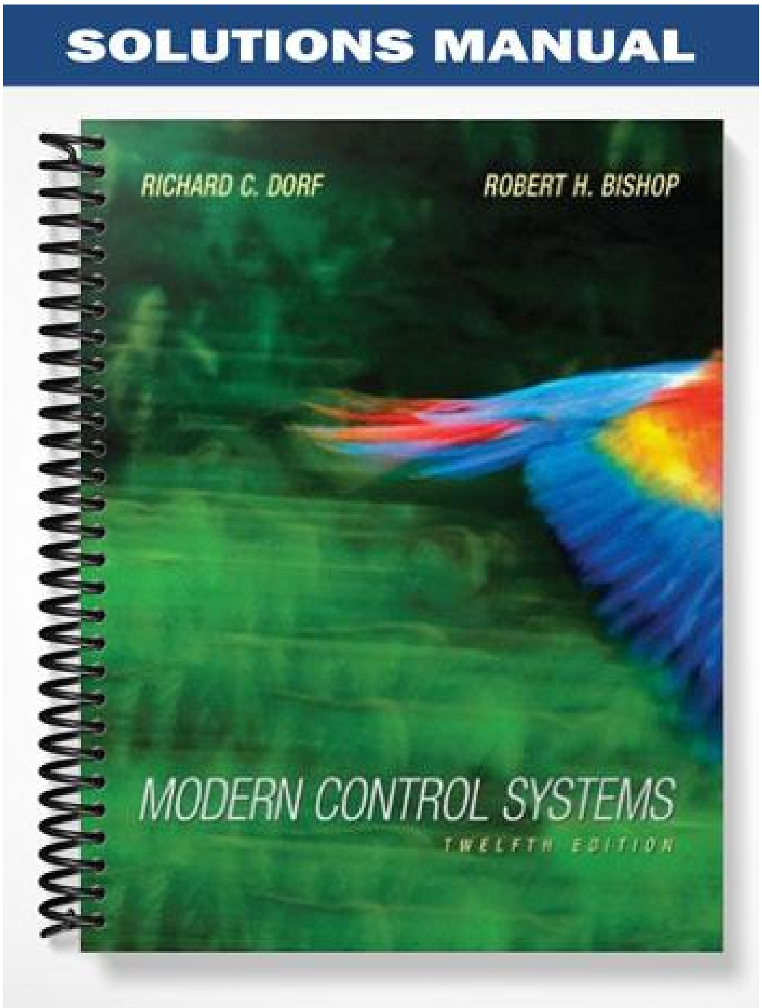
SOLUTIONS MANUAL
RICHARD C. DORF
ROBERT H. BISHOP
MODERN CONTROL SYSTEMS
TWELFTH EDITION

MODERN CONTROL SYSTEMS

SOLUTION MANUAL

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A companion to

MODERN CONTROL SYSTEMS
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Prentice Hall
Upper Saddle River Boston Columbus San Francisco New York
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PREFACE

In each chapter, there are five problem types:

- Exercises
- Problems
- Advanced Problems
- Design Problems/Continuous Design Problem
- Computer Problems

In total, there are over 1000 problems. The abundance of problems of increasing complexity gives students confidence in their problem-solving ability as they work their way from the exercises to the design and computer-based problems.

It is assumed that instructors (and students) have access to MATLAB and the Control System Toolbox or to LabVIEW and the MathScript RT Module. All of the computer solutions in this *Solution Manual* were developed and tested on an Apple MacBook Pro platform using MATLAB 7.6 Release 2008a and the Control System Toolbox Version 8.1 and LabVIEW 2009. It is not possible to verify each solution on all the available computer platforms that are compatible with MATLAB and LabVIEW MathScript RT Module. Please forward any incompatibilities you encounter with the scripts to Prof. Bishop at the email address given below.

The authors and the staff at Prentice Hall would like to establish an open line of communication with the instructors using *Modern Control Systems*. We encourage you to contact Prentice Hall with comments and suggestions for this and future editions.

Robert H. Bishop rhbishop@marquette.edu

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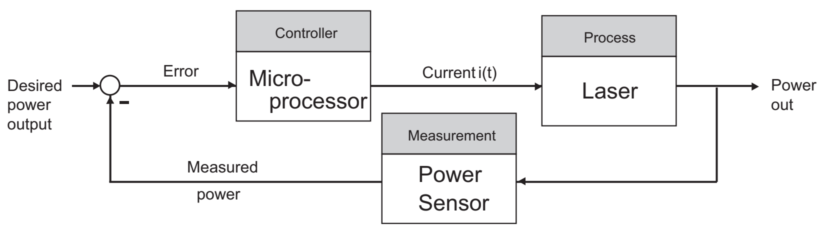
CHAPTER 1

Introduction to Control Systems

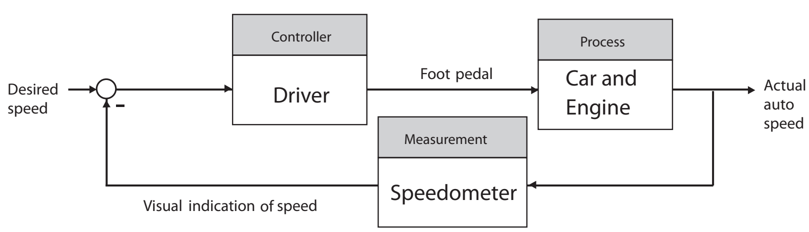
There are, in general, no unique solutions to the following exercises and problems. Other equally valid block diagrams may be submitted by the student.

Exercises

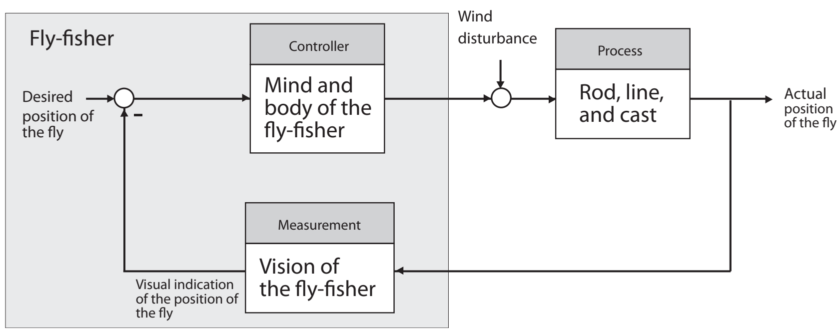
E1.1 A microprocessor controlled laser system:



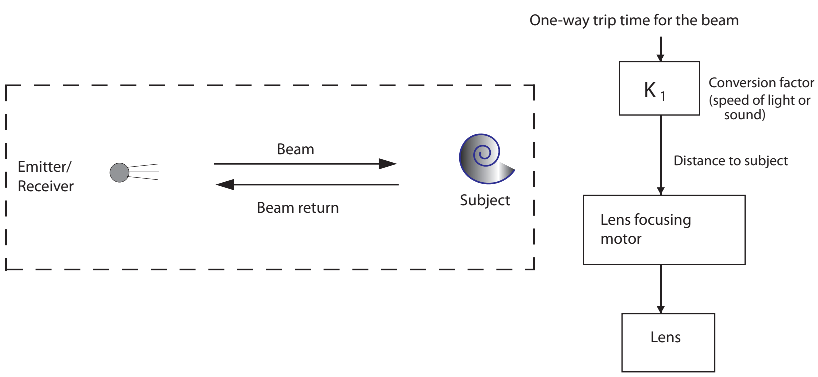
E1.2 A driver controlled cruise control system:



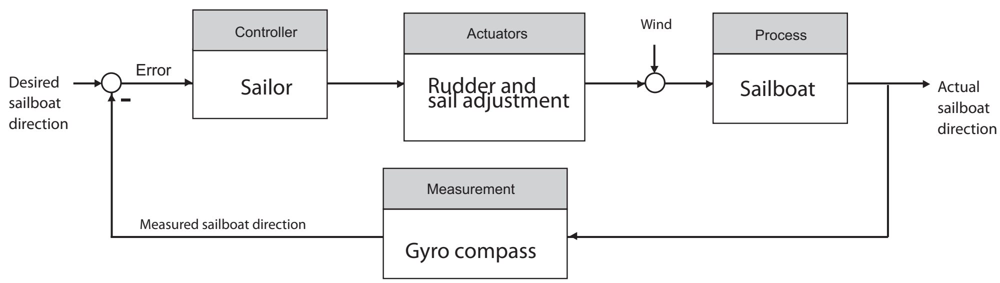
E1.3 Although the principle of conservation of momentum explains much of the process of fly-casting, there does not exist a comprehensive scientific explanation of how a fly-fisher uses the small backward and forward motion of the fly rod to cast an almost weightless fly lure long distances (the current world-record is 236 ft). The fly lure is attached to a short invisible leader about 15-ft long, which is in turn attached to a longer and thicker Dacron line. The objective is cast the fly lure to a distant spot with dead-eye accuracy so that the thicker part of the line touches the water first and then the fly gently settles on the water just as an insect might.



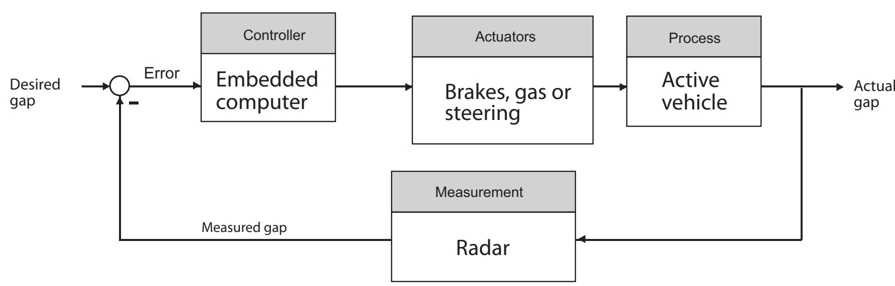
E1.4 An autofocus camera control system:



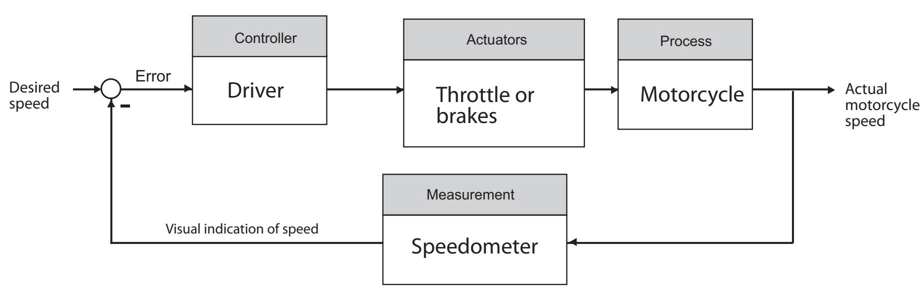
E1.5 Tacking a sailboat as the wind shifts:



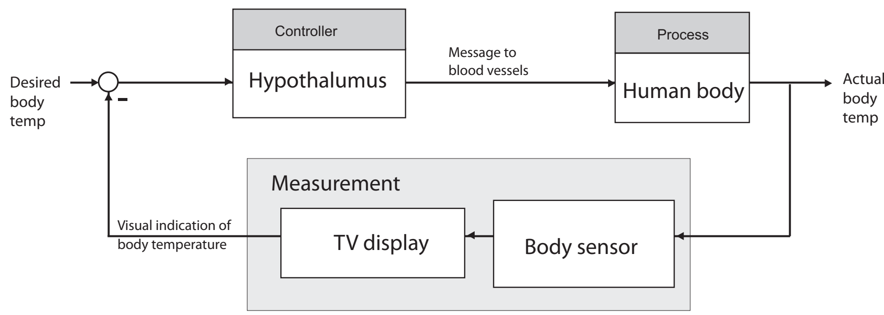
E1.6 An automated highway control system merging two lanes of traffic:



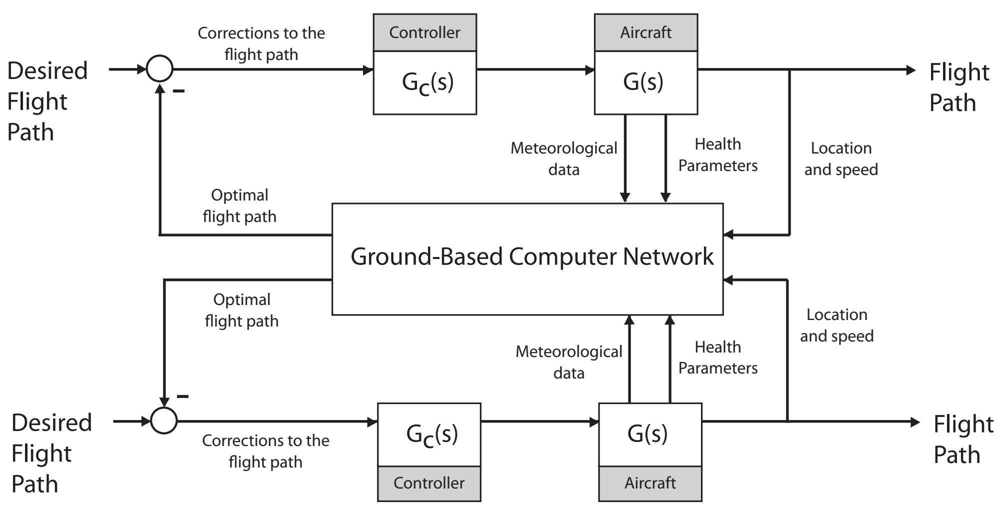
E1.7 Using the speedometer, the driver calculates the difference between the measured speed and the desired speed. The driver throotle knob or the brakes as necessary to adjust the speed. If the current speed is not too much over the desired speed, the driver may let friction and gravity slow the motorcycle down.



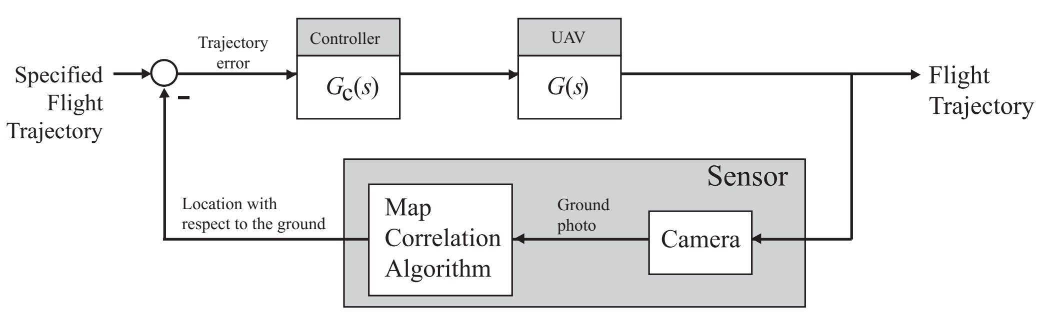
E1.8 Human biofeedback control system:



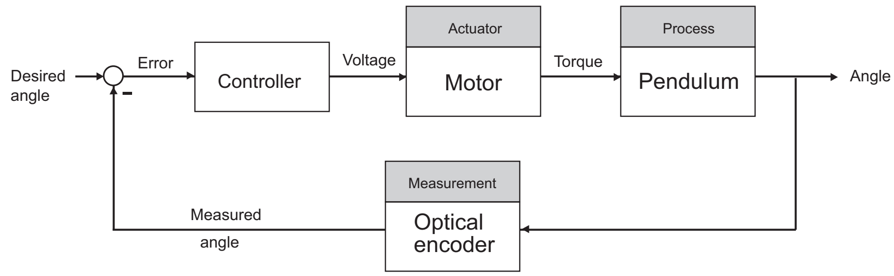
E1.9 E-enabled aircraft with ground-based flight path control:



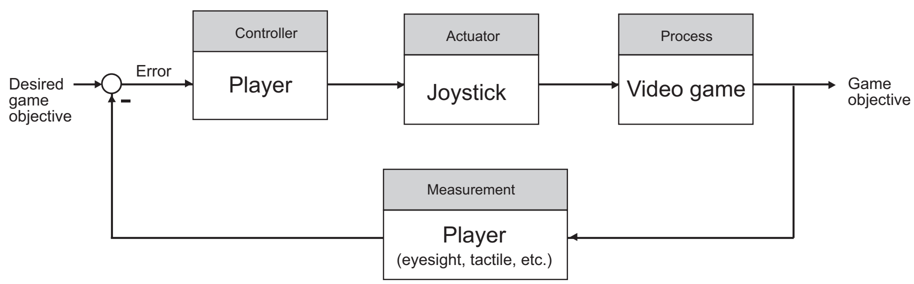
E1.10 Unmanned aerial vehicle used for crop monitoring in an autonomous mode:



E1.11 An inverted pendulum control system using an optical encoder to measure the angle of the pendulum and a motor producing a control torque:

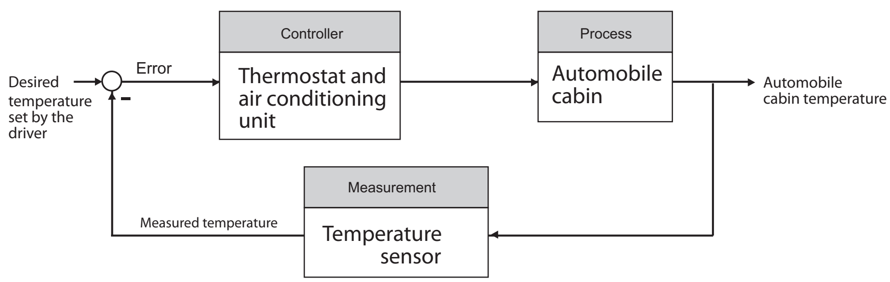


E1.12 In the video game, the player can serve as both the controller and the sensor. The objective of the game might be to drive a car along a prescribed path. The player controls the car trajectory using the joystick using the visual queues from the game displayed on the computer monitor.

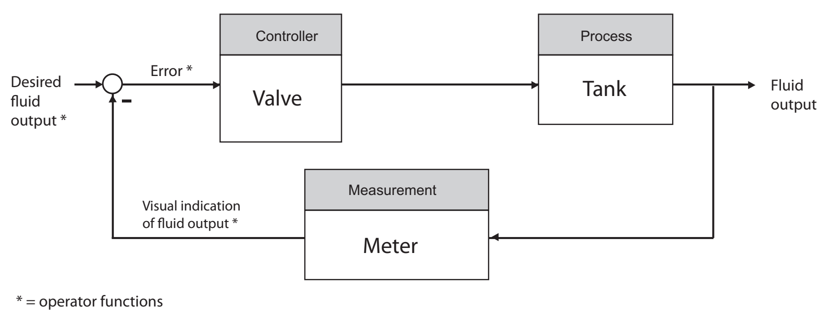


Problems

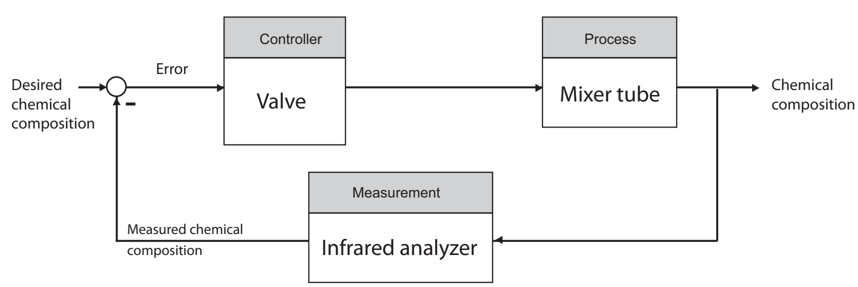
P1.1 An automobile interior cabin temperature control system block diagram:



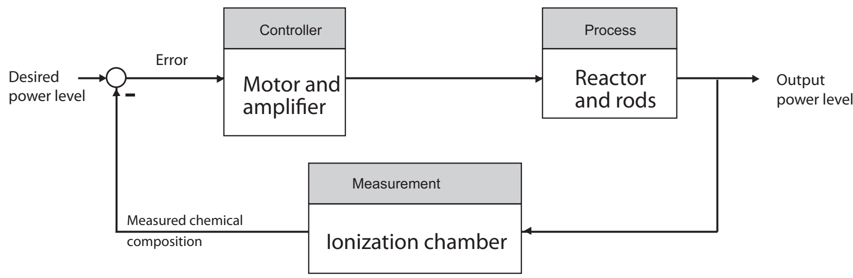
P1.2 A human operator controlled valve system:



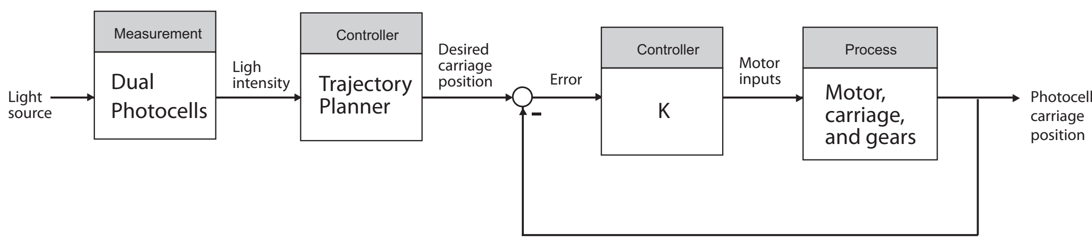
P1.3 A chemical composition control block diagram:



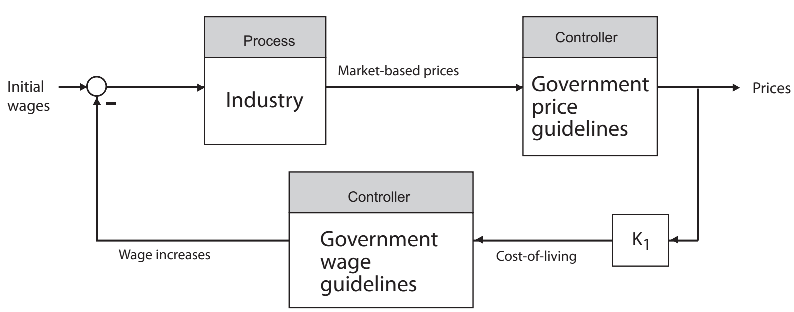
P1.4 A nuclear reactor control block diagram:



P1.5 A light seeking control system to track the sun:



P1.6 If you assume that increasing worker's wages results in increased prices, then by delaying or falsifying cost-of-living data you could reduce or eliminate the pressure to increase worker's wages, thus stabilizing prices. This would work only if there were no other factors forcing the cost-of-living up. Government price and wage economic guidelines would take the place of additional "controllers" in the block diagram, as shown in the block diagram.



