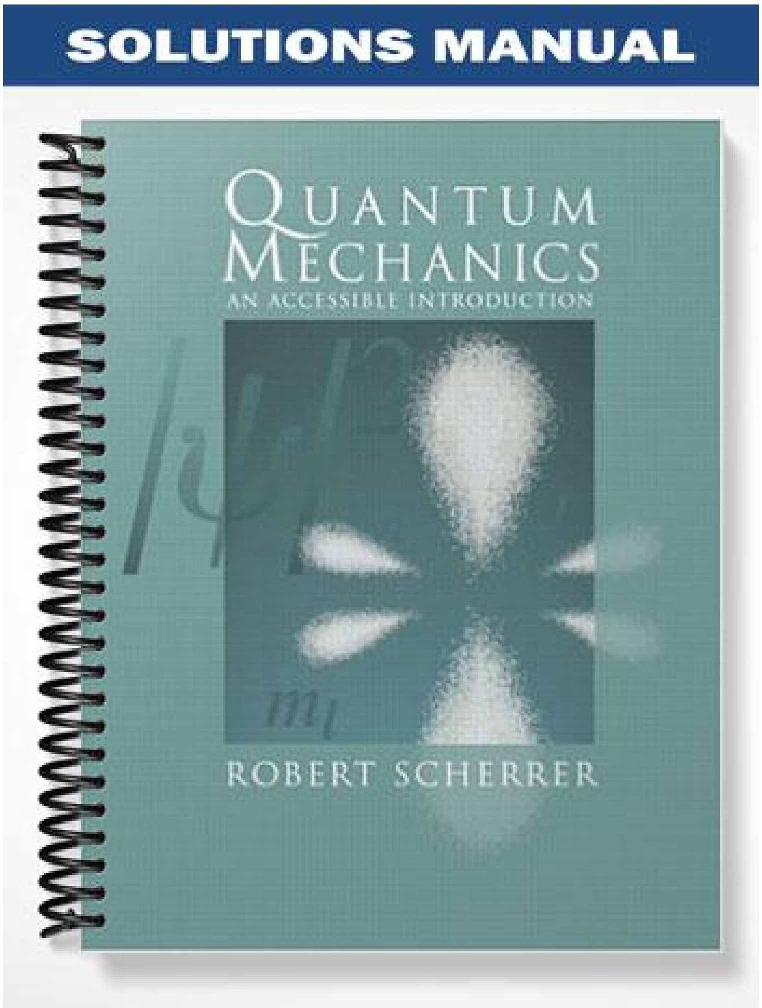
SOLUTIONS MANUAL
QUANTUM
MECHANICS
AN ACCESSIBLE INTRODUCTION
ROBERT SCHERRER

Solutions to the Exercises

I have given all numerical answers to two significant figures, for consistency. Note that I have assumed that all "difficult" definite integrals will be solved by looking them up in a table; most of these are ultimately derived using contour integration, but I certainly don't expect the students to solve them that way.

Chapter 1

1.1 (a) The power emitted by the human body is:

$$P = \sigma A T^4$$

so the total energy radiated in a time t is

$$E = \sigma A T^4 t$$

The surface area A of the human body is between 1 m^2 and 2 m^2 (depending on the person in question). Students should at least be able to estimate the order of magnitude of this number correctly. I will solve for the range $A = 1-2$ m^2. The temperature of the human body is:

$$T = 98.6^\text{o}\ \text{F} = 310\ \text{K}$$

So the total energy radiated in one hour is:

$$E = (5.67 \times 10^{-8}\ \text{J sec}^{-1}\ \text{m}^{-2}\ \text{K}^{-4})(1 - 2\text{m}^2)(310\ \text{K})^4(3600\ \text{sec}) = 1.9 - 3.8 \times 10^6\text{J}$$

(b) Wien's law gives:

$$\lambda_{peak} = w/T = (2.90 \times 10^{-3}\text{m K})/310\text{K} = 9.4 \times 10^{-6}\text{m} = 9.4 \times 10^4\text{Å}$$

1.2 Wien's law gives $\lambda_{peak} = w/T$ so that

$$T = w/\lambda_{peak} = (2.90 \times 10^{-3}\text{m K})/3.5 \times 10^{-7}\text{m} = 8300\text{K}$$

(Note the errata that this is not a red star - an error in the text of the problem! This is corrected in the second printing).

