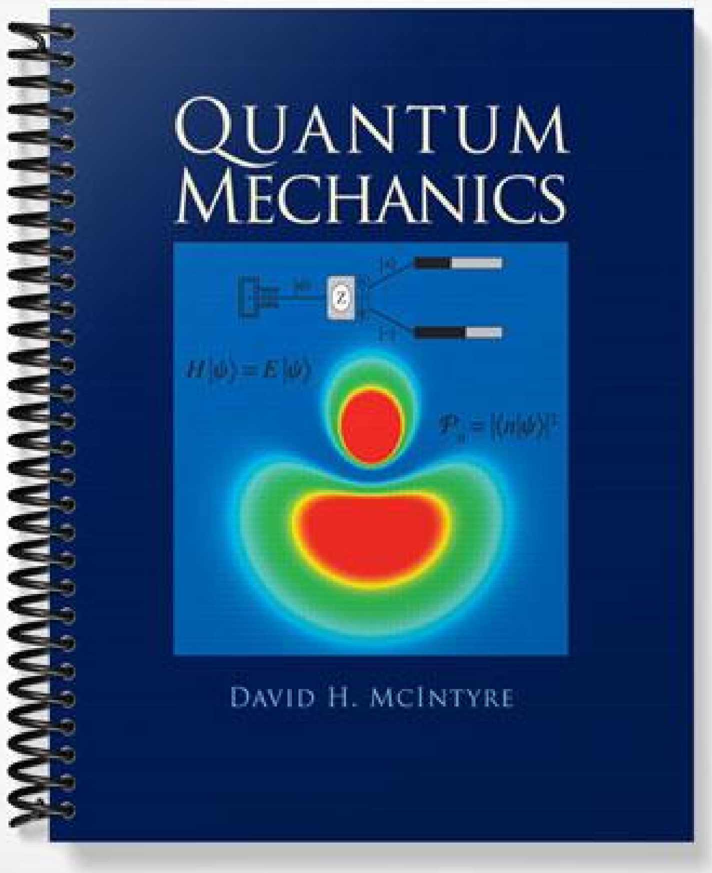
SOLUTIONS MANUAL



2.1 Let

$$S_x \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_x eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which yields

$$a+b=+\tfrac{\hbar}{2} \qquad c+d=+\tfrac{\hbar}{2}$$
$$a-b=-\tfrac{\hbar}{2} \qquad c-d=+\tfrac{\hbar}{2}$$

Solve by adding and subtracting the equations to get

$$a=0 \qquad b=\tfrac{\hbar}{2} \qquad c=\tfrac{\hbar}{2} \qquad d=0$$

Hence the matrix representing S_x in the S_z basis is

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let

$$S_y \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_y eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = +\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

which yields

$$a+ib=+\tfrac{\hbar}{2} \qquad c+id=+i\tfrac{\hbar}{2}$$
$$a-ib=-\tfrac{\hbar}{2} \qquad c-id=+i\tfrac{\hbar}{2}$$

Solve by adding and subtracting the equations to get

$$a=0 \qquad b=-i\tfrac{\hbar}{2} \qquad c=i\tfrac{\hbar}{2} \qquad d=0$$

Hence the matrix representing S_y in the S_z basis is

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2.2 Solve the secular equation

$$\det\left|S_x - \lambda I\right| = 0$$

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

Solve to find the eigenvalues

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\lambda = \pm\frac{\hbar}{2}$$

which was to be expected, because we know that the only possible results of a measurement of any spin component are $\pm\hbar/2$. Find the eigenvectors. For the positive eigenvalue:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

yields

$$b = a$$

The normalization condition yields

$$|a|^2 + |a|^2 = 1$$

$$|a|^2 = \tfrac{1}{2}$$

Choose a to be real and positive, resulting in

$$a = \tfrac{1}{\sqrt{2}}$$

$$b = \tfrac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the positive eigenvalue is

$$\left|+\right\rangle_x \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise, the eigenvector for the negative eigenvalue is

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = -a$$

The normalization condition yields

$$|a|^2 + |a|^2 = 1$$
$$|a|^2 = \tfrac{1}{2}$$

Choose a to be real and positive, resulting in

$$a = \tfrac{1}{\sqrt{2}}$$
$$b = -\tfrac{1}{\sqrt{2}}$$

so the eigenvector corresponding to the negative eigenvalue is

$$\left|-\right\rangle_x \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2.3 From Eq. (1.37), we know the S_z eigenstates in the S_x basis:

$$\left|+\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+\right\rangle_x + \left|-\right\rangle_x\right)$$
$$\left|-\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|+\right\rangle_x - \left|-\right\rangle_x\right)$$

Let the representation of S_z in the S_x basis be

$$S_z \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write the S_z eigenvalue equations in matrix notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

These yield

$$a+b=+\tfrac{\hbar}{2} \qquad c+d=+\tfrac{\hbar}{2}$$
$$a-b=-\tfrac{\hbar}{2} \qquad c-d=+\tfrac{\hbar}{2}$$

Solve by adding and subtracting the equations to get

$$a=0 \qquad b=\tfrac{\hbar}{2} \qquad c=\tfrac{\hbar}{2} \qquad d=0$$

Hence the matrix representing S_z in the S_x basis is

$$S_z \underset{S_x}{\doteq} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now diagonalize:

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \left(\tfrac{\hbar}{2}\right)^2 = 0 \quad \Rightarrow \quad \lambda = \pm\tfrac{\hbar}{2}$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad b=a$$

yielding

$$|+\rangle \underset{S_x}{\doteq} \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Likewise

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad b=-a$$

$$|-\rangle \underset{S_x}{\doteq} \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence the eigenvalue equations are

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = +\frac{\hbar}{2}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Rightarrow \quad S_z|+\rangle = +\frac{\hbar}{2}|+\rangle \quad OK$$
$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \Rightarrow \quad S_z|-\rangle = -\frac{\hbar}{2}|-\rangle \quad OK$$

2.4 The general matrix is

$$A \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix elements are

$$\langle + | A | + \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = a$$

$$\langle + | A | - \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = b$$

$$\langle - | A | + \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = c$$

$$\langle - | A | - \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = d$$

Hence we get

$$A \doteq \begin{pmatrix} \langle + | A | + \rangle & \langle + | A | - \rangle \\ \langle - | A | + \rangle & \langle - | A | - \rangle \end{pmatrix}$$

2.5 The commutators are

$$\begin{aligned}
[S_x, S_y] &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \doteq i\hbar \left(\frac{\hbar}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= i\hbar S_z
\end{aligned}$$

$$\begin{aligned}
[S_y, S_z] &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\
&\doteq \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \doteq i\hbar \left(\frac{\hbar}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= i\hbar S_x
\end{aligned}$$

$$
\begin{aligned}
[S_z,S_x] &\doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&\doteq \left(\frac{\hbar}{2}\right)^2\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \\
&\doteq \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \doteq i\hbar\left(\frac{\hbar}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= i\hbar S_y
\end{aligned}
$$

2.6 The spin component operator S_{n} is

$$
\begin{aligned}
S_n &= \mathbf{S}\bullet\hat{\mathbf{n}} \\
&= S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta
\end{aligned}
$$

Using the matrix representations for S_{x}, S_{y}, and S_{z} gives

$$
\begin{aligned}
S_n &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\
&\doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \\
&\doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}
\end{aligned}
$$

2.7 Diagonalize S_{n}:

$$
S_n \doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}
$$

Now diagonalize:

$$
\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = 0
$$

$$
\lambda^2 - \left(\tfrac{\hbar}{2}\right)^2\cos^2\theta - \left(\tfrac{\hbar}{2}\right)^2\sin^2\theta = 0 \quad \Rightarrow \quad \lambda^2 - \left(\tfrac{\hbar}{2}\right)^2 = 0 \quad \Rightarrow \quad \lambda = \pm\tfrac{\hbar}{2}
$$

as expected. Find the eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\cos\theta + b\sin\theta e^{-i\phi} = a \quad \Rightarrow \quad b = ae^{i\phi}\frac{1-\cos\theta}{\sin\theta}$$

The normalization condition yields

$$|a|^2 + |a|^2\left(\frac{1-\cos\theta}{\sin\theta}\right)^2 = 1$$

$$|a|^2 = \frac{\sin^2\theta}{2-2\cos\theta} = \frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \cos^2\tfrac{\theta}{2}$$

yielding

$$|+\rangle_n = \cos\tfrac{\theta}{2}|+\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-\rangle$$

Likewise

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a\cos\theta + b\sin\theta e^{-i\phi} = -a \quad \Rightarrow \quad b = -ae^{i\phi}\frac{1+\cos\theta}{\sin\theta}$$

The normalization condition yields

$$|a|^2 + |a|^2\left(\frac{1+\cos\theta}{\sin\theta}\right)^2 = 1$$

$$|a|^2 = \frac{\sin^2\theta}{2+2\cos\theta} = \frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{4\cos^2\frac{\theta}{2}} = \sin^2\tfrac{\theta}{2}$$

yielding

$$|-\rangle_n = \sin\tfrac{\theta}{2}|+\rangle - e^{i\phi}\cos\tfrac{\theta}{2}|-\rangle$$

2.8 The $|+\rangle_n$ eigenstate is

$$|+\rangle_n = \cos\tfrac{\theta}{2}|+\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-\rangle = \cos\tfrac{\pi}{8}|+\rangle + e^{i5\pi/3}\sin\tfrac{\pi}{8}|-\rangle$$