P1.7 Assume that the cannon fires initially at exactly 5:00 p.m.. We have a positive feedback system. Denote by Δt the time lost per day, and the net time error by E_T. Then the follwoing relationships hold:

$$\Delta t = 4/3 \text{ min.} + 3 \text{ min.} = 13/3 \text{ min.}$$

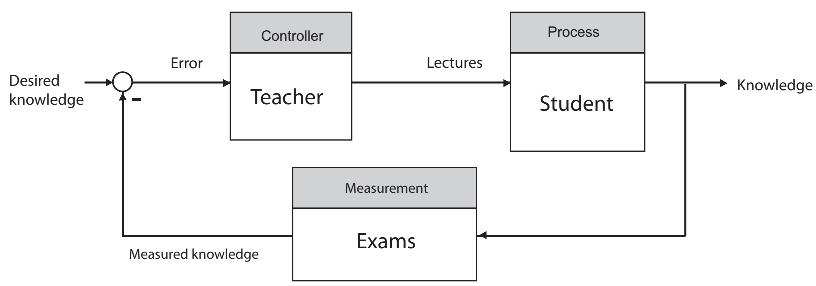
and

$$E_T = 12 \text{ days} \times 13/3 \text{ min./day} .$$

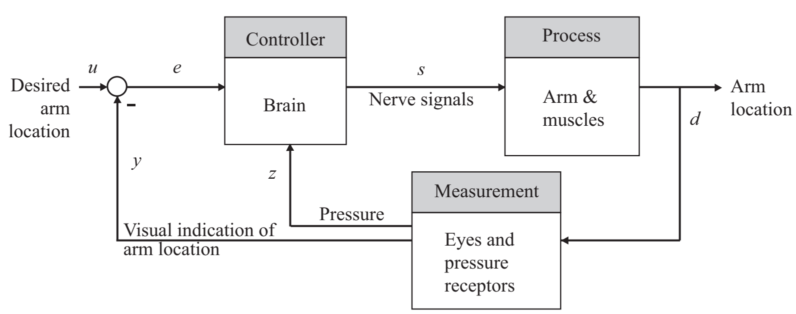
Therefore, the net time error after 15 days is

$$E_T = 52 \text{ min.}$$

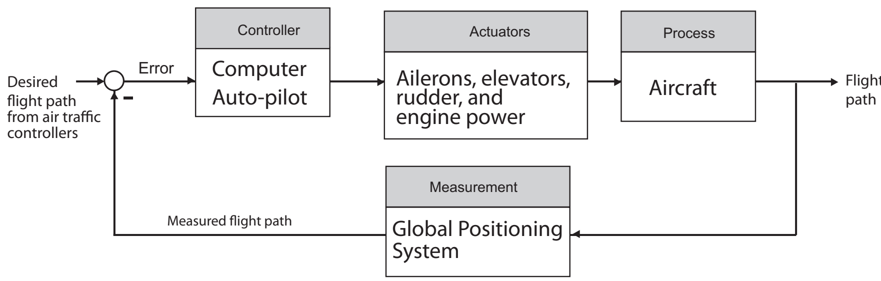
P1.8 The student-teacher learning process:



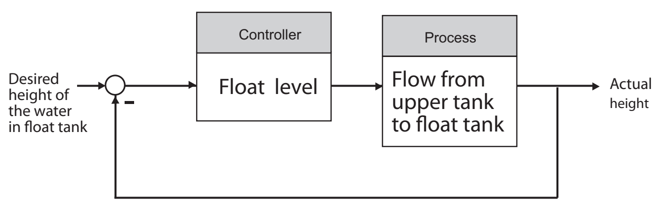
P1.9 A human arm control system:



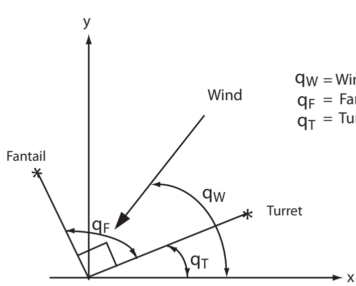
P1.10 An aircraft flight path control system using GPS:

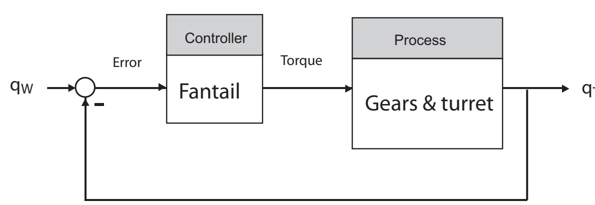


P1.11 The accuracy of the clock is dependent upon a constant flow from the orifice; the flow is dependent upon the height of the water in the float tank. The height of the water is controlled by the float. The control system controls only the height of the water. Any errors due to enlargement of the orifice or evaporation of the water in the lower tank is not accounted for. The control system can be seen as:

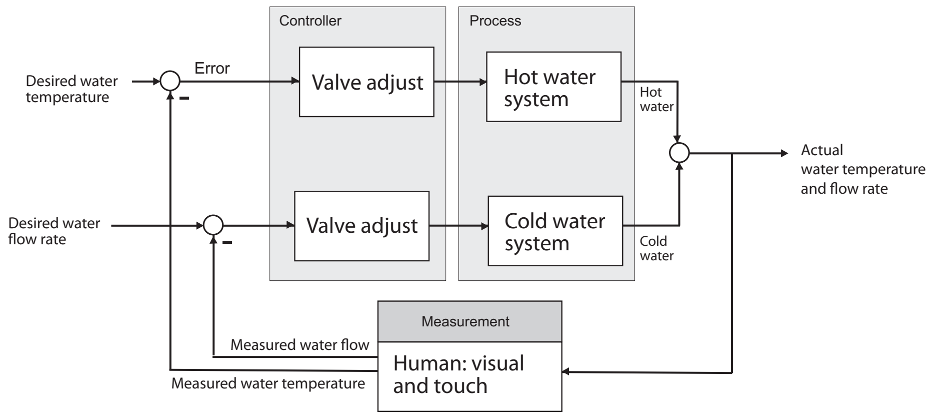


P1.12 Assume that the turret and fantail are at 90°, if $\theta_w \neq \theta_F$-90°. The fantail operates on the error signal θ_w - θ_T, and as the fantail turns, it drives the turret to turn.





P1.13 This scheme assumes the person adjusts the hot water for temperature control, and then adjusts the cold water for flow rate control.

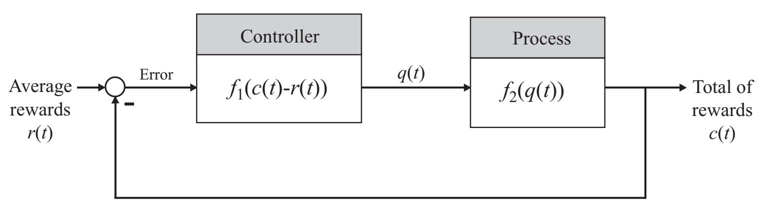


P1.14 If the rewards in a specific trade is greater than the average reward, there is a positive influx of workers, since

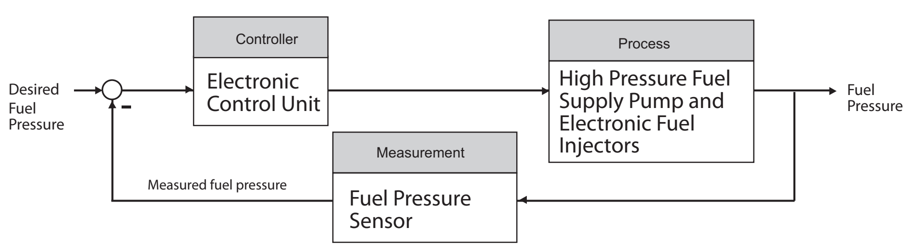
$$q(t) = f_1(c(t) - r(t)).$$

If an influx of workers occurs, then reward in specific trade decreases, since

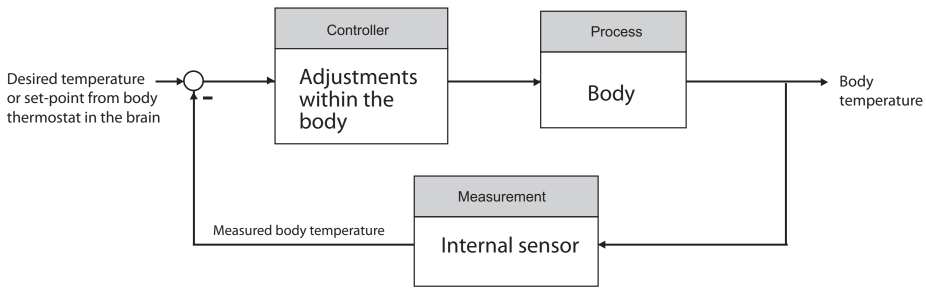
$$c(t) = -f_2(q(t)).$$



P1.15 A computer controlled fuel injection system:

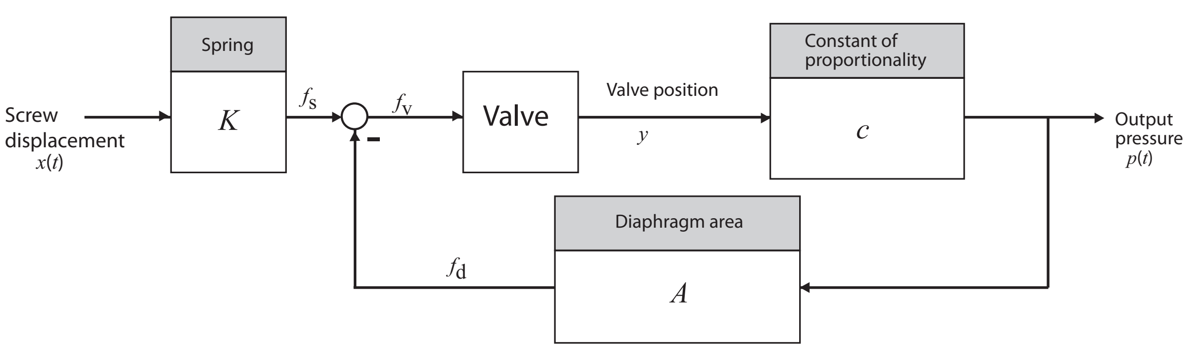


P1.16 With the onset of a fever, the body thermostat is turned up. The body adjusts by shivering and less blood flows to the skin surface. Aspirin acts to lowers the thermal set-point in the brain.

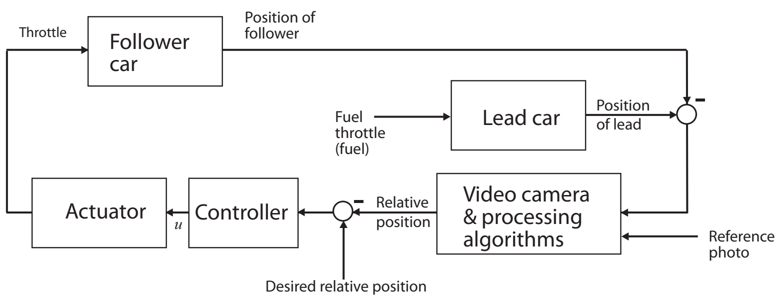


P1.17 Hitting a baseball is arguably one of the most difficult feats in all of sports. Given that pitchers may throw the ball at speeds of 90 mph (or higher!), batters have only about 0.1 second to make the decision to swing—with bat speeds aproaching 90 mph. The key to hitting a baseball a long distance is to make contact with the ball with a high bat velocity. This is more important than the bat's weight, which is usually around 33 ounces (compared to Ty Cobb's bat which was 41 ounces!). Since the pitcher can throw a variety of pitches (fast ball, curve ball, slider, etc.), a batter must decide if the ball is going to enter the strike zone and if possible, decide the type of pitch. The batter uses his/her vision as the sensor in the feedback loop. A high degree of eye-hand coordination is key to success—that is, an accurate feedback control system.

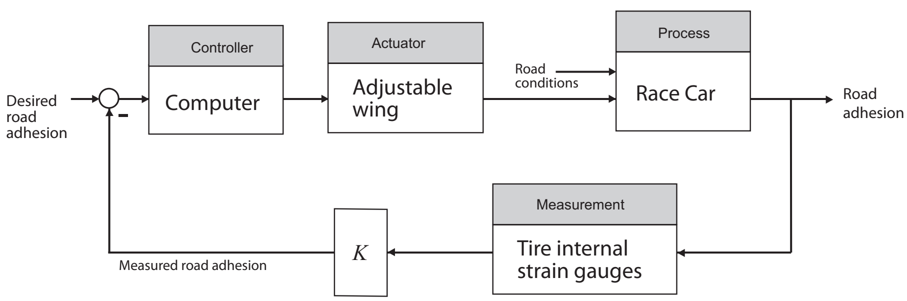
P1.18 Define the following variables: p = output pressure, f_s = spring force $= Kx$, f_d = diaphragm force $= Ap$, and f_v = valve force $= f_s - f_d$. The motion of the valve is described by $\ddot{y} = f_v/m$ where m is the valve mass. The output pressure is proportional to the valve displacement, thus $p = cy$, where c is the constant of proportionality.



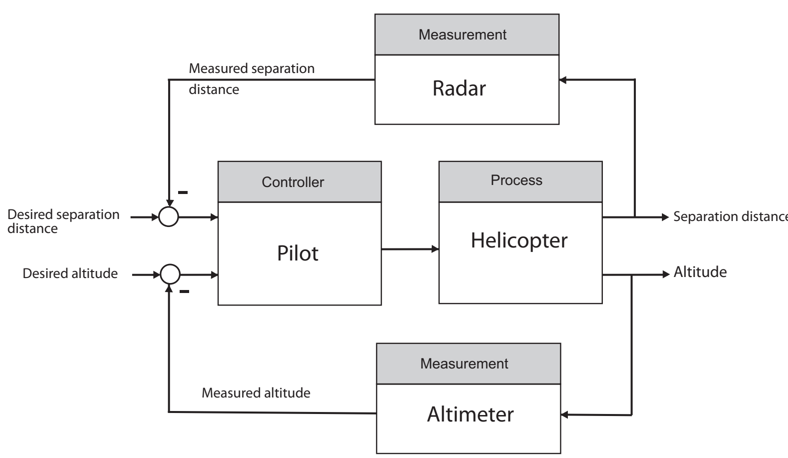
P1.19 A control system to keep a car at a given relative position offset from a lead car:



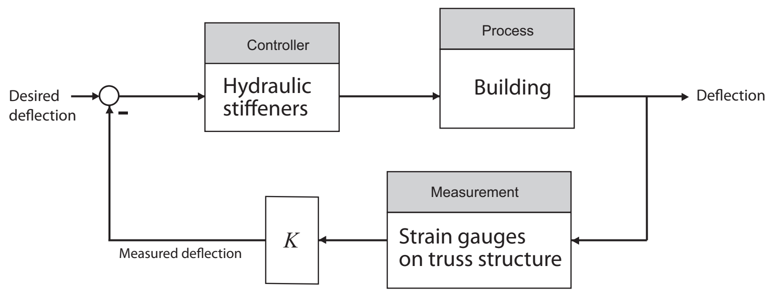
P1.20 A control system for a high-performance car with an adjustable wing:



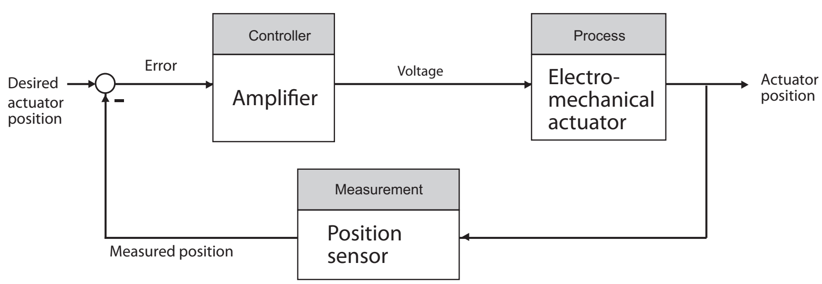
P1.21 A control system for a twin-lift helicopter system:



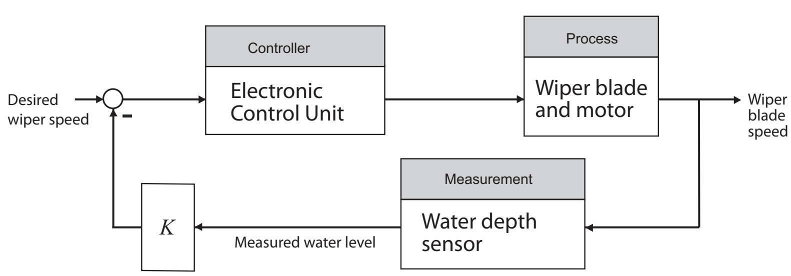
P1.22 The desired building deflection would not necessarily be zero. Rather it would be prescribed so that the building is allowed moderate movement up to a point, and then active control is applied if the movement is larger than some predetermined amount.



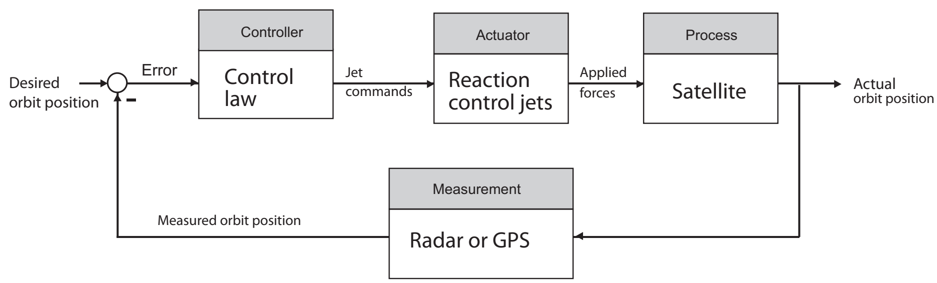
P1.23 The human-like face of the robot might have micro-actuators placed at strategic points on the interior of the malleable facial structure. Cooperative control of the micro-actuators would then enable the robot to achieve various facial expressions.



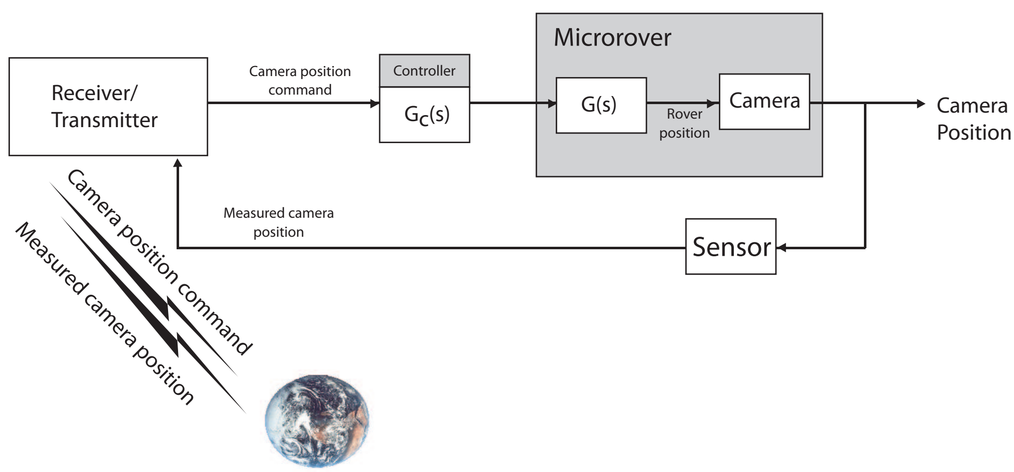
P1.24 We might envision a sensor embedded in a "gutter" at the base of the windshield which measures water levels—higher water levels corresponds to higher intensity rain. This information would be used to modulate the wiper blade speed.



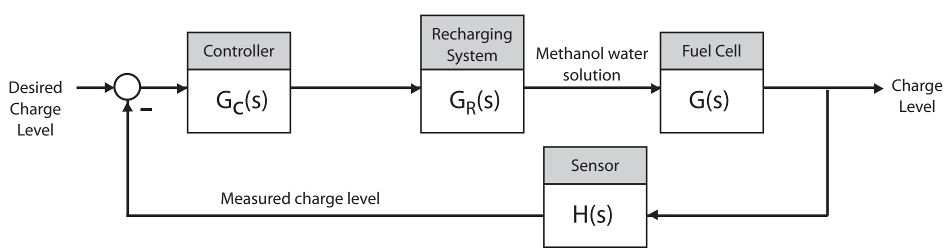
P1.25 A feedback control system for the space traffic control:



P1.26 Earth-based control of a microrover to point the camera:

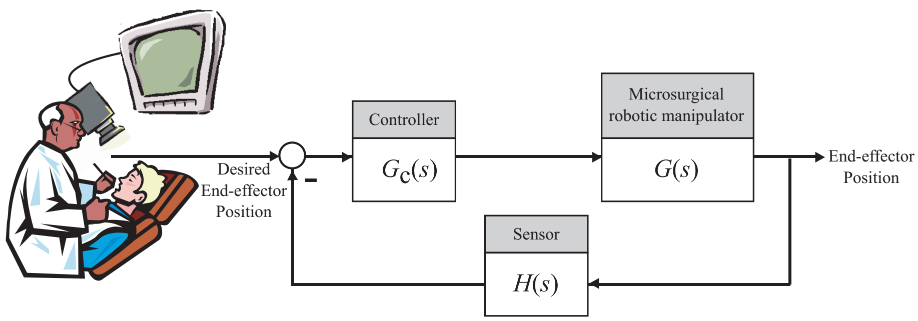


P1.27 Control of a methanol fuel cell:

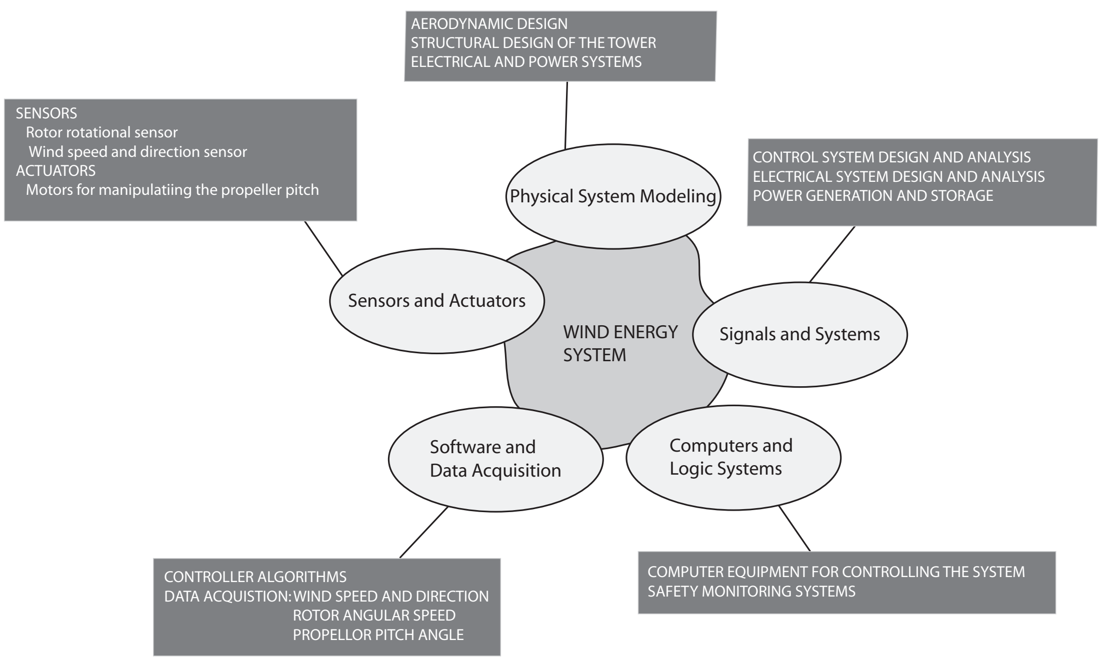


Advanced Problems

AP1.1 Control of a robotic microsurgical device:

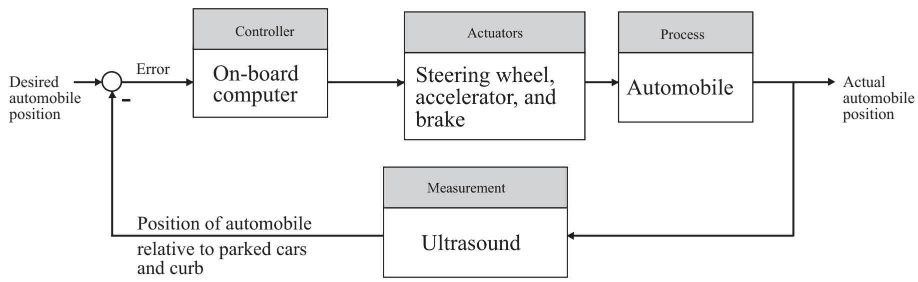


AP1.2 An advanced wind energy system viewed as a mechatronic system:

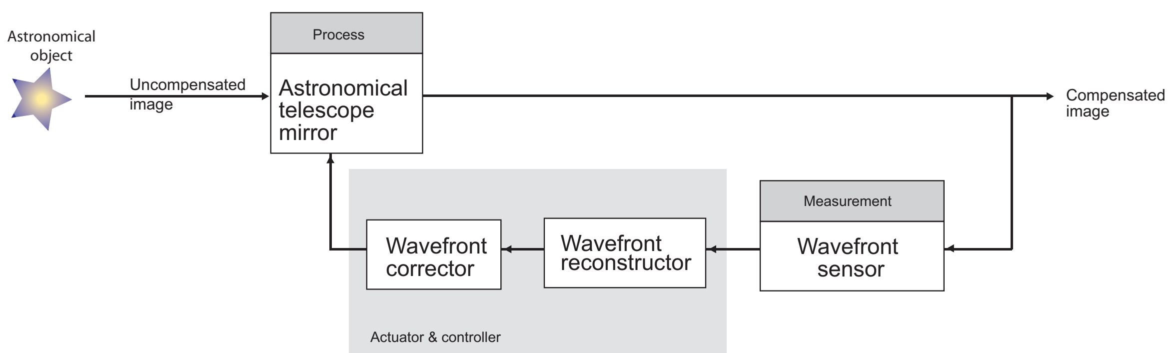


AP1.3 The automatic parallel parking system might use multiple ultrasound sensors to measure distances to the parked automobiles and the curb. The sensor measurements would be processed by an on-board computer to determine the steering wheel, accelerator, and brake inputs to avoid collision and to properly align the vehicle in the desired space.