1.3 (a) The total energy density is given by

$$\rho = aT^4 = (7.56 \times 10^{-16}\ \text{J m}^{-3}\ \text{K}^{-4})(2.7\ \text{K})^4 = 4.0 \times 10^{-14}\text{J m}^{-3}.$$

(b) The total energy density ρ between 1 mm and 1.01 mm is

$$\rho = \int_{\nu_1}^{\nu_2} \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu.$$

where $\nu_1 = c/(1.01\ \text{mm}) = 2.97$ Hz and $\nu_2 = c/(1\ \text{mm}) = 3.00$ Hz. This integral cannot be performed analytically, but because the range of integration is so small, it is a good approximation to take it to be given by

$$\rho = \frac{8\pi h}{c^3} \frac{\nu_2^3}{e^{h\nu_2/kT} - 1}(\nu_2 - \nu_1).$$

One could just as easily have substituted ν_1 instead of ν_2 in the integrand. The error in making this approximation is only about 1%. Note that $h\nu_2 = 1.99 \times 10^{-22}$J and $kT = 3.73 \times 10^{-23}$J. Then the desired density is:

$$\begin{aligned} \rho &= \frac{(8\pi)(6.63 \times 10^{-34}\text{J sec})}{(3 \times 10^8 \text{m sec}^{-1})^3} \frac{(3.0 \times 10^{11}\ \text{Hz})^3}{e^{5.34} - 1} (3 \times 10^{11}\text{Hz} - 2.97 \times 10^{11}\text{Hz}) \\ &= 2.4 \times 10^{-16}\ \text{J m}^{-3} \end{aligned}$$

The Rayleigh-Jeans formula is *not* a good approximation at these wavelengths. The easiest way to see this is to note that $h\nu/kT \approx 5$, and the Rayleigh-Jeans formula is only a good approximation when $h\nu/kT \ll 1$. Alternatively, one can re-do the calculation using the Rayleigh-Jeans formula

$$\begin{aligned} \rho &= \int_{\nu_1}^{\nu_2} \frac{8\pi kT}{c^3} \nu^2 d\nu \\ &= \frac{8\pi kT}{3c^3} (\nu_2^3 - \nu_1^3) \\ &= \frac{8\pi(3.73 \times 10^{-23}\text{J})}{(3 \times 10^8 \text{m sec}^{-1})^3} [(3 \times 10^{11}\text{Hz})^3 - (2.97 \times 10^{11}\text{Hz})^3] \\ &= 9.3 \times 10^{-15}\text{J m}^{-3} \end{aligned}$$

which disagrees significantly with the result obtained using the Planck spectrum.

1.4 Note that the Rayleigh-Jeans formula always gives a larger energy density than the Planck formula, so for 10% agreement between the two, we require

$$\rho_{\nu(Rayleigh-Jeans)} / \rho_{\nu(Planck)} \leq 1.1$$

Inserting the actual formulas gives

$$\rho_{\nu(Rayleigh-Jeans)} / \rho_{\nu(Planck)} = (kT/h\nu)(e^{h\nu/kT} - 1) \leq 1.1$$

This expression is a function only of $x = h\nu/kT$, so we want to solve

$$(e^x - 1)/x = 1.1$$

This equation can be solved numerically by plugging in numbers, but we can also note that $e^x - 1 = 1 + x + x^2/2 + ...$, so that $(e^x - 1)/x \approx 1 + x/2 = 1.1$, which gives $x = 0.2$. A numerical calculation gives an answer of roughly 0.19. So the correct result is $\nu < 0.19(kT/h)$.

1.5 We need to evaluate

$$\rho(< \nu_0) = \int_{\nu=0}^{\nu_0} \rho(\nu) d\nu.$$

Since $h\nu_0 \ll kT$, we know that $h\nu \ll kT$ over the entire range of integration, so we can just use the Rayleigh-Jeans formula:

$$\begin{aligned} \rho(< \nu_0) &= \int_{\nu=0}^{\nu_0} \frac{8\pi kT}{c^3} \nu^2 d\nu, \\ &= \frac{8\pi}{3} \left(\frac{\nu_0}{c}\right)^3 kT. \end{aligned}$$

1.6 The desired quantity is

$$\rho(> \nu_0) = \int_{\nu=\nu_0}^{\infty} \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

Following the hint in the back of the book, we take $e^{h\nu/kT} - 1 \approx e^{h\nu/kT}$, which is a good approximation for $h\nu \gg kT$. This gives

$$\rho(>\nu_0) = \int_{\nu=\nu_0}^{\infty} \frac{8\pi}{c^3} h\nu^3 e^{-h\nu/kT} d\nu$$