The probabilities are

$$\begin{aligned}
\mathcal{P}_{+y} &= \left|{}_y\langle +|+\rangle_n\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\cos\tfrac{\pi}{8}|+\rangle + e^{i5\pi/3}\sin\tfrac{\pi}{8}|-\rangle\right)\right|^2 \\
&= \tfrac{1}{2}\left|\cos\tfrac{\pi}{8} - ie^{i5\pi/3}\sin\tfrac{\pi}{8}\right|^2 = \tfrac{1}{2}\left|\cos\tfrac{\pi}{8} + \sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3} - i\sin\tfrac{\pi}{8}\cos\tfrac{5\pi}{3}\right|^2 \\
&= \tfrac{1}{2}\left(\cos^2\tfrac{\pi}{8} + \sin^2\tfrac{\pi}{8}\sin^2\tfrac{5\pi}{3} + \sin^2\tfrac{\pi}{8}\cos^2\tfrac{5\pi}{3} + 2\cos\tfrac{\pi}{8}\sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3}\right) \\
&= \tfrac{1}{2}\left(1 + 2\cos\tfrac{\pi}{8}\sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3}\right) \cong 0.194
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{-y} &= \left|{}_y\langle -|+\rangle_n\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{i}{\sqrt{2}}\langle -|\right)\left(\cos\tfrac{\pi}{8}|+\rangle + e^{i5\pi/3}\sin\tfrac{\pi}{8}|-\rangle\right)\right|^2 \\
&= \tfrac{1}{2}\left|\cos\tfrac{\pi}{8} + ie^{i5\pi/3}\sin\tfrac{\pi}{8}\right|^2 = \tfrac{1}{2}\left|\cos\tfrac{\pi}{8} - \sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3} + i\sin\tfrac{\pi}{8}\cos\tfrac{5\pi}{3}\right|^2 \\
&= \tfrac{1}{2}\left(\cos^2\tfrac{\pi}{8} + \sin^2\tfrac{\pi}{8}\sin^2\tfrac{5\pi}{3} + \sin^2\tfrac{\pi}{8}\cos^2\tfrac{5\pi}{3} - 2\cos\tfrac{\pi}{8}\sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3}\right) \\
&= \tfrac{1}{2}\left(1 - 2\cos\tfrac{\pi}{8}\sin\tfrac{\pi}{8}\sin\tfrac{5\pi}{3}\right) \cong 0.806
\end{aligned}$$

2.9 The expectation value of S_z is easy to do in Dirac notation:

$$\langle S_z\rangle = \langle +|S_z|+\rangle = \langle +|\frac{\hbar}{2}|+\rangle = \frac{\hbar}{2}\langle +|+\rangle = \frac{\hbar}{2}$$

The expectation values of S_x and S_y are easier in matrix notation:

$$\langle S_x\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_z^2\rangle = \langle +|S_z^2|+\rangle = \langle +|\frac{\hbar}{2}\frac{\hbar}{2}|+\rangle = \left(\frac{\hbar}{2}\right)^2\langle +|+\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_x^2\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_y^2\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

The uncertainties are

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

2.10 These expectation values are easier in matrix notation:

$$\langle S_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = 0\hbar$$

$$\langle S_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}$$

$$\langle S_z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = 0\hbar$$

To find the uncertainties, we need the expectation values of the squares:

$$\langle S_x^2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_y^2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_z^2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2$$

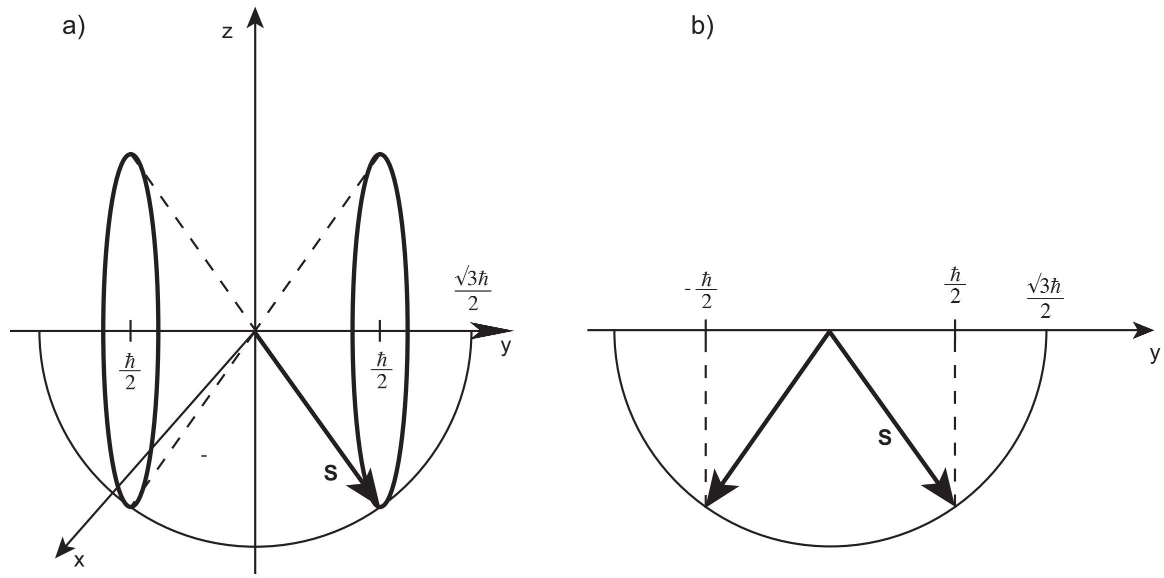
The uncertainties are

$$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

$$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2} = 0\hbar$$

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - 0} = \left(\frac{\hbar}{2}\right)$$

In the vector model, shown below, the spin is precessing around the y-axis at a constant angle such the y-component of the spin is constant and x- and z-components oscillate about zero.



2.11 The commutators in matrix notation are

$$[\mathbf{S}^2, S_x] \doteq \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\doteq \frac{3\hbar^3}{8}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$
$$= 0$$

$$[\mathbf{S}^2, S_y] \doteq \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\doteq \frac{3\hbar^3}{8}\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$
$$= 0$$

$$[\mathbf{S}^2, S_z] \doteq \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\doteq \frac{3\hbar^3}{8}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$
$$= 0$$

In abstract notation, the commutators are

$$
\begin{aligned}
[\mathbf{S}^2, S_x] &= \left[S_x^2 + S_y^2 + S_z^2, S_x\right] = \left[S_x^2, S_x\right] + \left[S_y^2, S_x\right] + \left[S_z^2, S_x\right] \\
&= S_x^2 S_x - S_x S_x^2 + S_y^2 S_x - S_x S_y^2 + S_z^2 S_x - S_x S_z^2 \\
&= S_x^3 - S_x^3 + S_y S_y S_x - S_x S_y S_y + S_z S_z S_x - S_x S_z S_z \\
&= 0 + S_y\left(S_x S_y - i\hbar S_z\right) - \left(S_y S_x + i\hbar S_z\right) S_y + S_z\left(S_x S_z + i\hbar S_y\right) - \left(S_z S_x - i\hbar S_y\right) S_z \\
&= S_y S_x S_y - i\hbar S_y S_z - S_y S_x S_y - i\hbar S_z S_y + S_z S_x S_z + i\hbar S_z S_y - S_z S_x S_z + i\hbar S_y S_z \\
&= 0 \\
[\mathbf{S}^2, S_y] &= \left[S_x^2 + S_y^2 + S_z^2, S_y\right] = \left[S_x^2, S_y\right] + \left[S_y^2, S_y\right] + \left[S_z^2, S_y\right] \\
&= S_x^2 S_y - S_y S_x^2 + S_y^2 S_y - S_y S_y^2 + S_z^2 S_y - S_y S_z^2 \\
&= S_x S_x S_y - S_y S_x S_x + S_y^3 - S_y^3 + S_z S_z S_y - S_y S_z S_z \\
&= S_x\left(S_y S_x + i\hbar S_z\right) - \left(S_x S_y - i\hbar S_z\right) S_x + 0 + S_z\left(S_y S_z - i\hbar S_x\right) - \left(S_z S_y + i\hbar S_x\right) S_z \\
&= S_x S_y S_x + i\hbar S_x S_z - S_x S_y S_x + i\hbar S_z S_x + S_z S_y S_z - i\hbar S_z S_x - S_z S_y S_z - i\hbar S_x S_z \\
&= 0 \\
[\mathbf{S}^2, S_z] &= \left[S_x^2 + S_y^2 + S_z^2, S_z\right] = \left[S_x^2, S_z\right] + \left[S_y^2, S_z\right] + \left[S_z^2, S_z\right] \\
&= S_x^2 S_z - S_z S_x^2 + S_y^2 S_z - S_z S_y^2 + S_z^2 S_z - S_z S_z^2 \\
&= S_x S_x S_z - S_z S_x S_x + S_y S_y S_z - S_z S_y S_y + S_z^3 - S_z^3 \\
&= S_x\left(S_z S_x - i\hbar S_y\right) - \left(S_x S_z + i\hbar S_y\right) S_x + S_y\left(S_z S_y + i\hbar S_x\right) - \left(S_y S_z - i\hbar S_x\right) S_y + 0 \\
&= S_x S_z S_x - i\hbar S_x S_y - S_x S_z S_x - i\hbar S_y S_x + S_y S_z S_y + i\hbar S_y S_x - S_y S_z S_y + i\hbar S_x S_y \\
&= 0
\end{aligned}
$$

2.12 For S_x the diagonalization yields the eigenvalues

$$
S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad -\lambda\left(\lambda^2 - \tfrac{\hbar^2}{2}\right) - \tfrac{\hbar}{\sqrt{2}}\left(-\lambda \tfrac{\hbar}{\sqrt{2}}\right) = 0
$$

$$
\lambda\left(\lambda^2 - \hbar^2\right) = 0 \quad \Rightarrow \quad \lambda = 1\hbar, 0, -1\hbar
$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b = a\sqrt{2} \\ a + c = b\sqrt{2} \\ b = c\sqrt{2} \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |b|^2\left(\tfrac{1}{2} + 1 + \tfrac{1}{2}\right) = 1 \quad \Rightarrow \quad b = \tfrac{1}{\sqrt{2}}, a = \tfrac{1}{2}, c = \tfrac{1}{2}$$

$$|1\rangle_x = \tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b = 0 \\ a + c = 0 \\ b = 0 \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |a|^2(1 + 1) = 1 \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}, b = 0, c = -\tfrac{1}{\sqrt{2}}$$

$$|0\rangle_x = \tfrac{1}{\sqrt{2}}|1\rangle - \tfrac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -1\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b = -a\sqrt{2} \\ a + c = -b\sqrt{2} \\ b = -c\sqrt{2} \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |b|^2\left(\tfrac{1}{2} + 1 + \tfrac{1}{2}\right) = 1 \quad \Rightarrow \quad b = -\tfrac{1}{\sqrt{2}}, a = \tfrac{1}{2}, c = \tfrac{1}{2}$$

$$|-1\rangle_x = \tfrac{1}{2}|1\rangle - \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{2}|-1\rangle$$

