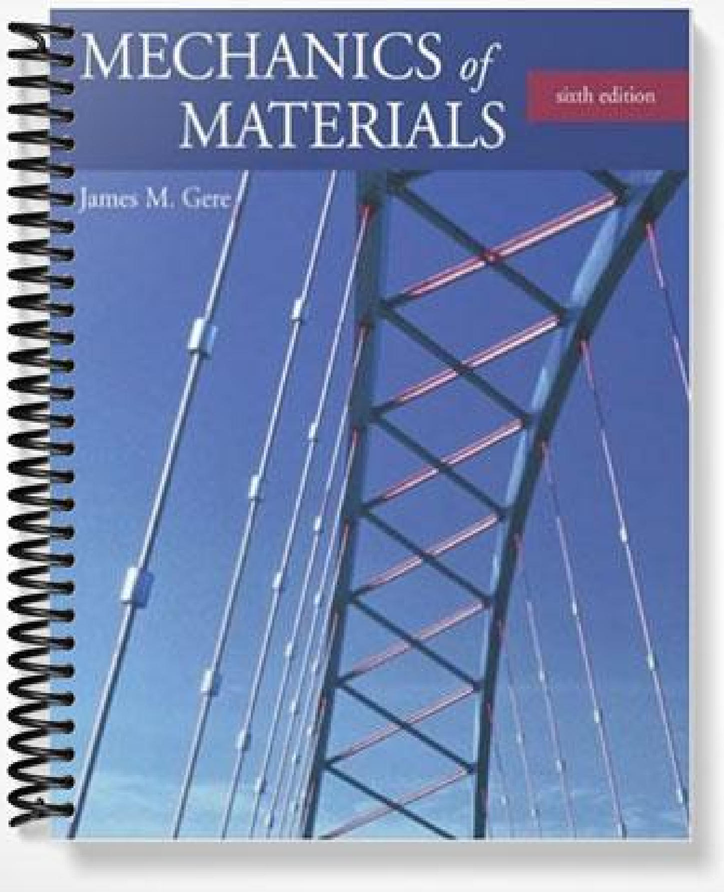
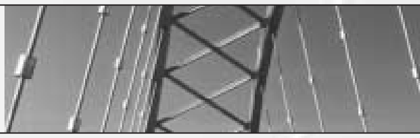
SOLUTIONS MANUAL





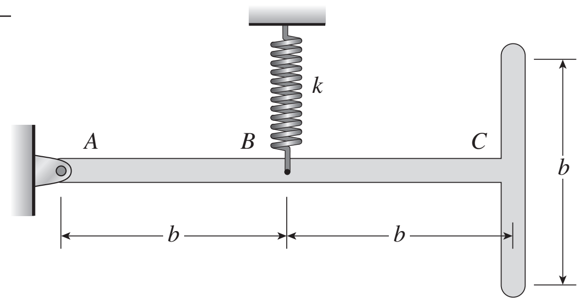
2

Axially Loaded Members

Changes in Lengths of Axially Loaded Members

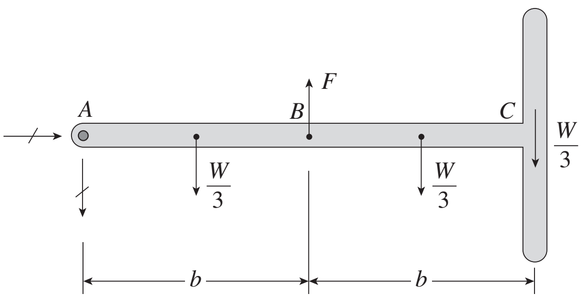
Problem 2.2-1 The T-shaped arm *ABC* shown in the figure lies in a vertical plane and pivots about a horizontal pin at *A*. The arm has constant cross-sectional area and total weight *W*. A vertical spring of stiffness *k* supports the arm at point *B*.

Obtain a formula for the elongation δ of the spring due to the weight of the arm.



Solution 2.2-1 **T-shaped arm**

FREE-BODY DIAGRAM OF ARM



F = tensile force in the spring

$\Sigma M_A = 0$ ↶+ ↷−

$$F(b) - \frac{W}{3}\left(\frac{b}{2}\right) - \frac{W}{3}\left(\frac{3b}{2}\right) - \frac{W}{3}(2b) = 0$$

$$F = \frac{4W}{3}$$

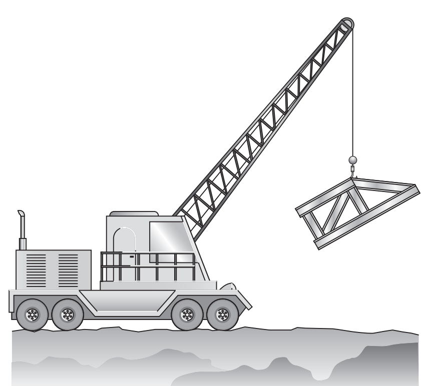
δ = elongation of the spring

$$\delta = \frac{F}{k} = \frac{4W}{3k} \quad \longleftarrow$$

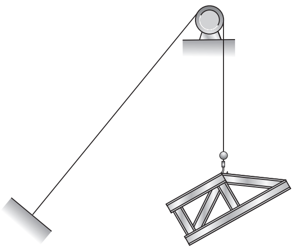
Problem 2.2-2 A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity E = 140 GPa.

(a) If the cable is 14 m long, how much will it stretch when the load is picked up?

(b) If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?



Solution 2.2-2 Bridge section lifted by a cable



$A = 304 \text{ mm}^2$ (from Table 2-1)

$W = 38 \text{ kN}$

$E = 140 \text{ GPa}$

$L = 14 \text{ m}$

(a) STRETCH OF CABLE

$$\delta = \frac{WL}{EA} = \frac{(38 \text{ kN})(14 \text{ m})}{(140 \text{ GPa})(304 \text{ mm}^2)}$$

$$= 12.5 \text{ mm} \longleftarrow$$

(b) FACTOR OF SAFETY

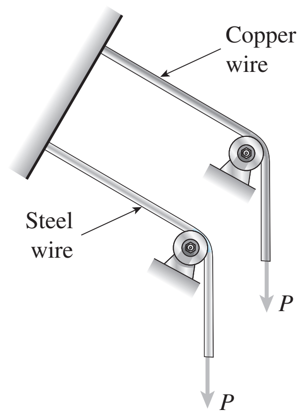
$P_{ULT} = 406 \text{ kN}$ (from Table 2-1)

$P_{\max} = 70 \text{ kN}$

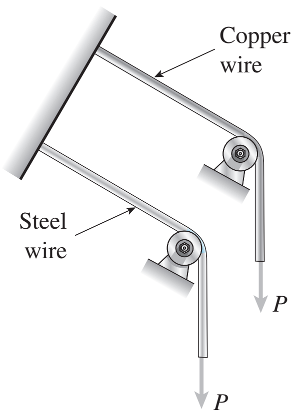
$$n = \frac{P_{ULT}}{P_{\max}} = \frac{406 \text{ kN}}{70 \text{ kN}} = 5.8 \longleftarrow$$

Problem 2.2-3 A steel wire and a copper wire have equal lengths and support equal loads P (see figure). The moduli of elasticity for the steel and copper are $E_s = 30{,}000$ ksi and $E_c = 18{,}000$ ksi, respectively.

(a) If the wires have the same diameters, what is the ratio of the elongation of the copper wire to the elongation of the steel wire?
(b) If the wires stretch the same amount, what is the ratio of the diameter of the copper wire to the diameter of the steel wire?



Solution 2.2-3 Steel wire and copper wire



Equal lengths and equal loads

Steel: $E_s = 30{,}000$ ksi

Copper: $E_c = 18{,}000$ ksi

(a) RATIO OF ELONGATIONS (EQUAL DIAMETERS)

$$\delta_c = \frac{PL}{E_c A} \qquad \delta_s = \frac{PL}{E_s A}$$

$$\frac{\delta_c}{\delta_s} = \frac{E_s}{E_c} = \frac{30}{18} = 1.67 \longleftarrow$$

(b) RATIO OF DIAMETERS (EQUAL ELONGATIONS)

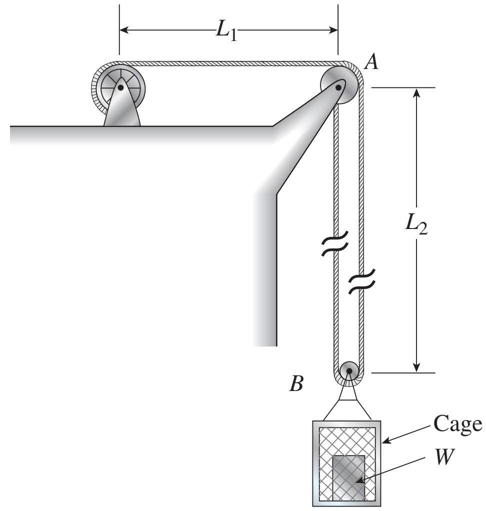
$$\delta_c = \delta_s \qquad \frac{PL}{E_c A_c} = \frac{PL}{E_s A_s} \quad \text{or} \quad E_c A_c = E_s A_s$$

$$E_c\left(\frac{\pi}{4}\right)d_c^2 = E_s\left(\frac{\pi}{4}\right)d_s^2$$

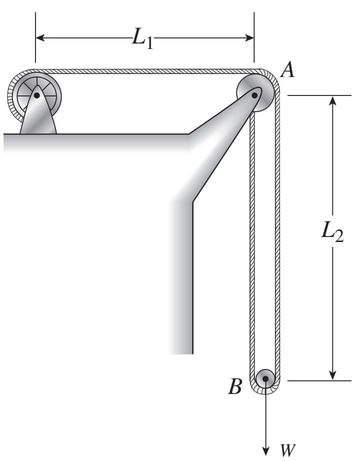
$$\frac{d_c^2}{d_s^2} = \frac{E_s}{E_c} \qquad \frac{d_c}{d_s} = \sqrt{\frac{E_s}{E_c}} = \sqrt{\frac{30}{18}} = 1.29 \longleftarrow$$

Problem 2.2-4 By what distance h does the cage shown in the figure move downward when the weight W is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10{,}700$ kN. The pulley at A has diameter $d_A = 300$ mm and the pulley at B has diameter $d_B = 150$ mm. Also, the distance $L_1 = 4.6$ m, the distance $L_2 = 10.5$ m, and the weight $W = 22$ kN. (*Note:* When calculating the length of the cable, include the parts of the cable that go around the pulleys at A and B.)



Solution 2.2-4 Cage supported by a cable



$d_A = 300$ mm

$d_B = 150$ mm

$L_1 = 4.6$ m

$L_2 = 10.5$ m

$EA = 10{,}700$ kN

$W = 22$ kN

TENSILE FORCE IN CABLE

$$T = \frac{W}{2} = 11 \text{ kN}$$

LENGTH OF CABLE

$$L = L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B)$$

$$= 4{,}600 \text{ mm} + 21{,}000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm}$$

$$= 26{,}072 \text{ mm}$$

ELONGATION OF CABLE

$$\delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26{,}072 \text{ mm})}{(10{,}700 \text{ kN})} = 26.8 \text{ mm}$$

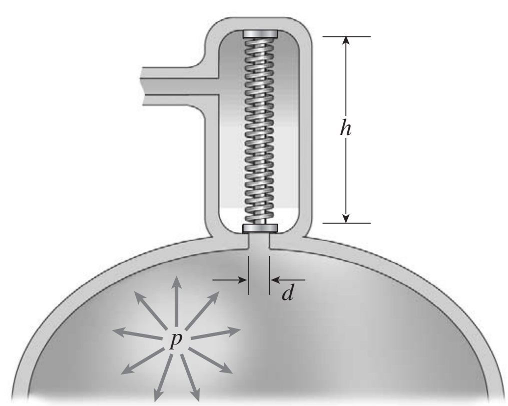
LOWERING OF THE CAGE

h = distance the cage moves downward

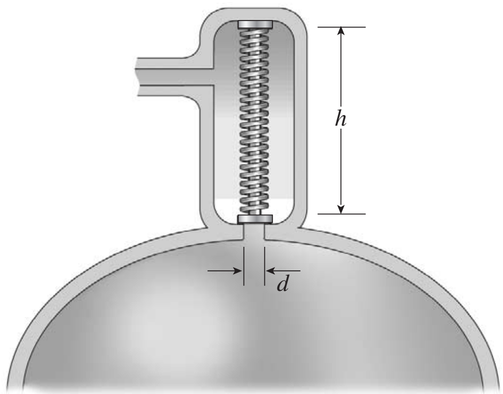
$$h = \frac{1}{2}\delta = 13.4 \text{ mm} \quad \longleftarrow$$

Problem 2.2-5 A safety valve on the top of a tank containing steam under pressure p has a discharge hole of diameter d (see figure). The valve is designed to release the steam when the pressure reaches the value p_{max}.