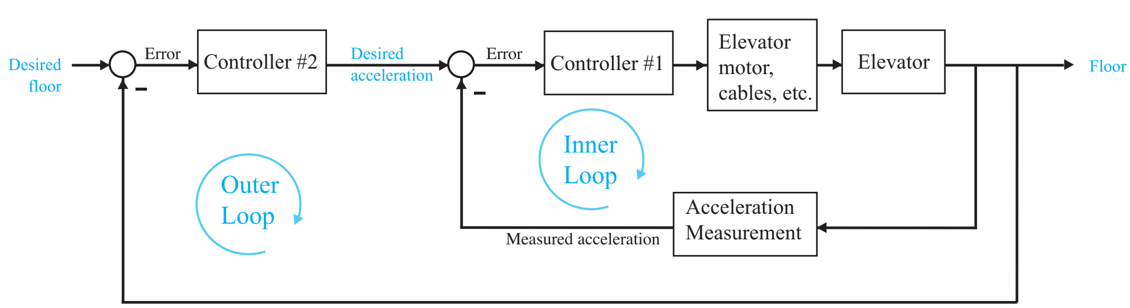
Even though the sensors may accurately measure the distance between the two parked vehicles, there will be a problem if the available space is not big enough to accommodate the parking car.



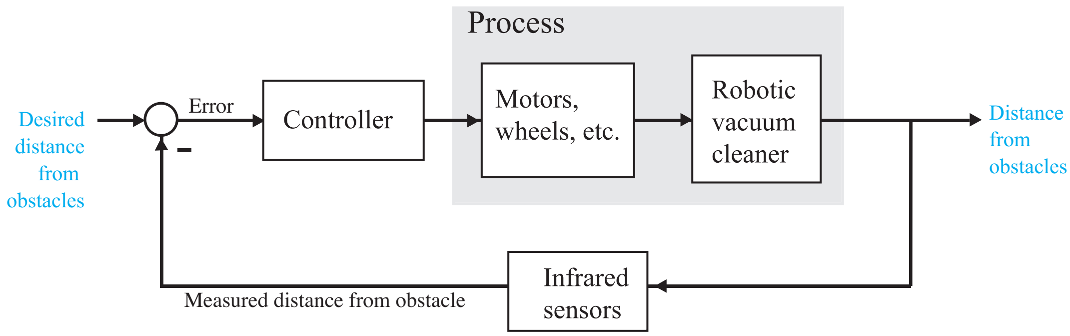
AP1.4 There are various control methods that can be considered, including placing the controller in the feedforward loop (as in Figure 1.3). The adaptive optics block diagram below shows the controller in the feedback loop, as an alternative control system architecture.



AP1.5 The control system might have an inner loop for controlling the acceleration and an outer loop to reach the desired floor level precisely.

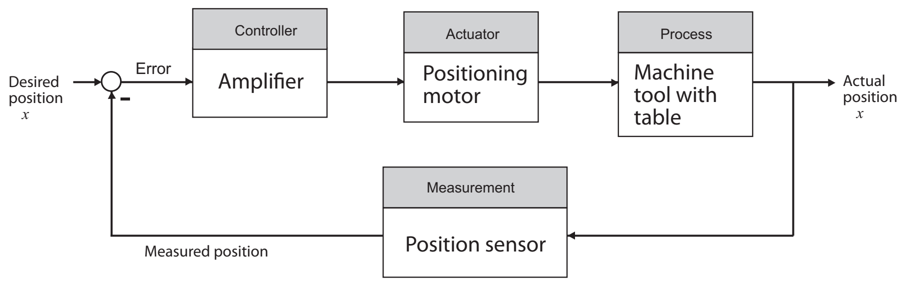


AP1.6 An obstacle avoidance control system would keep the robotic vacuum cleaner from colliding with furniture but it would not necessarily put the vacuum cleaner on an optimal path to reach the entire floor. This would require another sensor to measure position in the room, a digital map of the room layout, and a control system in the outer loop.

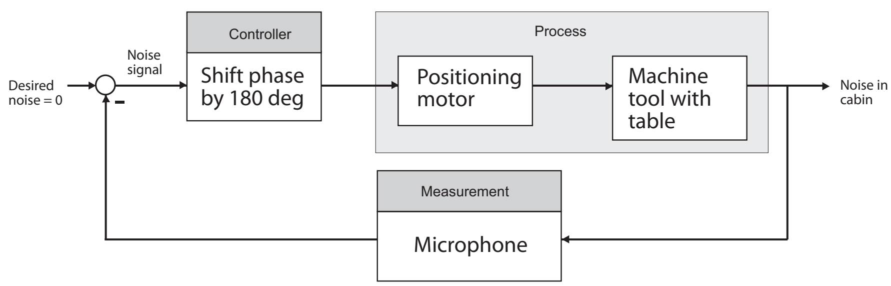


Design Problems

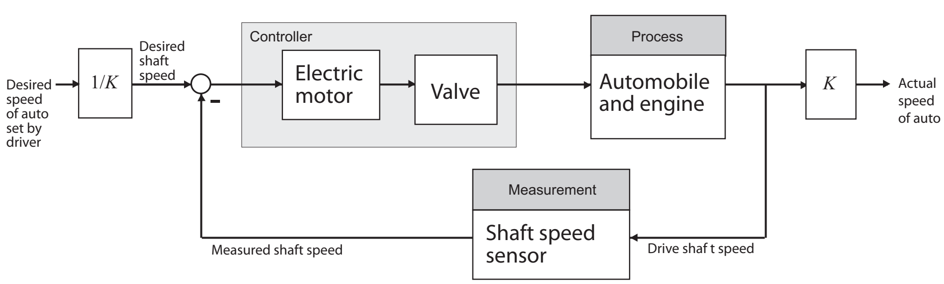
CDP1.1 The machine tool with the movable table in a feedback control configuration:



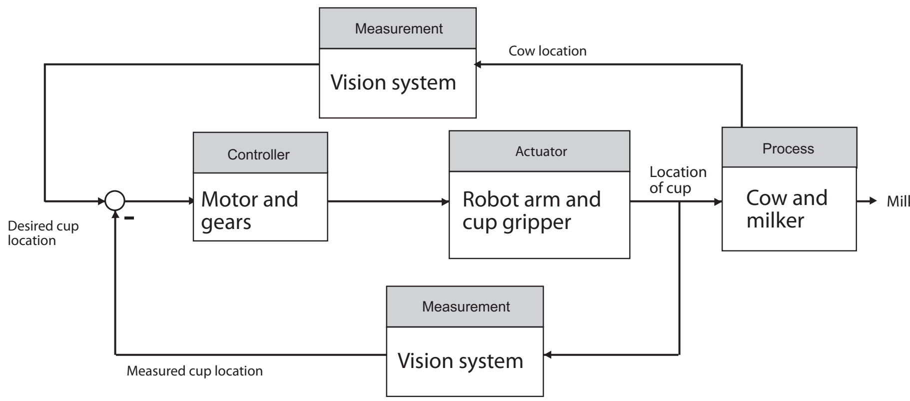
DP1.1 Use the stereo system and amplifiers to cancel out the noise by emitting signals 180° out of phase with the noise.



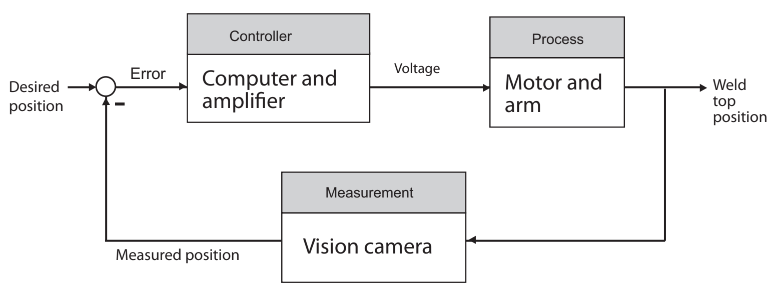
DP1.2 An automobile cruise control system:



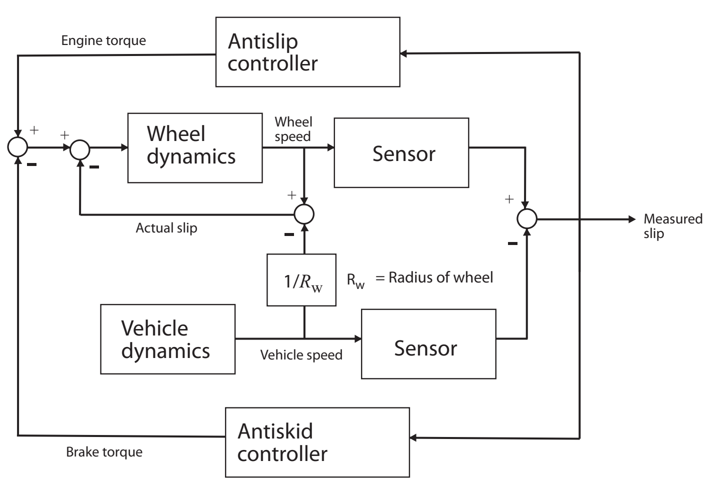
DP1.3 An automoted cow milking system:



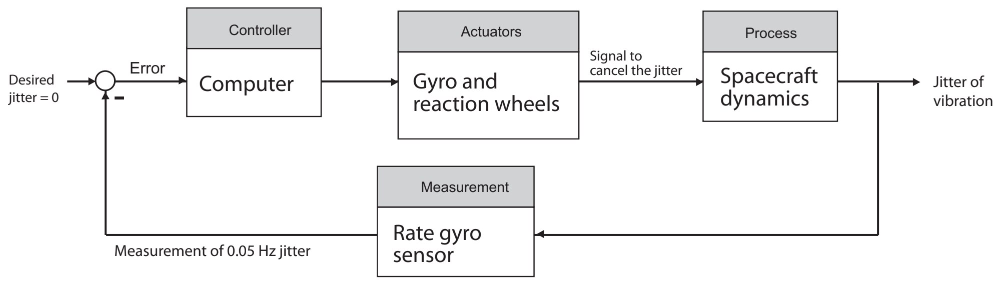
DP1.4 A feedback control system for a robot welder:



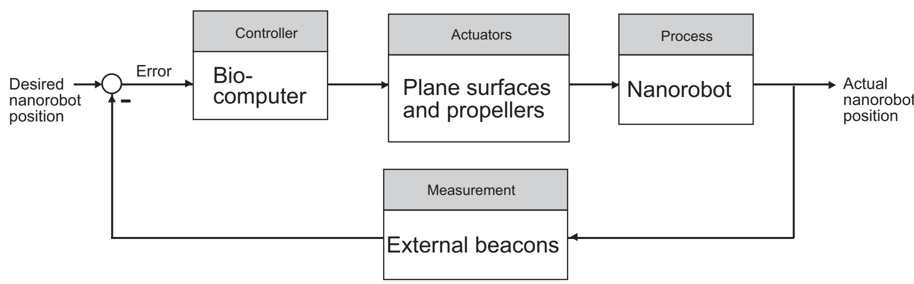
DP1.5 A control system for one wheel of a traction control system:



DP1.6 A vibration damping system for the Hubble Space Telescope:



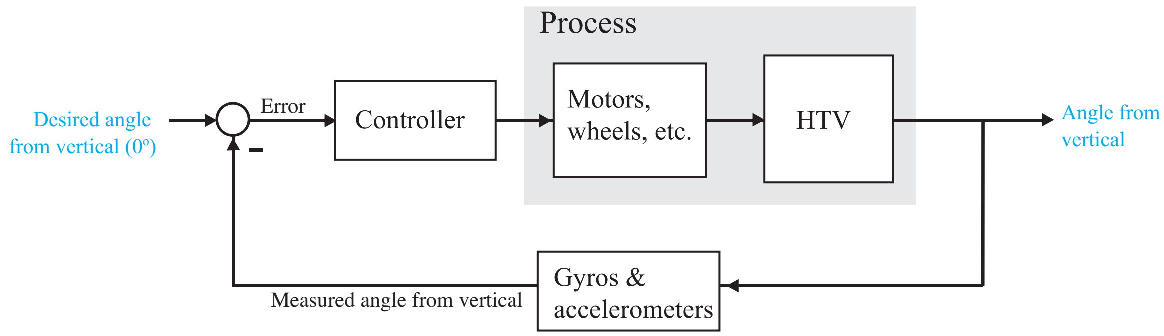
DP1.7 A control system for a nanorobot:



Many concepts from underwater robotics can be applied to nanorobotics within the bloodstream. For example, plane surfaces and propellers can provide the required actuation with screw drives providing the propulsion. The nanorobots can use signals from beacons located outside the skin as sensors to determine their position. The nanorobots use energy from the chemical reaction of oxygen and glucose available in the human body. The control system requires a bio-computer–an innovation that is not yet available.

For further reading, see A. Cavalcanti, L. Rosen, L. C. Kretly, M. Rosenfeld, and S. Einav, "Nanorobotic Challenges n Biomedical Application, Design, and Control," *IEEE ICECS Intl Conf. on Electronics, Circuits and Systems*, Tel-Aviv, Israel, December 2004.

DP1.8 The feedback control system might use gyros and/or accelerometers to measure angle change and assuming the HTV was originally in the vertical position, the feedback would retain the vertical position using commands to motors and other actuators that produced torques and could move the HTV forward and backward.


Process
Desired angle
from vertical (0°)
Error
Controller
Motors,
wheels, etc.
HTV
Angle from
vertical
–
Gyros &
accelerometers
Measured angle from vertical

CHAPTER 2

Mathematical Models of Systems

Exercises

E2.1 We have for the open-loop

$$y = r^2$$

and for the closed-loop

$$e = r - y \quad \text{and} \quad y = e^2 \ .$$

So, $e = r - e^2$ and $e^2 + e - r = 0$.

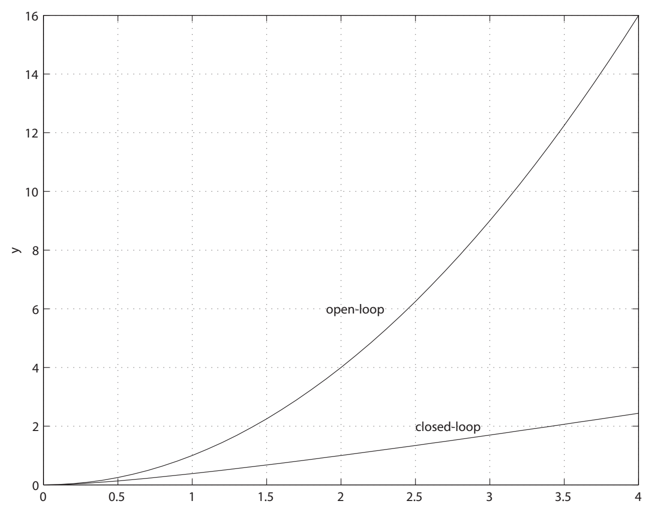


FIGURE E2.1
Plot of open-loop versus closed-loop.

For example, if $r = 1$, then $e^2 + e - 1 = 0$ implies that $e = 0.618$. Thus, $y = 0.382$. A plot y versus r is shown in Figure E2.1.

E2.2 Define

$$f(T) = R = R_0 e^{-0.1T}$$

and

$$\Delta R = f(T) - f(T_0) \ , \quad \Delta T = T - T_0 \ .$$

Then,

$$\Delta R = f(T) - f(T_0) = \left.\frac{\partial f}{\partial T}\right|_{T=T_0=20^\circ} \Delta T + \cdots$$

where

$$\left.\frac{\partial f}{\partial T}\right|_{T=T_0=20^\circ} = -0.1 R_0 e^{-0.1T_0} = -135,$$

when $R_0 = 10,000\Omega$. Thus, the linear approximation is computed by considering only the first-order terms in the Taylor series expansion, and is given by

$$\Delta R = -135 \Delta T \ .$$

E2.3 The spring constant for the equilibrium point is found graphically by estimating the slope of a line tangent to the force versus displacement curve at the point $y = 0.5$cm, see Figure E2.3. The slope of the line is $K \approx 1$.

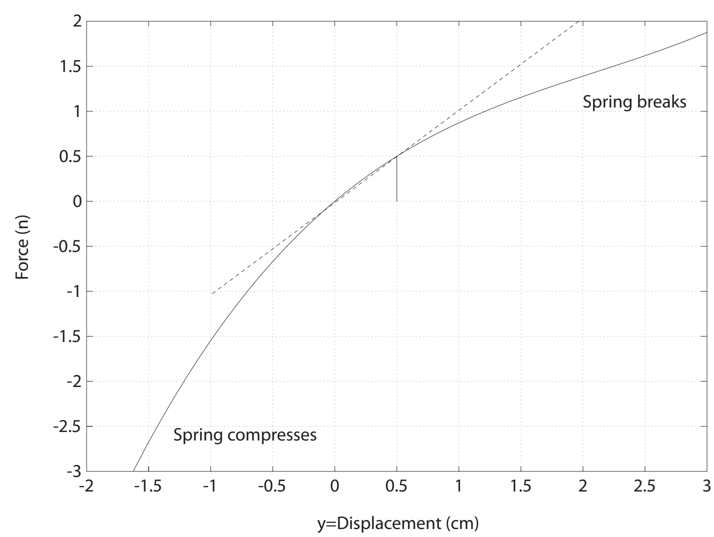


FIGURE E2.3
Spring force as a function of displacement.

E2.4 Since

$$R(s) = \frac{1}{s}$$

we have

$$Y(s) = \frac{4(s+50)}{s(s+20)(s+10)} .$$

The partial fraction expansion of $Y(s)$ is given by

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+20} + \frac{A_3}{s+10}$$

where

$$A_1 = 1 , \quad A_2 = 0.6 \text{ and } \quad A_3 = -1.6 .$$

Using the Laplace transform table, we find that

$$y(t) = 1 + 0.6e^{-20t} - 1.6e^{-10t} .$$

The final value is computed using the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \left[\frac{4(s+50)}{s(s^2+30s+200)} \right] = 1 .$$

E2.5 The circuit diagram is shown in Figure E2.5.

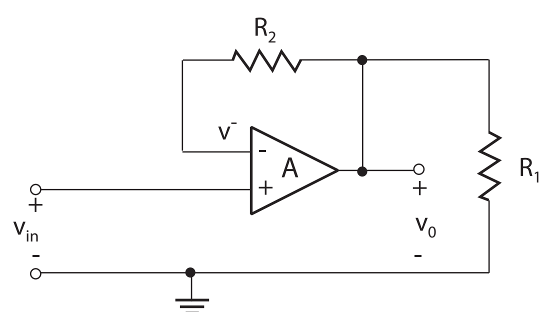


FIGURE E2.5
Noninverting op-amp circuit.

With an ideal op-amp, we have

$$v_o = A(v_{in} - v^-),$$

where A is very large. We have the relationship

$$v^- = \frac{R_1}{R_1 + R_2} v_o .$$

Therefore,

$$v_o = A(v_{in} - \frac{R_1}{R_1 + R_2} v_o),$$

and solving for v_o yields

$$v_o = \frac{A}{1 + \frac{AR_1}{R_1+R_2}} v_{in} .$$

Since $A \gg 1$, it follows that $1 + \frac{AR_1}{R_1+R_2} \approx \frac{AR_1}{R_1+R_2}$. Then the expression for v_o simplifies to

$$v_o = \frac{R_1 + R_2}{R_1} v_{in} .$$

E2.6 Given

$$y = f(x) = e^x$$

and the operating point $x_o = 1$, we have the linear approximation

$$y = f(x) = f(x_o) + \left.\frac{\partial f}{\partial x}\right|_{x=x_o} (x - x_o) + \cdots$$

where

$$f(x_o) = e, \quad \left.\frac{df}{dx}\right|_{x=x_o=1} = e, \quad \text{and} \quad x - x_o = x - 1 .$$

Therefore, we obtain the linear approximation $y = ex$.

E2.7 The block diagram is shown in Figure E2.7.

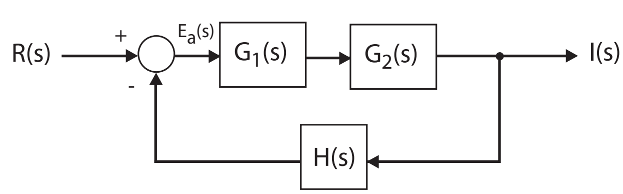


FIGURE E2.7
Block diagram model.

Starting at the output we obtain

$$I(s) = G_1(s)G_2(s)E(s).$$

But $E(s) = R(s) - H(s)I(s)$, so

$$I(s) = G_1(s)G_2(s)\left[R(s) - H(s)I(s)\right].$$

Solving for $I(s)$ yields the closed-loop transfer function

$$\frac{I(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} .$$

E2.8 The block diagram is shown in Figure E2.8.

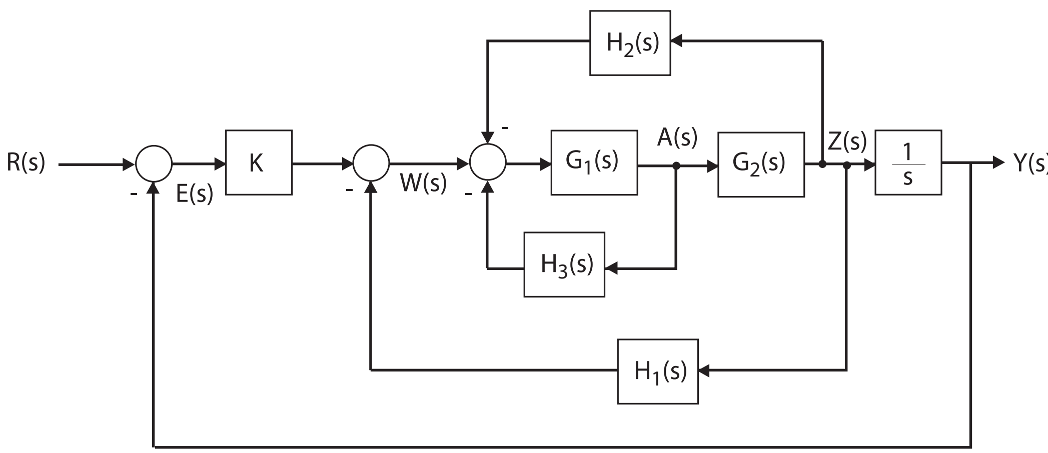


FIGURE E2.8
Block diagram model.