For S_y the diagonalization yields

$$S_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & \frac{-i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & -\lambda & \frac{-i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda\left(\lambda^2 - \tfrac{\hbar^2}{2}\right) - \tfrac{-i\hbar}{\sqrt{2}}\left(-\lambda\tfrac{i\hbar}{\sqrt{2}}\right) = 0$$

$$\lambda\left(\lambda^2 - \hbar^2\right) = 0 \quad \Rightarrow \quad \lambda = \hbar, 0, -\hbar$$

and the eigenvectors

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -ib = a\sqrt{2} \\ ia - ic = b\sqrt{2} \\ ib = c\sqrt{2} \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |b|^2\left(\tfrac{1}{2} + 1 + \tfrac{1}{2}\right) = 1 \quad \Rightarrow \quad b = \tfrac{i}{\sqrt{2}}, a = \tfrac{1}{2}, c = -\tfrac{1}{2}$$

$$|1\rangle_y = \tfrac{1}{2}|1\rangle + \tfrac{i}{\sqrt{2}}|0\rangle - \tfrac{1}{2}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -ib = 0 \\ ia - ic = 0 \\ ib = 0 \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |a|^2(1+1) = 1 \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}, b = 0, c = \tfrac{1}{\sqrt{2}}$$

$$|0\rangle_y = \tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -1\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -ib = -a\sqrt{2} \\ ia - ic = -b\sqrt{2} \\ ib = -c\sqrt{2} \end{matrix}$$

$$|a|^2 + |b|^2 + |c|^2 = 1 \quad \Rightarrow \quad |b|^2\left(\tfrac{1}{2} + 1 + \tfrac{1}{2}\right) = 1 \quad \Rightarrow \quad b = -\tfrac{i}{\sqrt{2}}, a = \tfrac{1}{2}, c = -\tfrac{1}{2}$$

$$|-1\rangle_y = \tfrac{1}{2}|1\rangle - \tfrac{i}{\sqrt{2}}|0\rangle - \tfrac{1}{2}|-1\rangle$$

2.13 The commutators are

$$[S_x, S_y] \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\doteq \frac{\hbar^2}{2}\left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right]$$

$$\doteq \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} \doteq i\hbar\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= i\hbar S_z$$

$$[S_y, S_z] \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\doteq \frac{\hbar^2}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}\right]$$

$$\doteq \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} \doteq i\hbar\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= i\hbar S_x$$

$$[S_z,S_x] \doteq \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\doteq \frac{\hbar^2}{\sqrt{2}}\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$\doteq \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \doteq i\hbar\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= i\hbar S_y$$

2.14 Using the component matrices we find

$$\mathbf{S}^2 = S_x^2 + S_y^2 + S_z^2$$

$$\doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$+\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\doteq \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\doteq 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalue equation is

$$\mathbf{S}^2\left|sm\right\rangle = \mathrm{s}(s+1)\hbar^2\left|sm\right\rangle$$

For spin-1 this is

$$\mathbf{S}^2\left|1m\right\rangle = 2\hbar^2\left|1m\right\rangle$$

Hence the $\mathbf{S}^2$ operator must be $2\hbar^2$ times the identity matrix:

$$\mathbf{S}^2 \doteq 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.15 a) The possible results of a measurement of the spin component S_z are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_{in}\rangle\right|^2 = \left|\langle 1|\left[\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right]\right|^2$$
$$= \left|\tfrac{2}{\sqrt{29}}\langle 1|1\rangle + \tfrac{3i}{\sqrt{29}}\langle 1|0\rangle - \tfrac{4}{\sqrt{29}}\langle 1|-1\rangle\right|^2 = \left|\tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{4}{29}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_{in}\rangle\right|^2 = \left|\langle 0|\left[\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right]\right|^2$$
$$= \left|\tfrac{2}{\sqrt{29}}\langle 0|1\rangle + \tfrac{3i}{\sqrt{29}}\langle 0|0\rangle - \tfrac{4}{\sqrt{29}}\langle 0|-1\rangle\right|^2 = \left|\tfrac{3i}{\sqrt{29}}\right|^2 = \tfrac{9}{29}$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_{in}\rangle\right|^2 = \left|\langle -1|\left[\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right]\right|^2$$
$$= \left|\tfrac{2}{\sqrt{29}}\langle -1|1\rangle + \tfrac{3i}{\sqrt{29}}\langle -1|0\rangle - \tfrac{4}{\sqrt{29}}\langle -1|-1\rangle\right|^2 = \left|-\tfrac{4}{\sqrt{29}}\right|^2 = \tfrac{16}{29}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_x are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_{1x} = \left|{}_x\langle 1|\psi_{in}\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 1| + \tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{2}\langle -1|\right)\left(\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right)\right|^2$$
$$= \left|\tfrac{1}{\sqrt{29}} + \tfrac{3i}{\sqrt{2}\sqrt{29}} - \tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{1}{58}\left|-\sqrt{2} + 3i\right|^2 = \tfrac{11}{58}$$

$$\mathcal{P}_{0x} = \left|{}_x\langle 0|\psi_{in}\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle 1| - \tfrac{1}{\sqrt{2}}\langle -1|\right)\left(\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right)\right|^2$$
$$= \left|\tfrac{2}{\sqrt{2}\sqrt{29}} + \tfrac{4}{\sqrt{2}\sqrt{29}}\right|^2 = \tfrac{36}{58}$$

$$\mathcal{P}_{-1x} = \left|{}_x\langle -1|\psi_{in}\rangle\right|^2 = \left|\left(\tfrac{1}{2}\langle 1| - \tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{2}\langle -1|\right)\left(\tfrac{2}{\sqrt{29}}|1\rangle + \tfrac{3i}{\sqrt{29}}|0\rangle - \tfrac{4}{\sqrt{29}}|-1\rangle\right)\right|^2$$
$$= \left|\tfrac{1}{\sqrt{29}} - \tfrac{3i}{\sqrt{2}\sqrt{29}} - \tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{1}{58}\left|-\sqrt{2} - 3i\right|^2 = \tfrac{11}{58}$$

The three probabilities add to unity, as they must.

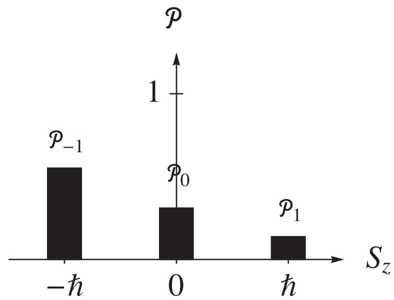
c) For the first measurement, the expectation value is

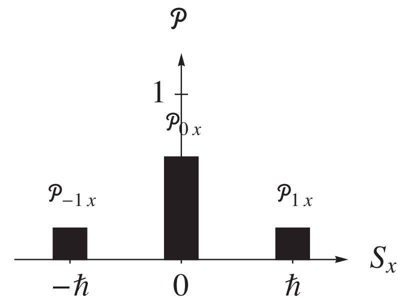
$$\langle S_z\rangle = \sum_m m\hbar\mathcal{P}_m = 1\hbar\tfrac{4}{29} + 0\hbar\tfrac{9}{29} + (-1)\hbar\tfrac{16}{29} = -\hbar\tfrac{12}{29}$$

For the second measurement, the expectation value is

$$\langle S_x\rangle = \sum_m m\hbar\mathcal{P}_{mx} = 1\hbar\tfrac{11}{58} + 0\hbar\tfrac{36}{58} + (-1)\hbar\tfrac{11}{58} = 0\hbar$$

The histograms are shown below.





2.16 a) The possible results of a measurement of the spin component S_z are always $+1\hbar,\, 0\hbar,\, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_1 &= \left|\langle 1|\psi_{in}\rangle\right|^2 = \left|\langle 1|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\langle 1|1\rangle_y + \tfrac{3i}{\sqrt{29}}\langle 1|0\rangle_y - \tfrac{4}{\sqrt{29}}\langle 1|-1\rangle_y\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\tfrac{1}{2} + \tfrac{3i}{\sqrt{29}}\tfrac{1}{\sqrt{2}} - \tfrac{4}{\sqrt{29}}\tfrac{1}{2}\right|^2 = \tfrac{1}{58}\left|-\sqrt{2}+3i\right|^2 = \tfrac{11}{58}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_0 &= \left|\langle 0|\psi_{in}\rangle\right|^2 = \left|\langle 0|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\langle 0|1\rangle_y + \tfrac{3i}{\sqrt{29}}\langle 0|0\rangle_y - \tfrac{4}{\sqrt{29}}\langle 0|-1\rangle_y\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\tfrac{i}{\sqrt{2}} + \tfrac{3i}{\sqrt{29}}0 - \tfrac{4}{\sqrt{29}}\tfrac{-i}{\sqrt{2}}\right|^2 = \tfrac{1}{58}\left|i(2+4)\right|^2 = \tfrac{36}{58}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{-1} &= \left|\langle -1|\psi_{in}\rangle\right|^2 = \left|\langle -1|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\langle -1|1\rangle_y + \tfrac{3i}{\sqrt{29}}\langle -1|0\rangle_y - \tfrac{4}{\sqrt{29}}\langle -1|-1\rangle_y\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}\tfrac{-1}{2} + \tfrac{3i}{\sqrt{29}}\tfrac{1}{\sqrt{2}} - \tfrac{4}{\sqrt{29}}\tfrac{-1}{2}\right|^2 = \tfrac{1}{58}\left|\sqrt{2}+3i\right|^2 = \tfrac{11}{58}
\end{aligned}$$