If the natural length of the spring is L and its stiffness is k, what should be the dimension h of the valve? (Express your result as a formula for h.)



Solution 2.2-5 Safety valve



h = height of valve (compressed length of the spring)

d = diameter of discharge hole

p = pressure in tank

p_{max} = pressure when valve opens

L = natural length of spring ($L > h$)

k = stiffness of spring

FORCE IN COMPRESSED SPRING

$F = k(L - h)$ (From Eq. 2-1a)

PRESSURE FORCE ON SPRING

$$P = p_{max}\left(\frac{\pi d^2}{4}\right)$$

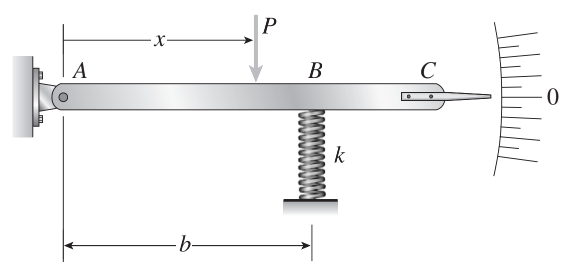
EQUATE FORCES AND SOLVE FOR h:

$$F = P \quad k(L - h) = \frac{\pi p_{max}\, d^2}{4}$$

$$h = L - \frac{\pi p_{max}\, d^2}{4k} \longleftarrow$$

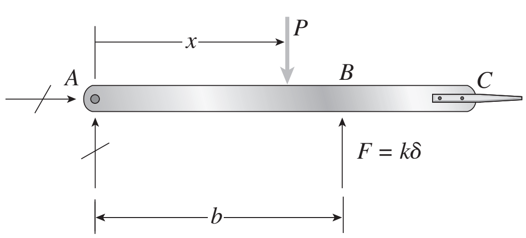
Problem 2.2-6 The device shown in the figure consists of a pointer ABC supported by a spring of stiffness $k = 800$ N/m. The spring is positioned at distance $b = 150$ mm from the pinned end A of the pointer. The device is adjusted so that when there is no load P, the pointer reads zero on the angular scale.

If the load $P = 8$ N, at what distance x should the load be placed so that the pointer will read 3° on the scale?



Solution 2.2-6 Pointer supported by a spring

FREE-BODY DIAGRAM OF POINTER



$P = 8$ N

$k = 800$ N/m

$b = 150$ mm

δ = displacement of spring

F = force in spring

$= k\delta$

$\Sigma M_A = 0$ ⟲⟳

$$-Px + (k\delta)b = 0 \quad \text{or} \quad \delta = \frac{Px}{kb}$$

Let α = angle of rotation of pointer

$$\tan\alpha = \frac{\delta}{b} = \frac{Px}{kb^2} \qquad x = \frac{kb^2}{P}\tan\alpha \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

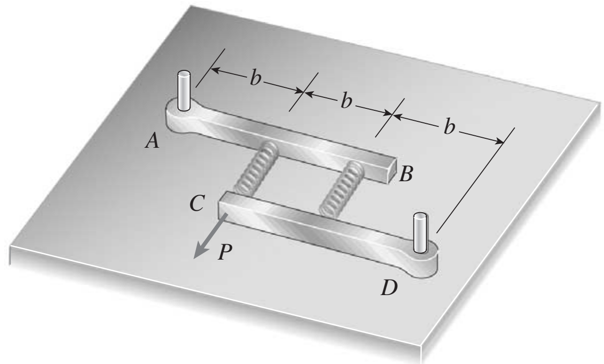
$\alpha = 3°$

$$x = \frac{(800\text{ N/m})(150\text{ mm})^2}{8\text{ N}}\tan 3°$$

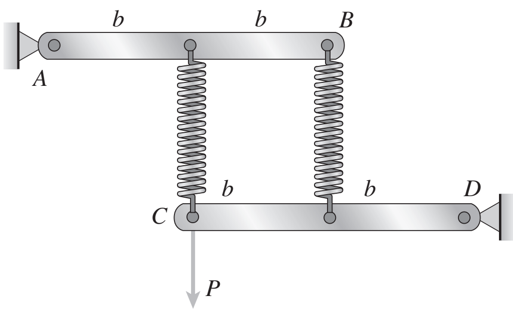
$$= 118\text{ mm} \longleftarrow$$

Problem 2.2-7 Two rigid bars, *AB* and *CD*, rest on a smooth horizontal surface (see figure). Bar *AB* is pivoted end *A* and bar *CD* is pivoted at end *D*. The bars are connected to each other by two linearly elastic springs of stiffness *k*. Before the load *P* is applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

Derive a formula for the displacement δ_C at point *C* when the load *P* is acting. (Assume that the bars rotate through very small angles under the action of the load *P*.)



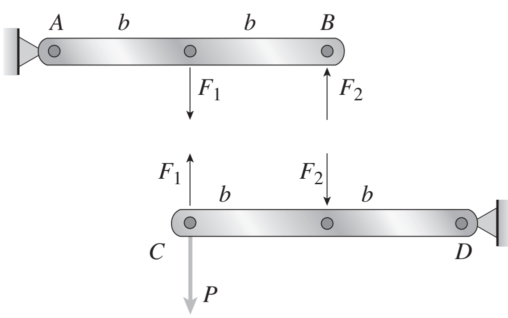
Solution 2.2-7 **Two bars connected by springs**



k = stiffness of springs

δ_C = displacement at point *C* due to load *P*

Free-body diagrams



F_1 = tensile force in first spring

F_2 = compressive force in second spring

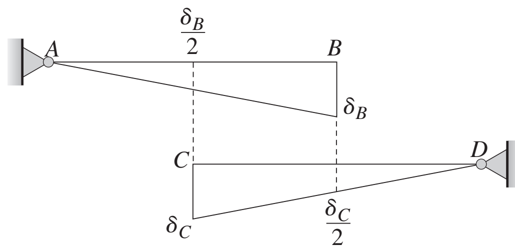
Equilibrium

$\Sigma M_A = 0 \quad -bF_1 + 2bF_2 = 0 \quad F_1 = 2F_2$

$\Sigma M_D = 0 \quad 2bP - 2bF_1 + bF_2 = 0 \quad F_2 = 2F_1 - 2P$

Solving, $F_1 = \dfrac{4P}{3} \quad F_2 = \dfrac{2P}{3}$

Displacement diagrams



δ_B = displacement of point *B*

δ_C = displacement of point *C*

Δ_1 = elongation of first spring

$= \delta_C - \dfrac{\delta_B}{2}$

Δ_2 = shortening of second spring

$= \delta_B - \dfrac{\delta_C}{2}$

Also, $\Delta_1 = \dfrac{F_1}{k} = \dfrac{4P}{3k}; \quad \Delta_2 = \dfrac{F_2}{k} = \dfrac{2P}{3k}$

Solve the equations:

$\Delta_1 = \Delta_1 \quad \delta_C - \dfrac{\delta_B}{2} = \dfrac{4P}{3k}$

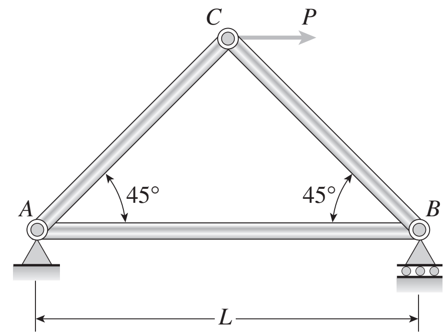
$\Delta_2 = \Delta_2 \quad \delta_B - \dfrac{\delta_C}{2} = \dfrac{2P}{3k}$

Eliminate δ_B and obtain δ_C:

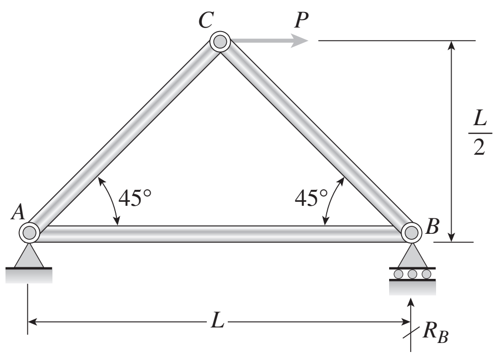
$$\delta_C = \frac{20P}{9k} \quad \longleftarrow$$

Problem 2.2-8 The three-bar truss ABC shown in the figure has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900$ mm^2 and modulus of elasticity $E = 200$ GPa. A load P acts horizontally to the right at joint C.

(a) If $P = 650$ kN, what is the horizontal displacement of joint B?
(b) What is the maximum permissible load P_{max} if the displacement of joint B is limited to 1.5 mm?



Solution 2.2-8 Truss with horizontal load



$L = 3$ m

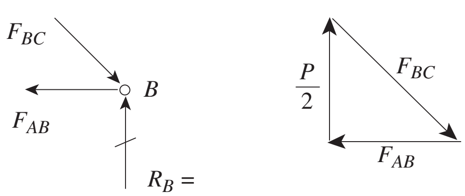
$A = 3900$ mm^2

$E = 200$ GPa

$\Sigma M_A = 0$ gives $R_B = \dfrac{P}{2}$

FREE-BODY DIAGRAM OF JOINT B

Force triangle:



From force triangle,

$$F_{AB} = \frac{P}{2} \text{ (tension)}$$

(a) HORIZONTAL DISPLACEMENT δ_B

$P = 650$ kN

$$\delta_B = \frac{F_{AB}\,L_{AB}}{EA} = \frac{PL}{2EA}$$

$$= \frac{(650 \text{ kN})(3 \text{ m})}{2(200 \text{ GPa})(3900 \text{ mm}^2)}$$

$$= 1.25 \text{ mm} \longleftarrow$$

(b) MAXIMUM LOAD P_{max}

$\delta_{max} = 1.5$ mm

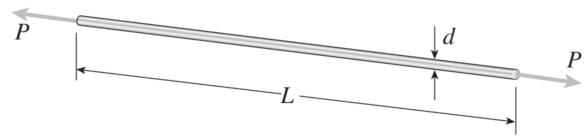
$$\frac{P_{max}}{\delta_{max}} = \frac{P}{\delta} \qquad P_{max} = P\left(\frac{\delta_{max}}{\delta}\right)$$

$$P_{max} = (650 \text{ kN})\left(\frac{1.5 \text{ mm}}{1.25 \text{ mm}}\right)$$

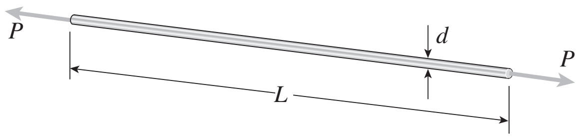
$$= 780 \text{ kN} \longleftarrow$$

Problem 2.2-9 An aluminum wire having a diameter $d = 2$ mm and length $L = 3.8$ m is subjected to a tensile load P (see figure). The aluminum has modulus of elasticity $E = 75$ GPa.

If the maximum permissible elongation of the wire is 3.0 mm and the allowable stress in tension is 60 MPa, what is the allowable load P_{max}?



Solution 2.2-9 **Aluminum wire in tension**



$d = 2$ mm

$L = 3.8$ m

$E = 75$ GPa

$$A = \frac{\pi d^2}{4} = 3.142 \text{ mm}^2$$

MAXIMUM LOAD BASED UPON ELONGATION

$$\delta_{max} = 3.0 \text{ mm} \qquad \delta = \frac{PL}{EA}$$

$$P_{max} = \frac{EA}{L}\delta_{max}$$

$$= \frac{(75 \text{ GPa})(3.142 \text{ mm}^2)}{3.8 \text{ m}}(3.0 \text{ mm})$$

$$= 186 \text{ N}$$

MAXIMUM LOAD BASED UPON STRESS

$$\sigma_{allow} = 60 \text{ MPa} \qquad \sigma = \frac{P}{A}$$

$$P_{max} = A\sigma_{allow} = (3.142 \text{ mm}^2)(60 \text{ MPa})$$

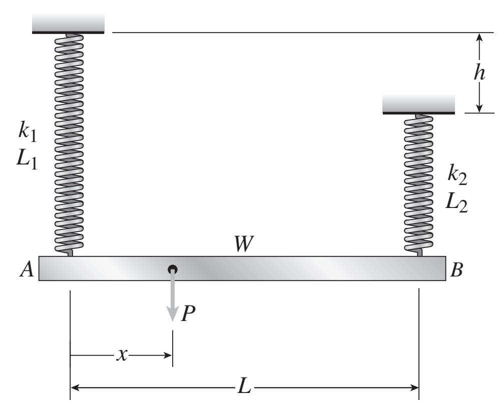
$$= 189 \text{ N}$$

ALLOWABLE LOAD

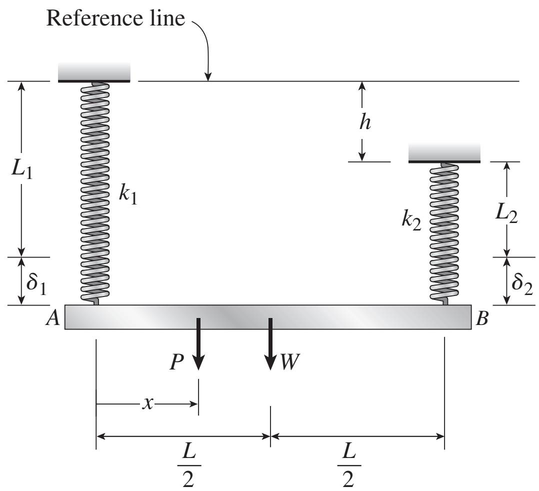
Elongation governs. $P_{max} = 186$ N ←

Problem 2.2-10 A uniform bar AB of weight $W = 25$ N is supported by two springs, as shown in the figure. The spring on the left has stiffness $k_1 = 300$ N/m and natural length $L_1 = 250$ mm. The corresponding quantities for the spring on the right are $k_2 = 400$ N/m and $L_2 = 200$ mm. The distance between the springs is $L = 350$ mm, and the spring on the right is suspended from a support that is distance $h = 80$ mm below the point of support for the spring on the left.