We substitute $h\nu/kT = x$, giving

$$\rho(>\nu_0) = \int_{x=h\nu_0/kT}^{\infty} \frac{8\pi}{c^3} h \left(\frac{kT}{h}\right)^4 x^3 e^{-x} dx$$

The integral of $x^3 e^{-x}$ can be obtained by repeated integration by parts or (better) looking it up in a table:

$$\int x^3 e^{-x} dx = -e^{-x}[x^3 + 3x^2 + 6x + 6]$$

so

$$\rho(>\nu_0) = \frac{8\pi}{c^3} h \left(\frac{kT}{h}\right)^4 e^{-h\nu_0/kT}[(h\nu_0/kT)^3 + 3(h\nu_0/kT)^2 + 6(h\nu_0/kT) + 6]$$

but $h\nu_0/kT \gg 1$, so the first term dominates, giving

$$\rho(>\nu_0) = \left(\frac{8\pi kT\nu_0^3}{c^3}\right) e^{-h\nu_0/kT}$$

1.7 (a) We have $\nu = c/\lambda$, and $d\nu = -(c/\lambda^2)d\lambda$. Substituting these expressions into Equation (1.7) in the book gives

$$\rho(\lambda)d\lambda = \frac{8\pi hc}{e^{hc/kT\lambda} - 1} \frac{d\lambda}{\lambda^5}$$

(b) We set $d\rho/d\lambda = 0$ at $\lambda = \lambda_{peak}$, which gives

$$\left(\frac{hc}{kT\lambda_{peak}} - 5\right) e^{hc/kT\lambda_{peak}} + 5 = 0.$$

The solution can be found numerically; it is $hc/kT\lambda_{peak} = 4.965$, so

$$\lambda_{peak} = 0.20(hc/kT)$$

(c) Using the expression for λ_{peak} from part (b), and the expression for ν_{peak} from page 9 of the book [$\nu_{max} = 2.8(kT/h)$], we get

$$\lambda_{peak}\nu_{peak} = [0.20(hc/kT)][2.8(kT/h)] = 0.56c,$$

so it is *not* the case that $\lambda_{peak}\nu_{peak} = c$.

1.8 We have

$$E_{max} = e\Phi_0 = h\nu - E_B$$

Solving for E_B gives

$$\begin{aligned} E_B &= h\nu - e\Phi_0 \\ &= hc/\lambda - e\Phi_0 \\ &= (6.6 \times 10^{-34}\text{J sec})(3 \times 10^8\text{m/sec})/(4000 \times 10^{-10}\text{m}) - (1.6 \times 10^{-19}\text{C})(0.5\text{ V}), \\ &= 4.15 \times 10^{-19}\text{J}. \end{aligned}$$

Photoelectric current flows when $h\nu > E_B$, so we set $h\nu = E_B$ to find the longest possible wavelength. Since $h\nu = E_B$ and $\lambda = c/\nu$, we have

$$\begin{aligned}\lambda &= ch/E_B \\ &= (3\times 10^8 \mathrm{m/sec})(6.6\times 10^{-34}\mathrm{J\ sec})/4.15\times 10^{-19}\mathrm{J} \\ &= 4.8\times 10^{-7}\mathrm{m} \\ &= 4800\ \text{Å}.\end{aligned}$$

1.9 We do part **(b)** first. The energy of a photon with a wavelength $\lambda = 6.0\times 10^{-7}$ m is:

$$E = h\nu = hc/\lambda = (6.6\times 10^{-34}\mathrm{J\ sec})(3\times 10^8 \mathrm{m\ sec^{-1}})/(6.0\times 10^{-7}\ \mathrm{m}) = 3.3\times 10^{-19}\ \mathrm{J}$$

Now we can solve part **(a)**. Let N be the number of photons emitted per second. Then N =power emitted/energy per photon. So

$$N = 40\ \mathrm{W}/3.3\times 10^{-19}\ \mathrm{J} = 1.2\times 10^{20}\mathrm{photons/sec}.$$

Note that this represents an idealization. A real lightbulb emits radiation over a range of frequencies, not just a single frequency. But it's OK for an order of magnitude estimate.