The three probabilities add to unity, as they must.

b) The possible results of a measurement of the spin component S_y are always $+1\hbar,\, 0\hbar,\, -1\hbar$ for a spin-1 particle. The probabilities are

$$\begin{aligned}
\mathcal{P}_{1y} &= \left|{}_y\langle 1|\psi_{in}\rangle\right|^2 = \left|{}_y\langle 1|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}{}_y\langle 1|1\rangle_y + \tfrac{3i}{\sqrt{29}}{}_y\langle 1|0\rangle_y - \tfrac{4}{\sqrt{29}}{}_y\langle 1|-1\rangle_y\right|^2 = \left|\tfrac{2}{\sqrt{29}}\right|^2 = \tfrac{4}{29}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{0y} &= \left|{}_y\langle 0|\psi_{in}\rangle\right|^2 = \left|{}_y\langle 0|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2 \\
&= \left|\tfrac{2}{\sqrt{29}}{}_y\langle 0|1\rangle_y + \tfrac{3i}{\sqrt{29}}{}_y\langle 0|0\rangle_y - \tfrac{4}{\sqrt{29}}{}_y\langle 0|-1\rangle_y\right|^2 = \left|\tfrac{3i}{\sqrt{29}}\right|^2 = \tfrac{9}{29}
\end{aligned}$$

$$\mathcal{P}_{-1y} = \left|{}_y\langle -1|\psi_{in}\rangle\right|^2 = \left|{}_y\langle -1|\left[\tfrac{2}{\sqrt{29}}|1\rangle_y + \tfrac{3i}{\sqrt{29}}|0\rangle_y - \tfrac{4}{\sqrt{29}}|-1\rangle_y\right]\right|^2$$
$$= \left|\tfrac{2}{\sqrt{29}}\,{}_y\langle -1|1\rangle_y + \tfrac{3i}{\sqrt{29}}\,{}_y\langle -1|0\rangle_y - \tfrac{4}{\sqrt{29}}\,{}_y\langle -1|-1\rangle_y\right|^2 = \left|-\tfrac{4}{\sqrt{29}}\right|^2 = \tfrac{16}{29}$$

The three probabilities add to unity, as they must.

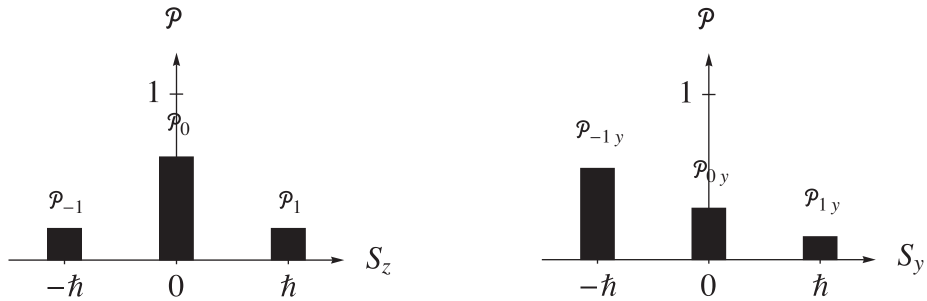
c) For the first measurement, the expectation value is

$$\langle S_z\rangle = \sum_m m\hbar\mathcal{P}_m = 1\hbar\,\tfrac{11}{58} + 0\hbar\,\tfrac{36}{58} + (-1)\hbar\,\tfrac{11}{58} = 0\hbar$$

For the second measurement, the expectation value is

$$\langle S_y\rangle = \sum_m m\hbar\mathcal{P}_{my} = 1\hbar\,\tfrac{4}{29} + 0\hbar\,\tfrac{9}{29} + (-1)\hbar\,\tfrac{16}{29} = -\hbar\,\tfrac{12}{29}$$

The histograms are shown below.



2.17 a) The possible results of a measurement of the spin component S_z are always $+1\hbar,\ 0\hbar,\ -1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_{\hbar} = \left|\langle 1|\psi\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0\end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5i\end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{30}}1\right|^2 = \frac{1}{30}$$

$$\mathcal{P}_{0} = \left|\langle 0|\psi\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0\end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5i\end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{30}}2\right|^2 = \frac{4}{30}$$

$$\mathcal{P}_{\hbar} = \left|\langle \text{-}1|\psi\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1\end{pmatrix}\tfrac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5i\end{pmatrix}\right|^2 = \left|\tfrac{1}{\sqrt{30}}5i\right|^2 = \frac{25}{30}$$

The expectation value of S_z is

$$\langle S_z\rangle = \mathcal{P}_{\hbar}\hbar + \mathcal{P}_0 0 + \mathcal{P}_{-\hbar}(-\hbar) = \frac{1}{30}\hbar + \frac{4}{30}0 + \frac{25}{30}(-\hbar) = -\frac{24}{30}\hbar = -\frac{4}{5}\hbar$$

b) The expectation value of S_x is

$$\langle S_x \rangle = \langle \psi | S_x | \psi \rangle = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}$$

$$= \frac{1}{30} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 2 \\ 1+5i \\ 2 \end{pmatrix} = \frac{1}{30} \frac{\hbar}{\sqrt{2}} \left(2 + 2(1+5i) - 5i \times 2 \right) = \frac{\sqrt{2}}{15} \hbar$$

2.18 a) The possible results of a measurement of the spin component S_z are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_1 = \left| \langle 1 | \psi_{in} \rangle \right|^2 = \left| \langle 1 | \left[\tfrac{1}{\sqrt{14}} |1\rangle - \tfrac{3}{\sqrt{14}} |0\rangle + \tfrac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2$$

$$= \left| \tfrac{1}{\sqrt{14}} \langle 1 | 1 \rangle - \tfrac{3}{\sqrt{14}} \langle 1 | 0 \rangle + \tfrac{2i}{\sqrt{14}} \langle 1 | -1 \rangle \right|^2 = \left| \tfrac{1}{\sqrt{14}} \right|^2 = \tfrac{1}{14}$$

$$\mathcal{P}_0 = \left| \langle 0 | \psi_{in} \rangle \right|^2 = \left| \langle 0 | \left[\tfrac{1}{\sqrt{14}} |1\rangle - \tfrac{3}{\sqrt{14}} |0\rangle + \tfrac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2$$

$$= \left| \tfrac{1}{\sqrt{14}} \langle 0 | 1 \rangle - \tfrac{3}{\sqrt{14}} \langle 0 | 0 \rangle + \tfrac{2i}{\sqrt{14}} \langle 0 | -1 \rangle \right|^2 = - \tfrac{3}{\sqrt{14}}^{2} = \tfrac{9}{14}$$

$$\mathcal{P}_{-1} = \left| \langle -1 | \psi_{in} \rangle \right|^2 = \left| \langle -1 | \left[\tfrac{1}{\sqrt{14}} |1\rangle - \tfrac{3}{\sqrt{14}} |0\rangle + \tfrac{2i}{\sqrt{14}} |-1\rangle \right] \right|^2$$

$$= \left| \tfrac{1}{\sqrt{14}} \langle -1 | 1 \rangle - \tfrac{3}{\sqrt{14}} \langle -1 | 0 \rangle + \tfrac{2i}{\sqrt{14}} \langle -1 | -1 \rangle \right|^2 = \left| \tfrac{2i}{\sqrt{14}} \right|^2 = \tfrac{4}{14}$$

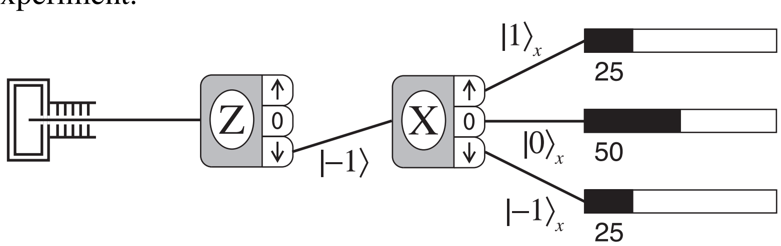
b) After the S_z measurement, the system is in the state $|-1\rangle$. The possible results of a measurement of the spin component S_x are always $+1\hbar, 0\hbar, -1\hbar$ for a spin-1 particle. The probabilities are

$$\mathcal{P}_{1x} = \left| {}_x\langle 1 | \psi_{in} \rangle \right|^2 = \left| \left(\tfrac{1}{2} \langle 1 | + \tfrac{1}{\sqrt{2}} \langle 0 | + \tfrac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \tfrac{1}{2} \right|^2 = \tfrac{1}{4}$$

$$\mathcal{P}_{0x} = \left| {}_x\langle 0 | \psi_{in} \rangle \right|^2 = \left| \left(\tfrac{1}{\sqrt{2}} \langle 1 | - \tfrac{1}{\sqrt{2}} \langle -1 | \right) |-1\rangle \right|^2 = \left| \tfrac{-1}{\sqrt{2}} \right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{-1x} = \left| {}_x\langle -1 | \psi_{in} \rangle \right|^2 = \left| \left(\tfrac{1}{2} \langle 1 | - \tfrac{1}{\sqrt{2}} \langle 0 | + \tfrac{1}{2} \langle -1 | \right) |-1\rangle \right|^2 = \left| \tfrac{1}{2} \right|^2 = \tfrac{1}{4}$$

c) Schematic of experiment.