Starting at the output we obtain

$$Y(s) = \frac{1}{s}Z(s) = \frac{1}{s}G_2(s)A(s).$$

But $A(s) = G_1(s)\left[-H_2(s)Z(s) - H_3(s)A(s) + W(s)\right]$ and $Z(s) = sY(s)$, so

$$Y(s) = -G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) + \frac{1}{s}G_1(s)G_2(s)W(s).$$

Substituting $W(s) = KE(s) - H_1(s)Z(s)$ into the above equation yields

$$Y(s) = -G_1(s)G_2(s)H_2(s)Y(s) - G_1(s)H_3(s)Y(s) + \frac{1}{s}G_1(s)G_2(s)\left[KE(s) - H_1(s)Z(s)\right]$$

and with $E(s) = R(s) - Y(s)$ and $Z(s) = sY(s)$ this reduces to

$$Y(s) = [-G_1(s)G_2(s)\,(H_2(s) + H_1(s)) - G_1(s)H_3(s) - \frac{1}{s}G_1(s)G_2(s)K]Y(s) + \frac{1}{s}G_1(s)G_2(s)KR(s).$$

Solving for $Y(s)$ yields the transfer function

$$Y(s) = T(s)R(s),$$

where

$$T(s) = \frac{KG_1(s)G_2(s)/s}{1 + G_1(s)G_2(s)\,[(H_2(s) + H_1(s)] + G_1(s)H_3(s) + KG_1(s)G_2(s)/s}.$$

E2.9 From Figure E2.9, we observe that

$$F_f(s) = G_2(s)U(s)$$

and

$$F_R(s) = G_3(s)U(s)\ .$$

Then, solving for $U(s)$ yields

$$U(s) = \frac{1}{G_2(s)}F_f(s)$$

and it follows that

$$F_R(s) = \frac{G_3(s)}{G_2(s)}U(s)\ .$$

Again, considering the block diagram in Figure E2.9 we determine

$$F_f(s) = G_1(s)G_2(s)[R(s) - H_2(s)F_f(s) - H_2(s)F_R(s)]\ .$$

But, from the previous result, we substitute for $F_R(s)$ resulting in

$$F_f(s) = G_1(s)G_2(s)R(s) - G_1(s)G_2(s)H_2(s)F_f(s) - G_1(s)H_2(s)G_3(s)F_f(s)\ .$$

Solving for $F_f(s)$ yields

$$F_f(s) = \left[\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_2(s) + G_1(s)G_3(s)H_2(s)}\right]R(s)\ .$$

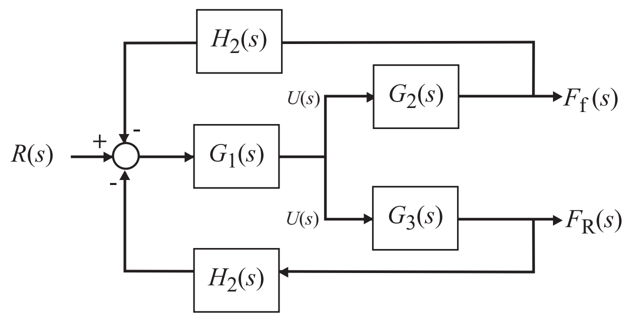


FIGURE E2.9
Block diagram model.

E2.10 The shock absorber block diagram is shown in Figure E2.10. The closed-loop transfer function model is

$$T(s) = \frac{G_c(s)G_p(s)G(s)}{1 + H(s)G_c(s)G_p(s)G(s)} .$$

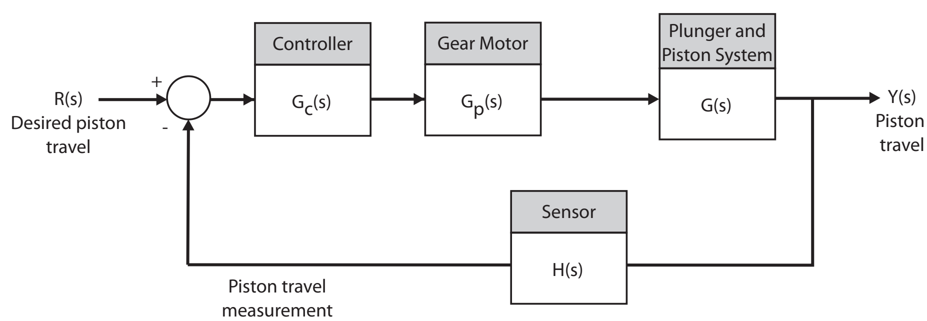


FIGURE E2.10
Shock absorber block diagram.

E2.11 Let f denote the spring force (n) and x denote the deflection (m). Then

$$K = \frac{\Delta f}{\Delta x} .$$

Computing the slope from the graph yields:

(a) $x_o = -0.14\text{m} \rightarrow K = \Delta f/\Delta x =$ 10 n / 0.04 m = 250 n/m

(b) $x_o = 0\text{m} \rightarrow K = \Delta f/\Delta x =$ 10 n / 0.05 m = 200 n/m

(c) $x_o = 0.35\text{m} \rightarrow K = \Delta f/\Delta x =$ 3n / 0.05 m = 60 n/m

E2.12 The signal flow graph is shown in Fig. E2.12. Find $Y(s)$ when $R(s) = 0$.

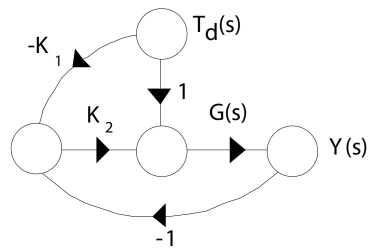


FIGURE E2.12
Signal flow graph.

The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s) = \frac{G(s)T_d(s) - K_1K_2G(s)T_d(s)}{1 - (-K_2G(s))} = \frac{G(s)(1 - K_1K_2)T_d(s)}{1 + K_2G(s)} .$$

If we set

$$K_1K_2 = 1 ,$$

then $Y(s) = 0$ for any $T_d(s)$.

E2.13 The transfer function from $R(s)$, $T_d(s)$, and $N(s)$ to $Y(s)$ is

$$Y(s) = \left[\frac{K}{s^2 + 10s + K}\right] R(s) + \left[\frac{1}{s^2 + 10s + K}\right] T_d(s) - \left[\frac{K}{s^2 + 10s + K}\right] N(s)$$

Therefore, we find that

$$Y(s)/T_d(s) = \frac{1}{s^2 + 10s + K} \quad \text{and} \quad Y(s)/N(s) = -\frac{K}{s^2 + 10s + K}$$

E2.14 Since we want to compute the transfer function from $R_2(s)$ to $Y_1(s)$, we can assume that $R_1 = 0$ (application of the principle of superposition). Then, starting at the output $Y_1(s)$ we obtain

$$Y_1(s) = G_3(s) \left[-H_1(s)Y_1(s) + G_2(s)G_8(s)W(s) + G_9(s)W(s)\right] ,$$

or

$$\left[1 + G_3(s)H_1(s)\right] Y_1(s) = \left[G_3(s)G_2(s)G_8(s)W(s) + G_3(s)G_9(s)\right] W(s).$$

Considering the signal $W(s)$ (see Figure E2.14), we determine that

$$W(s) = G_5(s) \left[G_4(s)R_2(s) - H_2(s)W(s)\right] ,$$

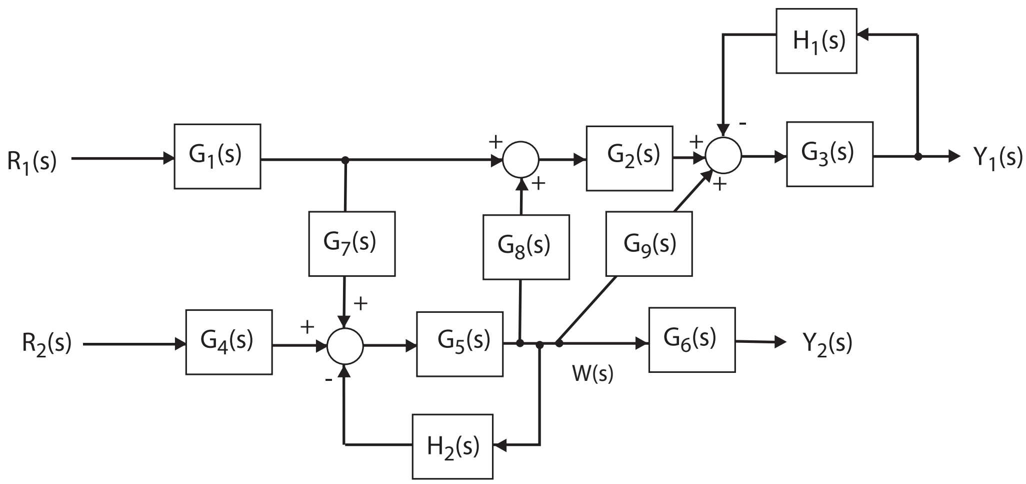


FIGURE E2.14
Block diagram model.

or

$$[1 + G_5(s)H_2(s)]\,W(s) = G_5(s)G_4(s)R_2(s).$$

Substituting the expression for $W(s)$ into the above equation for $Y_1(s)$ yields

$$\frac{Y_1(s)}{R_2(s)} = \frac{G_2(s)G_3(s)G_4(s)G_5(s)G_8(s) + G_3(s)G_4(s)G_5(s)G_9(s)}{1 + G_3(s)H_1(s) + G_5(s)H_2(s) + G_3(s)G_5(s)H_1(s)H_2(s)}.$$

E2.15 For loop 1, we have

$$R_1 i_1 + L_1\frac{di_1}{dt} + \frac{1}{C_1}\int (i_1 - i_2)dt + R_2(i_1 - i_2) = v(t) \ .$$

And for loop 2, we have

$$\frac{1}{C_2}\int i_2 dt + L_2\frac{di_2}{dt} + R_2(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)dt = 0 \ .$$

E2.16 The transfer function from $R(s)$ to $P(s)$ is

$$\frac{P(s)}{R(s)} = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} \ .$$

The block diagram is shown in Figure E2.16a. The corresponding signal flow graph is shown in Figure E2.16b for

$$P(s)/R(s) = \frac{4.2}{s^3 + 2s^2 + 4s + 4.2} \ .$$

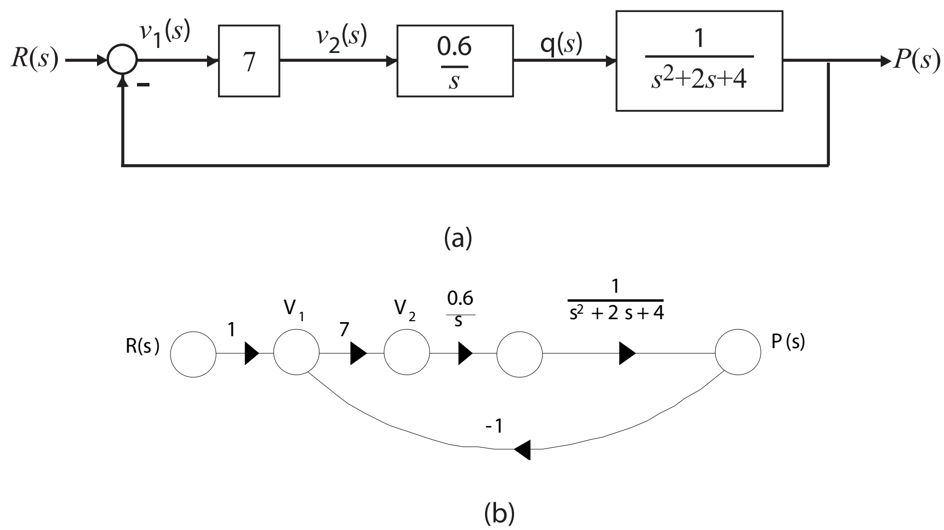


FIGURE E2.16
(a) Block diagram, (b) Signal flow graph.

E2.17 A linear approximation for f is given by

$$\Delta f = \left.\frac{\partial f}{\partial x}\right|_{x=x_o} \Delta x = 2kx_o\Delta x = k\Delta x$$

where $x_o = 1/2$, $\Delta f = f(x) - f(x_o)$, and $\Delta x = x - x_o$.

E2.18 The linear approximation is given by

$$\Delta y = m\Delta x$$

where

$$m = \left.\frac{\partial y}{\partial x}\right|_{x=x_o} .$$

(a) When $x_o = 1$, we find that $y_o = 2.4$, and $y_o = 13.2$ when $x_o = 2$.

(b) The slope m is computed as follows:

$$m = \left.\frac{\partial y}{\partial x}\right|_{x=x_o} = 1 + 4.2x_o^2 .$$

Therefore, $m = 5.2$ at $x_o = 1$, and $m = 18.8$ at $x_o = 2$.

E2.19 The output (with a step input) is

$$Y(s) = \frac{15(s+1)}{s(s+7)(s+2)} .$$

The partial fraction expansion is

$$Y(s) = \frac{15}{14s} - \frac{18}{7}\frac{1}{s+7} + \frac{3}{2}\frac{1}{s+2} .$$

Taking the inverse Laplace transform yields

$$y(t) = \frac{15}{14} - \frac{18}{7}e^{-7t} + \frac{3}{2}e^{-2t} .$$

E2.20 The input-output relationship is

$$\frac{V_o}{V} = \frac{A(K-1)}{1+AK}$$

where

$$K = \frac{Z_1}{Z_1+Z_2} .$$

Assume $A \gg 1$. Then,

$$\frac{V_o}{V} = \frac{K-1}{K} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1C_1s+1} \quad \text{and} \quad Z_2 = \frac{R_2}{R_2C_2s+1} .$$

Therefore,

$$\frac{V_o(s)}{V(s)} = -\frac{R_2(R_1C_1s+1)}{R_1(R_2C_2s+1)} = -\frac{2(s+1)}{s+2} .$$

E2.21 The equation of motion of the mass m_c is

$$m_c\ddot{x}_p + (b_d+b_s)\dot{x}_p + k_dx_p = b_d\dot{x}_{in} + k_dx_{in} .$$

Taking the Laplace transform with zero initial conditions yields

$$[m_cs^2 + (b_d+b_s)s + k_d]X_p(s) = [b_ds + k_d]X_{in}(s) .$$

So, the transfer function is

$$\frac{X_p(s)}{X_{in}(s)} = \frac{b_ds+k_d}{m_cs^2+(b_d+b_s)s+k_d} = \frac{0.7s+2}{s^2+2.8s+2} .$$

E2.22 The rotational velocity is

$$\omega(s) = \frac{2(s+4)}{(s+5)(s+1)^2} \frac{1}{s} .$$

Expanding in a partial fraction expansion yields

$$\omega(s) = \frac{8}{5}\frac{1}{s} + \frac{1}{40}\frac{1}{s+5} - \frac{3}{2}\frac{1}{(s+1)^2} - \frac{13}{8}\frac{1}{s+1} .$$

Taking the inverse Laplace transform yields

$$\omega(t) = \frac{8}{5} + \frac{1}{40}e^{-5t} - \frac{3}{2}te^{-t} - \frac{13}{8}e^{-t} .$$

E2.23 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{K_1K_2}{s^2 + (K_1 + K_2K_3 + K_1K_2)s + K_1K_2K_3} .$$

E2.24 The closed-loop tranfser function is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{10}{s^2 + 21s + 10} .$$

E2.25 Let $x = 0.6$ and $y = 0.8$. Then, with $y = ax^3$, we have

$$0.8 = a(0.6)^3 .$$

Solving for a yields $a = 3.704$. A linear approximation is

$$y - y_o = 3ax_o^2(x - x_o)$$

or $y = 4x - 1.6$, where $y_o = 0.8$ and $x_o = 0.6$.

E2.26 The equations of motion are

$$m_1\ddot{x}_1 + k(x_1 - x_2) = F$$
$$m_2\ddot{x}_2 + k(x_2 - x_1) = 0 .$$

Taking the Laplace transform (with zero initial conditions) and solving for $X_2(s)$ yields

$$X_2(s) = \frac{k}{(m_2s^2 + k)(m_1s^2 + k) - k^2}F(s) .$$

Then, with $m_1 = m_2 = k = 1$, we have

$$X_2(s)/F(s) = \frac{1}{s^2(s^2+2)} .$$

E2.27 The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s)/T_d(s) = \frac{G_2(s)}{1 + G_1G_2H(s)} .$$

E2.28 The transfer function is

$$\frac{V_o(s)}{V(s)} = \frac{R_2R_4C}{R_3}s + \frac{R_2R_4}{R_1R_3} = 24s + 144 .$$

E2.29 (a) If

$$G(s) = \frac{1}{s^2 + 15s + 50} \quad \text{and} \quad H(s) = 2s + 15 ,$$

then the closed-loop transfer function of Figure E2.28(a) and (b) (in Dorf & Bishop) are equivalent.

(b) The closed-loop transfer function is

$$T(s) = \frac{1}{s^2 + 17s + 65} .$$

E2.30 (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)}\frac{1}{s} = \frac{10}{s(s^2 + 2s + 20)} \quad \text{where} \quad G(s) = \frac{10}{s^2 + 2s + 10} .$$

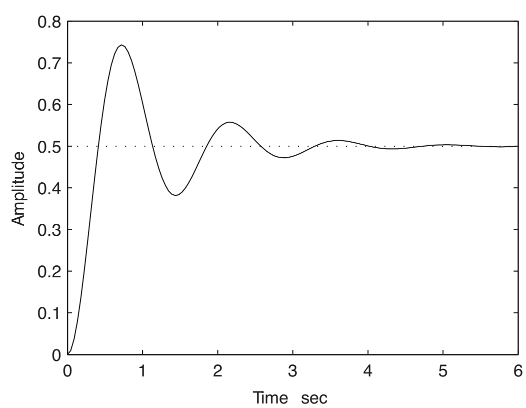


FIGURE E2.30
Step response.

(b) The output $Y(s)$ (when $R(s) = 1/s$) is

$$Y(s) = \frac{0.5}{s} - \frac{-0.25 + 0.0573j}{s + 1 - 4.3589j} + \frac{-0.25 - 0.0573j}{s + 1 + 4.3589j} \ .$$

(c) The plot of $y(t)$ is shown in Figure E2.30. The output is given by

$$y(t) = \frac{1}{2}\left[1 - e^{-t}\left(\cos\sqrt{19}t - \frac{1}{\sqrt{19}}\sin\sqrt{19}t\right)\right]$$

E2.31 The partial fraction expansion is

$$V(s) = \frac{a}{s + p_1} + \frac{b}{s + p_2}$$

where $p_1 = 4 - 19.6j$ and $p_2 = 4 + 19.6j$. Then, the residues are

$$a = -10.2j \quad b = 10.2j \ .$$

The inverse Laplace transform is

$$v(t) = -10.2je^{(-4+19.6j)t} + 10.2je^{(-4-19.6j)t} = 20.4e^{-4t}\sin 19.6t \ .$$

Problems

P2.1 The integrodifferential equations, obtained by Kirchoff's voltage law to each loop, are as follows:

$$R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_2(i_1 - i_2) = v(t) \quad \text{(loop 1)}$$

and

$$R_3 i_2 + \frac{1}{C_2}\int i_2 dt + R_2(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 \quad \text{(loop 2)} .$$

P2.2 The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$M_1\ddot{y}_1 + k_{12}(y_1 - y_2) + b\dot{y}_1 + k_1 y_1 = F(t)$$
$$M_2\ddot{y}_2 + k_{12}(y_2 - y_1) = 0 .$$

Using a force-current analogy, the analagous electric circuit is shown in Figure P2.2, where $C_i \to M_i$, $L_1 \to 1/k_1$, $L_{12} \to 1/k_{12}$, and $R \to 1/b$.

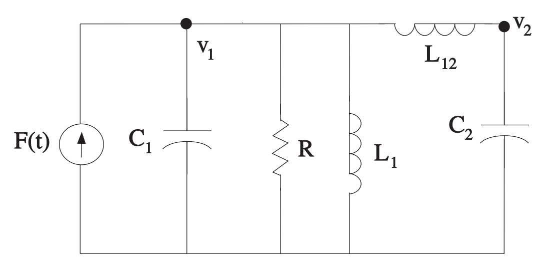


FIGURE P2.2
Analagous electric circuit.

P2.3 The differential equations describing the system can be obtained by using a free-body diagram analysis of each mass. For mass 1 and 2 we have

$$M\ddot{x}_1 + kx_1 + k(x_1 - x_2) = F(t)$$
$$M\ddot{x}_2 + k(x_2 - x_1) + b\dot{x}_2 = 0 .$$

Using a force-current analogy, the analagous electric circuit is shown in Figure P2.3, where

$$C \to M \qquad L \to 1/k \qquad R \to 1/b .$$

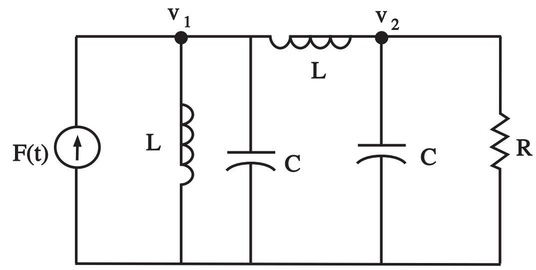


FIGURE P2.3
Analagous electric circuit.

P2.4 (a) The linear approximation around $v_{in} = 0$ is $v_o = 0v_{in}$, see Figure P2.4(a).

(b) The linear approximation around $v_{in} = 1$ is $v_o = 2v_{in} - 1$, see Figure P2.4(b).

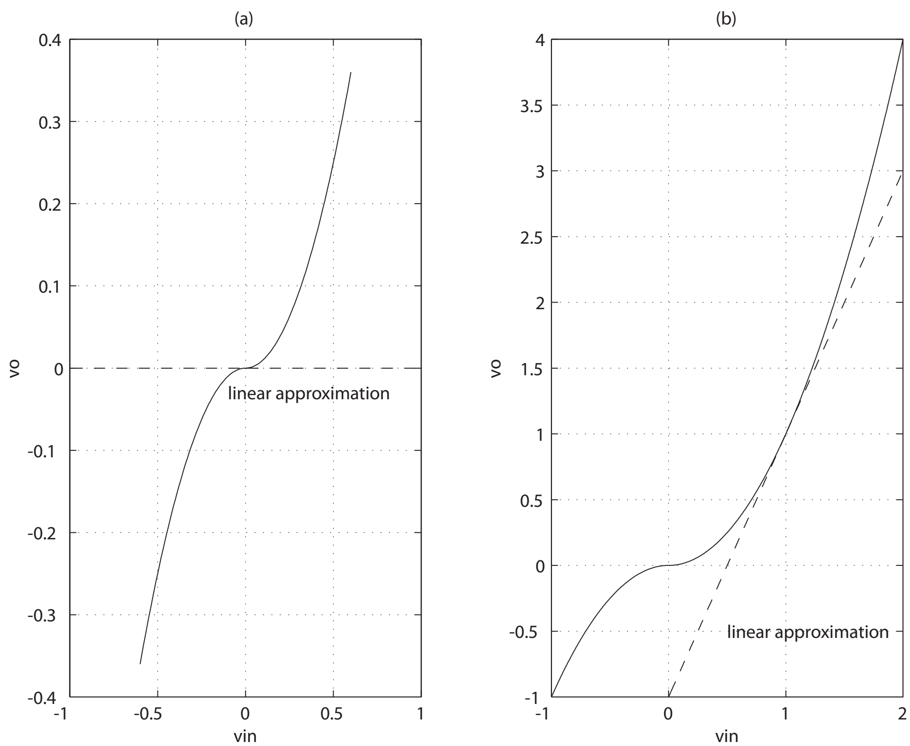


FIGURE P2.4
Nonlinear functions and approximations.

