angle BAC. \qquad \square$$

Exercise 1.27. Prove the Star Trek lemma for an obtuse angle.

The proof is identical to the one in the text. Just draw a different diagram.

Exercise 1.28†. Suppose $\angle ABC$ is a right angle inscribed in a circle. Prove that AC is a diameter. [S]

Exercise 1.29† (Bow Tie Lemma). Let A, A', B, and C lie on a circle, and suppose $\angle BAC$ and $\angle BA'C$ subtend the same arc (as in Figure 1.5(a)). Show that

$$\angle BAC = \angle BA'C.$$

Again, because of the diagram, this lemma is sometimes known as the *Bow Tie* lemma. We say 'The angles at A and A' are equal since they subtend the same arc.'

Solution. By the Star Trek lemma, $\angle BAC$ is half the measure of the arc BC that it subtends. Also by the Star Trek lemma, $\angle BA'C$ is half the measure of the arc BC that it subtends. Thus, the two angles are equal. $\square$

Exercise 1.30. If $|AB| = |AC| = |BC|$, what is the angle at D? (See Figure 1.5(b).) [A]

Solution. Since ΔABC is an equilateral triangle, $\angle ABC = 60°$. Since $\angle ADC$ subtends the same arc, it must also be $60°$. $\square$

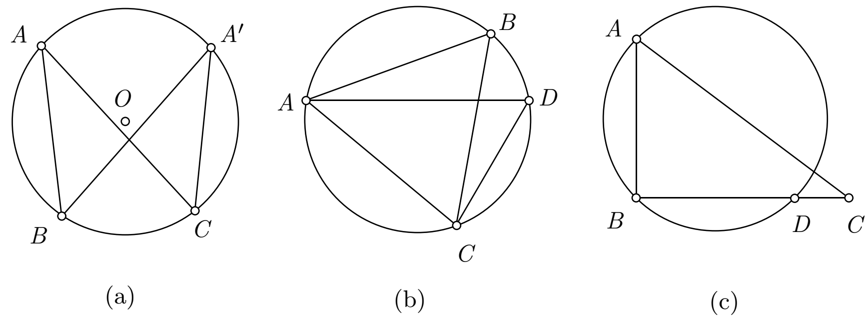

Figure 1.5. See Exercises 1.29, 1.30, and 1.31.

Exercise 1.31. If $|AB| = 12$, $|BD| = 9$, $|BC| = 16$, and $|AC| = 20$, then what is the length of the diameter? (See Figure 1.5(c).) [A]

Solution. Note that ΔABC has sides of length 12, 16, and 20, so $b^2 = a^2 + c^2$. Thus, by the converse of the Pythagorean theorem, the angle at B is a right angle. It therefore subtends a diameter (see Exercise 1.28), so AD is a diameter. By the Pythagorean theorem,

$$|AD|^2 = |AB|^2 + |BD|^2 = 12^2 + 9^2 = 15^2.$$

Thus, the length of the diameter is 15. □

Exercise 1.32. If $|AB| = |AC| = |BC|$ and AD is perpendicular to BC, then what is $\angle BCD$? (See Figure 1.6(a).)

Solution. Let AD and BC intersect at E. Since $|AB| = |AC|$, $|AE$ is shared, and $\angle AEC = \angle AEB = 90°$, we know $|BE| = |EC|$ by the Pythagorean theorem, and hence, $\Delta AEC \equiv \Delta AEB$ by SSS. Thus, $2\angle BAD = \angle BAC = 60°$, so $\angle BAD = 30°$. Since $\angle BAD$ and $\angle BCD$ subtend the same arc, we therefore get $\angle BCD = 30°$. □

Exercise 1.33† (The Tangential Case of the Star Trek Lemma). Suppose AT is a line segment that is tangent to a circle. Prove that $\angle ATB$ is half the measure of the arc TB which it subtends. Do this by picking a point C on the circle such that $\angle TCB$ subtends the arc TB (as in Figure 1.6(b)). Show that

$$\angle ATB = \angle TCB.$$

Solution. It doesn't matter where we pick C, since $\angle TCB$ is always half the arc it subtends. So, pick C so that BC is a diameter. Then $\angle CTB = 90°$, and $\angle TCB = \angle TCO = \angle CTO$. Thus, $\angle OTB = 90° - \angle TCB$. But the radius OT and the tangent TA intersect at right angles, so $\angle OTB = 90° - \angle BTA$. Thus,