2.19 The probability is

$$\mathcal{P}_{\psi_f} = \left|\langle\psi_f|\psi_i\rangle\right|^2 = \left|\left(\tfrac{1-i}{\sqrt{7}}\,{}_y\langle 1| + \tfrac{2}{\sqrt{7}}\,{}_y\langle 0| + \tfrac{i}{\sqrt{7}}\,{}_y\langle -1|\right)\left(\tfrac{1}{\sqrt{6}}|1\rangle - \tfrac{\sqrt{2}}{\sqrt{6}}|0\rangle + \tfrac{i\sqrt{3}}{\sqrt{6}}|-1\rangle\right)\right|^2$$

$$= \left|\begin{array}{l} \tfrac{1-i}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\,{}_y\langle 1|1\rangle - \tfrac{1-i}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}\,{}_y\langle 1|0\rangle + \tfrac{1-i}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\,{}_y\langle 1|-1\rangle + \tfrac{2}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\,{}_y\langle 0|1\rangle - \tfrac{2}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}\,{}_y\langle 0|0\rangle + \tfrac{2}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\,{}_y\langle 0|-1\rangle + \\ + \tfrac{i}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\,{}_y\langle -1|1\rangle - \tfrac{i}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}\,{}_y\langle -1|0\rangle + \tfrac{i}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\,{}_y\langle -1|-1\rangle \end{array}\right|^2$$

$$= \left|\begin{array}{l} \tfrac{1-i}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\tfrac{1}{2} + \tfrac{1-i}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}\tfrac{i}{\sqrt{2}} - \tfrac{1-i}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\tfrac{1}{2} + \tfrac{2}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}0 + \tfrac{2}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\tfrac{1}{\sqrt{2}} + \\ + \tfrac{i}{\sqrt{7}}\tfrac{1}{\sqrt{6}}\tfrac{1}{2} - \tfrac{i}{\sqrt{7}}\tfrac{\sqrt{2}}{\sqrt{6}}\tfrac{i}{\sqrt{2}} - \tfrac{i}{\sqrt{7}}\tfrac{i\sqrt{3}}{\sqrt{6}}\tfrac{1}{2} \end{array}\right|^2$$

$$= \tfrac{1}{168}\left|1 - i + 2 + 2i - \sqrt{3} - i\sqrt{3} + 2\sqrt{2} - 0 + i2\sqrt{6} + i + 2 + \sqrt{3}\right|^2$$

$$= \tfrac{1}{168}\left|5 + 2\sqrt{2} + 2i - i\sqrt{3} + i2\sqrt{6}\right|^2 = \tfrac{1}{168}\left\{\left(5 + 2\sqrt{2}\right)^2 + \left(2 - \sqrt{3} + 2\sqrt{6}\right)^2\right\}$$

$$= \tfrac{1}{168}\left(64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6}\right) \cong 0.524$$

or in matrix notation

$$\mathcal{P}_{\psi_f} = \left|\langle\psi_f|\psi_i\rangle\right|^2 = \left|\left\{\tfrac{1-i}{\sqrt{7}}\begin{pmatrix} \tfrac{1}{2} & \tfrac{-i}{\sqrt{2}} & \tfrac{-1}{2} \end{pmatrix} + \tfrac{2}{\sqrt{7}}\begin{pmatrix} \tfrac{1}{\sqrt{2}} & 0 & \tfrac{1}{\sqrt{2}} \end{pmatrix} + \tfrac{i}{\sqrt{7}}{}_y\begin{pmatrix} \tfrac{1}{2} & \tfrac{i}{\sqrt{2}} & \tfrac{-1}{2} \end{pmatrix}\right\}\tfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix}\right|^2$$

$$= \tfrac{1}{168}\left|\begin{pmatrix} 1 + 2\sqrt{2} & -2\sqrt{2} - i\sqrt{2} & -1 + 2\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix}\right|^2$$

$$= \tfrac{1}{168}\left|1 + 2\sqrt{2} + 4 + 2i - i\sqrt{3} + i2\sqrt{6}\right|^2 = \tfrac{1}{168}\left(64 + 8\sqrt{2} - 4\sqrt{3} + 8\sqrt{6}\right) \cong 0.524$$

2.20 Spin 1 unknowns. Follow the solution method given in the lab handout. (i) For unknown number 1, the measured probabilities are

$$\begin{array}{lll} \mathcal{P}_1 = \tfrac{1}{4} & \mathcal{P}_{1x} = \tfrac{1}{4} & \mathcal{P}_{1y} = 1 \\ \mathcal{P}_0 = \tfrac{1}{2} & \mathcal{P}_{0x} = \tfrac{1}{2} & \mathcal{P}_{0y} = 0 \\ \mathcal{P}_{-1} = \tfrac{1}{4} & \mathcal{P}_{-1x} = \tfrac{1}{4} & \mathcal{P}_{-1y} = 0 \end{array}$$

Write the unknown state as

$$|\psi_1\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_1 = |\langle 1|\psi_1\rangle|^2 = |\langle 1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |a|^2 = \tfrac{1}{4} \quad \Rightarrow \quad a = \tfrac{1}{2}$$
$$\mathcal{P}_0 = |\langle 0|\psi_1\rangle|^2 = |\langle 0|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |b|^2 = \tfrac{1}{2} \quad \Rightarrow \quad b = \tfrac{1}{\sqrt{2}}e^{i\alpha}$$
$$\mathcal{P}_{-1} = |\langle -1|\psi_1\rangle|^2 = |\langle -1|\{a|1\rangle + b|0\rangle + c|-1\rangle\}|^2 = |c|^2 = \tfrac{1}{4} \quad \Rightarrow \quad c = \tfrac{1}{4}e^{i\beta}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_1\rangle = \tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\mathcal{P}_{0x} = |{}_x\langle 0|\psi_1\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}\{\langle 1| - \langle -1|\}\left\{\tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{2\sqrt{2}}\{1 - e^{i\beta}\}\right|^2$$
$$= \tfrac{1}{8}\{1 - e^{i\beta}\}\{1 - e^{-i\beta}\} = \tfrac{1}{8}\{1 + 1 - e^{i\beta} - e^{-i\beta}\} = \tfrac{1}{4}\{1 - \cos\beta\} = \tfrac{1}{2} \Rightarrow \cos\beta = -1 \Rightarrow \beta = \pi$$

Giving the state

$$|\psi_1\rangle = \tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle - \tfrac{1}{2}|-1\rangle$$

Equating the predicted S_y probabilities and the experimental results gives

$$\mathcal{P}_{1y} = |{}_y\langle 1|\psi_1\rangle|^2 = \left|\left\{\tfrac{1}{2}\langle 1| - \tfrac{i}{\sqrt{2}}\langle 0| - \tfrac{1}{2}\langle -1|\right\}\left\{\tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle - \tfrac{1}{2}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{4} - \tfrac{i}{2}e^{i\alpha} + \tfrac{1}{4}\right|^2$$
$$= \tfrac{1}{4}\{1 - ie^{i\alpha}\}\{1 + ie^{-i\alpha}\} = \tfrac{1}{4}\{1 + 1 - ie^{i\alpha} + ie^{-i\alpha}\} = \tfrac{1}{2}\{1 + \sin\alpha\} = 1 \Rightarrow \sin\alpha = 1 \Rightarrow \alpha = \tfrac{\pi}{2}$$

Hence the unknown state is

$$|\psi_1\rangle = \tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}|0\rangle - \tfrac{1}{2}|-1\rangle = \tfrac{1}{2}|1\rangle + \tfrac{i}{\sqrt{2}}|0\rangle - \tfrac{1}{2}|-1\rangle = |1\rangle_y$$

(ii) For unknown number 2, the measured probabilities are

$$\begin{array}{lll} \mathcal{P}_1 = \tfrac{1}{4} & \mathcal{P}_{1x} = \tfrac{9}{16} & \mathcal{P}_{1y} = 0.870 \\ \mathcal{P}_0 = \tfrac{1}{2} & \mathcal{P}_{0x} = \tfrac{3}{8} & \mathcal{P}_{0y} = 0.125 \\ \mathcal{P}_{-1} = \tfrac{1}{4} & \mathcal{P}_{-1x} = \tfrac{1}{16} & \mathcal{P}_{-1y} = 0.005 \end{array}$$

Write the unknown state as

$$|\psi_2\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\langle 1|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |a|^2 = \tfrac{1}{4} \quad \Rightarrow \quad a = \tfrac{1}{2}$$
$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\langle 0|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |b|^2 = \tfrac{1}{2} \quad \Rightarrow \quad b = \tfrac{1}{\sqrt{2}}e^{i\alpha}$$
$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\langle -1|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |c|^2 = \tfrac{1}{4} \quad \Rightarrow \quad c = \tfrac{1}{2}e^{i\beta}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_2\rangle = \tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{0x} &= \left|{}_x\langle 0|\psi_2\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle 1| - \langle -1|\right\}\left\{\tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{2\sqrt{2}}\left\{1 - e^{i\beta}\right\}\right|^2 \\
&= \tfrac{1}{8}\left\{1 - e^{i\beta}\right\}\left\{1 - e^{-i\beta}\right\} = \tfrac{1}{8}\left\{1 + 1 - e^{i\beta} - e^{-i\beta}\right\} = \tfrac{1}{4}\left\{1 - \cos\beta\right\} = \tfrac{3}{8} \\
&\Rightarrow \cos\beta = -\tfrac{1}{2} \Rightarrow \beta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{1x} &= \left|{}_x\langle 1|\psi_2\rangle\right|^2 = \left|\left\{\tfrac{1}{2}\langle 1| + \tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{2}\langle -1|\right\}\left\{\tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{4} + \tfrac{1}{2}e^{i\alpha} + \tfrac{1}{4}e^{i\beta}\right|^2 \\
&= \tfrac{1}{16}\left\{1 + 2e^{i\alpha} + e^{i\beta}\right\}\left\{1 + 2e^{-i\alpha} + e^{-i\beta}\right\} = \tfrac{1}{16}\left\{6 + 4\cos\alpha + 2\cos\beta + 4\cos(\alpha - \beta)\right\} \\
&= \tfrac{1}{16}\left\{6 + 4\cos\alpha + 2\cos\beta + 4\cos\alpha\cos\beta + 4\sin\alpha\sin\beta\right\} \\
&= \tfrac{1}{16}\left\{5 + 2\cos\alpha + 4\sin\alpha\sin\beta\right\} = \tfrac{9}{16}
\end{aligned}$$