P2.5 Given

$$Q = K(P_1 - P_2)^{1/2} .$$

Let $\delta P = P_1 - P_2$ and δP_o = operating point. Using a Taylor series expansion of Q, we have

$$Q = Q_o + \left.\frac{\partial Q}{\partial \delta P}\right|_{\delta P = \delta P_o} (\delta P - \delta P_o) + \cdots$$

where

$$Q_o = K\delta P_o^{1/2} \quad \text{and} \quad \left.\frac{\partial Q}{\partial \delta P}\right|_{\delta P = \delta P_o} = \frac{K}{2}\delta P_o^{-1/2} .$$

Define $\Delta Q = Q - Q_o$ and $\Delta P = \delta P - \delta P_o$. Then, dropping higher-order terms in the Taylor series expansion yields

$$\Delta Q = m\Delta P$$

where

$$m = \frac{K}{2\delta P_o^{1/2}} .$$

P2.6 From P2.1 we have

$$R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d(i_1 - i_2)}{dt} + R_2(i_1 - i_2) = v(t)$$

and

$$R_3 i_2 + \frac{1}{C_2}\int i_2 dt + R_2(i_2 - i_1) + L_1 \frac{d(i_2 - i_1)}{dt} = 0 .$$

Taking the Laplace transform and using the fact that the initial voltage across C_2 is 10v yields

$$[R_1 + \frac{1}{C_1 s} + L_1 s + R_2]I_1(s) + [-R_2 - L_1 s]I_2(s) = 0$$

and

$$[-R_2 - L_1 s]I_1(s) + [L_1 s + R_3 + \frac{1}{C_2 s} + R_2]I_2(s) = -\frac{10}{s} .$$

Rewriting in matrix form we have

$$\begin{bmatrix} R_1 + \frac{1}{C_1 s} + L_1 s + R_2 & -R_2 - L_1 s \\ -R_2 - L_1 s & L_1 s + R_3 + \frac{1}{C_2 s} + R_2 \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -10/s \end{pmatrix} .$$

Solving for I_2 yields

$$\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} L_1 s + R_3 + \frac{1}{C_2 s} + R_2 & R_2 + L_1 s \\ R_2 + L_1 s & R_1 + \frac{1}{C_1 s} + L_1 s + R_2 \end{bmatrix} \begin{pmatrix} 0 \\ -10/s \end{pmatrix} .$$

or

$$I_2(s) = \frac{-10(R_1 + 1/C_1 s + L_1 s + R_2)}{s\Delta}$$

where

$$\Delta = (R_1 + \frac{1}{C_1 s} + L_1 s + R_2)(L_1 s + R_3 + \frac{1}{C_2 s} + R_2) - (R_2 + L_1 s)^2 .$$

P2.7 Consider the differentiating op-amp circuit in Figure P2.7. For an ideal op-amp, the voltage gain (as a function of frequency) is

$$V_2(s) = -\frac{Z_2(s)}{Z_1(s)} V_1(s),$$

where

$$Z_1 = \frac{R_1}{1 + R_1 C s}$$

and $Z_2 = R_2$ are the respective circuit impedances. Therefore, we obtain

$$V_2(s) = -\left[\frac{R_2(1 + R_1 C s)}{R_1}\right] V_1(s).$$

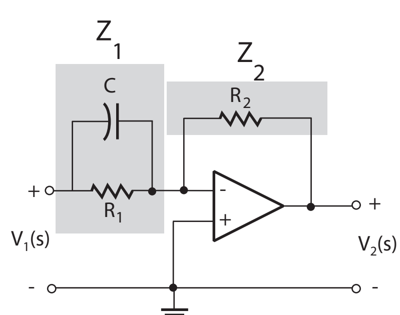


FIGURE P2.7
Differentiating op-amp circuit.

P2.8 Let

$$\Delta = \begin{vmatrix} G_2 + Cs & -Cs & -G_2 \\ -Cs & G_1 + 2Cs & -Cs \\ -G_2 & -Cs & Cs + G_2 \end{vmatrix} .$$

Then,

$$V_j = \frac{\Delta_{ij}}{\Delta} I_1 \qquad \text{or} \qquad or \frac{V_3}{V_1} = \frac{\Delta_{13} I_1/\Delta}{\Delta_{11} I_1/\Delta} .$$

Therefore, the transfer function is

$$T(s) = \frac{V_3}{V_1} = \frac{\Delta_{13}}{\Delta_{11}} = \frac{\begin{vmatrix} -Cs & 2Cs + G_1 \\ -G_2 & -Cs \end{vmatrix}}{\begin{vmatrix} 2Cs + G_1 & -Cs \\ -Cs & Cs + G_2 \end{vmatrix}}$$

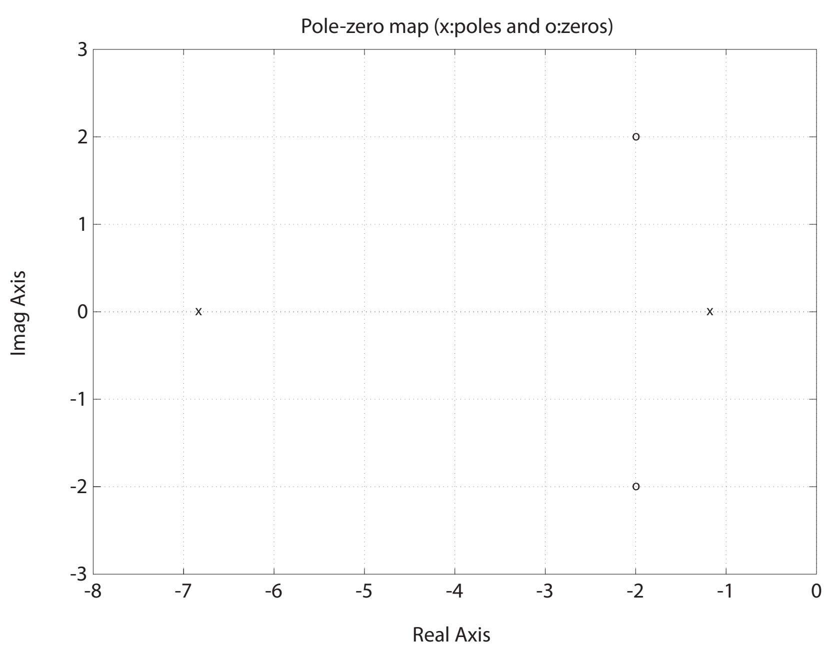


FIGURE P2.8
Pole-zero map.

$$= \frac{C^2R_1R_2s^2 + 2CR_1s + 1}{C^2R_1R_2s^2 + (2R_1 + R_2)Cs + 1} \ .$$

Using $R_1 = 0.5$, $R_2 = 1$, and $C = 0.5$, we have

$$T(s) = \frac{s^2 + 4s + 8}{s^2 + 8s + 8} = \frac{(s + 2 + 2j)(s + 2 - 2j)}{(s + 4 + \sqrt{8})(s + 4 - \sqrt{8})} \ .$$

The pole-zero map is shown in Figure P2.8.

P2.9 From P2.3 we have

$$M\ddot{x}_1 + kx_1 + k(x_1 - x_2) = F(t)$$
$$M\ddot{x}_2 + k(x_2 - x_1) + b\dot{x}_2 = 0 \ .$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} Ms^2 + 2k & -k \\ -k & Ms^2 + bs + k \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} F(s) \\ 0 \end{pmatrix} ,$$

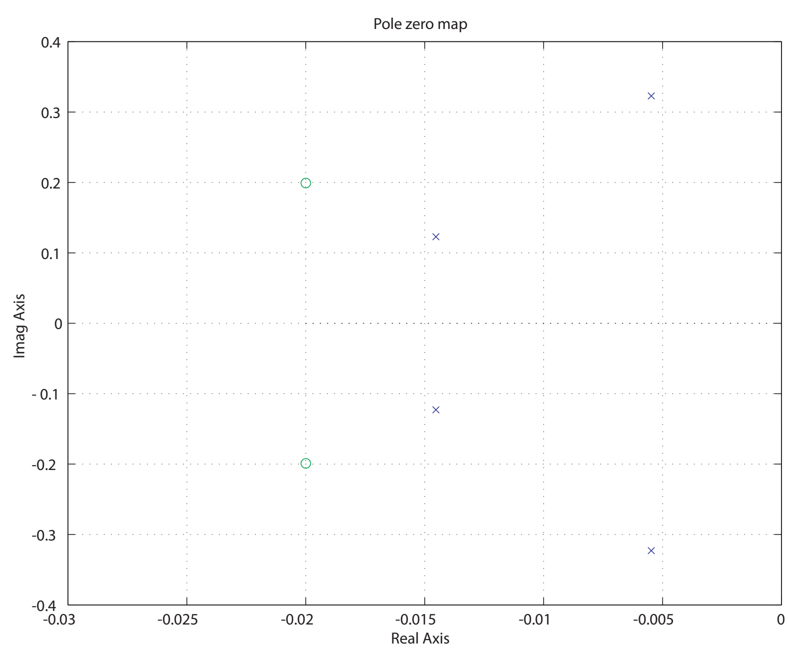


FIGURE P2.9
Pole-zero map.

or

$$\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} Ms^2 + bs + k & k \\ k & Ms^2 + 2k \end{bmatrix} \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

where $\Delta = (Ms^2 + bs + k)(Ms^2 + 2k) - k^2$. So,

$$G(s) = \frac{X_1(s)}{F(s)} = \frac{Ms^2 + bs + k}{\Delta} .$$

When $b/k = 1$, $M = 1$, $b^2/Mk = 0.04$, we have

$$G(s) = \frac{s^2 + 0.04s + 0.04}{s^4 + 0.04s^3 + 0.12s^2 + 0.0032s + 0.0016} .$$

The pole-zero map is shown in Figure P2.9.

P2.10 From P2.2 we have

$$M_1\ddot{y}_1 + k_{12}(y_1 - y_2) + b\dot{y}_1 + k_1 y_1 = F(t)$$
$$M_2\ddot{y}_2 + k_{12}(y_2 - y_1) = 0 .$$

Taking the Laplace transform of both equations and writing the result in matrix form, it follows that

$$\begin{bmatrix} M_1 s^2 + bs + k_1 + k_{12} & -k_{12} \\ -k_{12} & M_2 s^2 + k_{12} \end{bmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} M_2 s^2 + k_{12} & k_{12} \\ k_{12} & M_1 s^2 + bs + k_1 + k_{12} \end{bmatrix} \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

where

$$\Delta = (M_2 s^2 + k_{12})(M_1 s^2 + bs + k_1 + k_{12}) - k_{12}^2 .$$

So, when $f(t) = a \sin \omega_o t$, we have that $Y_1(s)$ is given by

$$Y_1(s) = \frac{aM_2\omega_o(s^2 + k_{12}/M_2)}{(s^2 + \omega_o^2)\Delta(s)} .$$

For motionless response (in the steady-state), set the zero of the transfer function so that

$$(s^2 + \frac{k_{12}}{M_2}) = s^2 + \omega_o^2 \quad \text{or} \quad \omega_o^2 = \frac{k_{12}}{M_2} .$$

P2.11 The transfer functions from $V_c(s)$ to $V_d(s)$ and from $V_d(s)$ to $\theta(s)$ are:

$$V_d(s)/V_c(s) = \frac{K_1K_2}{(L_qs+R_q)(L_cs+R_c)} \text{ , and}$$

$$\theta(s)/V_d(s) = \frac{K_m}{(Js^2+fs)((L_d+L_a)s+R_d+R_a)+K_3K_ms} .$$

The block diagram for $\theta(s)/V_c(s)$ is shown in Figure P2.11, where

$$\theta(s)/V_c(s) = \frac{\theta(s)}{V_d(s)}\frac{V_d(s)}{V_c(s)} = \frac{K_1K_2K_m}{\Delta(s)} ,$$

where

$$\Delta(s) = s(L_cs+R_c)(L_qs+R_q)((Js+b)((L_d+L_a)s+R_d+R_a)+K_mK_3) .$$

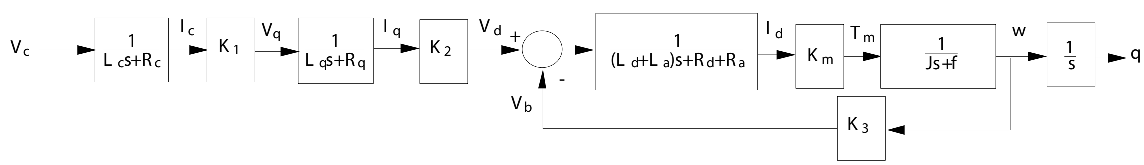


FIGURE P2.11
Block diagram.

P2.12 The open-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K}{s+20} .$$

With $R(s) = 1/s$, we have

$$Y(s) = \frac{K}{s(s+20)} .$$

The partial fraction expansion is

$$Y(s) = \frac{K}{20}\left(\frac{1}{s} - \frac{1}{s+20}\right) ,$$

and the inverse Laplace transform is

$$y(t) = \frac{K}{20}\left(1 - e^{-20t}\right) ,$$

As $t \to \infty$, it follows that $y(t) \to K/20$. So we choose $K = 20$ so that $y(t)$

approaches 1. Alternatively we can use the final value theorem to obtain

$$y(t)_{t\to\infty} = \lim_{s\to 0} sY(s) = \frac{K}{20} = 1 \ .$$

It follows that choosing $K = 20$ leads to $y(t) \to 1$ as $t \to \infty$.

P2.13 The motor torque is given by

$$\begin{aligned} T_m(s) &= (J_m s^2 + b_m s)\theta_m(s) + (J_L s^2 + b_L s)n\theta_L(s) \\ &= n((J_m s^2 + b_m s)/n^2 + J_L s^2 + b_L s)\theta_L(s) \end{aligned}$$

where

$$n = \theta_L(s)/\theta_m(s) = \text{ gear ratio } .$$

But

$$T_m(s) = K_m I_g(s)$$

and

$$I_g(s) = \frac{1}{(L_g + L_f)s + R_g + R_f} V_g(s) \ ,$$

and

$$V_g(s) = K_g I_f(s) = \frac{K_g}{R_f + L_f s} V_f(s) \ .$$

Combining the above expressions yields

$$\frac{\theta_L(s)}{V_f(s)} = \frac{K_g K_m}{n\Delta_1(s)\Delta_2(s)} \ .$$

where

$$\Delta_1(s) = J_L s^2 + b_L s + \frac{J_m s^2 + b_m s}{n^2}$$

and

$$\Delta_2(s) = (L_g s + L_f s + R_g + R_f)(R_f + L_f s) \ .$$

P2.14 For a field-controlled dc electric motor we have

$$\omega(s)/V_f(s) = \frac{K_m/R_f}{Js + b} \ .$$

With a step input of $V_f(s) = 80/s$, the final value of $\omega(t)$ is

$$\omega(t)_{t\to\infty} = \lim_{s\to 0} s\omega(s) = \frac{80K_m}{R_f b} = 2.4 \quad \text{or} \quad \frac{K_m}{R_f b} = 0.03 \ .$$

Solving for $\omega(t)$ yields

$$\omega(t) = \frac{80K_m}{R_f J}\mathcal{L}^{-1}\left\{\frac{1}{s(s+b/J)}\right\} = \frac{80K_m}{R_f b}(1-e^{-(b/J)t}) = 2.4(1-e^{-(b/J)t}) \ .$$

At $t = 1/2$, $\omega(t) = 1$, so

$$\omega(1/2) = 2.4(1-e^{-(b/J)t}) = 1 \quad \text{implies} \quad b/J = 1.08 \text{ sec} \ .$$

Therefore,

$$\omega(s)/V_f(s) = \frac{0.0324}{s+1.08} \ .$$

P2.15 Summing the forces in the vertical direction and using Newton's Second Law we obtain

$$\ddot{x} + \frac{k}{m}x = 0 \ .$$

The system has no damping and no external inputs. Taking the Laplace transform yields

$$X(s) = \frac{x_0 s}{s^2 + k/m} \ ,$$

where we used the fact that $x(0) = x_0$ and $\dot{x}(0) = 0$. Then taking the inverse Laplace transform yields

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}t \ .$$

P2.16 Using Cramer's rule, we have

$$\begin{bmatrix} 1 & 1.5 \\ 2 & 4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\Delta}\begin{bmatrix} 4 & -1.5 \\ -2 & 1 \end{bmatrix}\begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

where $\Delta = 4(1) - 2(1.5) = 1$. Therefore,

$$x_1 = \frac{4(6) - 1.5(11)}{1} = 7.5 \quad \text{and} \quad x_2 = \frac{-2(6) + 1(11)}{1} = -1 \ .$$

The signal flow graph is shown in Figure P2.16.

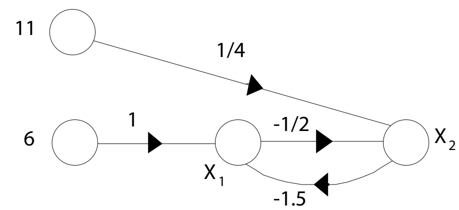


FIGURE P2.16
Signal flow graph.

So,

$$x_1 = \frac{6(1) - 1.5(\frac{11}{4})}{1 - \frac{3}{4}} = 7.5 \quad \text{and} \quad x_2 = \frac{11(\frac{1}{4}) + \frac{-1}{2}(6)}{1 - \frac{3}{4}} = -1 \ .$$

P2.17 (a) For mass 1 and 2, we have

$$M_1\ddot{x}_1 + K_1(x_1 - x_2) + b_1(\dot{x}_3 - \dot{x}_1) = 0$$
$$M_2\ddot{x}_2 + K_2(x_2 - x_3) + b_2(\dot{x}_3 - \dot{x}_2) + K_1(x_2 - x_1) = 0 \ .$$

(b) Taking the Laplace transform yields

$$(M_1s^2 + b_1s + K_1)X_1(s) - K_1X_2(s) = b_1sX_3(s)$$
$$-K_1X_1(s) + (M_2s^2 + b_2s + K_1 + K_2)X_2(s) = (b_2s + K_2)X_3(s) \ .$$

(c) Let

$$G_1(s) = K_2 + b_2s$$
$$G_2(s) = 1/p(s)$$
$$G_3(s) = 1/q(s)$$
$$G_4(s) = sb_1 \ ,$$

where

$$p(s) = s^2M_2 + sf_2 + K_1 + K_2$$

and

$$q(s) = s^2M_1 + sf_1 + K_1 \ .$$

The signal flow graph is shown in Figure P2.17.

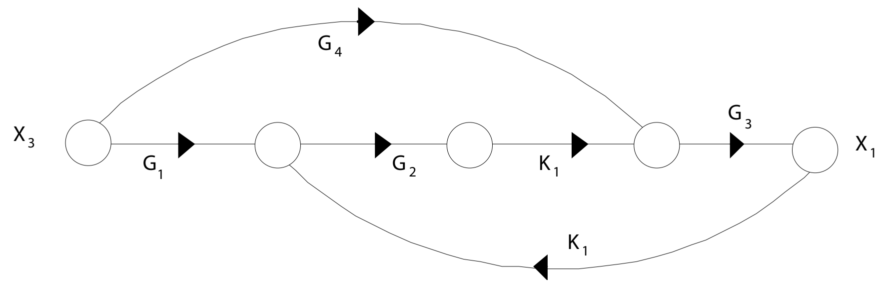


FIGURE P2.17
Signal flow graph.

(d) The transfer function from $X_3(s)$ to $X_1(s)$ is

$$\frac{X_1(s)}{X_3(s)} = \frac{K_1G_1(s)G_2(s)G_3(s) + G_4(s)G_3(s)}{1 - K_1^2G_2(s)G_3(s)} .$$

P2.18 The signal flow graph is shown in Figure P2.18.

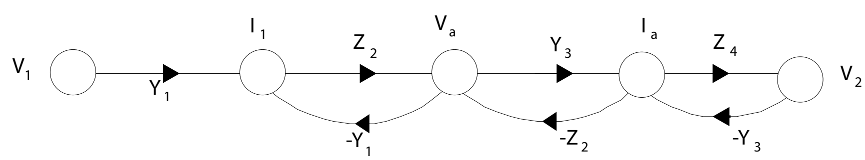


FIGURE P2.18
Signal flow graph.

The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{Y_1Z_2Y_3Z_4}{1 + Y_1Z_2 + Y_3Z_2 + Y_3Z_4 + Y_1Z_2Z_4Y_3} .$$

P2.19 For a noninerting op-amp circuit, depicted in Figure P2.19a, the voltage gain (as a function of frequency) is

$$V_o(s) = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}V_{in}(s),$$

where $Z_1(s)$ and $Z_2(s)$ are the impedances of the respective circuits. In the case of the voltage follower circuit, shown in Figure P2.19b, we have

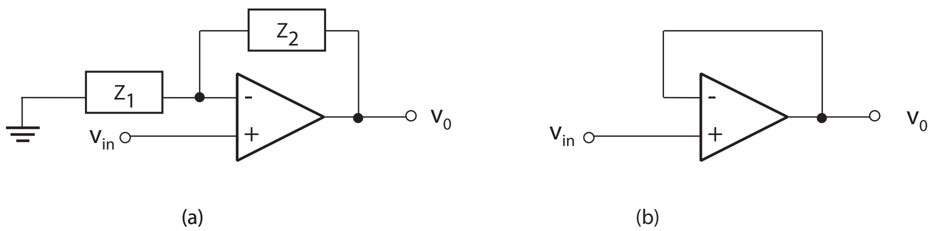


FIGURE P2.19
(a) Noninverting op-amp circuit. (b) Voltage follower circuit.

$Z_1 = \infty$ (open circuit) and $Z_2 = 0$. Therefore, the transfer function is

$$\frac{V_o(s)}{V_{in}(s)} = \frac{Z_1}{Z_1} = 1.$$

P2.20 (a) Assume $R_g \gg R_s$ and $R_s \gg R_1$. Then $R_s = R_1 + R_2 \approx R_2$, and

$$v_{gs} = v_{in} - v_o \ ,$$

where we neglect i_{in}, since $R_g \gg R_s$. At node S, we have

$$\frac{v_o}{R_s} = g_m v_{gs} = g_m(v_{in} - v_o) \quad \text{or} \quad \frac{v_o}{v_{in}} = \frac{g_m R_s}{1 + g_m R_s} \ .$$

(b) With $g_m R_s = 20$, we have

$$\frac{v_o}{v_{in}} = \frac{20}{21} = 0.95 \ .$$

(c) The block diagram is shown in Figure P2.20.

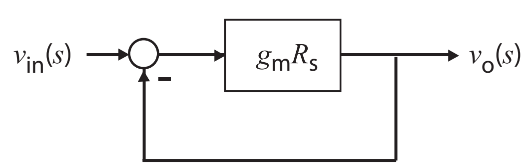


FIGURE P2.20
Block diagram model.