At what distance x from the left-hand spring should a load $P = 18$ N be placed in order to bring the bar to a horizontal position?



Solution 2.2-10 Bar supported by two springs



$W = 25$ N

$k_1 = 300$ N/m

$k_2 = 400$ N/m

$L = 350$ mm

$h = 80$ mm

$P = 18$ N

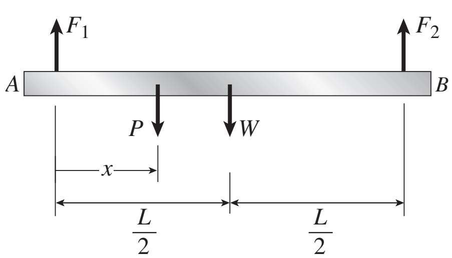
NATURAL LENGTHS OF SPRINGS

$L_1 = 250$ mm $\qquad L_2 = 200$ mm

OBJECTIVE

Find distance x for bar AB to be horizontal.

FREE-BODY DIAGRAM OF BAR AB



$\Sigma M_A = 0$

$$F_2 L - P_X - \frac{WL}{2} = 0 \qquad \text{(Eq. 1)}$$

$\Sigma F_{\text{vert}} = 0 \quad \uparrow_+ \quad \downarrow^-$

$$F_1 + F_2 - P - W = 0 \qquad \text{(Eq. 2)}$$

SOLVE EQS. (1) AND (2):

$$F_1 = P\left(1 - \frac{x}{L}\right) + \frac{W}{2} \qquad F_2 = \frac{P_x}{L} + \frac{W}{2}$$

SUBSTITUTE NUMERICAL VALUES:

UNITS: Newtons and meters

$$F_1 = (18)\left(1 - \frac{x}{0.350}\right) + 12.5 = 30.5 - 51.429x$$

$$F_2 = (18)\left(\frac{x}{0.350}\right) + 12.5 = 51.429x + 12.5$$

ELONGATIONS OF THE SPRINGS

$$\delta_1 = \frac{F_1}{k_1} = \frac{F_1}{300} = 0.10167 - 0.17143x$$

$$\delta_2 = \frac{F_2}{k_2} = \frac{F_2}{400} = 0.12857x + 0.031250$$

BAR AB REMAINS HORIZONTAL

Points A and B are the same distance below the reference line (see figure above).

$$\therefore L_1 + \delta_1 = h + L_2 + \delta_2$$

or $\quad 0.250 + 0.10167 - 0.17143\,x$
$\quad = 0.080 + 0.200 + 0.12857\,x + 0.031250$

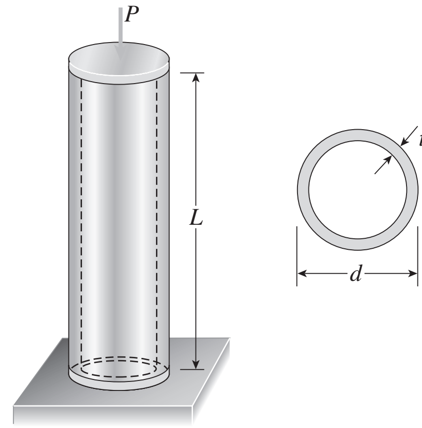
SOLVE FOR x:

$0.300\,x = 0.040420 \qquad x = 0.1347$ m

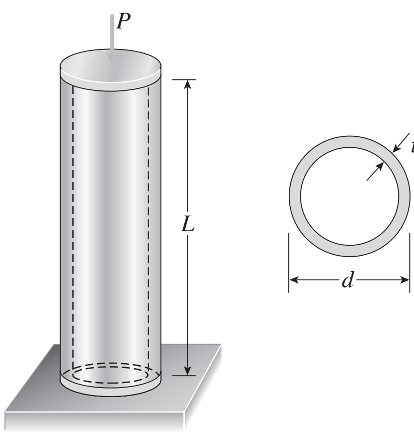
$x = 135$ mm $\quad \longleftarrow$

Problem 2.2-11 A hollow, circular, steel column ($E = 30{,}000$ ksi) is subjected to a compressive load P, as shown in the figure. The column has length $L = 8.0$ ft and outside diameter $d = 7.5$ in. The load $P = 85$ k.

If the allowable compressive stress is 7000 psi and the allowable shortening of the column is 0.02 in., what is the minimum required wall thickness t_{min}?



Solution 2.2-11 **Column in compression**



$P = 85$ k

$E = 30{,}000$ ksi

$L = 8.0$ ft

$d = 7.5$ in.

$\sigma_{allow} = 7{,}000$ psi

$\delta_{allow} = 0.02$ in.

Required area based upon allowable stress

$$\sigma = \frac{P}{A} \qquad A = \frac{P}{\sigma_{allow}} = \frac{85 \text{ k}}{7{,}000 \text{ psi}} = 12.14 \text{ in.}^2$$

Required area based upon allowable shortening

$$\delta = \frac{PL}{EA} \qquad A = \frac{PL}{E\delta_{allow}} = \frac{(85 \text{ k})(96 \text{ in.})}{(30{,}000 \text{ ksi})(0.02 \text{ in.})}$$

$$= 13.60 \text{ in.}^2$$

Shortening governs

$$A_{min} = 13.60 \text{ in.}^2$$

Minimum thickness t_{min}

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] \quad \text{or}$$

$$\frac{4A}{\pi} - d^2 = -(d - 2t)^2$$

$$(d - 2t)^2 = d^2 - \frac{4A}{\pi} \quad \text{or} \quad d - 2t = \sqrt{d^2 - \frac{4A}{\pi}}$$

$$t = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A}{\pi}} \quad \text{or}$$

$$t_{min} = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A_{min}}{\pi}}$$

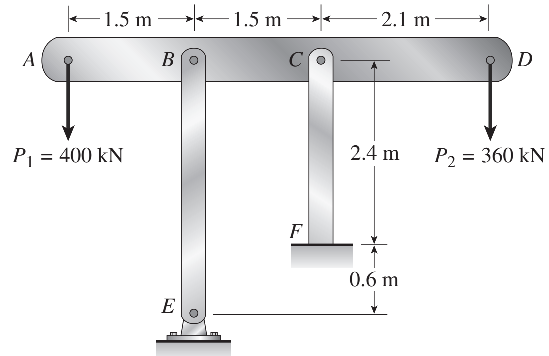
Substitute numerical values

$$t_{min} = \frac{7.5 \text{ in.}}{2} - \sqrt{\left(\frac{7.5 \text{ in.}}{2}\right)^2 - \frac{13.60 \text{ in.}^2}{\pi}}$$

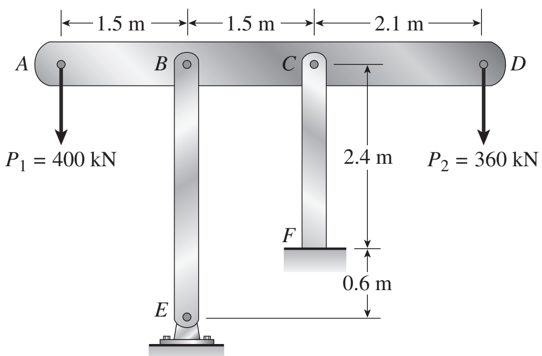
$$t_{min} = 0.63 \text{ in.} \quad \longleftarrow$$

Problem 2.2-12 The horizontal rigid beam *ABCD* is supported by vertical bars *BE* and *CF* and is loaded by vertical forces $P_1 = 400$ kN and $P_2 = 360$ kN acting at points *A* and *D*, respectively (see figure). Bars *BE* and *CF* are made of steel ($E = 200$ GPa) and have cross-sectional areas $A_{BE} = 11{,}100$ mm^2 and $A_{CF} = 9{,}280$ mm^2. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements δ_A and δ_D of points *A* and *D*, respectively.



Solution 2.2-12 Rigid beam supported by vertical bars



$A_{BE} = 11{,}100$ mm^2

$A_{CF} = 9{,}280$ mm^2

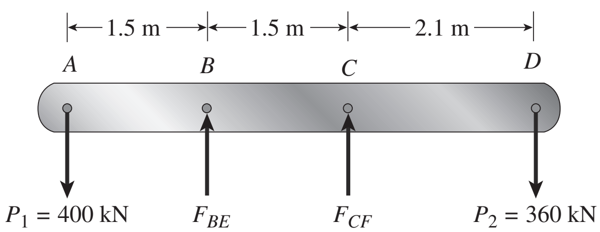
$E = 200$ GPa

$L_{BE} = 3.0$ m

$L_{CF} = 2.4$ m

$P_1 = 400$ kN; $P_2 = 360$ kN

Free-body diagram of bar *ABCD*



$\Sigma M_B = 0$

$(400\text{ kN})(1.5\text{ m}) + F_{CF}(1.5\text{ m}) - (360\text{ kN})(3.6\text{ m}) = 0$

$F_{CF} = 464$ kN

$\Sigma M_C = 0$

$(400\text{ kN})(3.0\text{ m}) - F_{BE}(1.5\text{ m}) - (360\text{ kN})(2.1\text{ m}) = 0$

$F_{BE} = 296$ kN

Shortening of bar *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296\text{ kN})(3.0\text{ m})}{(200\text{ GPa})(11{,}100\text{ mm}^2)}$$

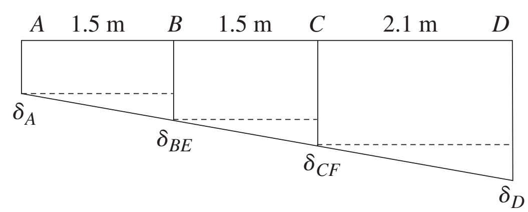
$$= 0.400\text{ mm}$$

Shortening of bar *CF*

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464\text{ kN})(2.4\text{ m})}{(200\text{ GPa})(9{,}280\text{ mm}^2)}$$

$$= 0.600\text{ mm}$$

Displacement diagram



$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE}$ or $\delta_A = 2\delta_{BE} - \delta_{CF}$

$\delta_A = 2(0.400\text{ mm}) - 0.600\text{ m}$

$= 0.200$ mm ←

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

or $$\delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$

$$= \frac{12}{5}(0.600\text{ mm}) - \frac{7}{5}(0.400\text{ mm})$$

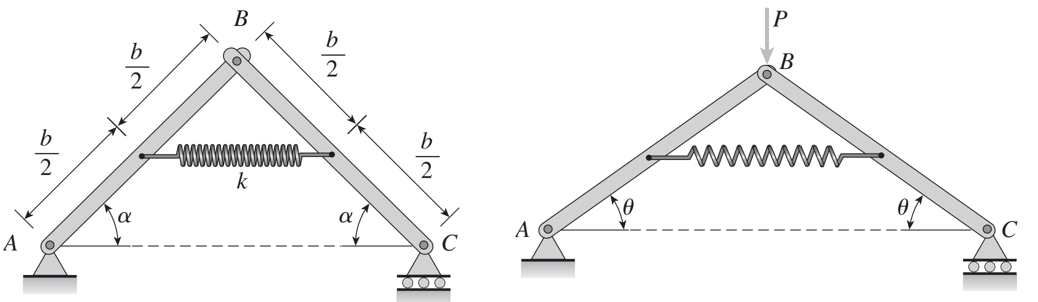
$= 0.880$ mm ←

(Downward)

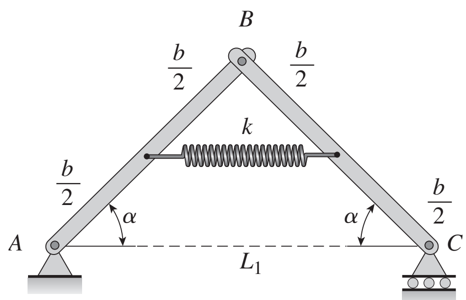
Problem 2.2-13 A framework ABC consists of two rigid bars AB and BC, each having length b (see the first part of the figure). The bars have pin connections at A, B, and C and are joined by a spring of stiffness k. The spring is attached at the midpoints of the bars. The framework has a pin support at A and a roller support at C, and the bars are at an angle α to the hoizontal.