$$90° - \angle BTA = 90° - \angle TCB$$

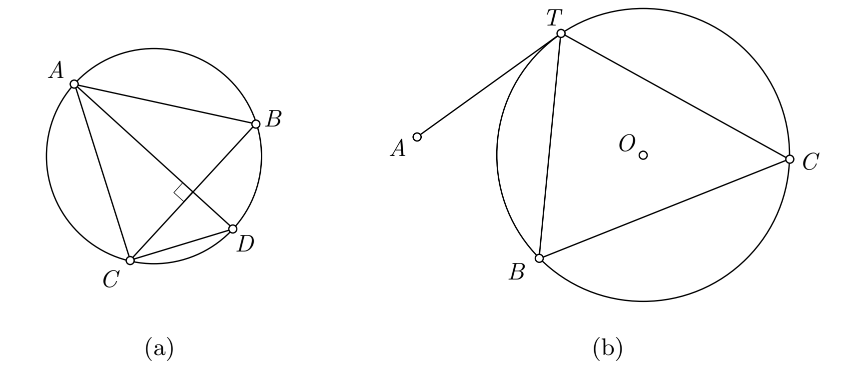

Figure 1.6. See Exercises 1.32 and 1.33.

$$\angle BTA = \angle TCB.$$

Hence, $\angle BTA$ is half the the measure of the arc it subtends. $\square$

Exercise 1.34†. Suppose two lines intersect at P inside a circle and meet the circle at A and A' and at B and B', as shown in Figure 1.7(a). Let α and β be the measures of the arcs $A'B'$ and AB respectively. Prove that

$$\angle APB = \frac{\alpha + \beta}{2}.$$

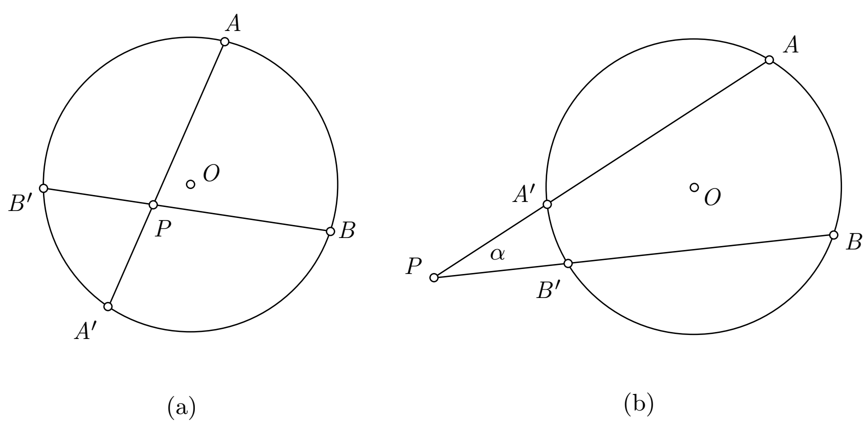

Figure 1.7

Solution. Draw the segment AB', and consider the triangle $\Delta AB'P$. By the Star Trek lemma, $\angle AB'P = \beta/2$, and $\angle B'AP = \alpha/2$. Since $\angle APB$ is the exterior angle of $\Delta AB'P$ at P, we have

$$\angle APB = \angle AB'P + \angle B'AP = \frac{\alpha+\beta}{2}. \qquad \square$$

Exercise 1.35†. Suppose an angle α is defined by two rays which intersect a circle at four points, as in Figure 1.8(b). Suppose the angular measure of the outside arc it subtends is β, and the angular measure of the inside arc it subtends is γ. (So, in Figure 1.8(b), $\angle AOB = \beta$ and $\angle A'OB' = \gamma$.) Show that

$$\alpha = \frac{\beta-\gamma}{2}.$$ [S]