P2.21 From the geometry we find that

$$\Delta z = k\frac{l_1 - l_2}{l_1}(x - y) - \frac{l_2}{l_1}y \ .$$

The flow rate balance yields

$$A\frac{dy}{dt} = p\Delta z \quad \text{which implies} \quad Y(s) = \frac{p\Delta Z(s)}{As} .$$

By combining the above results it follows that

$$Y(s) = \frac{p}{As}\left[k\left(\frac{l_1 - l_2}{l_1}\right)(X(s) - Y(s)) - \frac{l_2}{l_1}Y(s)\right] .$$

Therefore, the signal flow graph is shown in Figure P2.21. Using Mason's

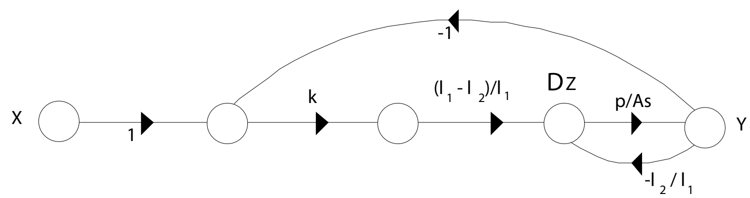


FIGURE P2.21
Signal flow graph.

gain formula we find that the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{\frac{k(l_1-l_2)p}{l_1 As}}{1 + \frac{l_2 p}{l_1 As} + \frac{k(l_1-l_2)p}{l_1 As}} = \frac{K_1}{s + K_2 + K_1} ,$$

where

$$K_1 = \frac{k(l_1 - l_2)p}{l_1 A}p \quad \text{and} \quad K_2 = \frac{l_2 p}{l_1 A} .$$

P2.22 (a) The equations of motion for the two masses are

$$ML^2\ddot{\theta}_1 + MgL\theta_1 + k\left(\frac{L}{2}\right)^2(\theta_1 - \theta_2) = \frac{L}{2}f(t)$$

$$ML^2\ddot{\theta}_2 + MgL\theta_2 + k\left(\frac{L}{2}\right)^2(\theta_2 - \theta_1) = 0 .$$

With $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_2 = \omega_2$, we have

$$\dot{\omega}_1 = -\left(\frac{g}{L} + \frac{k}{4M}\right)\theta_1 + \frac{k}{4M}\theta_2 + \frac{f(t)}{2ML}$$

$$\dot{\omega}_2 = \frac{k}{4M}\theta_1 - \left(\frac{g}{L} + \frac{k}{4M}\right)\theta_2 .$$

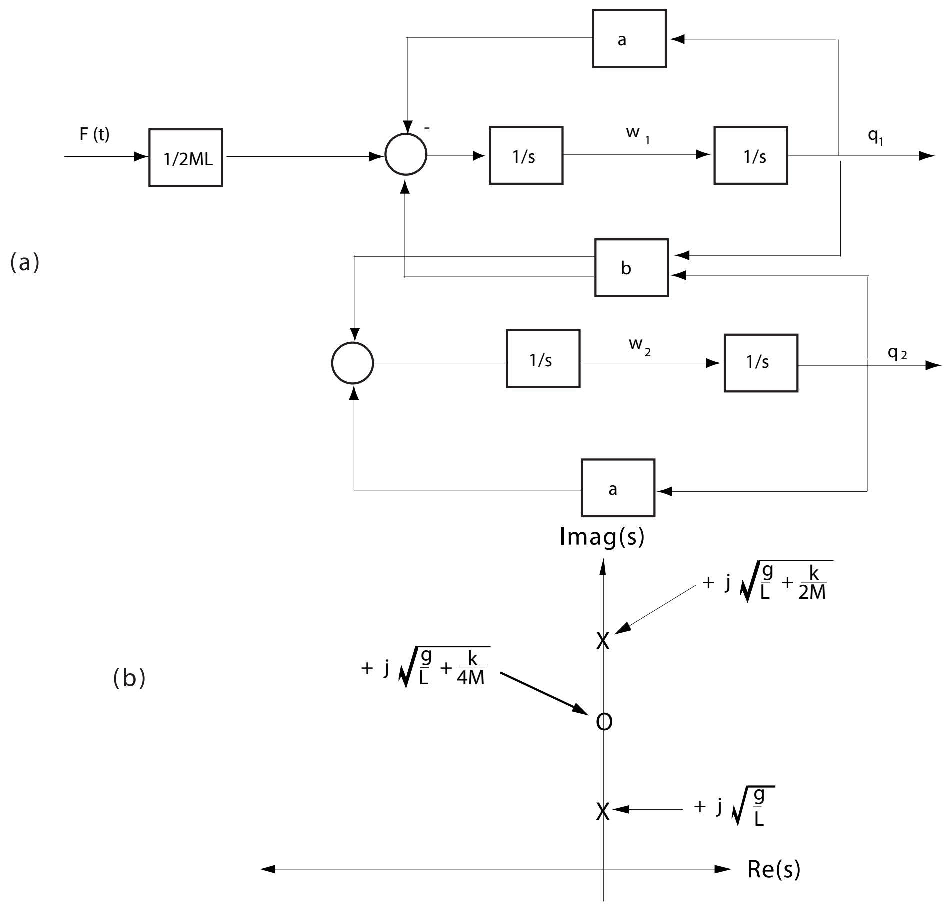


FIGURE P2.22
(a) Block diagram. (b) Pole-zero map.

(b) Define $a = g/L + k/4M$ and $b = k/4M$. Then

$$\frac{\theta_1(s)}{F(s)} = \frac{1}{2ML} \frac{s^2 + a}{(s^2 + a)^2 - b^2} \ .$$

(c) The block diagram and pole-zero map are shown in Figure P2.22.

P2.23 The input-output ratio, V_{ce}/V_{in}, is found to be

$$\frac{V_{ce}}{V_{in}} = \frac{\beta(R - 1) + h_{ie}R_f}{-\beta h_{re} + h_{ie}(-h_{oe} + R_f)} \ .$$

P2.24 (a) The voltage gain is given by

$$\frac{v_o}{v_{in}} = \frac{R_L\beta_1\beta_2(R_1 + R_2)}{(R_1 + R_2)(R_g + h_{ie1}) + R_1(R_1 + R_2)(1 + \beta_1) + R_1R_L\beta_1\beta_2} \ .$$

(b) The current gain is found to be

$$\frac{i_{c2}}{i_{b1}} = \beta_1\beta_2 \ .$$

(c) The input impedance is

$$\frac{v_{in}}{i_{b1}} = \frac{(R_1+R_2)(R_g+h_{ie1}) + R_1(R_1+R_2)(1+\beta_1) + R_1R_L\beta_1\beta_2}{R_1+R_2} \ ,$$

and when $\beta_1\beta_2$ is very large, we have the approximation

$$\frac{v_{in}}{i_{b1}} \approx \frac{R_LR_1\beta_1\beta_2}{R_1+R_2} \ .$$

P2.25 The transfer function from $R(s)$ and $T_d(s)$ to $Y(s)$ is given by

$$\begin{aligned} Y(s) &= G(s)\left(R(s) - \frac{1}{G(s)}\left(G(s)R(s)+T_d(s)\right)\right) + T_d(s) + G(s)R(s) \\ &= G(s)R(s) \ . \end{aligned}$$

Thus,

$$Y(s)/R(s) = G(s) \ .$$

Also, we have that

$$Y(s) = 0 \ .$$

when $R(s) = 0$. Therefore, the effect of the disturbance, $T_d(s)$, is eliminated.

P2.26 The equations of motion for the two mass model of the robot are

$$\begin{aligned} M\ddot{x} + b(\dot{x}-\dot{y}) + k(x-y) &= F(t) \\ m\ddot{y} + b(\dot{y}-\dot{x}) + k(y-x) &= 0 \ . \end{aligned}$$

Taking the Laplace transform and writing the result in matrix form yields

$$\begin{bmatrix} Ms^2+bs+k & -(bs+k) \\ -(bs+k) & ms^2+bs+k \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix} \ .$$

Solving for $Y(s)$ we find that

$$\frac{Y(s)}{F(s)} = \frac{\frac{1}{mM}(bs+k)}{s^2[s^2 + (1+\frac{m}{M})\left(\frac{b}{m}s+\frac{k}{m}\right)]} \ .$$

P2.27 The describing equation of motion is

$$m\ddot{z} = mg - k\frac{i^2}{z^2} \ .$$

Defining

$$f(z,i) = g - \frac{ki^2}{mz^2}$$

leads to

$$\ddot{z} = f(z,i) \ .$$

The equilibrium condition for i_o and z_o, found by solving the equation of motion when

$$\dot{z} = \ddot{z} = 0 \ ,$$

is

$$\frac{ki_o^2}{mg} = z_o^2 \ .$$

We linearize the equation of motion using a Taylor series approximation. With the definitions

$$\Delta z = z - z_o \quad \text{and} \quad \Delta i = i - i_o \ ,$$

we have $\dot{\Delta} z = \dot{z}$ and $\ddot{\Delta} z = \ddot{z}$. Therefore,

$$\ddot{\Delta} z = f(z,i) = f(z_o,i_o) + \left.\frac{\partial f}{\partial z}\right|_{\substack{z=z_o \\ i=i_o}} \Delta z + \left.\frac{\partial f}{\partial i}\right|_{\substack{z=z_o \\ i=i_o}} \Delta i + \cdots$$

But $f(z_o,i_o) = 0$, and neglecting higher-order terms in the expansion yields

$$\ddot{\Delta} z = \frac{2ki_o^2}{mz_o^3}\Delta z - \frac{2ki_o}{mz_o^2}\Delta i \ .$$

Using the equilibrium condition which relates z_o to i_o, we determine that

$$\ddot{\Delta} z = \frac{2g}{z_o}\Delta z - \frac{g}{i_o}\Delta i \ .$$

Taking the Laplace transform yields the transfer function (valid around the equilibrium point)

$$\frac{\Delta Z(s)}{\Delta I(s)} = \frac{-g/i_o}{s^2 - 2g/z_o} \ .$$

P2.28 The signal flow graph is shown in Figure P2.28.

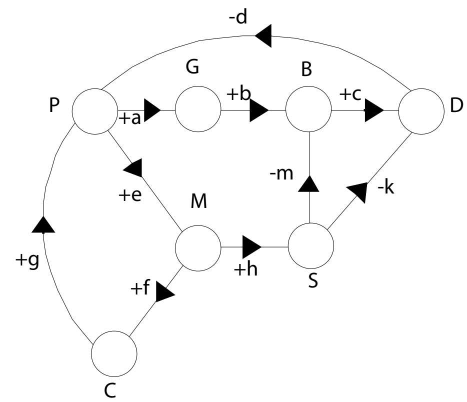


FIGURE P2.28
Signal flow graph.

(a) The PGBDP loop gain is equal to -abcd. This is a **negative** transmission since the population produces garbage which increases bacteria and leads to diseases, thus reducing the population.

(b) The PMCP loop gain is equal to +efg. This is a **positive** transmission since the population leads to modernization which encourages immigration, thus increasing the population.

(c) The PMSDP loop gain is equal to +ehkd. This is a **positive** transmission since the population leads to modernization and an increase in sanitation facilities which reduces diseases, thus reducing the rate of decreasing population.

(d) The PMSBDP loop gain is equal to +ehmcd. This is a **positive** transmission by similar argument as in (3).

P2.29 Assume the motor torque is proportional to the input current

$$T_m = ki \ .$$

Then, the equation of motion of the beam is

$$J\ddot{\phi} = ki \ ,$$

where J is the moment of inertia of the beam and shaft (neglecting the inertia of the ball). We assume that forces acting on the ball are due to gravity and friction. Hence, the motion of the ball is described by

$$m\ddot{x} = mg\phi - b\dot{x}$$

where m is the mass of the ball, b is the coefficient of friction, and we have assumed small angles, so that $\sin\phi \approx \phi$. Taking the Laplace transfor of both equations of motion and solving for $X(s)$ yields

$$X(s)/I(s) = \frac{gk/J}{s^2(s^2+b/m)} .$$

P2.30 Given

$$H(s) = \frac{k}{\tau s+1}$$

where $\tau = 4\mu s = 4 \times 10^{-6}$ seconds and $0.999 \leq k < 1.001$. The step response is

$$Y(s) = \frac{k}{\tau s+1} \cdot \frac{1}{s} = \frac{k}{s} - \frac{k}{s+1/\tau} .$$

Taking the inverse Laplace transform yields

$$y(t) = k - ke^{-t/\tau} = k(1-e^{-t/\tau}) .$$

The final value is k. The time it takes to reach 98% of the final value is $t = 15.6\mu s$ independent of k.

P2.31 From the block diagram we have

$$\begin{aligned} Y_1(s) &= G_2(s)[G_1(s)E_1(s) + G_3(s)E_2(s)] \\ &= G_2(s)G_1(s)[R_1(s) - H_1(s)Y_1(s)] + G_2(s)G_3(s)E_2(s) . \end{aligned}$$

Therefore,

$$Y_1(s) = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)}R_1(s) + \frac{G_2(s)G_3(s)}{1+G_1(s)G_2(s)H_1(s)}E_2(s) .$$

And, computing $E_2(s)$ (with $R_2(s) = 0$) we find

$$E_2(s) = H_2(s)Y_2(s) = H_2(s)G_6(s)\left[\frac{G_4(s)}{G_2(s)}Y_1(s) + G_5(s)E_2(s)\right]$$

or

$$E_2(s) = \frac{G_4(s)G_6(s)H_2(s)}{G_2(s)(1-G_5(s)G_6(s)H_2(s))}Y_1(s) .$$

Substituting $E_2(s)$ into equation for $Y_1(s)$ yields

$$Y_1(s) = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H_1(s)}R_1(s)$$

$$+ \frac{G_3(s)G_4(s)G_6(s)H_2(s)}{(1+G_1(s)G_2(s)H_1(s))(1-G_5(s)G_6(s)H_2(s))} Y_1(s) \ .$$

Finally, solving for $Y_1(s)$ yields

$$Y_1(s) = T_1(s)R_1(s)$$

where

$$T_1(s) = \left[\frac{G_1(s)G_2(s)(1-G_5(s)G_6(s)H_2(s))}{(1+G_1(s)G_2(s)H_1(s))(1-G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)}\right] \ .$$

Similarly, for $Y_2(s)$ we obtain

$$Y_2(s) = T_2(s)R_1(s) \ .$$

where

$$T_2(s) = \left[\frac{G_1(s)G_4(s)G_6(s)}{(1+G_1(s)G_2(s)H_1(s))(1-G_5(s)G_6(s)H_2(s)) - G_3(s)G_4(s)G_6(s)H_2(s)}\right] \ .$$

P2.32 The signal flow graph shows three loops:

$$L_1 = -G_1G_3G_4H_2$$
$$L_2 = -G_2G_5G_6H_1$$
$$L_3 = -H_1G_8G_6G_2G_7G_4H_2G_1 \ .$$

The transfer function Y_2/R_1 is found to be

$$\frac{Y_2(s)}{R_1(s)} = \frac{G_1G_8G_6\Delta_1 - G_2G_5G_6\Delta_2}{1-(L_1+L_2+L_3)+(L_1L_2)} \ ,$$

where for path 1

$$\Delta_1 = 1$$

and for path 2

$$\Delta_2 = 1 - L_1 \ .$$

Since we want Y_2 to be independent of R_1, we need $Y_2/R_1 = 0$. Therefore, we require

$$G_1G_8G_6 - G_2G_5G_6(1+G_1G_3G_4H_2) = 0 \ .$$

P2.33 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_3(s)G_1(s)(G_2(s)+K_5K_6)}{1-G_3(s)(H_1(s)+K_6)+G_3(s)G_1(s)(G_2(s)+K_5K_6)(H_2(s)+K_4)} .$$

P2.34 The equations of motion are

$$m_1\ddot{y}_1 + b(\dot{y}_1-\dot{y}_2) + k_1(y_1-y_2) = 0$$
$$m_2\ddot{y}_2 + b(\dot{y}_2-\dot{y}_1) + k_1(y_2-y_1) + k_2y_2 = k_2x$$

Taking the Laplace transform yields

$$(m_1s^2+bs+k_1)Y_1(s) - (bs+k_1)Y_2(s) = 0$$
$$(m_2s^2+bs+k_1+k_2)Y_2(s) - (bs+k_1)Y_1(s) = k_2X(s)$$

Therefore, after solving for $Y_1(s)/X(s)$, we have

$$\frac{Y_2(s)}{X(s)} = \frac{k_2(bs+k_1)}{(m_1s^2+bs+k_1)(m_2s^2+bs+k_1+k_2)-(bs+k_1)^2} .$$

P2.35 (a) We can redraw the block diagram as shown in Figure P2.35. Then,

$$T(s) = \frac{K_1/s(s+1)}{1+K_1(1+K_2s)/s(s+1)} = \frac{K_1}{s^2+(1+K_2K_1)s+K_2} .$$

(b) The signal flow graph reveals two loops (both touching):

$$L_1 = \frac{-K_1}{s(s+1)} \quad \text{and} \quad L_2 = \frac{-K_1K_2}{s+1} .$$

Therefore,

$$T(s) = \frac{K_1/s(s+1)}{1+K_1/s(s+1)+K_1K_2/(s+1)} = \frac{K_1}{s^2+(1+K_2K_1)s+K_1} .$$

(c) We want to choose K_1 and K_2 such that

$$s^2+(1+K_2K_1)s+K_1 = s^2+20s+100 = (s+10)^2 .$$

Therefore, $K_1 = 100$ and $1+K_2K_1 = 20$ or $K_2 = 0.19$.

(d) The step response is shown in Figure P2.35.

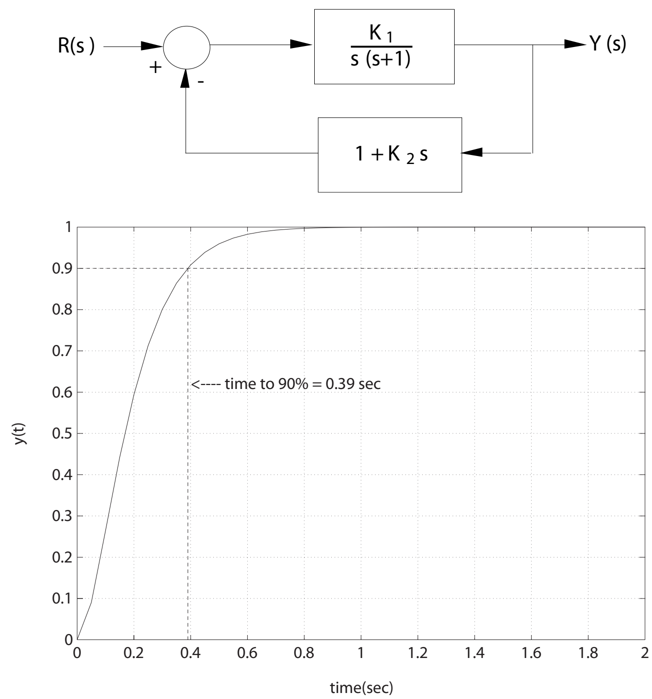


FIGURE P2.35
The equivalent block diagram and the system step response.

P2.36 (a) Given $R(s) = 1/s^2$, the partial fraction expansion is

$$Y(s) = \frac{24}{s^2(s+2)(s+3)(s+4)} = \frac{3}{s+2} - \frac{8/3}{s+3} + \frac{3/4}{s+4} + \frac{1}{s^2} - \frac{13/12}{s} .$$

Therefore, using the Laplace transform table, we determine that the ramp response is

$$y(t) = 3e^{-2t} - \frac{8}{3}e^{-3t} + \frac{3}{4}e^{-4t} + t - \frac{13}{12} , \quad t \geq 0 .$$

(b) For the ramp input, $y(t) \approx 0.21$ at $t = 1.$ second (see Figure P2.36a).

(c) Given $R(s) = 1$, the partial fraction expansion is

$$Y(s) = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4} .$$

Therefore, using the Laplace transform table, we determine that the impulse response is

$$y(t) = 12e^{-2t} - 24e^{-3t} + 412e^{-4t} , \quad t \geq 0 .$$

(d) For the impulse input, $y(t) \approx 0.65$ at $t = 1$ seconds (see Figure P2.36b).

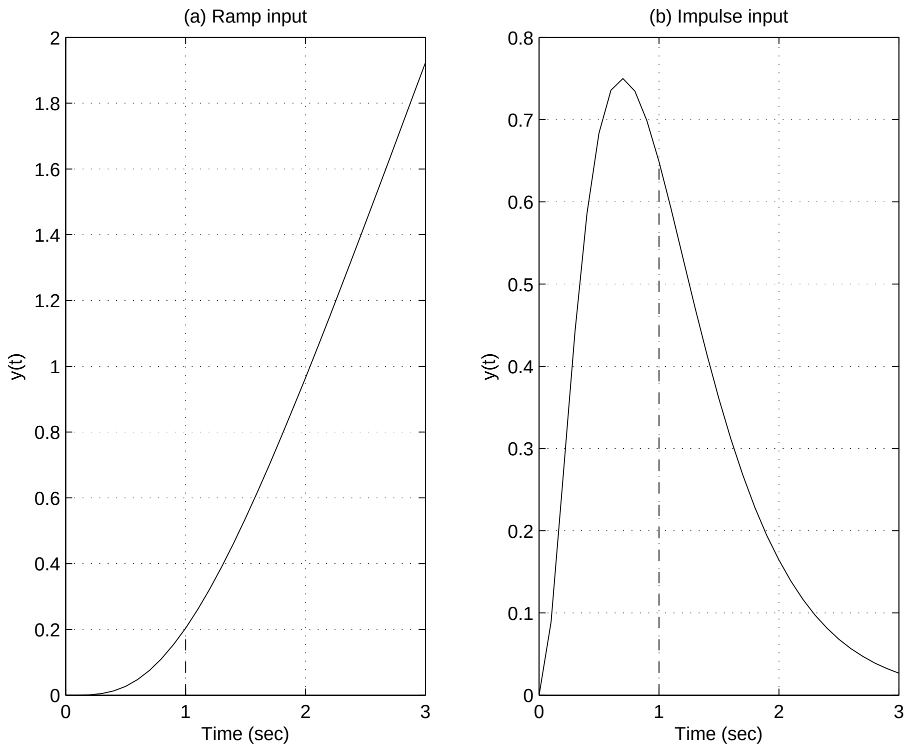


FIGURE P2.36
(a) Ramp input response. (b) Impulse input response.

P2.37 The equations of motion are

$$m_1 \frac{d^2x}{dt^2} = -(k_1 + k_2)x + k_2 y \quad \text{and} \quad m_2 \frac{d^2y}{dt^2} = k_2(x - y) + u \ .$$

When $m_1 = m_2 = 1$ and $k_1 = k_2 = 1$, we have

$$\frac{d^2x}{dt^2} = -2x + y \quad \text{and} \quad \frac{d^2y}{dt^2} = x - y + u \ .$$

P2.38 The equation of motion for the system is

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\theta = 0 \ ,$$

where k is the rotational spring constant and b is the viscous friction coefficient. The initial conditions are $\theta(0) = \theta_o$ and $\dot{\theta}(0) = 0$. Taking the

Laplace transform yields

$$J(s^2\theta(s) - s\theta_o) + b(s\theta(s) - \theta_o) + k\theta(s) = 0 \ .$$

Therefore,

$$\theta(s) = \frac{(s + \frac{b}{J}\theta_o)}{(s^2 + \frac{b}{J}s + \frac{K}{J})} = \frac{(s + 2\zeta\omega_n)\theta_o}{s^2 + 2\zeta\omega_n s + \omega_n^2} \ .$$

Neglecting the mass of the rod, the moment of inertia is detemined to be

$$J = 2Mr^2 = 0.5 \text{ kg} \cdot \text{m}^2 \ .$$

Also,

$$\omega_n = \sqrt{\frac{k}{J}} = 0.02 \text{ rad/s} \quad \text{and} \quad \zeta = \frac{b}{2J\omega_n} = 0.01 \ .$$

Solving for $\theta(t)$, we find that

$$\theta(t) = \frac{\theta_o}{\sqrt{1-\zeta^2}} \, e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} \ t + \phi) \ ,$$

where $\tan\phi = \sqrt{1-\zeta^2}/\zeta)$. Therefore, the envelope decay is

$$\theta_e = \frac{\theta_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \ .$$

So, with $\zeta\omega_n = 2 \times 10^{-4}$, $\theta_o = 4000^o$ and $\theta_f = 10^o$, the elapsed time is computed as

$$t = \frac{1}{\zeta\omega_n} ln \frac{\theta_o}{\sqrt{1-\zeta^2}\theta_f} = 8.32 \text{ hours} \ .$$

P2.39 When $t < 0$, we have the steady-state conditions

$$i_1(0) = 1A \ , \quad v_a(0) = 2V \quad \text{and} \quad v_c(0) = 5V \ ,$$

where $v_c(0)$ is associated with the 1F capacitor. After $t \geq 0$, we have

$$2\frac{di_1}{dt} + 2i_1 + 4(i_1 - i_2) = 10e^{-2t}$$

and

$$\int i_2 dt + 10i_2 + 4(i_2 - i_1) - i_1 = 0 \ .$$

Taking the Laplace transform (using the initial conditions) yields

$$2(sI_1 - i_1(0)) + 2I_1 + 4I_1 - 4I_2 = \frac{10}{s+2} \quad \text{or} \quad (s+3)I_1(s) - 2I_2(s) = \frac{s+7}{s+2}$$

and

$$[\frac{1}{s}I_2 - v_c(0)] + 10I_2 + 4(I_2 - I_1) = I_1(s) \quad \text{or} \quad -5sI_1(s) + (14s+1)I_2(s) = 5s \ .$$

Solving for $I_2(s)$ yields

$$I_2 = \frac{5s(s^2+6s+13)}{14(s+2)\Delta(s)} \ ,$$

where

$$\Delta(s) = \begin{vmatrix} s+3 & -2 \\ -5s & 14s+1 \end{vmatrix} = 14s^2 + 33s + 3 \ .$$

Then,

$$V_o(s) = 10I_2(s) \ .$$

P2.40 The equations of motion are

$$J_1\ddot{\theta}_1 = K(\theta_2 - \theta_1) - b(\dot{\theta}_1 - \dot{\theta}_2) + T \quad \text{and} \quad J_2\ddot{\theta}_2 = b(\dot{\theta}_1 - \dot{\theta}_2) \ .$$

Taking the Laplace transform yields

$$(J_1s^2 + bs + K)\theta_1(s) - bs\theta_2(s) = K\theta_2(s) + T(s)$$

and

$$(J_2s^2 + bs)\theta_2(s) - bs\theta_1(s) = 0 \ .$$

Solving for $\theta_1(s)$ and $\theta_2(s)$, we find that

$$\theta_1(s) = \frac{(K\theta_2(s) + T(s))(J_2s + b)}{\Delta(s)} \quad \text{and} \quad \theta_2(s) = \frac{b(K\theta_2(s) + T(s))}{\Delta(s)} \ ,$$

where

$$\Delta(s) = J_1J_2s^3 + b(J_1 + J_2)s^2 + J_2Ks + bK \ .$$

P2.41 Assume that the only external torques acting on the rocket are control torques, T_c and disturbance torques, T_d, and assume small angles, $\theta(t)$. Using the small angle approximation, we have

$$\dot{h} = V\theta$$

$$J\ddot{\theta} = T_c + T_d \ ,$$

where J is the moment of inertia of the rocket and V is the rocket velocity (assumed constant). Now, suppose that the control torque is proportional to the lateral displacement, as

$$T_c(s) = -KH(s) \ ,$$

where the negative sign denotes a negative feedback system. The corresponding block diagram is shown in Figure P2.41.