When a vertical load P is applied at joint B (see the second part of the figure) the roller support C moves to the right, the spring is stretched, and the angle of the bars decreases from α to the angle θ.

Determine the angle θ and the increase δ in the distance between points A and C. (Use the following data; $b = 8.0$ in., $k = 16$ lb/in., $\alpha = 45°$, and $P = 10$ lb.)



Solution 2.2-13 **Framework with rigid bars and a spring**



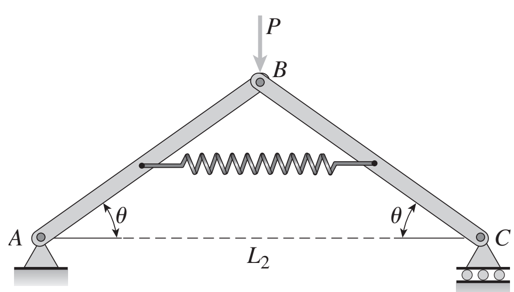
WITH NO LOAD

L_1 = span from A to C

$= 2b \cos \alpha$

S_1 = length of spring

$= \dfrac{L_1}{2} = b \cos \alpha$



WITH LOAD P

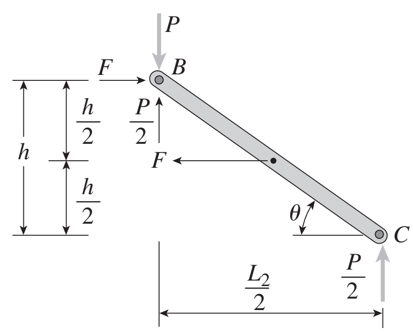
L_2 = span from A to C

$= 2b \cos \theta$

S_2 = length of spring

$= \dfrac{L_2}{2} = b \cos \theta$

FREE-BODY DIAGRAM OF BC



h = height from C to B = $b \sin \theta$

$\dfrac{L_2}{2} = b \cos \theta$

F = force in spring due to load P

$\Sigma M_B = 0$

$$\frac{P}{2}\left(\frac{L_2}{2}\right) - F\left(\frac{h}{2}\right) = 0 \text{ or } P \cos \theta = F \sin \theta \qquad \text{(Eq. 1)}$$

(Continued)

DETERMINE THE ANGLE θ

ΔS = elongation of spring

$= S_2 - S_1 = b(\cos\theta - \cos\alpha)$

For the spring: $F = k(\Delta S)$

$F = bk(\cos\theta - \cos\alpha)$

Substitute F into Eq. (1):

$P\cos\theta = bk(\cos\theta - \cos\alpha)(\sin\theta)$

or $\dfrac{\mathrm{P}}{\mathrm{bk}}\cot\theta - \cos\theta + \cos\alpha = 0$ ⟵ (Eq. 2)

This equation must be solved numerically for the angle θ.

DETERMINE THE DISTANCE δ

$\delta = L_2 - L_1 = 2b\cos\theta - 2b\cos\alpha$

$= 2b(\cos\theta - \cos\alpha)$

From Eq. (2): $\cos\alpha = \cos\theta - \dfrac{P\cot\theta}{bk}$

Therefore,

$$\delta = 2b\left(\cos\theta - \cos\theta + \frac{P\cot\theta}{bk}\right)$$

$$= \frac{2P}{b}\cot\theta \longleftarrow \qquad \text{(Eq. 3)}$$

NUMERICAL RESULTS

$b = 8.0$ in. $\quad k = 16$ lb/in. $\quad \alpha = 45°$ $\quad P = 10$ lb

Substitute into Eq. (2):

$0.078125\cot\theta - \cos\theta + 0.707107 = 0$ (Eq. 4)

Solve Eq. (4) numerically:

$\theta = 35.1°$ ⟵

Substitute into Eq. (3):

$\delta = 1.78$ in. ⟵

Problem 2.2-14 Solve the preceding problem for the following data: $b = 200$ mm, $k = 3.2$ kN/m, $\alpha = 45°$, and $P = 50$ N.

Solution 2.2-14 Framework with rigid bars and a spring

See the solution to the preceding problem.

Eq. (2): $\dfrac{P}{bk}\cot\theta - \cos\theta + \cos\alpha = 0$

Eq. (3): $\delta = \dfrac{2P}{k}\cot\theta$

NUMERICAL RESULTS

$b = 200$ mm $\quad k = 3.2$ kN/m $\quad \alpha = 45°$ $\quad P = 50$ N

Substitute into Eq. (2):

$0.078125\cot\theta - \cos\theta + 0.707107 = 0$ (Eq. 4)

Solve Eq. (4) numerically:

$\theta = 35.1°$ ⟵

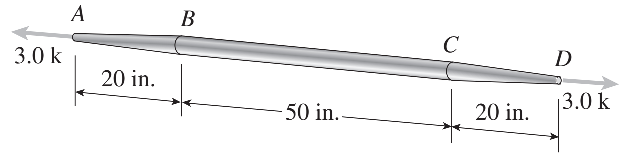
Substitute into Eq. (3):

$\delta = 44.5$ mm ⟵

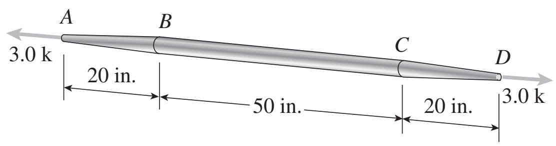
Changes in Lengths under Nonuniform Conditions

Problem 2.3-1 Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections A, B, C, and D are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi. (*Hint:* Use the result of Example 2-4.)



Solution 2.3-1 **Bar with tapered ends**



$d_A = d_D = 0.5 \text{ in.} \quad P = 3.0 \text{ k}$

$d_B = d_C = 1.0 \text{ in.} \quad E = 18{,}000 \text{ ksi}$

End segment ($L = 20$ in.)

From Example 2-4:

$$\delta = \frac{4PL}{\pi E\, d_A\, d_B}$$

$$\delta_1 = \frac{4(3.0 \text{ k})(20 \text{ in.})}{\pi(18{,}000 \text{ ksi})(0.5 \text{ in.})(1.0 \text{ in.})} = 0.008488 \text{ in.}$$

Middle segment ($L = 50$ in.)

$$\delta_2 = \frac{PL}{EA} = \frac{(3.0 \text{ k})(50 \text{ in.})}{(18{,}000 \text{ ksi})(\frac{\pi}{4})(1.0 \text{ in.})^2}$$

$$= 0.01061 \text{in.}$$

Elongation of bar

$$\delta = \sum \frac{NL}{EA} = 2\delta_1 + \delta_2$$

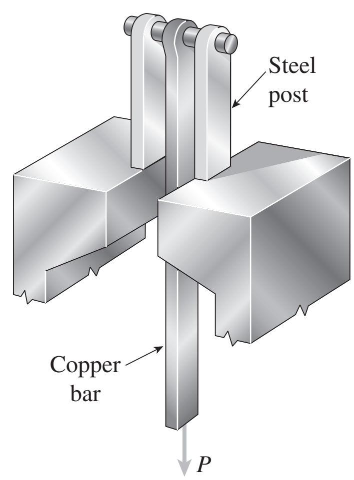
$$= 2(0.008488 \text{ in.}) + (0.01061 \text{ in.})$$

$$= 0.0276 \text{ in.} \quad \longleftarrow$$

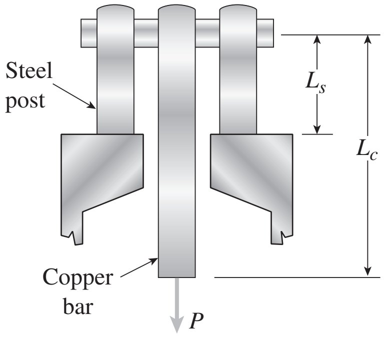
Problem 2.3-2 A long, rectangular copper bar under a tensile load P hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm^2, and a modulus of elasticity $E_c = 120$ GPa. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm^2, and a modulus of elasticity $E_s = 200$ GPa.

(a) Determine the downward displacement δ of the lower end of the copper bar due to a load $P = 180$ kN.

(b) What is the maximum permissible load P_{max} if the displacement δ is limited to 1.0 mm?



Solution 2.3-2 **Copper bar with a tensile load**



$L_c = 2.0$ m

$A_c = 4800$ mm^2

$E_c = 120$ GPa

$L_s = 0.5$ m

$A_s = 4500$ mm^2

$E_s = 200$ GPa

(a) Downward displacement δ ($P = 180$ kN)

$$\delta_c = \frac{PL_c}{E_c A_c} = \frac{(180\text{ kN})(2.0\text{ m})}{(120\text{ GPa})(4800\text{ mm}^2)}$$

$$= 0.625\text{ mm}$$

$$\delta_s = \frac{(P/2)L_s}{E_s A_s} = \frac{(90\text{ kN})(0.5\text{ m})}{(200\text{ GPa})(4500\text{ mm}^2)}$$

$$= 0.050\text{ mm}$$

$$\delta = \delta_c + \delta_s = 0.625\text{ mm} + 0.050\text{ mm}$$

$$= 0.675\text{ mm} \longleftarrow$$

(b) Maximum load P_{max} ($\delta_{max} = 1.0$ mm)

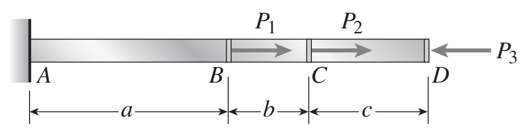
$$\frac{P_{max}}{P} = \frac{\delta_{max}}{\delta} \qquad P_{max} = P\left(\frac{\delta_{max}}{\delta}\right)$$

$$P_{max} = (180\text{ kN})\left(\frac{1.0\text{ mm}}{0.675\text{ mm}}\right) = 267\text{ kN} \longleftarrow$$

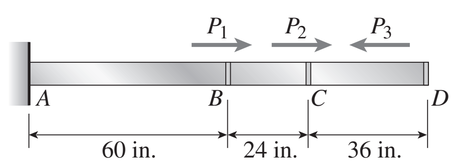
Problem 2.3-3 A steel bar AD (see figure) has a cross-sectional area of 0.40 in.^{2} and is loaded by forces $P_1 = 2700$ lb, $P_2 = 1800$ lb, and $P_3 = 1300$ lb. The lengths of the segments of the bar are $a = 60$ in., $b = 24$ in., and $c = 36$ in.

(a) Assuming that the modulus of elasticity $E = 30 \times 10^6$ psi, calculate the change in length δ of the bar. Does the bar elongate or shorten?

(b) By what amount P should the load P_3 be increased so that the bar does not change in length when the three loads are applied?



Solution 2.3-3 **Steel bar loaded by three forces**



$A = 0.40$ in.^{2} $\quad P_1 = 2700$ lb $\quad P_2 = 1800$ lb

$P_3 = 1300$ lb $\quad E = 30 \times 10^6$ psi

Axial forces

$N_{AB} = P_1 + P_2 - P_3 = 3200$ lb

$N_{BC} = P_2 - P_3 = 500$ lb

$N_{CD} = -P_3 = -1300$ lb

(a) Change in length

$$\delta = \sum \frac{N_i L_i}{E_i A_i}$$

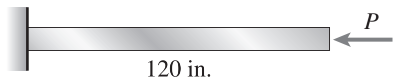
$$= \frac{1}{EA}(N_{AB}L_{AB} + N_{BC}L_{BC} + N_{CD}L_{CD})$$

$$= \frac{1}{(30 \times 10^6\text{ psi})(0.40\text{ in.}^2)}[(3200\text{ lb})(60\text{ in.})$$

$$+ (500\text{ lb})(24\text{ in.}) - (1300\text{ lb})(36\text{ in.})]$$

$$= 0.0131\text{ in. (elongation)} \longleftarrow$$

(b) INCREASE IN P_3 FOR NO CHANGE IN LENGTH



P = increase in force P_3

The force P must produce a shortening equal to 0.0131 in. in order to have no change in length.