1.10 (a) The desired expression is

$$\begin{aligned}n(\nu)d\nu &= (1/h\nu)\rho(\nu)d\nu \\ &= (1/h\nu)\frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}d\nu \\ &= \frac{8\pi}{c^3}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu.\end{aligned}$$

(b) To find the total number, we integrate the expression in part (a) over frequency:

$$n = \int_{\nu=0}^{\infty} n(\nu)d\nu = \frac{8\pi}{c^3}\int_{\nu=0}^{\infty}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu.$$

This is most easily simplified (and the dependence on the various physical parameters is most obvious) if we make the change of variables $x = h\nu/kT$, which gives

$$n = \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_{x=0}^{\infty}\frac{x^2}{e^x-1}.$$

The integral cannot be done analytically. It can be expressed as a zeta function [it is a multiple of $\zeta(3)$] by multiplying numerator and denominator by e^{-x}, expanding out the denominator in a power series, and integrating term by term. But you probably don't want to go there. Instead, we use the recommendation in the exercises and simply set it equal to 2.4. This gives

$$n = \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3(2.4) = 60\left(\frac{kT}{hc}\right)^3.$$

1.11 The derivation of the Compton scattering formula can be applied to any particle; there is nothing unique about electrons. Thus, we have

$$\lambda_f - \lambda_i = \frac{h}{m_0 c}(1-\cos\theta)$$

where m_0 is the mass of the unknown particle. Expressing λ_f and λ_i in terms of the photon energies gives: $E_f = h\nu_f = hc/\lambda_f$ and $E_i = h\nu_i = hc/\lambda_i$. Substituting these expressions for the λ_f and λ_i in the Compton scattering formula, and solving for m_0c^2, we get

$$m_0c^2 = \frac{1-\cos\theta}{1/E_f - 1/E_i}$$

Backward scattering corresponds to a scattering angle of 180°, so we have

$$m_0c^2 = \frac{1-\cos(180°)}{1/0.98 \text{ MeV} - 1/1.0 \text{ MeV}} = 98 \text{ MeV}.$$

Note that this does not correspond to the mass of any known particle; I did not want the students randomly guessing the masses of such particles.

1.12 (a) Since the kinetic energy in this case is much less than the rest energy of the proton we have $E = p^2/2m_p$, so $p = \sqrt{2m_pE}$, where m_p is the proton mass. Then the de Broglie wavelength is

$$\begin{aligned}\lambda &= h/p \\ &= h/\sqrt{2m_pE} = 6.6 \times 10^{-34} \text{ J sec}/\sqrt{(2)(1.67\times10^{-27}\text{ kg})(0.1\times10^6\text{eV})(1.6\times10^{-19}\text{J/eV})} \\ &= 9.0\times10^{-14}\text{ m.}\end{aligned}$$

(b) The relation between total energy E and momentum p is

$$E^2 = p^2c^2 + m_p^2c^4$$

Then

$$pc = \sqrt{E^2 - m_p^2c^4} = \sqrt{(3\times10^3\text{ MeV})^2 - (938\text{ MeV})^2} = 2850\text{ MeV}$$

Converting to MKS units and solving for p gives $pc = 4.56 \times 10^{-10}$ J, and $p = 1.52 \times 10^{-18}$ kg m sec^{-1}. Then the de Broglie wavelength is

$$\lambda = h/p = 6.6\times10^{-34}\text{J sec}/1.52\times10^{-18}\text{kg m sec}^{-1} = 4.3\times10^{-16}\text{m}$$

1.13 The Balmer series lies in the visible. The Lyman series (for example) lies in the ultraviolet.

1.14 Note that $\hbar$ has units of (energy)(time), but energy has units of (mass)(velocity)^2 = (mass) (distance)^2/(time)^2. Thus, $\hbar$ has units of (mass)(distance)^2/(time). Now recall that angular momentum is (momentum)(distance), which has units of (mass)(velocity)(distance) = (mass)(distance)^2 /(time).

1.15 The wavelengths of the spectral lines are given by

$$\frac{1}{\lambda} = \nu/c = (E_{n_1} - E_{n_2})/hc$$

Inserting the energies given by equation (1.24) in the book gives

$$\frac{1}{\lambda} = \frac{1}{hc}\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

This looks just like equation (1.18) in the book with $m = n_2$ and $n = n_1$, and R given by

$$R = \frac{1}{hc}\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{m}{4\pi\hbar^3 c}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$$

Plugging in the values of all of the constants gives

$$R = \frac{9.11\times10^{-31}\text{kg}}{4\pi(1.05\times10^{-34}\text{J sec})^3(3\times10^8\text{m/sec})}\left(\frac{(1.6\times10^{-19}\text{C})^2}{4\pi(8.85\times10^{-12}\text{farad/m})}\right)^2 = 1.1\times10^7\ \text{m}$$

which is the desired answer.