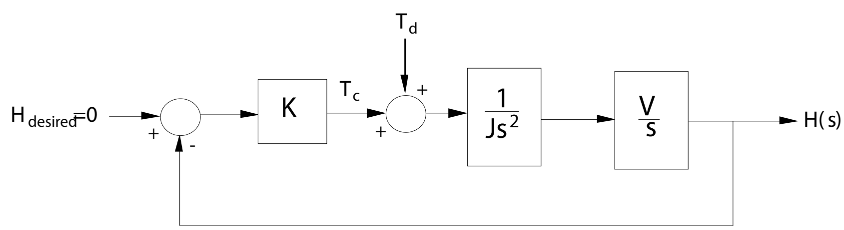


FIGURE P2.41
Block diagram.

P2.42 (a) The equation of motion of the motor is

$$J\frac{d\omega}{dt} = T_m - b\omega \ ,$$

where $J = 0.1$, $b = 0.06$, and T_m is the motor input torque.

(b) Given $T_m(s) = 1/s$, and $\omega(0) = 0.7$, we take the Laplace transform of the equation of motion yielding

$$s\omega(s) - \omega(0) + 0.6\omega(s) = 10T_m$$

or

$$\omega(s) = \frac{0.7s + 10}{s(s + 0.6)} \ .$$

Then, computing the partial fraction expansion, we find that

$$\omega(s) = \frac{A}{s} + \frac{B}{s + 0.6} = \frac{16.67}{s} - \frac{15.97}{s + 0.6} \ .$$

The step response, determined by taking the inverse Laplace transform, is

$$\omega(t) = 16.67 - 15.97e^{-0.6t} \ , \quad t \geq 0 \ .$$

P2.43 The work done by each gear is equal to that of the other, therefore

$$T_m\theta_m = T_L\theta_L \ .$$

Also, the travel distance is the same for each gear, so

$$r_1\theta_m = r_2\theta_L \ .$$

The number of teeth on each gear is proportional to the radius, or

$$r_1 N_2 = r_2 N_1 \ .$$

So,

$$\frac{\theta_m}{\theta_L} = \frac{r_2}{r_1} = \frac{N_2}{N_1} \ ,$$

and

$$\begin{aligned} N_1\theta_m &= N_2\theta_L \\ \theta_L &= \frac{N_1}{N_2}\theta_m = n\theta_m \ , \end{aligned}$$

where

$$n = N_1/N_2 \ .$$

Finally,

$$\frac{T_m}{T_L} = \frac{\theta_L}{\theta_m} = \frac{N_1}{N_2} = n \ .$$

P2.44 The inertia of the load is

$$J_L = \frac{\pi\rho L r^4}{2} \ .$$

Also, from the dynamics we have

$$T_2 = J_L\dot{\omega}_2 + b_L\omega_2$$

and

$$T_1 = nT_2 = n(J_L\dot{\omega}_2 + b_L\omega_2) \ .$$

So,

$$T_1 = n^2(J_L\dot{\omega}_1 + b_L\omega_1) \ ,$$

since

$$\omega_2 = n\omega_1 \ .$$

Therefore, the torque at the motor shaft is

$$T = T_1 + T_m = n^2(J_L\dot{\omega}_1 + b_L\omega_1) + J_m\dot{\omega}_1 + b_m\omega_1 \ .$$

P2.45 Let $U(s)$ denote the human input and $F(s)$ the load input. The transfer function is

$$P(s) = \frac{G(s) + KG_1(s)}{\Delta(s)}U(s) + \frac{G_c(s) + KG_1(s)}{\Delta(s)}F(s) \ ,$$

where

$$\Delta = 1 + GH(s) + G_1KBH(s) + G_cE(s) + G_1KE(s) \ .$$

P2.46 Consider the application of Newton's law ($\sum F = m\ddot{x}$). From the mass m_v we obtain

$$m_v\ddot{x}_1 = F - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_1(s)$ yields

$$X_1(s) = \frac{1}{\Delta_1(s)}F(s) + \frac{b_1s + k_1}{\Delta_1(s)}X_2(s),$$

where

$$\Delta_1 := m_vs^2 + b_1s + k_1.$$

From the mass m_t we obtain

$$m_t\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2).$$

Taking the Laplace transform, and solving for $X_2(s)$ yields

$$X_2(s) = \frac{b_1s + k_1}{\Delta_2(s)}X_1(s),$$

where

$$\Delta_2 := m_ts^2 + (b_1 + b_2)s + k_1 + k_2.$$

Substituting $X_2(s)$ above into the relationship fpr $X_1(s)$ yields the transfer function

$$\frac{X_1(s)}{F(s)} = \frac{\Delta_2(s)}{\Delta_1(s)\Delta_2(s) - (b_1s + k_1)^2}.$$

P2.47 Using the following relationships

$$\begin{aligned} h(t) &= \int (1.6\theta(t) - h(t))dt \\ \omega(t) &= \dot{\theta}(t) \\ J\dot{\omega}(t) &= K_m i_a(t) \\ v_a(t) = 50v_i(t) &= 10i_a(t) + v_b(t) \\ \dot{\theta} &= K v_b \end{aligned}$$

we find the differential equation is

$$\frac{d^3h}{dt^3} + \left(1 + \frac{K_m}{10JK}\right)\frac{d^2h}{dt^2} + \frac{K_m}{10JK}\frac{dh}{dt} = \frac{8K_m}{J}v_i \, .$$

P2.48 (a) The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + sR_1C_1)(1 + sR_2C_2)}{R_1C_2s} \, .$$

(b) When $R_1 = 100\ k\Omega$, $R_2 = 200\ k\Omega$, $C_1 = 1\ \mu F$ and $C_2 = 0.1\ \mu F$, we have

$$\frac{V_2(s)}{V_1(s)} = \frac{0.2(s + 10)(s + 50)}{s} \, .$$

P2.49 (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{6205}{s^3 + 13s^2 + 1281s + 6205} \, .$$

(b) The poles of $T(s)$ are $s_1 = -5$ and $s_{2,3} = -4 \pm j35$.

(c) The partial fraction expansion (with a step input) is

$$Y(s) = 1 - \frac{1.0122}{s + 5} + \frac{0.0061 + 0.0716j}{s + 4 + j35} + \frac{0.0061 - 0.0716j}{s + 4 - j35} \, .$$

(d) The step response is shown in Figure P2.49. The real and complex roots are close together and by looking at the poles in the s-plane we have difficulty deciding which is dominant. However, the residue at the real pole is much larger and thus dominates the response.

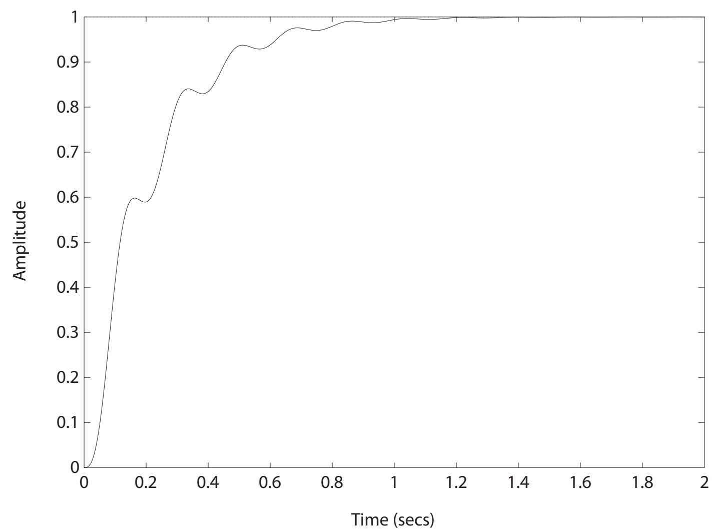


FIGURE P2.49
Step response.

P2.50 (a) The closed-loop transfer function is

$$T(s) = \frac{14000}{s^3 + 45s^2 + 3100s + 14500} .$$

(b) The poles of $T(s)$ are

$$s_1 = -5 \quad \text{and} \quad s_{2,3} = -20 \pm j50.$$

(c) The partial fraction expansion (with a step input) is

$$Y(s) = \frac{0.9655}{s} - \frac{1.0275}{s+5} + \frac{0.0310 - 0.0390j}{s + 20 + j50} + \frac{0.0310 + 0.0390j}{s + 20 - j50} .$$

(d) The step response is shown in Figure P2.50. The real root dominates the response.

(e) The final value of $y(t)$ is

$$y_{ss} = \lim_{s \to 0} sY(s) = 0.9655 .$$

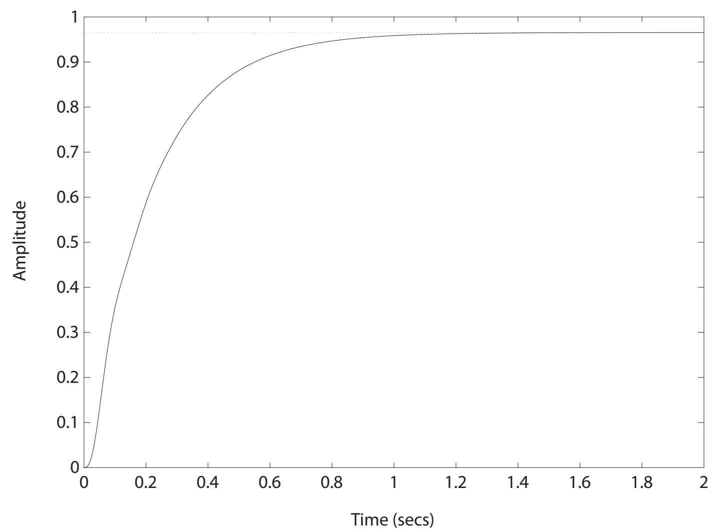


FIGURE P2.50
Step response.

P2.51 Consider the free body diagram in Figure P2.51. Using Newton's Law and summing the forces on the two masses yields

$$M_1\ddot{x}(t) + b_1\dot{x}(t) + k_1x(t) = b_1\dot{y}(t)$$
$$M_2\ddot{y}(t) + b_1\dot{y}(t) + k_2y(t) = b_1\dot{x}(t) + u(t)$$

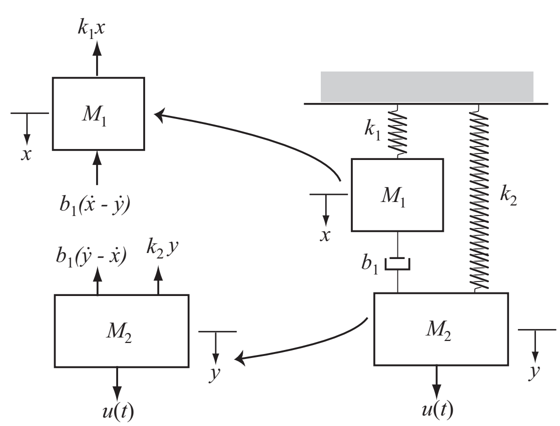


FIGURE P2.51
Free body diagram.

Advanced Problems

AP2.1 The transfer function from $V(s)$ to $\omega(s)$ has the form

$$\frac{\omega(s)}{V(s)} = \frac{K_m}{\tau_m s + 1} .$$

In the steady-state,

$$\omega_{ss} = \lim_{s \to 0} s \left[\frac{K_m}{\tau_m s + 1} \right] \frac{5}{s} = 5K_m .$$

So,

$$K_m = 70/5 = 14 .$$

Also,

$$\omega(t) = V_m K_m (1 - e^{-t/\tau_m})$$

where $V(s) = V_m/s$. Solving for τ_m yields

$$\tau_m = \frac{-t}{\ln(1 - \omega(t)/\omega_{ss})} .$$

When $t = 2$, we have

$$\tau_m = \frac{-2}{\ln(1 - 30/70)} = 3.57 .$$

Therefore, the transfer function is

$$\frac{\omega(s)}{V(s)} = \frac{14}{3.57s + 1} .$$

AP2.2 The closed-loop transfer function form $R_1(s)$ to $Y_2(s)$ is

$$\frac{Y_2(s)}{R_1(s)} = \frac{G_1 G_4 G_5(s) + G_1 G_2 G_3 G_4 G_6(s)}{\Delta}$$

where

$$\Delta = [1 + G_3 G_4 H_2(s)][1 + G_1 G_2 H_3(s)] .$$

If we select

$$G_5(s) = -G_2 G_3 G_6(s)$$

then the numerator is zero, and $Y_2(s)/R_1(s) = 0$. The system is now decoupled.

AP2.3 (a) Computing the closed-loop transfer function:

$$Y(s) = \left[\frac{G(s)G_c(s)}{1 + G_c(s)G(s)H(s)}\right] R(s) \ .$$

Then, with $E(s) = R(s) - Y(s)$ we obtain

$$E(s) = \left[\frac{1 + G_c(s)G(s)(H(s) - 1)}{1 + G_c(s)G(s)H(s)}\right] R(s) \ .$$

If we require that $E(s) \equiv 0$ for any input, we need $1 + G_c(s)G(s)(H(s) - 1) = 0$ or

$$H(s) = \frac{G_c(s)G(s) - 1}{G_c(s)G(s)} = \frac{n(s)}{d(s)} \ .$$

Since we require $H(s)$ to be a causal system, the order of the numerator polynomial, $n(s)$, must be less than or equal to the order of the denominator polynomial, $d(s)$. This will be true, in general, only if both $G_c(s)$ and $G(s)$ are proper rational functions (that is, the numerator and denominator polynomials have the same order). Therefore, making $E \equiv 0$ for any input $R(s)$ is possible only in certain circumstances.
(b) The transfer function from $T_d(s)$ to $Y(s)$ is

$$Y(s) = \left[\frac{G_d(s)G(s)}{1 + G_c(s)G(s)H(s)}\right] T_d(s) \ .$$

With $H(s)$ as in part (a) we have

$$Y(s) = \left[\frac{G_d(s)}{G_c(s)}\right] T_d(s) \ .$$

(c) No. Since

$$Y(s) = \left[\frac{G_d(s)G(s)}{1 + G_c(s)G(s)H(s)}\right] T_d(s) = T(s)T_d(s) \ ,$$

the only way to have $Y(s) \equiv 0$ for any $T_d(s)$ is for the transfer function $T(s) \equiv 0$ which is not possible in general (since $G(s) \neq 0$).

AP2.4 (a) With $q(s) = 1/s$ we obtain

$$\tau(s) = \frac{1/C_t}{s + \frac{QS+1/R}{C_t}} \cdot \frac{1}{s} \ .$$

Define

$$\alpha := \frac{QS + 1/R}{C_t} \quad \text{and} \quad \beta := 1/\mathrm{C_t} \ .$$

Then, it follows that

$$\tau(s) = \frac{\beta}{s+\alpha} \cdot \frac{1}{s} = \frac{-\beta/\alpha}{s+\alpha} + \frac{\beta/\alpha}{s} .$$

Taking the inverse Laplace transform yields

$$\tau(t) = \frac{-\beta}{\alpha} e^{-\alpha t} + \frac{\beta}{\alpha} = \frac{\beta}{\alpha}[1 - e^{-\alpha t}] .$$

(b) As $t \to \infty$, $\tau(t) \to \frac{\beta}{\alpha} = \frac{1}{Qs+1/R}$.

(c) To increase the speed of response, you want to choose C_t, Q, S and R such that

$$\alpha := \frac{Qs + 1/R}{C_t}$$

is "large."

AP2.5 Considering the motion of each mass, we have

$$\begin{aligned} M_3\ddot{x}_3 + b_3\dot{x}_3 + k_3x_3 &= u_3 + b_3\dot{x}_2 + k_3x_2 \\ M_2\ddot{x}_2 + (b_2+b_3)\dot{x}_2 + (k_2+k_3)x_2 &= u_2 + b_3\dot{x}_3 + k_3x_3 + b_2\dot{x}_1 + k_2x_1 \\ M_1\ddot{x}_1 + (b_1+b_2)\dot{x}_1 + (k_1+k_2)x_1 &= u_1 + b_2\dot{x}_2 + k_2x_2 \end{aligned}$$

In matrix form the three equations can be written as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{bmatrix} b_1+b_2 & -b_2 & 0 \\ -b_2 & b_2+b_3 & -b_3 \\ 0 & -b_3 & b_3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$
$$+ \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} .$$

AP2.6 Considering the cart mass and using Newton's Law we obtain

$$M\ddot{x} = u - b\dot{x} - F\sin\varphi$$

where F is the reaction force between the cart and the pendulum. Considering the pendulum we obtain

$$m\frac{d^2(x + L\sin\varphi)}{dt^2} = F\sin\varphi$$

$$m\frac{d^2(L\cos\varphi)}{dt^2} = F\cos\varphi + mg$$

Eliminating the reaction force F yields the two equations

$$(m+M)\ddot{x} + b\dot{x} + mL\ddot{\varphi}\cos\varphi - mL\dot{\varphi}^2\sin\varphi = u$$
$$mL^2\ddot{\varphi} + mgL\sin\varphi + mL\ddot{x}\cos\varphi = 0$$

If we assume that the angle $\varphi \approx 0$, then we have the linear model

$$(m+M)\ddot{x} + b\dot{x} + mL\ddot{\varphi} = u$$
$$mL^2\ddot{\varphi} + mgL\varphi = -mL\ddot{x}$$

AP2.7 The transfer function from the disturbance input to the output is

$$Y(s) = \frac{1}{s+20+K}T_d(s) \ .$$

When $T_d(s) = 1$, we obtain

$$y(t) = e^{-(20+K)t} \ .$$

Solving for t when $y(t) < 0.1$ yields

$$t > \frac{2.3}{20+K} \ .$$

When $t = 0.05$ and $y(0.05) = 0.1$, we find $K = 26.05$.

AP2.8 The closed-loop transfer function is

$$T(s) = \frac{200K(0.25s+1)}{(0.25s+1)(s+1)(s+8)+200K}$$

The final value due to a step input of $R(s) = A/s$ is

$$v(t) \to A\frac{200K}{200K+8}.$$

We need to select K so that $v(t) \to 50$. However, to keep the percent overshoot to less than 10%, we need to limit the magnitude of K. Figure AP2.8a shows the percent overshoot as a function of K. Let $K = 0.06$ and select the magnitude of the input to be $A = 83.3$. The inverse Laplace transform of the closed-loop response with $R(s) = 83.3/s$ is

$$v(t) = 50 + 9.85e^{-9.15t} - e^{-1.93t}\left(59.85\cos(2.24t) + 11.27\sin(2.24t)\right)$$

The result is $P.O. = 9.74\%$ and the steady-state value of the output is approximately 50 m/s, as shown in Figure AP2.8b.

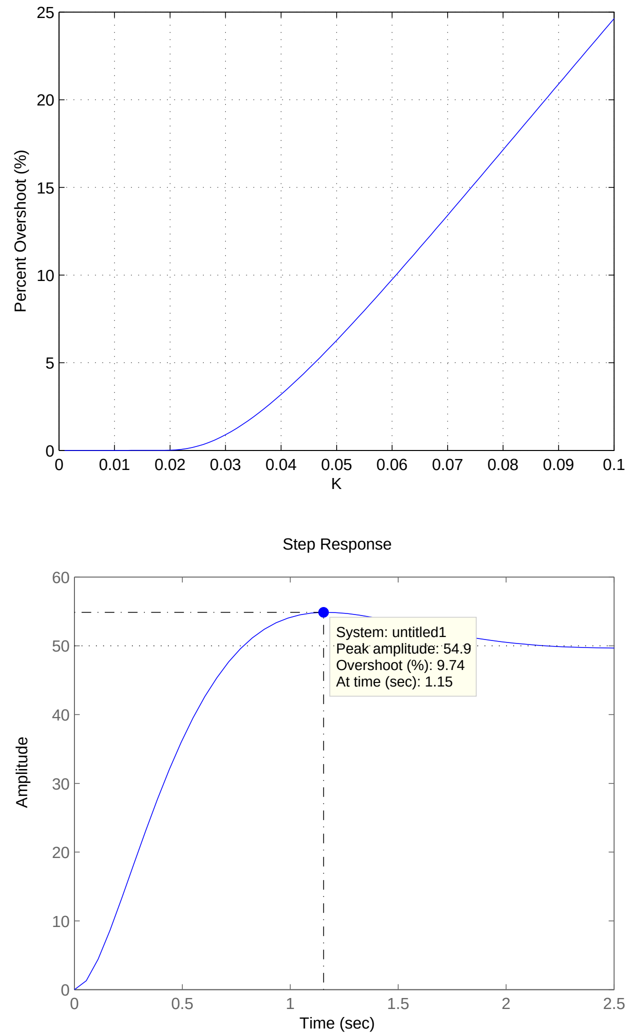


FIGURE AP2.8
(a) Percent overshoot versus the gain K. (b) Step response.