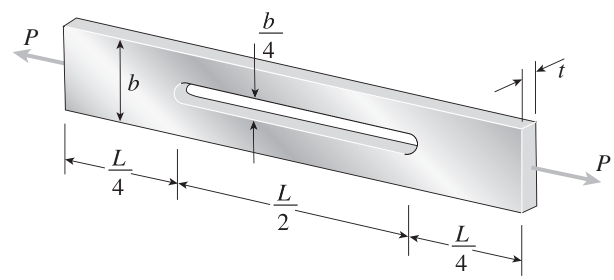
$$\therefore\ 0.0131\text{ in.} = \delta = \frac{PL}{EA}$$

$$= \frac{P(120\text{ in.})}{(30 \times 10^6\text{ psi})(0.40\text{ in.}^2)}$$

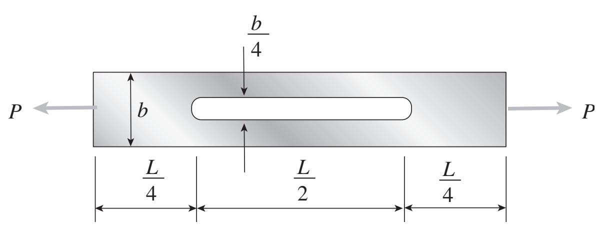
$$P = 1310\text{ lb} \quad \longleftarrow$$

Problem 2.3-4 A rectangular bar of length L has a slot in the middle half of its length (see figure). The bar has width b, thickness t, and modulus of elasticity E. The slot has width $b/4$.

(a) Obtain a formula for the elongation δ of the bar due to the axial loads P.
(b) Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.



Solution 2.3-4 Bar with a slot



t = thickness $\qquad L$ = length of bar

(a) ELONGATION OF BAR

$$\delta = \sum \frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E(\frac{3}{4}bt)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \longleftarrow$$

STRESS IN MIDDLE REGION

$$\sigma = \frac{P}{A} = \frac{P}{(\frac{3}{4}bt)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

$$= \frac{7\sigma L}{8E}$$

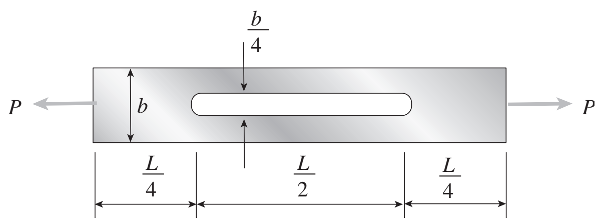
(b) SUBSTITUTE NUMERICAL VALUES:

$\sigma = 160$ MPa $\quad L = 750$ mm $\quad E = 210$ GPa

$$\delta = \frac{7(160\text{ MPa})(750\text{ mm})}{8\,(210\text{ GPa})} = 0.500\text{ mm} \quad \longleftarrow$$

Problem 2.3-5 Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is 30×10^6 psi.

Solution 2.3-5 Bar with a slot



t = thickness $\quad L$ = length of bar

(a) Elongation of bar

$$\delta = \sum \frac{N_i L_i}{EA_i} = \frac{P(L/4)}{E(bt)} + \frac{P(L/2)}{E\,(\frac{3}{4}\,bt)} + \frac{P(L/4)}{E(bt)}$$

$$= \frac{PL}{Ebt}\left(\frac{1}{4} + \frac{4}{6} + \frac{1}{4}\right) = \frac{7PL}{6Ebt} \quad \longleftarrow$$

Stress in middle region

$$\sigma = \frac{P}{A} = \frac{P}{(\frac{3}{4}\,bt)} = \frac{4P}{3bt} \quad \text{or} \quad \frac{P}{bt} = \frac{3\sigma}{4}$$

Substitute into the equation for δ:

$$\delta = \frac{7PL}{6Ebt} = \frac{7L}{6E}\left(\frac{P}{bt}\right) = \frac{7L}{6E}\left(\frac{3\sigma}{4}\right)$$

$$= \frac{7\sigma L}{8E}$$

(b) Substitute numerical values:

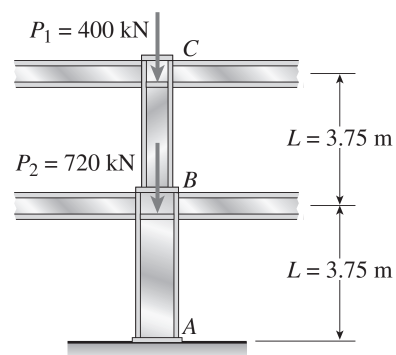
$\sigma = 24{,}000$ psi $\quad L = 30$ in.

$E = 30 \times 10^6$ psi

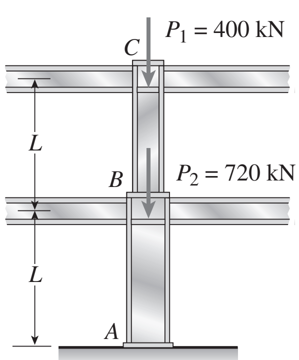
$$\delta = \frac{7(24{,}000 \text{ psi})(30 \text{ in.})}{8(30 \times 10^6 \text{ psi})} = 0.0210 \text{ in.} \quad \longleftarrow$$

Problem 2.3-6 A two-story building has steel columns AB in the first floor and BC in the second floor, as shown in the figure. The roof load P_1 equals 400 kN and the second-floor load P_2 equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are 11,000 mm^2 and 3,900 mm^2, respectively.

(a) Assuming that $E = 206$ GPa, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2.
(b) How much additional load P_0 can be placed at the top of the column (point C) if the total shortening δ_{AC} is not to exceed 4.0 mm?



Solution 2.3-6 Steel columns in a building



L = length of each column

$= 3.75$ m

$E = 206$ GPa

$A_{AB} = 11{,}000$ mm^2

$A_{BC} = 3{,}900$ mm^2

(a) Shortening δ_{AC} of the two columns

$$\delta_{AC} = \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB}L}{EA_{AB}} + \frac{N_{BC}L}{EA_{BC}}$$

$$= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11{,}000 \text{ mm}^2)} + \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3{,}900 \text{ mm}^2)}$$

$$= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm}$$

$$\delta_{AC} = 3.72 \text{ mm} \quad \longleftarrow$$

(b) ADDITIONAL LOAD P_0 AT POINT C

$(\delta_{AC})_{max} = 4.0$ mm

δ_0 = additional shortening of the two columns due to the load P_0

$$\delta_0 = (\delta_{AC})_{max} - \delta_{AC} = 4.0 \text{ mm} - 3.7206 \text{ mm}$$
$$= 0.2794 \text{ mm}$$

Also, $\delta_0 = \dfrac{P_0 L}{EA_{AB}} + \dfrac{P_0 L}{EA_{BC}} = \dfrac{P_0 L}{E}\left(\dfrac{1}{A_{AB}} + \dfrac{1}{A_{BC}}\right)$

Solve for P_0:

$$P_0 = \frac{E\delta_0}{L}\left(\frac{A_{AB}A_{BC}}{A_{AB} + A_{BC}}\right)$$

SUBSTITUTE NUMERICAL VALUES:

$E = 206 \times 10^9 \text{ N/m}^2 \quad \delta_0 = 0.2794 \times 10^{-3}$ m

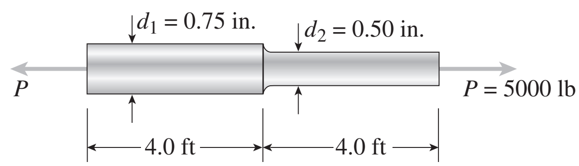
$L = 3.75 \text{ m} \quad A_{AB} = 11{,}000 \times 10^{-6} \text{ m}^2$

$A_{BC} = 3{,}900 \times 10^{-6} \text{ m}^2$

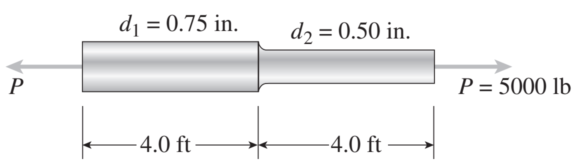
$P_0 = 44{,}200 \text{ N} = 44.2 \text{ kN}$ ←

Problem 2.3-7 A steel bar 8.0 ft long has a circular cross section of diameter $d_1 = 0.75$ in. over one-half of its length and diameter $d_2 = 0.5$ in. over the other half (see figure). The modulus of elasticity $E = 30 \times 10^6$ psi.

(a) How much will the bar elongate under a tensile load $P = 5000$ lb?
(b) If the same volume of material is made into a bar of constant diameter d and length 8.0 ft, what will be the elongation under the same load P?



Solution 2.3-7 **Bar in tension**



$P = 5000$ lb

$E = 30 \times 10^6$ psi

$L = 4 \text{ ft} = 48$ in.

(a) ELONGATION OF NONPRISMATIC BAR

$$\delta = \sum \frac{N_i L_i}{E_i A_i} = \frac{PL}{E}\sum \frac{1}{A_i}$$

$$\delta = \frac{(5000 \text{ lb})(48 \text{ in.})}{30 \times 10^6 \text{ psi}} \times \left[\frac{1}{\frac{\pi}{4}(0.75 \text{ in})^2} + \frac{1}{\frac{\pi}{4}(0.50 \text{ in.})^2}\right]$$

$= 0.0589$ in. ←

(b) ELONGATION OF PRISMATIC BAR OF SAME VOLUME

Original bar: $V_o = A_1L + A_2L = L(A_1 + A_2)$

Prismatic bar: $V_p = A_p(2L)$

Equate volumes and solve for A_p:

$V_o = V_p \qquad L(A_1 + A_2) = A_p(2L)$

$$A_p = \frac{A_1 + A_2}{2} = \frac{1}{2}\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= \frac{\pi}{8}[(0.75 \text{ in.})^2 + (0.50 \text{ in.})^2] = 0.3191 \text{ in.}^2$$

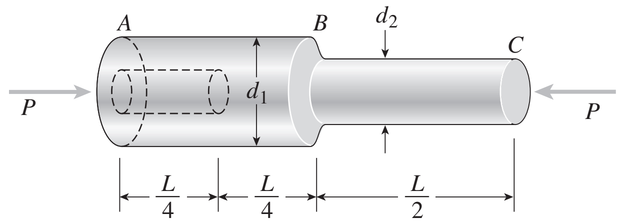
$$\delta = \frac{P(2L)}{EA_p} = \frac{(5000 \text{ lb})(2)(48 \text{ in.})}{(30 \times 10^6 \text{ psi})(0.3191 \text{ in.}^2)}$$

$= 0.0501$ in. ←

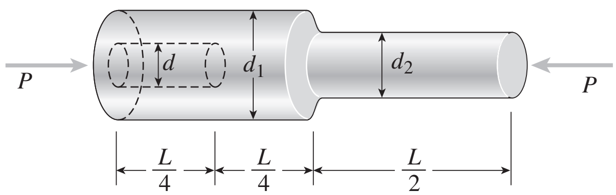
NOTE: A prismatic bar of the same volume will *always* have a smaller change in length than will a nonprismatic bar, provided the constant axial load P, modulus E, and total length L are the same.

Problem 2.3-8 A bar *ABC* of length *L* consists of two parts of equal lengths but different diameters (see figure). Segment *AB* has diameter $d_1 = 100$ mm and segment *BC* has diameter $d_2 = 60$ mm. Both segments have length $L/2 = 0.6$ m. A longitudinal hole of diameter *d* is drilled through segment *AB* for one-half of its length (distance $L/4 = 0.3$ m). The bar is made of plastic having modulus of elasticity $E = 4.0$ GPa. Compressive loads $P = 110$ kN act at the ends of the bar.

If the shortening of the bar is limited to 8.0 mm, what is the maximum allowable diameter d_{max} of the hole?



Solution 2.3-8 **Bar with a hole**



d = diameter of hole

SHORTENING δ OF THE BAR

$$\delta = \sum \frac{N_i L_i}{E_i A_i} = \frac{P}{E} \sum \frac{L_i}{A_i}$$

$$= \frac{P}{E}\left[\frac{L/4}{\frac{\pi}{4}(d_1^2 - d^2)} + \frac{L/4}{\frac{\pi}{4}d_1^2} + \frac{L/2}{\frac{\pi}{4}d_2^2}\right]$$

$$= \frac{PL}{\pi E}\left(\frac{1}{d_1^2 - d^2} + \frac{1}{d_1^2} + \frac{2}{d_2^2}\right) \qquad \text{(Eq. 1)}$$

NUMERICAL VALUES (DATA):

δ = maximum allowable shortening of the bar $= 8.0$ mm

$P = 110$ kN $\quad L = 1.2$ m $\quad E = 4.0$ GPa

$d_1 = 100$ mm

d_{max} = maximum allowable diameter of the hole

$d_2 = 60$ mm

SUBSTITUTE NUMERICAL VALUES INTO EQ. (1) FOR δ AND SOLVE FOR $d = d_{max}$:

UNITS: Newtons and meters

$$0.008 = \frac{(110{,}000)(1.2)}{\pi(4.0 \times 10^9)} \times \left[\frac{1}{(0.1)^2 - d^2} + \frac{1}{(0.1)^2} + \frac{2}{(0.06)^2}\right]$$

$$761.598 = \frac{1}{0.01 - d^2} + \frac{1}{0.01} + \frac{2}{0.0036}$$

$$\frac{1}{0.01 - d^2} = 761.598 - 100 - 555.556 = 106.042$$

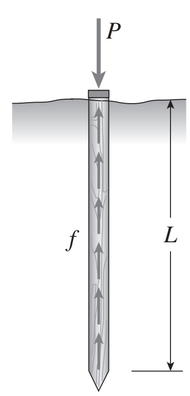
$$d^2 = 569.81 \times 10^{-6} \text{ m}^2$$

$$d = 0.02387 \text{ m}$$

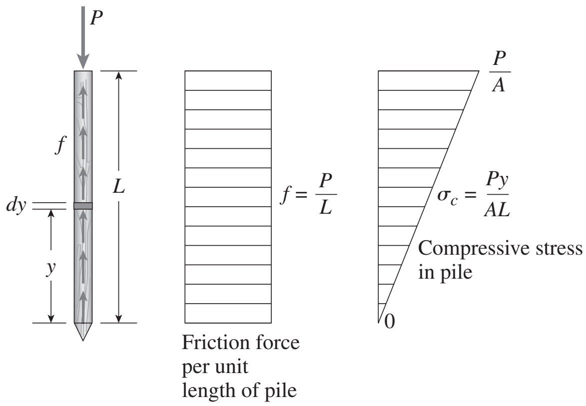
$$d_{max} = 23.9 \text{ mm} \longleftarrow$$

Problem 2.3-9 A wood pile, driven into the earth, supports a load *P* entirely by friction along its sides (see figure). The friction force *f* per unit length of pile is assumed to be uniformly distributed over the surface of the pile. The pile has length *L*, cross-sectional area *A*, and modulus of elasticity *E*.