1.16 Following Bohr's argument for the Coulomb potential, we set centripetal force equal to the attractive force:

$$kr^\beta = \frac{mv^2}{r}$$

Also, the Bohr quantization rule gives

$$L = mvr = n\hbar$$

so that

$$v = \frac{n\hbar}{mr}$$

Substituting this expression for v in the first equation above gives

$$kr^\beta = m\left(\frac{n\hbar}{mr}\right)^2\frac{1}{r}$$

and solving for r gives

$$r = \left(\frac{n^2\hbar^2}{km}\right)^{1/(\beta+3)}$$

Now the total energy is

$$E = \text{kinetic energy} + \text{potential energy}$$

The kinetic energy is just $mv^2/2$. Substituting our expression for v from the equation above, the kinetic energy becomes $(1/2)m(n\hbar/mr)^2$. To find the potential energy, we use $V = -\int F dr$. Note also that since the force is attractive, we must take $F = -kr^\beta$ in this integral, giving

$$V = -\int -kr^\beta = \frac{k}{\beta+1}r^{\beta+1}.$$

This assumes that $\beta \neq 1$, as given in the problem. (Note that many students will forget that an attractive force should have a minus sign, and they may also forget the minus sign in going from force to potential. Unfortunately, a student making both mistakes will obtain the correct result). Then the total energy is

$$E = \frac{1}{2}m\left(\frac{n\hbar}{mr}\right)^2 + \frac{k}{\beta+1}r^{\beta+1}$$

We substitute our expression for r into this equation and get

$$E = \left(\frac{\hbar^2 n^2}{m}\right)^{(\beta+1)/(\beta+3)} k^{2/(\beta+3)}\left(\frac{1}{2} + \frac{1}{\beta+1}\right)$$

Why does this answer break down for $\beta = -3$? Going back to the equation in which we set centripetal force equal to the attractive force, and setting $\beta = -3$, we get

$$kr^{-3} = \frac{mv^2}{r}$$

which can be written as

$$k = mv^2r^2 = L^2/m$$

Thus, in this case, $L = \sqrt{km}$ is a constant, and all of the orbits have the same angular momentum. Hence, we cannot apply the Bohr quantization condition ($L = n\hbar$), since there is only one L.

Chapter 2

2.1 (a) Brute force multiplication yields

$$i(2-3i)(3+5i) = i(6-9i+10i+15) = i(21+i) = -1+21i$$

(b) Here it is easier to do the problem in polar form: $i = e^{i\pi/2}$ and $i-1 = \sqrt{2}e^{i3\pi/4}$, so

$$\frac{i}{i-1} = \frac{e^{i\pi/2}}{\sqrt{2}e^{i3\pi/4}} = \frac{1}{\sqrt{2}}e^{-i\pi/4}.$$

Converting back to the form $a+bi$:

$$\frac{1}{\sqrt{2}}e^{-i\pi/4} = \frac{1}{\sqrt{2}}[\cos(-\pi/4) + i\sin(-\pi/4)] = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2}i$$

(c) This could be solved by expanding out using the binomial theorem, but that would be a Bad Idea. Instead, we use the polar form: $1+i = \sqrt{2}e^{i\pi/4}$. Then

$$(1+i)^{30} = (\sqrt{2}e^{i\pi/4})^{30} = 2^{15}e^{i15\pi/2}$$

Using $e^{i2\pi} = 1$, we can factor out $e^{i12\pi/2} = 1$, giving

$$(1+i)^{30} = 32768e^{3\pi/2} = -32768i$$

2.2 One fifth root of i has a polar angle of $2\pi/5$, and the others have polar angles that are all integer multiples of $2\pi/5$. So the 5 different fifth roots are:

$$e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}, 1$$

2.3

$$z^* = 1 + e^{-i\theta}$$

$$z^2 = (1+e^{i\theta})(1+e^{i\theta}) = 1 + 2e^{i\theta} + e^{2i\theta}$$

$$|z|^2 = z^*z = (1+e^{-i\theta})(1+e^{i\theta}) = 1 + e^{i\theta} + e^{-i\theta} + 1 = 2 + 2\cos\theta$$

2.4 Suppose $z = a+bi$. Then $a - bi = a + bi$ so $b = 0$, and z is real.

2.5 Note that $i = e^{i\pi/2}$. So $i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2} \approx 0.21$

2.6 The problem arises because every complex number has two different square roots. In going from the second line to the third line of the argument, if we take $\sqrt{-1} = -i$ instead of $\sqrt{-1} = i$, then the final result makes sense.