which yields

$$\begin{aligned}
\cos\alpha + 2\sin\alpha\sin\beta &= 2 \\
2\sin\alpha\sin\beta &= 2 - \cos\alpha \\
4\sin^2\alpha\sin^2\beta &= 4 - 4\cos\alpha + \cos^2\alpha \\
4\left(1 - \cos^2\alpha\right)\tfrac{3}{4} &= 4 - 4\cos\alpha + \cos^2\alpha \\
4\cos^2\alpha - 4\cos\alpha + 1 &= 0 \\
2\cos\alpha - 1 &= 0 \\
\cos\alpha &= \tfrac{1}{2} \Rightarrow \alpha = \tfrac{\pi}{3}, \tfrac{5\pi}{3}
\end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned}
\mathcal{P}_{1y} &= \left|{}_y\langle 1|\psi_2\rangle\right|^2 = \left|\left\{\tfrac{1}{2}\langle 1| - \tfrac{i}{\sqrt{2}}\langle 0| - \tfrac{1}{2}\langle -1|\right\}\left\{\tfrac{1}{2}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\alpha}|0\rangle + \tfrac{1}{2}e^{i\beta}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{4} - \tfrac{i}{2}e^{i\alpha} - \tfrac{1}{4}e^{i\beta}\right|^2 \\
&= \tfrac{1}{16}\left\{1 - 2ie^{i\alpha} - e^{i\beta}\right\}\left\{1 + 2ie^{-i\alpha} - e^{-i\beta}\right\} = \tfrac{1}{16}\left\{6 + 4\sin\alpha - 2\cos\beta - 4\sin(\alpha - \beta)\right\} \\
&= \tfrac{1}{16}\left\{6 + 4\sin\alpha - 2\cos\beta - 4\sin\alpha\cos\beta + 4\cos\alpha\sin\beta\right\} \\
&= \tfrac{1}{16}\left\{7 + 6\sin\alpha + 4\cos\alpha\sin\beta\right\} = 0.87
\end{aligned}$$

which gives

$$
\begin{aligned}
3\sin\alpha + 2\cos\alpha\sin\beta &= 3.46 \cong 2\sqrt{3} \\
2\cos\alpha\sin\beta &= 2\sqrt{3} - 3\sin\alpha \\
4\cos^2\alpha\sin^2\beta &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
4\left(1-\sin^2\alpha\right)\tfrac{3}{4} &= 12 - 12\sqrt{3}\sin\alpha + 9\sin^2\alpha \\
\sin^2\alpha - \sqrt{3}\sin\alpha + \tfrac{3}{4} &= 0 \\
\sin\alpha - \tfrac{\sqrt{3}}{2} &= 0 \\
\sin\alpha = \tfrac{\sqrt{3}}{2} \;\Rightarrow\; \alpha = \tfrac{\pi}{3}, \tfrac{2\pi}{3} \;&\Rightarrow\; \alpha = \tfrac{\pi}{3} \;\Rightarrow\; \beta = \tfrac{2\pi}{3}
\end{aligned}
$$

Hence the unknown state is

$$
\left|\psi_2\right\rangle = \tfrac{1}{2}\left|1\right\rangle + \tfrac{1}{\sqrt{2}}e^{i\frac{\pi}{3}}\left|0\right\rangle + \tfrac{1}{2}e^{i\frac{2\pi}{3}}\left|-1\right\rangle = \left|1\right\rangle_{n\left(\theta=\frac{\pi}{2},\phi=\frac{\pi}{3}\right)}
$$

(iii) For unknown number 3, the measured probabilities are

$$
\begin{array}{lll}
\mathcal{P}_1 = \tfrac{1}{3} & \mathcal{P}_{1x} = \tfrac{1}{6} & \mathcal{P}_{1y} = 0.0286 \\
\mathcal{P}_0 = \tfrac{1}{3} & \mathcal{P}_{0x} = \tfrac{2}{3} & \mathcal{P}_{0y} = 0 \\
\mathcal{P}_{-1} = \tfrac{1}{3} & \mathcal{P}_{-1x} = \tfrac{1}{6} & \mathcal{P}_{-1y} = 0.9714
\end{array}
$$

Write the unknown state as

$$
\left|\psi_3\right\rangle = a\left|1\right\rangle + b\left|0\right\rangle + c\left|-1\right\rangle
$$

Equating the predicted S_z probabilities and the experimental results gives

$$
\begin{aligned}
\mathcal{P}_1 &= \left|\left\langle 1|\psi_3\right\rangle\right|^2 = \left|\left\langle 1\right|\left\{a\left|1\right\rangle + b\left|0\right\rangle + c\left|-1\right\rangle\right\}\right|^2 = \left|a\right|^2 = \tfrac{1}{3} \quad\Rightarrow\quad a = \tfrac{1}{\sqrt{3}} \\
\mathcal{P}_0 &= \left|\left\langle 0|\psi_3\right\rangle\right|^2 = \left|\left\langle 0\right|\left\{a\left|1\right\rangle + b\left|0\right\rangle + c\left|-1\right\rangle\right\}\right|^2 = \left|b\right|^2 = \tfrac{1}{3} \quad\Rightarrow\quad b = \tfrac{1}{\sqrt{3}}e^{i\alpha} \\
\mathcal{P}_{-1} &= \left|\left\langle -1|\psi_3\right\rangle\right|^2 = \left|\left\langle -1\right|\left\{a\left|1\right\rangle + b\left|0\right\rangle + c\left|-1\right\rangle\right\}\right|^2 = \left|c\right|^2 = \tfrac{1}{3} \quad\Rightarrow\quad c = \tfrac{1}{\sqrt{3}}e^{i\beta}
\end{aligned}
$$

allowing for possible relative phases. So now the unknown state is

$$
\left|\psi_3\right\rangle = \tfrac{1}{\sqrt{3}}\left|1\right\rangle + \tfrac{1}{\sqrt{3}}e^{i\alpha}\left|0\right\rangle + \tfrac{1}{\sqrt{3}}e^{i\beta}\left|-1\right\rangle
$$

Equating the predicted S_x probabilities and the experimental results gives

$$
\begin{aligned}
\mathcal{P}_{0x} &= \left|{}_x\!\left\langle 0|\psi_3\right\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\left\langle 1\right| - \left\langle -1\right|\right\}\tfrac{1}{\sqrt{3}}\left\{\left|1\right\rangle + e^{i\alpha}\left|0\right\rangle + e^{i\beta}\left|-1\right\rangle\right\}\right|^2 = \left|\tfrac{1}{\sqrt{6}}\left\{1 - e^{i\beta}\right\}\right|^2 \\
&= \tfrac{1}{6}\left\{1 - e^{i\beta}\right\}\left\{1 - e^{-i\beta}\right\} = \tfrac{1}{6}\left\{1 + 1 - e^{i\beta} - e^{-i\beta}\right\} = \tfrac{1}{3}\left\{1 - \cos\beta\right\} = \tfrac{2}{3} \\
&\Rightarrow\; \cos\beta = -1 \;\Rightarrow\; \beta = \pi
\end{aligned}
$$

Equating the predicted S_y probabilities and the experimental results gives

$$\mathcal{P}_{1y} = \left|{}_y\langle 1|\psi_3\rangle\right|^2 = \left|\left\{\tfrac{1}{2}\langle 1| - \tfrac{i}{\sqrt{2}}\langle 0| - \tfrac{1}{2}\langle -1|\right\}\tfrac{1}{\sqrt{3}}\left\{|1\rangle + e^{i\alpha}|0\rangle - |-1\rangle\right\}\right|^2 = \left|\tfrac{1}{\sqrt{3}}\left(\tfrac{1}{2} - \tfrac{i}{\sqrt{2}}e^{i\alpha} + \tfrac{1}{2}\right)\right|^2$$
$$= \tfrac{1}{3}\left\{1 - \tfrac{i}{\sqrt{2}}e^{i\alpha}\right\}\left\{1 + \tfrac{i}{\sqrt{2}}e^{-i\alpha}\right\} = \tfrac{1}{3}\left\{\tfrac{3}{2} + \sqrt{2}\sin\alpha\right\} = 0.0286$$
$$\Rightarrow \sin\alpha = -1 \Rightarrow \alpha = \tfrac{3\pi}{2}$$

Hence the unknown state is

$$|\psi_3\rangle = \tfrac{1}{\sqrt{3}}|1\rangle - \tfrac{i}{\sqrt{3}}|0\rangle - \tfrac{1}{\sqrt{3}}|-1\rangle \quad \left\{\neq |m\rangle_n\right\}$$

(iv) For unknown number 4, the measured probabilities are

$$\begin{array}{lll} \mathcal{P}_1 = \tfrac{1}{2} & \mathcal{P}_{1x} = \tfrac{1}{4} & \mathcal{P}_{1y} = \tfrac{1}{4} \\ \mathcal{P}_0 = 0 & \mathcal{P}_{0x} = \tfrac{1}{2} & \mathcal{P}_{0y} = \tfrac{1}{2} \\ \mathcal{P}_{-1} = \tfrac{1}{2} & \mathcal{P}_{-1x} = \tfrac{1}{4} & \mathcal{P}_{-1y} = \tfrac{1}{4} \end{array}$$

Write the unknown state as

$$|\psi_4\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\mathcal{P}_1 = \left|\langle 1|\psi_4\rangle\right|^2 = \left|\langle 1|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |a|^2 = \tfrac{1}{2} \quad \Rightarrow \quad a = \tfrac{1}{\sqrt{2}}$$
$$\mathcal{P}_0 = \left|\langle 0|\psi_4\rangle\right|^2 = \left|\langle 0|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |b|^2 = 0 \quad \Rightarrow \quad b = 0$$
$$\mathcal{P}_{-1} = \left|\langle -1|\psi_4\rangle\right|^2 = \left|\langle -1|\left\{a|1\rangle + b|0\rangle + c|-1\rangle\right\}\right|^2 = |c|^2 = \tfrac{1}{2} \quad \Rightarrow \quad c = \tfrac{1}{\sqrt{2}}e^{i\beta}$$

allowing for possible relative phases. So now the unknown state is

$$|\psi_4\rangle = \tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}e^{i\beta}|-1\rangle$$