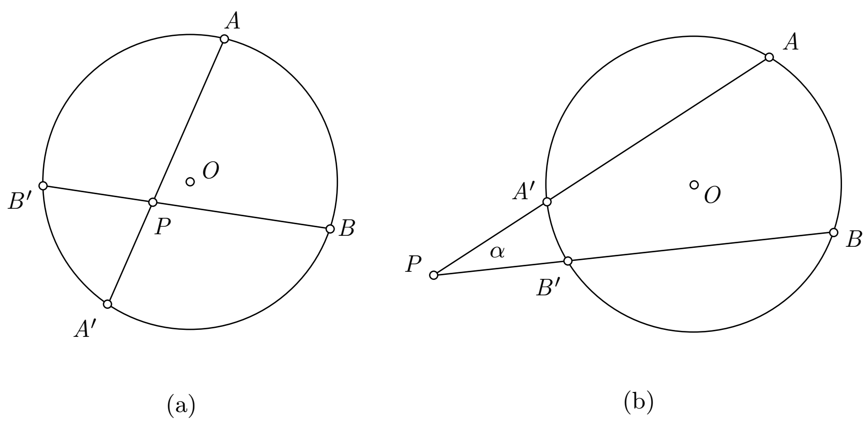

Figure 1.8. See Exercises 1.34 and 1.35.

Exercise 1.36. Prove that the opposite angles in a convex quadrilateral (called a convex quadrangle in the first printing) inscribed in a circle sum to $180°$. Conversely, prove that if the opposite angles in a convex quadrilateral sum to $180°$, then the quadrilateral can be inscribed in a circle. Such a quadrilateral is called a *cyclic quadrilateral*.

Solution. Let $ABCD$ be a convex quadrilateral inscribed in circle. Then the arc subtended by the angles at B and D, together, form the whole circle, so measure $360°$. Thus, the sum of the angles B and D is half that, so is $180°$.

Conversely, suppose the opposite angles B and D sum to $180°$ in a convex quadrilateral $ABCD$. Consider the circumcircle of ΔABC. If D lies inside this circle, then by Exercise 1.34, the angle D is larger than half the measure of the arc $ABC = 360° - 2\angle ABC$, so the sum of the angles at B and D is greater than $180°$. Similarly, if D lies outside the circle, then by Exercise 1.35, the sum of the angles is less than $180°$. Thus, D must lie on the circle. That is, the quadrilateral $ABCD$ is cyclic. $\square$

Exercise 1.37†. Let two circles Γ and Γ' intersect at A and B, as in Figure 1.9. Let CD be a chord on Γ. Let AC and BD intersect Γ' again at E and F. Prove that CD and EF are parallel. [S]

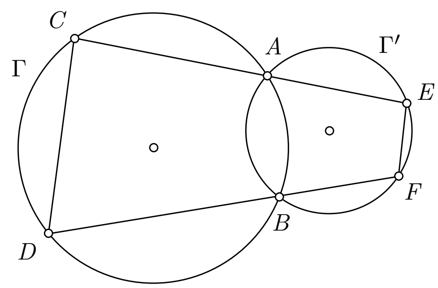

Figure 1.9. See Exercise 1.37.

This exercise is restated and proved in Lemma 11.2.2.

Exercise 1.38†. Let $ABCD$ be a *nonconvex cyclic quadrilateral*. That is, $ABCD$ is inscribed in a circle and two of its opposite sides intersect (as in Figure 1.10(a)). Prove that $\angle ABC = \angle CDA$ and $\angle DAB = \angle BCD$. Conversely, suppose $\angle ABC = \angle DAB$ and $\angle BCD = \angle DAB$ in a quadrilateral with intersecting opposite sides. Show that $ABCD$ is cyclic.

Solution. The solution is essentially the same as the solution of Exercise 1.36. The angles are equal since they subtend the same arcs. Conversely, consider the circumcircle of ΔABC. If D is inside this circle, then by Exercise 1.34, the angle at D must be larger than the angle at B; and if D is outside this circle, then by Exercise 1.35, the angle at D is smaller than the angle at B. Thus, D must lie on the circle. □

Exercise 1.39. Suppose $ABCDEF$ is a hexagon inscribed in a circle. Show that

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$

Prove that the converse is not true. That is, find an example of a hexagon $ABCDEF$ whose angles B, D, and F sum to 360° but which cannot be inscribed in a circle.

Solution. The sum of the arcs subtended by the angles $\angle ABC$, $\angle CDE$, and $\angle EFA$ is twice the entire circle, so 720°. Thus, the sum of these angles is half that, or 360°.

The converse, though, is not true. To see this, let us first consider a regular hexagon $ABCDEF$ inscribed in a circle. Note that the sides AB and DE are parallel. Thus, we may stretch these sides without changing the angles. By doing so, we will have created a hexagon which is not inscribed in a circle, but whose angles satisfy the condition. □