2.7 (a)

$$\Pi[f(x) + g(x)] = f(-x) + g(-x) = \Pi[f(x)] + \Pi[g(x)]$$

and

$$\Pi[cf(x)] = cf(-x) = c\Pi[f(x)]$$

so Π *is* linear.

(b)

$$T[f(x) + g(x)] = f(x+1) + g(x+1) = T[f(x)] + T[g(x)]$$

and

$$T[cf(x)] = cf(x+1) = cT[f(x)]$$

so T *is* linear.

(c)

$$L[f(x) + g(x)] = f(x) + g(x) + 1$$

and

$$L[f(x)] + L[g(x)] = f(x) + 1 + g(x) + 1$$

so

$$L[f(x) + g(x)] \neq L[f(x)] + L[g(x)]$$

and L is *not* linear.

2.8 (a) For any functions $f(x)$ and $g(x)$ and constant c, we have

$$I[f(x) + g(x)] = f(x) + g(x) = I[f(x)] + I[g(x)]$$

and

$$I[cf(x)] = cf(x) = cI[f(x)]$$

(b) We need to solve the equation $I[f(x)] = cf(x)$. Applying the identity operator gives $I[f(x)] = f(x) = cf(x)$. This is satisfied for $c = 1$ and for any function $f(x)$. Thus, *all* functions are eigenfunctions of the operator I, and they all have eigenvalue of 1.

2.9

$$PQ[f(x)] = P[qf(x)] = qP[f(x)] = qpf(x)$$

and

$$QP[f(x)] = Q[pf(x)] = pQ[f(x)] = pqf(x)$$

and $qpf(x) = pqf(x)$, so $PQ[f(x)] = QP[f(x)]$.

2.10 (a) For any functions $f(x)$ and $g(x)$ and constant c, we have

$$D^2[f(x) + g(x)] = D[D[f(x) + g(x)]] = D[\frac{df}{dx} + \frac{dg}{dx}] = \frac{d^2f}{dx^2} + \frac{d^2g}{dx^2} = D^2[f(x)] + D^2[g(x)]$$

and

$$D^2[cf(x)] = D[D[cf(x)]] = D[cD[f(x)] = cD[D[f(x)]] = cD^2[f(x)]$$

(b) We need to solve $D^2[f(x)] = cf(x)$, which can be written as

$$\frac{d^2 f}{dx^2} = cf(x)$$

The general solution to this equation is

$$f(x) = A_1 e^{\sqrt{c}x} + A_2 e^{-\sqrt{c}x}$$

where A_1 and A_2 are arbitrary constants. This gives the most general possible eigenfunction of D^2 with eigenvalue c, and any complex number c can be an eigenvalue.

(c) Examples include the sine and cosine functions, e.g.,

$$D^2[\sin(ax)] = -a^2 \sin(ax)$$

but

$$D[\sin(ax)] = a\cos(ax)$$

and similarly for the cosine function.

2.11 Let $f(x)$ be an eigenfunction of L with eigenvalue a, so that $L[f(x)] = af(x)$. Now consider what happens when L operates on $cf(x)$, where c is a constant:

$$L[cf(x)] = cL[f(x)] = c(af(x)) = a(cf(x))$$

Thus, $cf(x)$ is also an eigenfunction of L with eigenvalue a.

2.12 (a)

$$L[f(x) + g(x)] = \int_0^x \Big(f(s) + g(s)\Big) ds = \int_0^x f(s)ds + \int_0^x g(s)ds = L[f(x)] + L[g(x)]$$

and

$$L[cf(x)] = \int_0^x cf(s)ds = c\int_0^x f(s)ds = cL[f(x)]$$

(b) Suppose that $f(x)$ is an eigenfunction of L, with eigenvalue a. Then we have

$$L[f(x)] = \int_0^x f(s)ds = cf(x)$$

Taking the derivative of both sides gives

$$f(x) = c\frac{df}{dx}$$

The general solution is

$$f(x) = Ae^{x/c}$$

However, substituting this back into our original eigenfunction equation gives

$$L[Ae^{x/c}] = \int_0^x Ae^{s/c}ds = cA(e^{x/c} - 1) \neq cAe^{x/c}$$

so L has *no* eigenfunctions. Note that the case $A = 0$ does give a trivial eigenfunction (the "zero function") but this is not normally considered a legitimate eigenfunction.