(a) Derive a formula for the shortening δ of the pile in terms of *P*, *L*, *E*, and *A*.
(b) Draw a diagram showing how the compressive stress σ_c varies throughout the length of the pile.



Solution 2.3-9 Wood pile with friction



From free-body diagram of pile:

$$\Sigma F_{\text{vert}} = 0 \quad \uparrow_+ \ \downarrow^- \quad fL - P = 0 \quad f = \frac{P}{L} \qquad \text{(Eq. 1)}$$

(a) Shortening δ of pile:

At distance y from the base:

$$N(y) = \text{axial force} \qquad N(y) = fy \qquad \text{(Eq. 2)}$$

$$d\delta = \frac{N(y)\,dy}{EA} = \frac{fy\,dy}{EA}$$

$$\delta = \int_0^L d\delta = \frac{f}{EA}\int_0^L y\,dy = \frac{fL^2}{2EA} = \frac{PL}{2EA}$$

$$\delta = \frac{PL}{2EA} \quad \longleftarrow$$

(b) Compressive stress σ_c in pile

$$\sigma_c = \frac{N(y)}{A} = \frac{fy}{A} = \frac{Py}{AL} \quad \longleftarrow$$

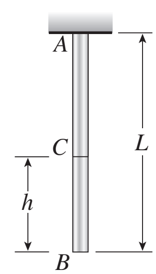
At the base ($y = 0$): $\sigma_c = 0$

At the top($y = L$): $\sigma_c = \dfrac{P}{A}$

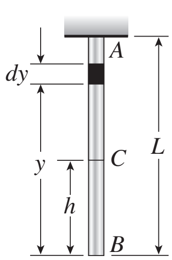
See the diagram above.

Problem 2.3-10 A prismatic bar AB of length L, cross-sectional area A, modulus of elasticity E, and weight W hangs vertically under its own weight (see figure).

(a) Derive a formula for the downward displacement δ_C of point C, located at distance h from the lower end of the bar.
(b) What is the elongation δ_B of the entire bar?
(c) What is the ratio β of the elongation of the upper half of the bar to the elongation of the lower half of the bar?



Solution 2.3-10 Prismatic bar hanging vertically



W = Weight of bar

(a) Downward displacement δ_C

Consider an element at distance y from the lower end.

$$N(y) = \frac{Wy}{L} \qquad d\delta = \frac{N(y)dy}{EA} = \frac{Wydy}{EAL}$$

$$\delta_C = \int_h^L d\delta = \int_h^L \frac{Wydy}{EAL} = \frac{W}{2EAL}(L^2 - h^2)$$

$$\delta_C = \frac{W}{2EAL}(L^2 - h^2) \quad \longleftarrow$$

(b) Elongation of bar ($h = 0$)

$$\delta_B = \frac{WL}{2EA} \quad \longleftarrow$$

(c) Ratio of elongations

Elongation of upper half of bar $\left(h = \dfrac{L}{2}\right)$:

$$\delta_{\text{upper}} = \frac{3WL}{8EA}$$

Elongation of lower half of bar:

$$\delta_{\text{lower}} = \delta_B - \delta_{\text{upper}} = \frac{WL}{2EA} - \frac{3WL}{8EA} = \frac{WL}{8EA}$$

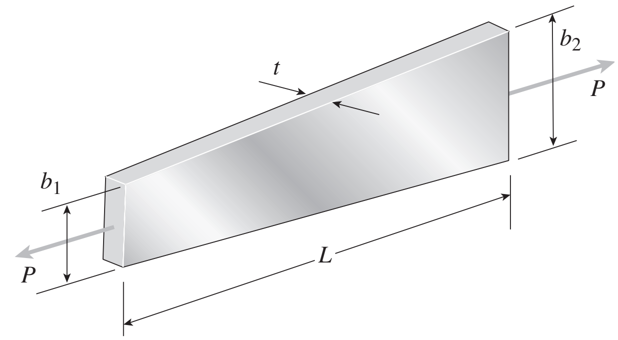
$$\beta = \frac{\delta_{\text{upper}}}{\delta_{\text{lower}}} = \frac{3/8}{1/8} = 3 \quad \longleftarrow$$

Problem 2.3-11 A flat bar of rectangular cross section, length L, and constant thickness t is subjected to tension by forces P (see figure). The width of the bar varies linearly from b_1 at the smaller end to b_2 at the larger end. Assume that the angle of taper is small.

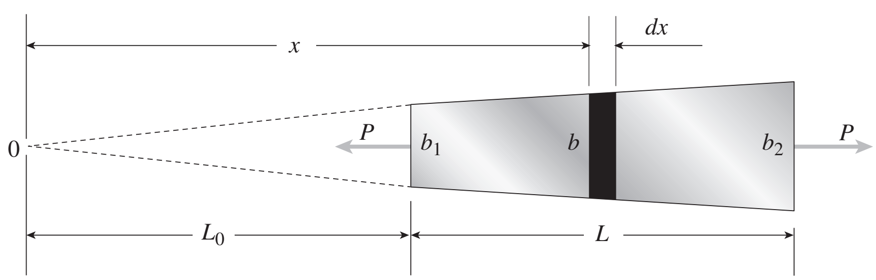
(a) Derive the following formula for the elongation of the bar:

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$$

(b) Calculate the elongation, assuming $L = 5$ ft, $t = 1.0$ in., $P = 25$ k, $b_1 = 4.0$ in., $b_2 = 6.0$ in., and $E = 30 \times 10^6$ psi.



Solution 2.3-11 **Tapered bar (rectangular cross section)**



t = thickness (constant)

$$b = b_1\left(\frac{x}{L_0}\right) \quad b_2 = b_1\left(\frac{L_0 + L}{L_0}\right) \qquad \text{(Eq. 1)}$$

$$A(x) = bt = b_1 t\left(\frac{x}{L_0}\right)$$

(a) Elongation of the bar

$$d\delta = \frac{Pdx}{EA(x)} = \frac{PL_0\,dx}{Eb_1tx}$$

$$\delta = \int_{L_0}^{L_0+L} d\delta = \frac{PL_0}{Eb_1t}\int_{L_0}^{L_0+L}\frac{dx}{x}$$

$$= \frac{PL_0}{Eb_1t}\ln x\Big|_{L_0}^{L_0+L} = \frac{PL_0}{Eb_1t}\ln\frac{L_0+L}{L_0} \qquad \text{(Eq. 2)}$$

From Eq. (1): $$\frac{L_0 + L}{L_0} = \frac{b_2}{b_1} \qquad \text{(Eq. 3)}$$

Solve Eq. (3) for L_0: $$L_0 = L\left(\frac{b_1}{b_2 - b_1}\right) \qquad \text{(Eq. 4)}$$

Substitute Eqs. (3) and (4) into Eq. (2):

$$\delta = \frac{PL}{Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \longleftarrow \qquad \text{(Eq. 5)}$$

(b) Substitute numerical values:

$L = 5$ ft $= 60$ in. $\qquad t = 10$ in.

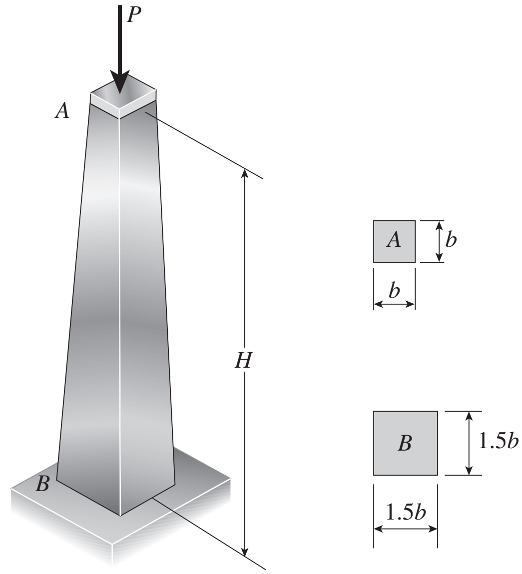
$P = 25$ k $\qquad b_1 = 4.0$ in.

$b_2 = 6.0$ in. $\qquad E = 30 \times 10^6$ psi

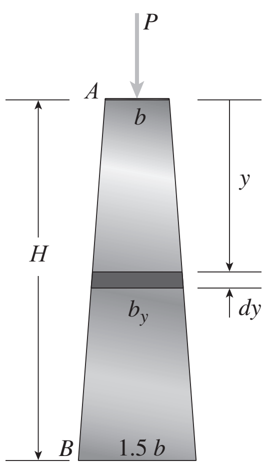
From Eq. (5): $\delta = 0.010$ in. $\longleftarrow$

Problem 2.3-12 A post AB supporting equipment in a laboratory is tapered uniformly throughout its height H (see figure). The cross sections of the post are square, with dimensions $b \times b$ at the top and $1.5b \times 1.5b$ at the base.

Derive a formula for the shortening δ of the post due to the compressive load P acting at the top. (Assume that the angle of taper is small and disregard the weight of the post itself.)



Solution 2.3-12 **Tapered post**



Square cross sections

b = width at A

$1.5b$ = width at B

b_y = width at distance y

$$= b + (1.5b - b)\frac{y}{H}$$

$$= \frac{b}{H}(H + 0.5y)$$

A_y = cross-sectional area at distance y

$$= (b_y)^2 = \frac{b^2}{H^2}(H + 0.5y)^2$$

Shortening of element dy

$$d\delta = \frac{Pdy}{EA_y} = \frac{Pdy}{E\left(\frac{b^2}{H^2}\right)(H + 0.5y)^2}$$

Shortening of entire post

$$\delta = \int d\delta = \frac{PH^2}{Eb^2}\int_0^H \frac{dy}{(H + 0.5y)^2}$$

From Appendix C: $\displaystyle\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

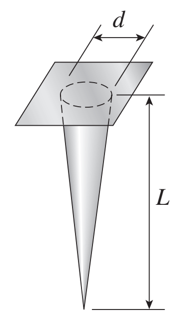
$$\delta = \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(H + 0.5y)}\right]_0^H$$

$$= \frac{PH^2}{Eb^2}\left[-\frac{1}{(0.5)(1.5H)} + \frac{1}{0.5H}\right]$$

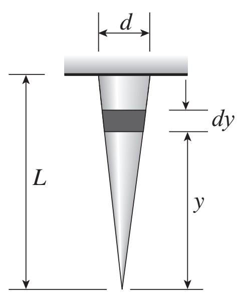
$$= \frac{2PH}{3Eb^2} \quad \longleftarrow$$

Problem 2.3-13 A long, slender bar in the shape of a right circular cone with length L and base diameter d hangs vertically under the action of its own weight (see figure). The weight of the cone is W and the modulus of elasticity of the material is E.

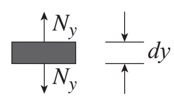
Derive a formula for the increase δ in the length of the bar due to its own weight. (Assume that the angle of taper of the cone is small.)