AP2.9 The transfer function is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \ ,$$

where

$$Z_1(s) = \frac{R_1}{R_1C_1s+1} \quad \text{and} \quad Z_2(s) = \frac{R_2C_2s+1}{C_2s} \ .$$

Then we can write

$$\frac{V_o(s)}{V_i(s)} = K_p + \frac{K_I}{s} + K_Ds$$

where

$$K_P = -\left(\frac{R_1C_1}{R_2C_2}+1\right), \quad K_I = -\frac{1}{R_1C_2}, \quad K_D = -R_2C_1 \ .$$

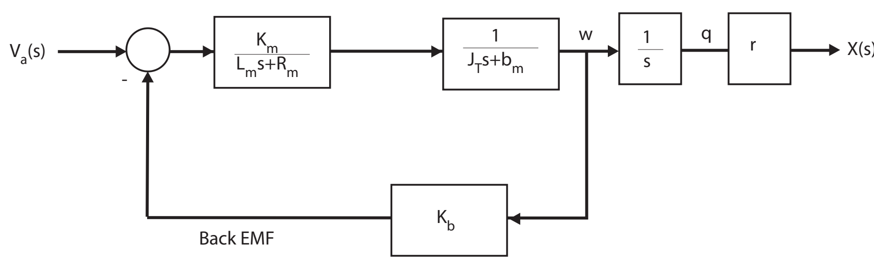
Design Problems

CDP2.1 The model of the traction drive, capstan roller, and linear slide follows closely the armature-controlled dc motor model depicted in Figure 2.18 in Dorf and Bishop. The transfer function is

$$T(s) = \frac{rK_m}{s\left[(L_m s + R_m)(J_T s + b_m) + K_b K_m\right]} ,$$

where

$$J_T = J_m + r^2(M_s + M_b) .$$



DP2.1 The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)H_1(s) - G_2(s)H_2(s)} .$$

When $G_1 H_1 = G_2 H_2$ and $G_1 G_2 = 1$, then $Y(s)/R(s) = 1$. Therefore, select

$$G_1(s) = \frac{1}{G_2(s)} \quad \text{and} \quad H_1(s) = \frac{G_2(s)H_2(s)}{G_1(s)} = G_2^2(s)H_2(s) .$$

DP2.2 At the lower node we have

$$v\left(\frac{1}{4} + \frac{1}{3} + G\right) + 2i_2 - 20 = 0 .$$

Also, we have $v = 24$ and $i_2 = Gv$. So

$$v\left(\frac{1}{4} + \frac{1}{3} + G\right) + 2Gv - 20 = 0$$

and

$$G = \frac{20 - v\left(\frac{1}{4} + \frac{1}{3}\right)}{3v} = \frac{1}{12}\ S\ .$$

DP2.3 Taking the Laplace transform of

$$y(t) = e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4} + \frac{1}{2}t$$

yields

$$Y(s) = \frac{1}{s+1} - \frac{1}{4(s+2)} - \frac{3}{4s} + \frac{1}{2s^2}\ .$$

Similarly, taking the Laplace transform of the ramp input yields

$$R(s) = \frac{1}{s^2}\ .$$

Therefore

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{(s+1)(s+2)}\ .$$

DP2.4 For an ideal op-amp, at node a we have

$$\frac{v_{in} - v_a}{R_1} + \frac{v_o - v_a}{R_1} = 0\ ,$$

and at node b

$$\frac{v_{in} - v_b}{R_2} = C\dot{v}_b\ ,$$

from it follows that

$$\left[\frac{1}{R_2} + Cs\right] V_b = \frac{1}{R_2}V_{in}\ .$$

Also, for an ideal op-amp, $V_b - V_a = 0$. Then solving for V_b in the above equation and substituting the result into the node a equation for V_a yields

$$\frac{V_o}{V_{in}} = \frac{2}{\frac{1}{R_2} + Cs}\left[\frac{1}{R_2} - \frac{\frac{1}{R_2} + Cs}{2}\right]$$

or

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1}\ .$$

For $v_{in}(t) = At$, we have $V_{in}(s) = A/s^2$, therefore

$$v_o(t) = A\left[\frac{2}{\beta}e^{-\beta t} + t - \frac{2}{\beta}\right]$$

where $\beta = 1/R_2C$.

DP2.5 The equation of motion describing the motion of the inverted pendulum (assuming small angles) is

$$\ddot{\varphi} + \frac{g}{L}\varphi = 0 \ .$$

Assuming a solution of the form $\varphi = k\cos\varphi$, taking the appropriate derivatives and substituting the result into the equation of motion yields the relationship

$$\dot{\varphi} = \sqrt{\frac{g}{L}} \ .$$

If the period is $T = 2$ seconds, we compute $\dot{\varphi} = 2\pi/T$. Then solving for L yields $L = 0.99$ meters when $g = 9.81$ m/s^2. So, to fit the pendulum into the grandfather clock, the dimensions are generally about 1.5 meters or more.

Computer Problems

CP2.1 The m-file script is shown in Figure CP2.1.

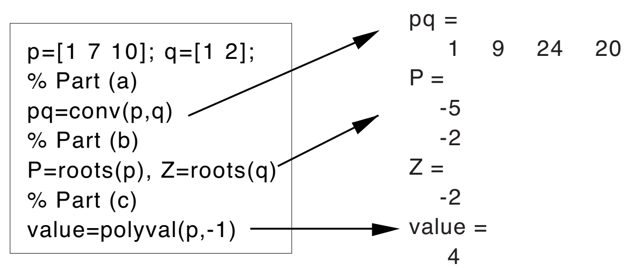


FIGURE CP2.1
Script for various polynomial evaluations.

CP2.2 The m-file script and step response is shown in Figure CP2.2.

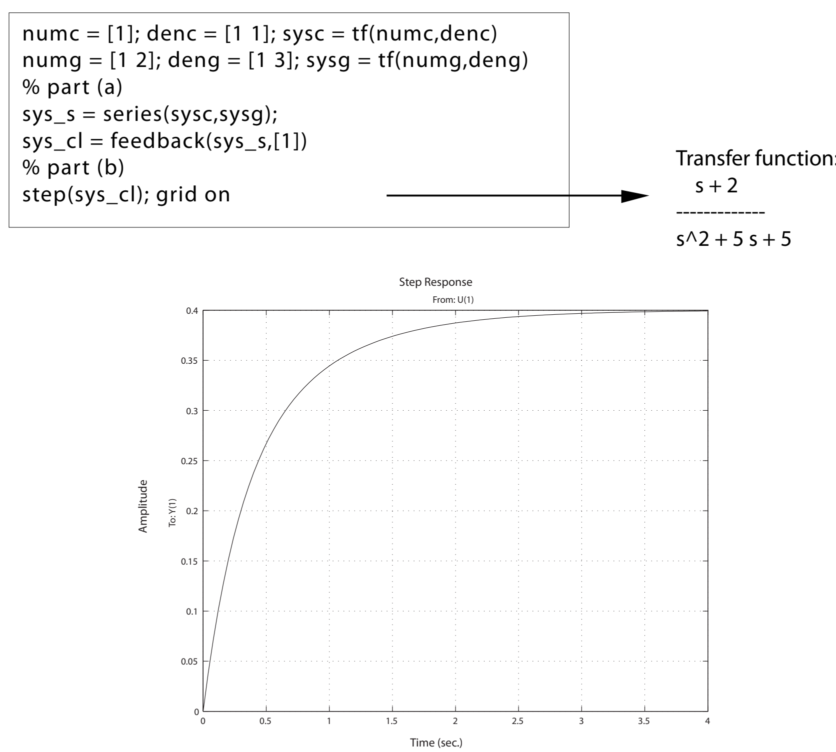


FIGURE CP2.2
Step response.

CP2.3 Given

$$\ddot{y} + 4\dot{y} + 3y = u$$

with $y(0) = \dot{y} = 0$ and $U(s) = 1/s$, we obtain (via Laplace transform)

$$Y(s) = \frac{1}{s(s^2 + 4s + 3)} = \frac{1}{s(s+3)(s+1)} .$$

Expanding in a partial fraction expansion yields

$$Y(s) = \frac{1}{3s} - \frac{1}{6(s+3)} - \frac{1}{2(s+1)} .$$

Taking the inverse Laplace transform we obtain the solution

$$y(t) = 0.3333 + 0.1667e^{-3t} - 0.5e^{-t} .$$

The m-file script and step response is shown in Figure CP2.3.

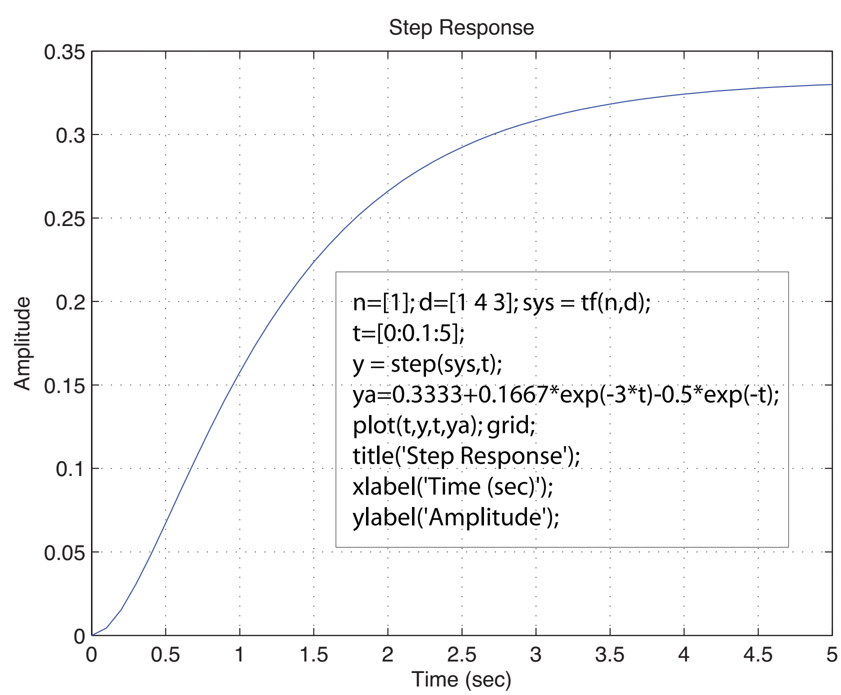


FIGURE CP2.3
Step response.

CP2.4 The mass-spring-damper system is represented by

$$m\ddot{x} + b\dot{x} + kx = f \ .$$

Taking the Laplace transform (with zero initial conditions) yields the transfer function

$$X(s)/F(s) = \frac{1/m}{s^2 + bs/m + k/m} \ .$$

The m-file script and step response is shown in Figure CP2.4.

```
m=10; k=1; b=0.5;
num=[1/m]; den=[1 b/m k/m];
sys = tf(num,den);
t=[0:0.1:150];
step(sys,t)
```

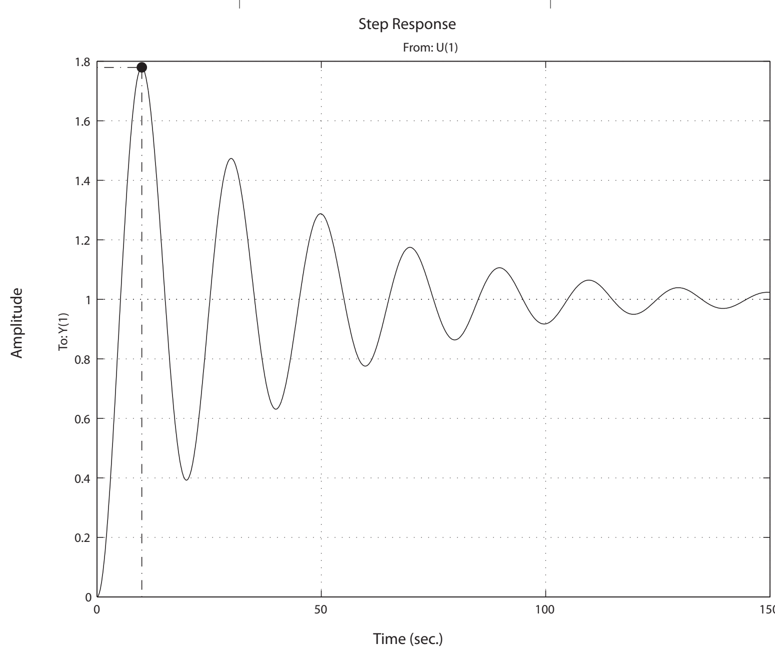
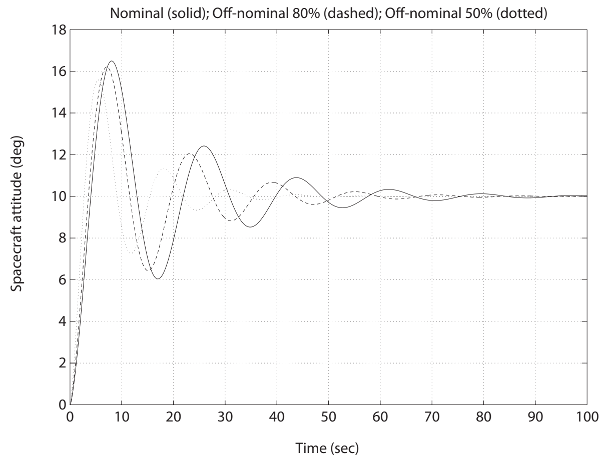


FIGURE CP2.4
Step response.

CP2.5 The spacecraft simulations are shown in Figure CP2.5. We see that as J is decreased, the time to settle down decreases. Also, the overhoot from 10^o decreases as J decreases. Thus, the performance seems to get better (in some sense) as J decreases.




```
%Part (a)
a=1; b=8; k=10.8e+08; J=10.8e+08;
num=k*[1 a];
den=J*[1  b 0 0]; sys=tf(num,den);
sys_cl=feedback(sys,[1]);
%
% Part (b) and (c)
t=[0:0.1:100];
%
% Nominal case
f=10*pi/180; sysf=sys_cl*f;
y=step(sysf,t);
%
% Off-nominal case 80%
J=10.8e+08*0.8; den=J*[1 b 0 0];
sys=tf(num,den); sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y1=step(sysf,t);
%
% Off-nominal case 50%
J=10.8e+08*0.5; den=J*[1 b 0 0];
sys=tf(num,den); sys_cl=feedback(sys,[1]);
sysf=sys_cl*f;
y2=step(sysf,t);
%
plot(t,y*180/pi,t,y1*180/pi,'--',t,y2*180/pi,':'),grid
xlabel('Time (sec)')
ylabel('Spacecraft attitude (deg)')
title('Nominal (solid); Off-nominal 80% (dashed); Off-nominal 50% (dotted)')
```

FIGURE CP2.5
Step responses for the nominal and off-nominal spacecraft parameters.

CP2.6 The closed-loop transfer function is

$$T(s) = \frac{4s^6 + 8s^5 + 4s^4 + 56s^3 + 112s^2 + 56s}{\Delta(s)} ,$$

```
num1=[4]; den1=[1]; sys1 = tf(num1,den1);
num2=[1]; den2=[1 1]; sys2 = tf(num2,den2);
num3=[1 0]; den3=[1 0 2]; sys3 = tf(num3,den3);
num4=[1]; den4=[1 0 0]; sys4 = tf(num4,den4);
num5=[4 2]; den5=[1 2 1]; sys5 = tf(num5,den5);
num6=[50]; den6=[1]; sys6 = tf(num6,den6);
num7=[1 0 2]; den7=[1 0 0 14]; sys7 = tf(num7,den7);
sysa = feedback(sys4,sys6,+1);
sysb = series(sys2,sys3);
sysc = feedback(sysb,sys5);
sysd = series(sysc,sysa);
syse = feedback(sysd,sys7);
sys = series(sys1,syse)
%
pzmap(sys)
%
p=pole(sys)
z=zero(sys)
```

poles

```
p =

  7.0709
 -7.0713
  1.2051 + 2.0863i
  1.2051 - 2.0863i
  0.1219 + 1.8374i
  0.1219 - 1.8374i
 -2.3933
 -2.3333
 -0.4635 + 0.1997i
 -0.4635 - 0.1997i

z =
      0
  1.2051 + 2.0872i
  1.2051 - 2.0872i
 -2.4101
 -1.0000 + 0.0000i
 -1.0000 - 0.0000i
```

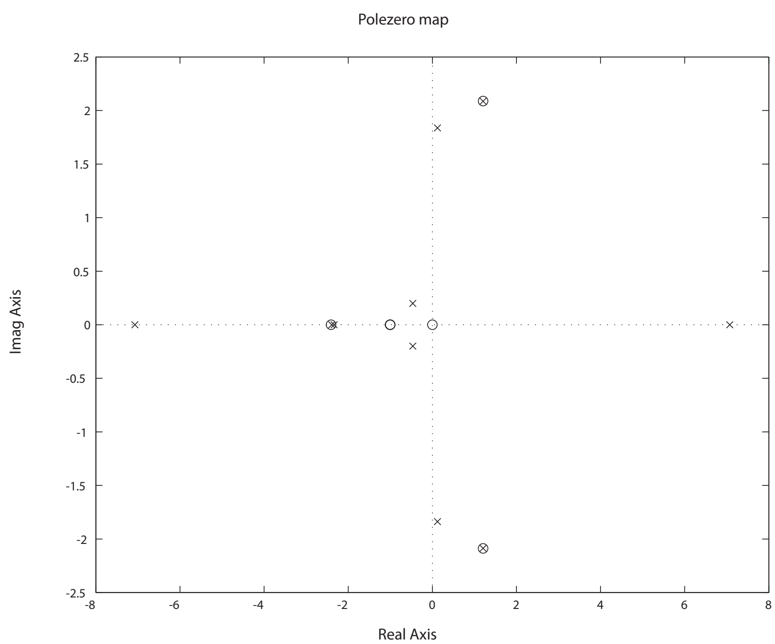


FIGURE CP2.6
Pole-zero map.

where

$$\Delta(s) = s^{10} + 3s^9 - 45s^8 - 125s^7 - 200s^6 - 1177s^5 - 2344s^4 - 3485s^3 - 7668s^2 - 5598s - 1400 \ .$$

CP2.7 The m-file script and plot of the pendulum angle is shown in Figure CP2.7. With the initial conditions, the Laplace transform of the linear system is

$$\theta(s) = \frac{\theta_0 s}{s^2 + g/L} .$$

To use the step function with the m-file, we can multiply the transfer function as follows:

$$\theta(s) = \frac{s^2}{s^2 + g/L}\frac{\theta_0}{s} ,$$

which is equivalent to the original transfer function except that we can use the step function input with magnitude θ_0. The nonlinear response is shown as the solid line and the linear response is shown as the dashed line. The difference between the two responses is not great since the initial condition of $\theta_0 = 30°$ is not that large.

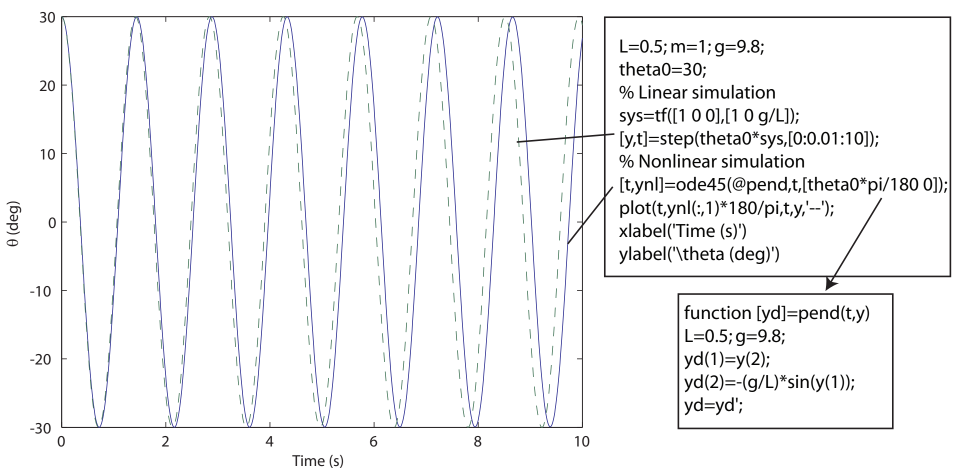


FIGURE CP2.7
Plot of θ versus xt when $\theta_0 = 30°$.

CP2.8 The system step responses for $z = 5, 10$, and 15 are shown in Figure CP2.8.

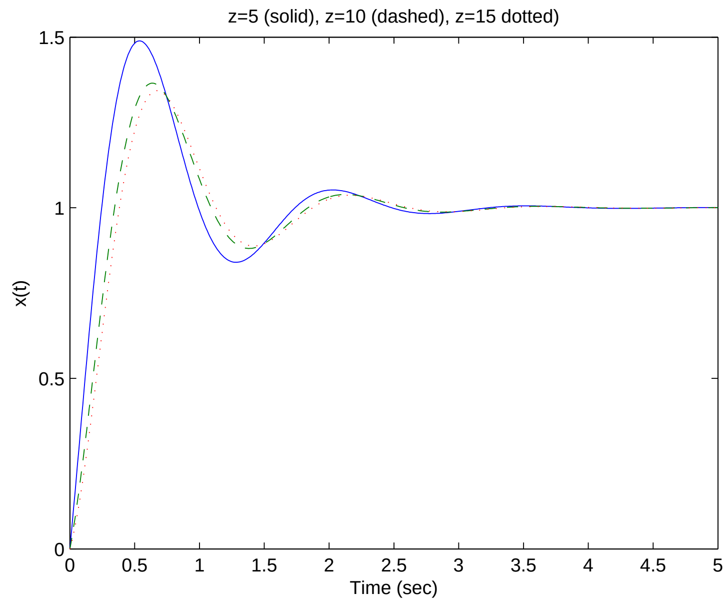


FIGURE CP2.8
The system response.

CP2.9 (a,b) Computing the closed-loop transfer function yields

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{s^2 + 2s + 1}{s^2 + 4s + 3} .$$

The poles are $s = -3, -1$ and the zeros are $s = -1, -1$.
(c) Yes, there is one pole-zero cancellation. The transfer function (after pole-zero cancellation) is

$$T(s) = \frac{s + 1}{s + 3} .$$

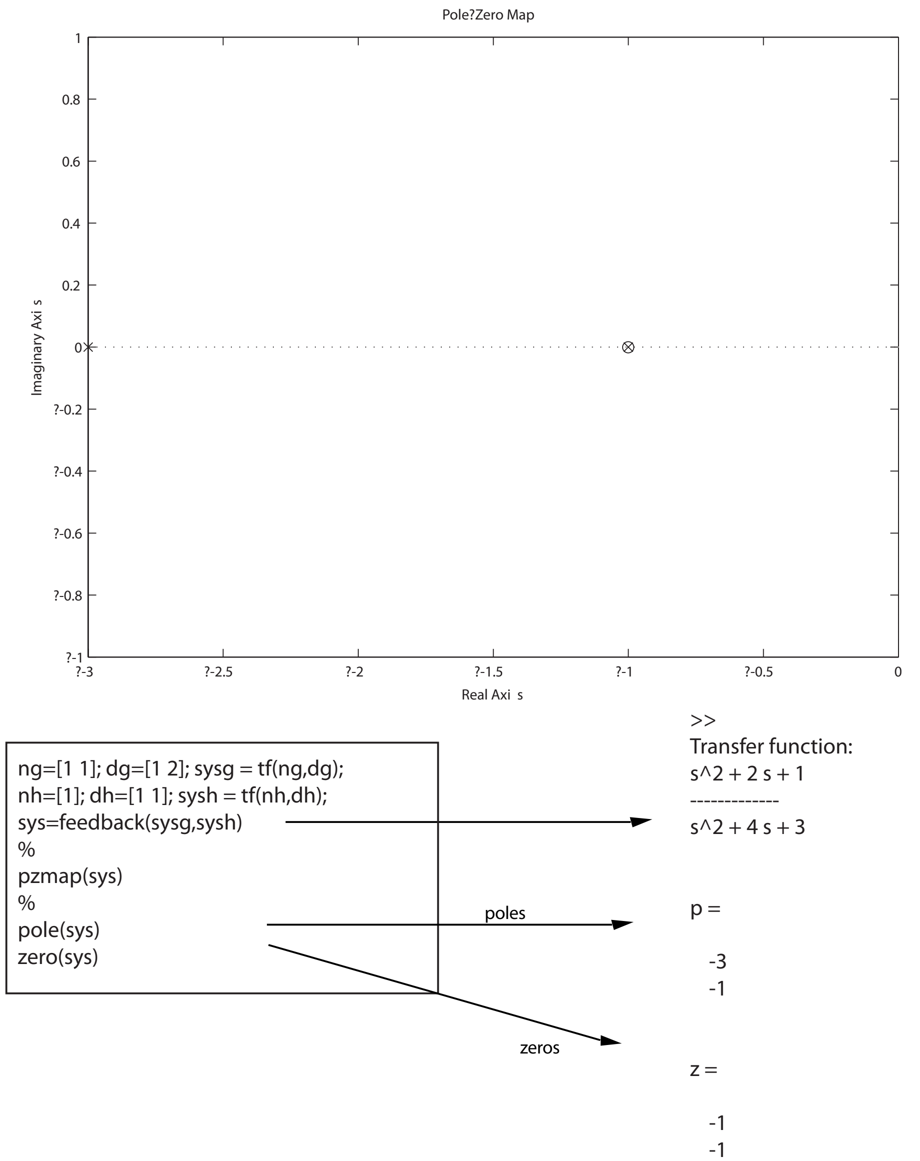


FIGURE CP2.9
Pole-zero map.

CP2.10 Figure CP2.10 shows the steady-state response to a unit step input and a unit step disturbance. We see that $K = 1$ leads to the same steady-state response.