Equating the predicted S_x probabilities and the experimental results gives

$$\mathcal{P}_{0x} = \left|{}_x\langle 0|\psi_4\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\{\langle 1| - \langle -1|\right\}\tfrac{1}{\sqrt{2}}\left\{|1\rangle + e^{i\beta}|-1\rangle\right\}\right|^2 = \left|\tfrac{1}{2}\left\{1 - e^{i\beta}\right\}\right|^2$$
$$= \tfrac{1}{4}\left\{1 - e^{i\beta}\right\}\left\{1 - e^{-i\beta}\right\} = \tfrac{1}{4}\left\{1 + 1 - e^{i\beta} - e^{-i\beta}\right\} = \tfrac{1}{2}\left\{1 - \cos\beta\right\} = \tfrac{1}{2}$$
$$\Rightarrow \cos\beta = 0 \Rightarrow \beta = \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$

Giving

$$|\psi_4\rangle = \tfrac{1}{\sqrt{2}}\left\{|1\rangle \pm i|-1\rangle\right\}$$

with no more info from S_x, S_y, S_z measurements. In the SPINS program choose **n** at angles θ = 90°, ϕ = 45°, 225° to see that the unknown state is

$$|\psi_4\rangle = \tfrac{1}{\sqrt{2}}\left\{|1\rangle - i|-1\rangle\right\} = |0\rangle_{n\left(\theta=\frac{\pi}{2},\phi=\frac{\pi}{4}\right)} = |0\rangle_{n\left(\theta=\frac{\pi}{2},\phi=\frac{5\pi}{4}\right)}$$

2.21. The spin-1 interferometer had an S_z SG device, an S_x deveice, and an S_z SG device. The S_z eigenstates are $|1\rangle, |0\rangle, |-1\rangle$. The S_x eigenstates are

$$\begin{aligned}
|1\rangle_x &= \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle \\
|0\rangle_x &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle \qquad . \\
|-1\rangle_x &= \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle
\end{aligned}$$

Let $|\psi_i\rangle$ be the quantum state after the i^{th} Stern-Gerlach device. The first SG device transmit particles with $S_z = +\hbar$, so the state $|\psi_1\rangle$ is

$$|\psi_1\rangle = |1\rangle .$$

The second SG device transmits particles in 1, 2, or 3 of the S_x eigenstates $|1\rangle_x, |0\rangle_x, |-1\rangle_x$. To find $|\psi_2\rangle$, we use the projection postulate:

$$|\psi_2\rangle = \frac{P_n|\psi_1\rangle}{\sqrt{\langle\psi_1|P_n|\psi_1\rangle}}$$

where P_n is the projection operator onto the measured states. For example, if the second SG device transmits particles with $S_x = +\hbar$, we get

$$|\psi_2\rangle = \frac{P_{1x}|\psi_1\rangle}{\sqrt{\langle\psi_1|P_{1x}|\psi_1\rangle}} = \frac{|1\rangle_{x\ x}\langle 1||1\rangle}{\sqrt{\langle 1||1\rangle_{x\ x}\langle 1||1\rangle}} = |1\rangle_x$$

as expected. In matrix notation, the S_x eigenstates are

$$|1\rangle_x \doteq \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \qquad |0\rangle_x \doteq \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \qquad |-1\rangle_x \doteq \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

and the projection operators are

$$P_{1x} = |1\rangle_{x\;x}\langle 1| \doteq \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

$$P_{0x} = |0\rangle_{x\;x}\langle 0| \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P_{-1x} = |-1\rangle_{x\;x}\langle -1| \doteq \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{-1}{2\sqrt{2}} & \frac{1}{2} & \frac{-1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}$$

The probability of measuring a result after the third SG device is $\mathcal{P} = \left|\langle \psi_2 | \psi_3 \rangle\right|^2$. We want to calculate the three probabilities

$$\mathcal{P}_1 = \left|\langle 1 | \psi_2 \rangle\right|^2$$
$$\mathcal{P}_0 = \left|\langle 0 | \psi_2 \rangle\right|^2$$
$$\mathcal{P}_{-1} = \left|\langle -1 | \psi_2 \rangle\right|^2$$

for all possible (7) cases of 1 beam, 2 beams, or 3 beams from SG2.
(i) When the second SG device transmits particles with $S_x = +1\hbar$ only:

$$|\psi_2\rangle = \frac{P_{1x}|\psi_1\rangle}{\sqrt{\langle \psi_1 | P_{1x} | \psi_1 \rangle}} = \frac{P_{1x}|1\rangle}{\sqrt{\langle 1 | P_{1x} | 1 \rangle}}$$

$$P_{1x}|1\rangle \doteq \begin{pmatrix} \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix}$$

$$\langle 1 | P_{1x} | 1 \rangle \doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} = 1/4$$

$$|\psi_2\rangle \doteq \frac{1}{\sqrt{1/4}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

which is $|1\rangle_x$ as expected. The three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{4}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{4}$$

(ii) When the second SG device transmits particles with $S_x = 0\hbar$, $|\psi_2\rangle = |0\rangle_x$ and the three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = 0$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = \frac{1}{2}$$

(iii) When the second SG device transmits particles with $S_x = -1\hbar$, $|\psi_2\rangle = |-1\rangle_x$ and the three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{4}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\right|^2 = \frac{1}{4}$$

(iv) When the second SG device transmits particles with $S_x = +\hbar$ **and** $S_x = 0\hbar$ in a coherent beam

$$|\psi_2\rangle = \frac{\left(P_{1x} + P_{0x}\right)|\psi_1\rangle}{\sqrt{\langle\psi_1|\left(P_{1x} + P_{0x}\right)|\psi_1\rangle}} = \frac{\left(P_{1x} + P_{0x}\right)|1\rangle}{\sqrt{\langle 1|\left(P_{1x} + P_{0x}\right)|1\rangle}}$$

$$P_{0x}|1\rangle \doteq \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$\left(P_{1x} + P_{0x}\right)|1\rangle \doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix}$$

$$\langle 1|\left(P_{1x} + P_{0x}\right)|1\rangle \doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4$$

$$|\psi_2\rangle \doteq \frac{1}{\sqrt{3/4}}\begin{pmatrix} 3/4 \\ 1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}$$

The three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{3}{4}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{1}{6}$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ 1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{1}{12}$$

(v) When the second SG device transmits particles with $S_x = +\hbar$ **and** $S_x = -1\hbar$ in a coherent beam

$$|\psi_2\rangle = \frac{\left(P_{1x} + P_{-1x}\right)|\psi_1\rangle}{\sqrt{\langle\psi_1|\left(P_{1x} + P_{-1x}\right)|\psi_1\rangle}} = \frac{\left(P_{1x} + P_{-1x}\right)|1\rangle}{\sqrt{\langle 1|\left(P_{1x} + P_{-1x}\right)|1\rangle}}$$

$$P_{-1x}|1\rangle \doteq \begin{pmatrix} \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{-1}{2\sqrt{2}} & \frac{1}{2} & \frac{-1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix}$$

$$\left(P_{1x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\langle 1|\left(P_{1x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = 1/2$$

$$|\psi_2\rangle \doteq \frac{1}{\sqrt{1/2}}\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

The three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right|^2 = \frac{1}{2}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right|^2 = 0$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}\right|^2 = \frac{1}{2}$$

(vi) When the second SG device transmits particles with $S_x = 0\hbar$ **and** $S_x = -1\hbar$ in a coherent beam

$$|\psi_2\rangle = \frac{\left(P_{0x} + P_{-1x}\right)|\psi_1\rangle}{\sqrt{\langle\psi_1|\left(P_{0x} + P_{-1x}\right)|\psi_1\rangle}} = \frac{\left(P_{0x} + P_{-1x}\right)|1\rangle}{\sqrt{\langle 1|\left(P_{0x} + P_{-1x}\right)|1\rangle}}$$

$$\left(P_{0x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix}$$

$$\langle 1|\left(P_{0x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = 3/4$$

$$|\psi_2\rangle \doteq \frac{1}{\sqrt{3/4}}\begin{pmatrix} 3/4 \\ -1/2\sqrt{2} \\ -1/4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}$$

The three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{3}{4}$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{1}{6}$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ -1/\sqrt{6} \\ -1/2\sqrt{3} \end{pmatrix}\right|^2 = \frac{1}{12}$$

(vii) When the second SG device transmits particles with $S_x = +\hbar$, $S_x = 0\hbar$, **and** $S_x = -1\hbar$ in a coherent beam.

$$|\psi_2\rangle = \frac{\left(P_{1x} + P_{0x} + P_{-1x}\right)|\psi_1\rangle}{\sqrt{\langle\psi_1|\left(P_{1x} + P_{0x} + P_{-1x}\right)|\psi_1\rangle}} = \frac{\left(P_{1x} + P_{0x} + P_{-1x}\right)|1\rangle}{\sqrt{\langle 1|\left(P_{1x} + P_{0x} + P_{-1x}\right)|1\rangle}}$$

$$P_{-1x}|1\rangle \doteq \begin{pmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix}$$

$$\left(P_{1x} + P_{0x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1/4 \\ 1/2\sqrt{2} \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ -1/2\sqrt{2} \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle 1|\left(P_{1x} + P_{0x} + P_{-1x}\right)|1\rangle \doteq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$|\psi_2\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which is $|1\rangle$ as expected. The three probabilities are

$$\mathcal{P}_1 = \left|\langle 1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right|^2 = 1$$

$$\mathcal{P}_0 = \left|\langle 0|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right|^2 = 0$$

$$\mathcal{P}_{-1} = \left|\langle -1|\psi_2\rangle\right|^2 = \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = 0$$