Solution 2.3-13 **Conical bar hanging vertically**



ELEMENT OF BAR



W=weight of cone

TERMINOLOGY

N_y = axial force acting on element dy

A_y = cross-sectional area at element dy

A_B = cross-sectional area at base of cone

$$= \frac{\pi d^2}{4}$$

V = volume of cone

$$= \frac{1}{3} A_B L$$

V_y = volume of cone below element dy

$$= \frac{1}{3} A_y y$$

W_y = weight of cone below element dy

$$= \frac{V_y}{V}(W) = \frac{A_y y W}{A_B L}$$

$N_y = W_y$

ELONGATION OF ELEMENT dy

$$d\delta = \frac{N_y\, dy}{E A_y} = \frac{Wy\, dy}{E A_B L} = \frac{4W}{\pi d^2 EL}\, y\, dy$$

ELONGATION OF CONICAL BAR

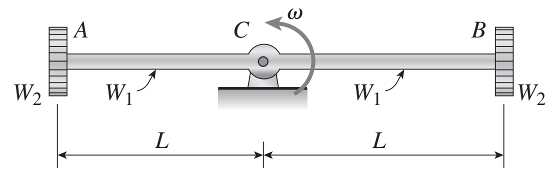
$$\delta = \int d\delta = \frac{4W}{\pi d^2 EL} \int_0^L y\, dy = \frac{2WL}{\pi d^2 E} \quad \longleftarrow$$

Problem 2.3-14 A bar *ABC* revolves in a horizontal plane about a vertical axis at the midpoint *C* (see figure). The bar, which has length $2L$ and cross-sectional area *A*, revolves at constant angular speed ω. Each half of the bar (*AC* and *BC*) has weight W_1 and supports a weight W_2 at its end.

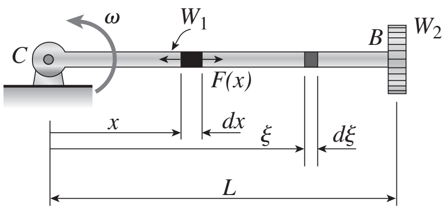
Derive the following formula for the elongation of one-half of the bar (that is, the elongation of either *AC* or *BC*):

$$\delta = \frac{L^2\omega^2}{3gEA}(W_1 + 3W_2)$$

in which *E* is the modulus of elasticity of the material of the bar and *g* is the acceleration of gravity.



Solution 2.3-14 Rotating bar



ω = angular speed

A = cross-sectional area

E = modulus of elasticity

g = acceleration of gravity

$F(x)$ = axial force in bar at distance x from point C

Consider an element of length dx at distance x from point C.

To find the force $F(x)$ acting on this element, we must find the inertia force of the part of the bar from distance x to distance L, plus the inertia force of the weight W_2.

Since the inertia force varies with distance from point C, we now must consider an element of length $d\xi$ at distance ξ, where ξ varies from x to L.

Mass of element $d\xi = \frac{d\xi}{L}\left(\frac{W_1}{g}\right)$

Acceleration of element $= \xi\omega^2$

Centrifugal force produced by element

$$= (\text{mass})(\text{acceleration}) = \frac{W_1\omega^2}{gL}\xi d\xi$$

Centrifugal force produced by weight W_2

$$= \left(\frac{W_2}{g}\right)(L\omega^2)$$

Axial force $F(x)$

$$F(x) = \int_{\xi=x}^{\xi=L} \frac{W_1\omega^2}{gL}\xi d\xi + \frac{W_2 L\omega^2}{g}$$

$$= \frac{W_1\omega^2}{2gL}(L^2 - x^2) + \frac{W_2 L\omega^2}{g}$$

Elongation of bar *BC*

$$\delta = \int_0^L \frac{F(x)\,dx}{EA}$$

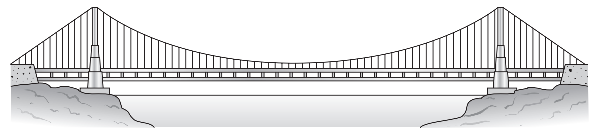
$$= \int_0^L \frac{W_1\omega^2}{2gLEA}(L^2 - x^2)dx + \int_0^L \frac{W_2 L\omega^2 dx}{gEA}$$

$$= \frac{W_1\omega^2}{2gLEA}\left[\int_0^L L^2\,dx - \int_0^L x^2\,dx\right] + \frac{W_2 L\omega^2}{gEA}\int_0^L dx$$

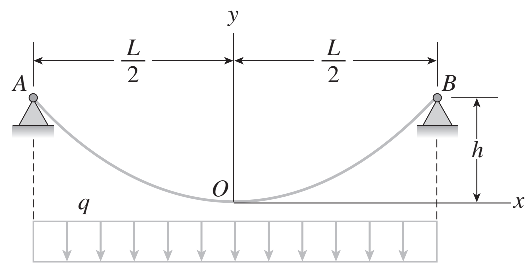
$$= \frac{W_1 L^2\omega^2}{3gEA} + \frac{W_2 L^2\omega^2}{gEA}$$

$$= \frac{L^2\omega^2}{3gEA}(W_1 + 3W_2) \longleftarrow$$

Problem 2.3-15 The main cables of a suspension bridge [see part (a) of the figure] follow a curve that is nearly parabolic because the primary load on the cables is the weight of the bridge deck, which is uniform in intensity along the horizontal. Therefore, let us represent the central region *AOB* of one of the main cables [see part (b) of the figure] as a parabolic cable supported at points *A* and *B* and carrying a uniform load of intensity q along the horizontal. The span of the cable is L, the sag is h, the axial rigidity is EA, and the origin of coordinates is at midspan.



(a)



(b)

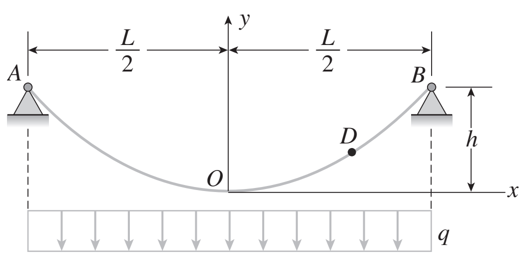
(a) Derive the following formula for the elongation of cable *AOB* shown in part (b) of the figure:

$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^2}\right)$$

(b) Calculate the elongation δ of the central span of one of the main cables of the Golden Gate Bridge, for which the dimensions and properties are $L = 4200$ ft, $h = 470$ ft, $q = 12{,}700$ lb/ft, and $E = 28{,}800{,}000$ psi. The cable consists of 27,572 parallel wires of diameter 0.196 in.

Hint: Determine the tensile force T at any point in the cable from a free-body diagram of part of the cable; then determine the elongation of an element of the cable of length ds; finally, integrate along the curve of the cable to obtain an equation for the elongation δ.

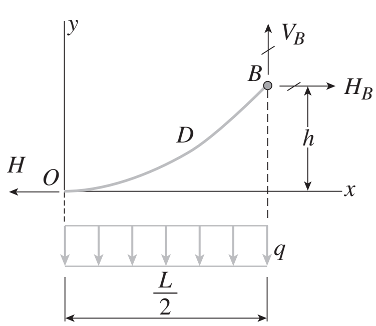
Solution 2.3-15 Cable of a suspension bridge



Equation of parabolic curve:

$$y = \frac{4hx^2}{L^2}$$

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$



FREE-BODY DIAGRAM OF HALF OF CABLE

$\Sigma M_B = 0$

$$-Hh + \frac{qL}{2}\left(\frac{L}{4}\right) = 0$$

$$H = \frac{qL^2}{8h}$$

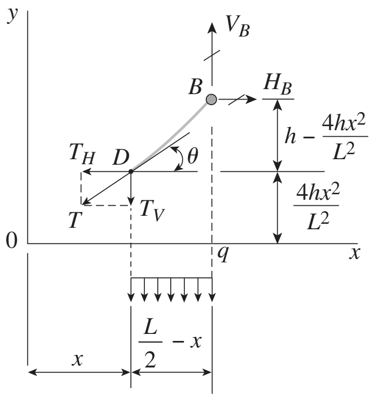
$$\Sigma F_{\text{horizontal}} = 0$$

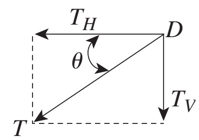
$$H_B = H = \frac{qL^2}{8h} \qquad \text{(Eq. 1)}$$

$$\Sigma F_{\text{vertical}} = 0$$

$$V_B = \frac{qL}{2} \qquad \text{(Eq. 2)}$$

FREE-BODY DIAGRAM OF SEGMENT DB OF CABLE





$$\Sigma F_{\text{horiz}} = 0 \qquad T_H = H_B = \frac{qL^2}{8h} \qquad \text{(Eq. 3)}$$

$$\Sigma F_{\text{vert}} = 0 \quad V_B - T_V - q\left(\frac{L}{2} - x\right) = 0$$

$$T_V = V_B - q\left(\frac{L}{2} - x\right) = \frac{qL}{2} - \frac{qL}{2} + qx$$

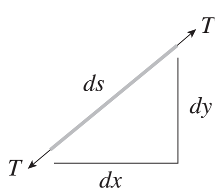
$$= qx \qquad \text{(Eq. 4)}$$

TENSILE FORCE T IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{\left(\frac{qL^2}{8h}\right)^2 + (qx)^2}$$

$$= \frac{qL^2}{8h}\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 5)}$$

ELONGATION $d\delta$ OF AN ELEMENT OF LENGTH ds



$$d\delta = \frac{T ds}{EA}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{1 + \left(\frac{8hx}{L^2}\right)^2}$$

$$= dx\sqrt{1 + \frac{64h^2x^2}{L^4}} \qquad \text{(Eq. 6)}$$

(a) ELONGATION δ OF CABLE AOB

$$\delta = \int d\delta = \int \frac{T\,ds}{EA}$$

Substitute for T from Eq. (5) and for ds from Eq. (6):

$$\delta = \frac{1}{EA}\int \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right)dx$$

For both halves of cable:

$$\delta = \frac{2}{EA}\int_0^{L/2} \frac{qL^2}{8h}\left(1 + \frac{64h^2x^2}{L^4}\right)dx$$

$$\delta = \frac{qL^3}{8hEA}\left(1 + \frac{16h^2}{3L^2}\right) \longleftarrow \qquad \text{(Eq. 7)}$$

(b) GOLDEN GATE BRIDGE CABLE

$L = 4200$ ft $\qquad h = 470$ ft
$q = 12{,}700$ lb/ft $\qquad E = 28{,}800{,}000$ psi

27,572 wires of diameter $d = 0.196$ in.

$$A = (27{,}572)\left(\frac{\pi}{4}\right)(0.196 \text{ in.})^2 = 831.90 \text{ in.}^2$$

Substitute into Eq. (7):

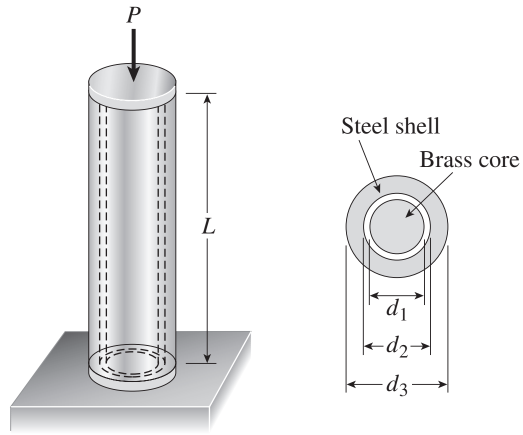
$\delta = 133.7$ in $= 11.14$ ft $\longleftarrow$

Statically Indeterminate Structures

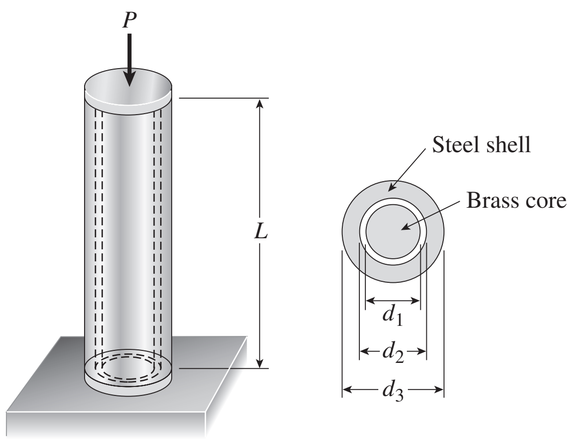
Problem 2.4-1 The assembly shown in the figure consists of a brass core (diameter $d_1 = 0.25$ in.) surrounded by a steel shell (inner diameter $d_2 = 0.28$ in., outer diameter $d_3 = 0.35$ in.). A load P compresses the core and shell, which have length $L = 4.0$ in. The moduli of elasticity of the brass and steel are $E_b = 15 \times 10^6$ psi and $E_s = 30 \times 10^6$ psi, respectively.

(a) What load P will compress the assembly by 0.003 in.?

(b) If the allowable stress in the steel is 22 ksi and the allowable stress in the brass is 16 ksi, what is the allowable compressive load P_{allow}? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-1 **Cylindrical assembly in compression**



$d_1 = 0.25$ in. $\quad E_b = 15 \times 10^6$ psi

$d_2 = 0.28$ in. $\quad E_s = 30 \times 10^6$ psi

$d_3 = 0.35$ in. $\quad A_s = \dfrac{\pi}{4}(d_3^2 - d_2^2) = 0.03464 \text{ in.}^2$

$L = 4.0$ in. $\quad A_b = \dfrac{\pi}{4} d_1^2 = 0.04909 \text{ in.}^2$

(a) Decrease in length ($\delta = 0.003$ in.)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_s A_s + E_b A_b} \quad \text{or}$$

$$P = (E_s A_s + E_b A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_s A_s + E_b A_b = (30 \times 10^6 \text{ psi})(0.03464 \text{ in.}^2) + (15 \times 10^6 \text{ psi})(0.04909 \text{ in.}^2) = 1.776 \times 10^6 \text{ lb}$$

$$P = (1.776 \times 10^6 \text{ lb})\left(\frac{0.003 \text{ in.}}{4.0 \text{ in.}}\right) = 1330 \text{ lb} \longleftarrow$$

(b) Allowable load

$\sigma_s = 22$ ksi $\qquad \sigma_b = 16$ ksi

Use Eqs. (2-12a and b) of Example 2-5.