The cumulated results are

$\lvert\psi_2\rangle$	$\mathcal{P}_1$	$\mathcal{P}_0$	$\mathcal{P}_{-1}$
$\lvert 1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\lvert 0\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\lvert -1\rangle_x$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\lvert 1\rangle_x \,\&\, \lvert 0\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$\lvert 1\rangle_x \,\&\, \lvert -1\rangle_x$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\lvert 0\rangle_x \,\&\, \lvert -1\rangle_x$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
$\lvert 1\rangle_x \,\&\, \lvert 0\rangle_x \,\&\, \lvert -1\rangle_x$	1	0	0

2.22 a) The probability of measuring spin up at the 2nd Stern-Gerlach analyzer and spin down at the 3rd Stern-Gerlach analyzer is the product of the individual probabilities:

$$\mathcal{P}_{+\to+n\to-z} = \mathcal{P}_{+\to+n}\mathcal{P}_{+n\to-z} = \left|{}_n\langle +|+\rangle\right|^2 \left|\langle -|+\rangle_n\right|^2$$

The $\lvert +\rangle_n$ eigenstate is

$$\lvert +\rangle_n = \cos\tfrac{\theta}{2}\lvert +\rangle + e^{i\phi}\sin\tfrac{\theta}{2}\lvert -\rangle$$

so the probability is

$$\begin{aligned}\mathcal{P}_{+\to+n\to-z} &= \left|\left(\cos\tfrac{\theta}{2}\langle +| + e^{-i\phi}\sin\tfrac{\theta}{2}\langle -|\right)\lvert +\rangle\right|^2 \left|\langle -|\left(\cos\tfrac{\theta}{2}\lvert +\rangle + e^{i\phi}\sin\tfrac{\theta}{2}\lvert -\rangle\right)\right|^2 \\ &= \cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2} = \tfrac{1}{4}\sin^2\theta\end{aligned}$$

b) To maximize the probability requires that $\theta = \pi/2$, and the probability is

$$\mathcal{P}_{+\to+n\to-z} = \tfrac{1}{4}\sin^2\tfrac{\pi}{2} = \tfrac{1}{4}$$

c) If the 2nd Stern-Gerlach analyzer is removed, then the probability is

$$\mathcal{P}_{+\to -z} = \left|\langle -|+\rangle\right|^2 = 0$$

because the two states are orthogonal.

2.23 (a) The commutator is

$$[A,B] = AB - BA \doteq \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} - \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$\doteq \begin{pmatrix} a_1b_1 & 0 & 0 \\ 0 & 0 & a_2b_2 \\ 0 & a_3b_2 & 0 \end{pmatrix} - \begin{pmatrix} a_1b_1 & 0 & 0 \\ 0 & 0 & a_3b_2 \\ 0 & a_2b_2 & 0 \end{pmatrix}$$

$$\doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_2(a_2 - a_3) \\ 0 & b_2(a_3 - a_2) & 0 \end{pmatrix} \neq 0$$

so they do not commute.

(b) A is already diagonal, so the eigenvalues and eigenvectors are obtained by inspection. The eigenvalues are

$$a_1, a_2, a_3$$

and the eigenvectors are

$$|a_1\rangle = |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |a_2\rangle = |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |a_3\rangle = |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For B, diagonalization yields the eigenvalues

$$\begin{pmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (b_1 - \lambda)(\lambda^2 - b_2^2) = 0$$

$$\Rightarrow \quad \lambda = b_1, b_2, -b_2$$

and the eigenvectors

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = b_1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b_1 u = b_1 u \\ b_2 w = b_1 v \\ b_2 v = b_1 w \end{matrix} \quad \Rightarrow \quad w = v = 0$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad |u|^2 = 1 \quad \Rightarrow \quad u = 1, v = 0, w = 0 \quad \Rightarrow \quad |b_1\rangle = |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = b_2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b_1 u = b_2 u \\ b_2 w = b_2 v \\ b_2 v = b_2 w \end{matrix} \quad \Rightarrow \quad u = 0, \;\; w = v$$

$$\langle b_2 | b_2 \rangle = 1 \quad \Rightarrow \quad |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad u = 0, v = \tfrac{1}{\sqrt{2}}, w = \tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad |b_2\rangle = \tfrac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right) \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -b_2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} b_1 u = -b_2 u \\ b_2 w = -b_2 v \\ b_2 v = -b_2 w \end{matrix} \quad \Rightarrow \quad u = 0, \;\; w = -v$$

$$\langle -b_2 | -b_2 \rangle = 1 \quad \Rightarrow \quad |v|^2 + |w|^2 = 1 \quad \Rightarrow \quad u = 0, v = \tfrac{1}{\sqrt{2}}, w = -\tfrac{1}{\sqrt{2}} \quad \Rightarrow \quad |-b_2\rangle = \tfrac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right) \doteq \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

c) If B is measured, the possible results are the allowed eigenvalues $b_1, b_2, -b_2$. If the initial state is $|\psi_i\rangle = |2\rangle$, then the probabilities are

$$\mathcal{P}_{b_1} = \left|\langle b_1 | \psi_i \rangle\right|^2 = \left|\langle 1 | 2 \rangle\right|^2 = 0$$

$$\mathcal{P}_{b_2} = \left|\langle b_2 | \psi_i \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle 2| + \langle 3|\right)|2\rangle\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{-b_2} = \left|\langle -b_2 | \psi_i \rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left(\langle 2| - \langle 3|\right)|2\rangle\right|^2 = \tfrac{1}{2}$$

If A is then measured, the possible results are the allowed eigenvalues a_1, a_2, a_3. If b_2 was the first result, then the new state is $|b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = \left|\langle a_1|b_2\rangle\right|^2 = \left|\langle 1|\tfrac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right)\right|^2 = 0$$
$$\mathcal{P}_{a_2} = \left|\langle a_2|b_2\rangle\right|^2 = \left|\langle 2|\tfrac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right)\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{a_3} = \left|\langle a_3|b_2\rangle\right|^2 = \left|\langle 3|\tfrac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right)\right|^2 = \tfrac{1}{2}$$

If $-b_2$ was the first result, then the new state is $|-b_2\rangle$ and when A is measured the subsequent probabilities are

$$\mathcal{P}_{a_1} = \left|\langle a_1|-b_2\rangle\right|^2 = \left|\langle 1|\tfrac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right)\right|^2 = 0$$
$$\mathcal{P}_{a_2} = \left|\langle a_2|-b_2\rangle\right|^2 = \left|\langle 2|\tfrac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right)\right|^2 = \tfrac{1}{2}$$
$$\mathcal{P}_{a_3} = \left|\langle a_3|-b_2\rangle\right|^2 = \left|\langle 3|\tfrac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right)\right|^2 = \tfrac{1}{2}$$

d) If two operators do not commute, then the corresponding observables cannot be measured simultaneously. Part (a) tells us that the operators A and B not commute. Part (c) tells us that measurement B "disturbs" the measurement of A so the two measurements are not compatible (cannot be made simultaneously).

2.24 (a) The eigenvalue equations for the S_z operator and the four eigenstates are

$$S_z\left|+\tfrac{3}{2}\right\rangle = +\tfrac{3}{2}\hbar\left|+\tfrac{3}{2}\right\rangle$$
$$S_z\left|+\tfrac{1}{2}\right\rangle = +\tfrac{1}{2}\hbar\left|+\tfrac{1}{2}\right\rangle$$
$$S_z\left|-\tfrac{1}{2}\right\rangle = -\tfrac{1}{2}\hbar\left|-\tfrac{1}{2}\right\rangle$$
$$S_z\left|-\tfrac{3}{2}\right\rangle = -\tfrac{3}{2}\hbar\left|-\tfrac{3}{2}\right\rangle$$

(b) The matrix representations of the S_z eigenstates are the unit vectors

$$\left|+\tfrac{3}{2}\right\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \left|+\tfrac{1}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \left|-\tfrac{1}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \left|-\tfrac{3}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) The matrix representation of the S_z operator has the eigenvalues along the diagonal:

$$S_z \doteq \begin{pmatrix} +\tfrac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & +\tfrac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\tfrac{3}{2}\hbar \end{pmatrix}$$

(d) The eigenvalue equations for the $\mathbf{S}^2$ operator follow from the general equation $\mathbf{S}^2|sm_s\rangle = s(s+1)\hbar^2|sm_s\rangle$

$$\mathbf{S}^2\left|+\tfrac{3}{2}\right\rangle = \tfrac{15}{4}\hbar^2\left|+\tfrac{3}{2}\right\rangle$$
$$\mathbf{S}^2\left|+\tfrac{1}{2}\right\rangle = \tfrac{15}{4}\hbar^2\left|+\tfrac{1}{2}\right\rangle$$
$$\mathbf{S}^2\left|-\tfrac{1}{2}\right\rangle = \tfrac{15}{4}\hbar^2\left|-\tfrac{1}{2}\right\rangle$$
$$\mathbf{S}^2\left|-\tfrac{3}{2}\right\rangle = \tfrac{15}{4}\hbar^2\left|-\tfrac{3}{2}\right\rangle$$

where we have suppressed the s label.

(e) The matrix representation of the $\mathbf{S}^2$ operator has the eigenvalues along the diagonal:

$$\mathbf{S}^2 \doteq \begin{pmatrix} \tfrac{15}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & \tfrac{15}{4}\hbar^2 & 0 & 0 \\ 0 & 0 & \tfrac{15}{4}\hbar^2 & 0 \\ 0 & 0 & 0 & \tfrac{15}{4}\hbar^2 \end{pmatrix}$$

2.25 The projection operators P_+ and P_- are represented by the matrices

$$P_+ \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The Hermitian adjoints of these matrices are obtained by transposing and complex conjugating them, yielding

$$P_+^\dagger \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_-^\dagger \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the Hermitian adjoints are equal to the original matrices, these operators are Hermitian.