For steel:

$$\sigma_s = \frac{PE_s}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_s}{E_s}$$

$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{22 \text{ ksi}}{30 \times 10^6 \text{ psi}}\right) = 1300 \text{ lb}$$

For brass:

$$\sigma_b = \frac{PE_b}{E_s A_s + E_b A_b} \qquad P_s = (E_s A_s + E_b A_b)\frac{\sigma_b}{E_b}$$

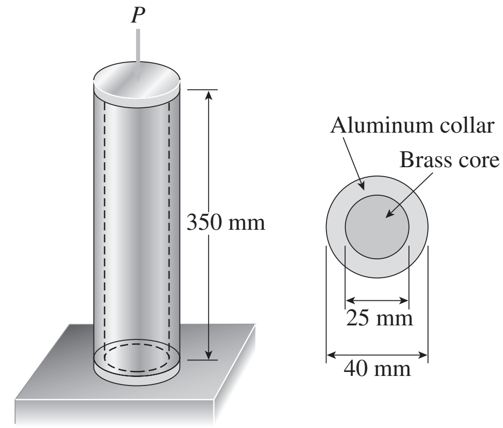
$$P_s = (1.776 \times 10^6 \text{ lb})\left(\frac{16 \text{ ksi}}{15 \times 10^6 \text{ psi}}\right) = 1890 \text{ lb}$$

Steel governs. $\quad P_{allow} = 1300 \text{ lb} \longleftarrow$

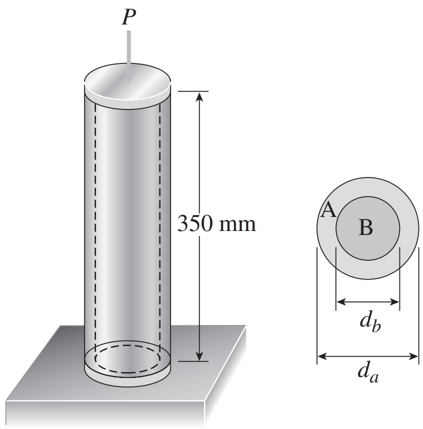
Problem 2.4-2 A cylindrical assembly consisting of a brass core and an aluminum collar is compressed by a load P (see figure). The length of the aluminum collar and brass core is 350 mm, the diameter of the core is 25 mm, and the outside diameter of the collar is 40 mm. Also, the moduli of elasticity of the aluminum and brass are 72 GPa and 100 GPa, respectively.

(a) If the length of the assembly decreases by 0.1% when the load P is applied, what is the magnitude of the load?

(b) What is the maximum permissible load P_{max} if the allowable stresses in the aluminum and brass are 80 MPa and 120 MPa, respectively? (*Suggestion:* Use the equations derived in Example 2-5.)



Solution 2.4-2 **Cylindrical assembly in compression**



A = aluminum

B = brass

L = 350 mm

d_a = 40 mm

d_b = 25 mm

$$A_a = \frac{\pi}{4}(d_a^2 - d_b^2)$$

$$= 765.8 \text{ mm}^2$$

E_a = 72 GPa E_b = 100 GPa $A_b = \frac{\pi}{4}d_b^2$

$$= 490.9 \text{ mm}^2$$

(a) DECREASE IN LENGTH

(δ = 0.1% of L = 0.350 mm)

Use Eq. (2-13) of Example 2-5.

$$\delta = \frac{PL}{E_a A_a + E_b A_b} \quad \text{or}$$

$$P = (E_a A_a + E_b A_b)\left(\frac{\delta}{L}\right)$$

Substitute numerical values:

$$E_a A_a + E_b A_b = (72 \text{ GPa})(765.8 \text{ mm}^2) + (100 \text{ GPa})(490.9 \text{ mm}^2)$$

$$= 55.135 \text{ MN} + 49.090 \text{ MN}$$

$$= 104.23 \text{ MN}$$

$$P = (104.23 \text{ MN})\left(\frac{0.350 \text{ mm}}{350 \text{ mm}}\right)$$

$$= 104.2 \text{ kN} \quad \longleftarrow$$

(b) ALLOWABLE LOAD

σ_a = 80 MPa σ_b = 120 MPa

Use Eqs. (2-12a and b) of Example 2-5.

For aluminum:

$$\sigma_a = \frac{PE_a}{E_a A_a + E_b A_b} \qquad P_a = (E_a A_a + E_b A_b)\left(\frac{\sigma_a}{E_a}\right)$$

$$P_a = (104.23 \text{ MN})\left(\frac{80 \text{ MPa}}{72 \text{ GPa}}\right) = 115.8 \text{ kN}$$

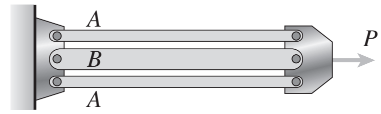
For brass:

$$\sigma_b = \frac{PE_b}{E_a A_a + E_b A_b} \qquad P_b = (E_a A_a + E_b A_b)\left(\frac{\sigma_b}{E_b}\right)$$

$$P_b = (104.23 \text{ MN})\left(\frac{120 \text{ MPa}}{100 \text{ GPa}}\right) = 125.1 \text{ kN}$$

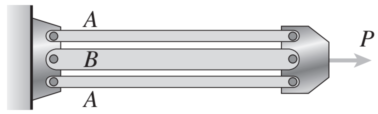
Aluminum governs. P_{max} = 116 kN $\longleftarrow$

Problem 2.4-3 Three prismatic bars, two of material A and one of material B, transmit a tensile load P (see figure). The two outer bars (material A) are identical. The cross-sectional area of the middle bar (material B) is 50% larger than the cross-sectional area of one of the outer bars. Also, the modulus of elasticity of material A is twice that of material B.

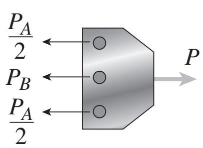


(a) What fraction of the load P is transmitted by the middle bar?
(b) What is the ratio of the stress in the middle bar to the stress in the outer bars?
(c) What is the ratio of the strain in the middle bar to the strain in the outer bars?

Solution 2.4-3 Prismatic bars in tension



FREE-BODY DIAGRAM OF END PLATE



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \qquad P_A + P_B - P = 0 \qquad (1)$$

EQUATION OF COMPATIBILITY

$$\delta_A = \delta_B \qquad (2)$$

FORCE-DISPLACEMENT RELATIONS

A_A = total area of both outer bars

$$\delta_A = \frac{P_A L}{E_A A_A} \qquad \delta_B = \frac{P_B L}{E_B A_B} \qquad (3)$$

Substitute into Eq. (2):

$$\frac{P_A L}{E_A A_A} = \frac{P_B L}{E_B A_B} \qquad (4)$$

SOLUTION OF THE EQUATIONS

Solve simultaneously Eqs. (1) and (4):

$$P_A = \frac{E_A A_A P}{E_A A_A + E_B A_B} \qquad P_B = \frac{E_B A_B P}{E_A A_A + E_B A_B} \qquad (5)$$

Substitute into Eq. (3):

$$\delta = \delta_A = \delta_B = \frac{PL}{E_A A_A + E_B A_B} \qquad (6)$$

STRESSES:

$$\sigma_A = \frac{P_A}{A_A} = \frac{E_A P}{E_A A_A + E_B A_B}$$
$$\sigma_B = \frac{P_B}{A_B} = \frac{E_B P}{E_A A_A + E_B A_B} \qquad (7)$$

(a) LOAD IN MIDDLE BAR

$$\frac{P_B}{P} = \frac{E_B A_B}{E_A A_A + E_B A_B} = \frac{1}{\dfrac{E_A A_A}{E_B A_B} + 1}$$

$$\text{Given: } \frac{E_A}{E_B} = 2 \qquad \frac{A_A}{A_B} = \frac{1+1}{1.5} = \frac{4}{3}$$

$$\therefore \frac{P_B}{P} = \frac{1}{\left(\dfrac{E_A}{E_B}\right)\left(\dfrac{A_A}{A_B}\right) + 1} = \frac{1}{\dfrac{8}{3} + 1} = \frac{3}{11} \longleftarrow$$

(b) RATIO OF STRESSES

$$\frac{\sigma_B}{\sigma_A} = \frac{E_B}{E_A} = \frac{1}{2} \longleftarrow$$

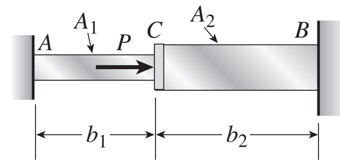
(c) RATIO OF STRAINS

All bars have the same strain

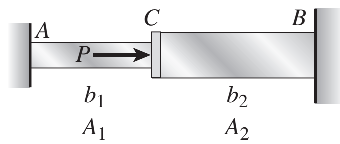
Ratio = 1 $\longleftarrow$

Problem 2.4-4 A bar *ACB* having two different cross-sectional areas A_1 and A_2 is held between rigid supports at *A* and *B* (see figure). A load *P* acts at point *C*, which is distance b_1 from end *A* and distance b_2 from end *B*.

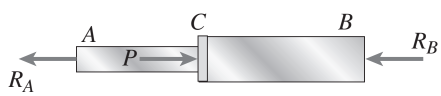
(a) Obtain formulas for the reactions R_A and R_B at supports *A* and *B*, respectively, due to the load *P*.
(b) Obtain a formula for the displacement δ_C of point *C*.
(c) What is the ratio of the stress σ_1 in region *AC* to the stress σ_2 in region *CB*?



Solution 2.4-4 **Bar with intermediate load**



FREE-BODY DIAGRAM



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \qquad R_A + R_B = P \qquad$ (Eq. 1)

EQUATION OF COMPATIBILITY

δ_{AC} = elongation of *AC*

δ_{CB} = shortening of *CB*

$\delta_{AC} = \delta_{CB} \qquad$ (Eq. 2)

FORCE DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A b_1}{EA_1} \qquad \delta_{CB} = \frac{R_B b_2}{EA_2} \qquad \text{(Eqs. 3\&4)}$$

(a) SOLUTION OF EQUATIONS

Substitute Eq. (3) and Eq. (4) into Eq. (2):

$$\frac{R_A b_1}{EA_1} = \frac{R_B b_2}{EA_2} \qquad \text{(Eq. 5)}$$

Solve Eq. (1) and Eq. (5) simultaneously:

$$R_A = \frac{b_2 A_1 P}{b_1 A_2 + b_2 A_1} \qquad R_B = \frac{b_1 A_2 P}{b_1 A_2 + b_2 A_1} \longleftarrow$$

(b) DISPLACEMENT OF POINT *C*

$$\delta_C = \delta_{AC} = \frac{R_A b_1}{EA_1} = \frac{b_1 b_2 P}{E(b_1 A_2 + b_2 A_1)} \longleftarrow$$

(c) RATIO OF STRESSES

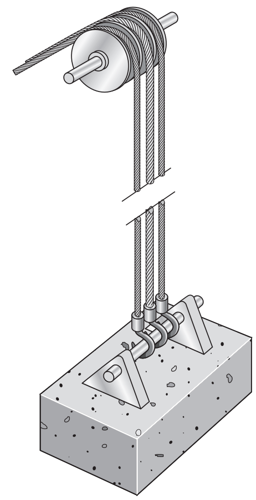
$$\sigma_1 = \frac{R_A}{A_1} \text{ (tension)} \qquad \sigma_2 = \frac{R_B}{A_2} \text{ (compression)}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{b_2}{b_1} \longleftarrow$$

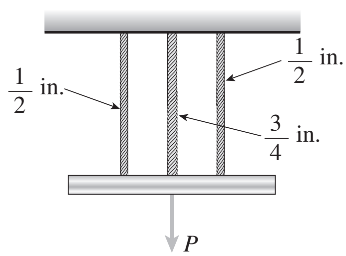
(Note that if $b_1 = b_2$, the stresses are numerically equal regardless of the areas A_1 and A_2.)

Problem 2.4-5 Three steel cables jointly support a load of 12 k (see figure). The diameter of the middle cable is ¾ in. and the diameter of each outer cable is ½ in. The tensions in the cables are adjusted so that each cable carries one-third of the load (i.e., 4 k). Later, the load is increased by 9 k to a total load of 21 k.

(a) What percent of the total load is now carried by the middle cable?
(b) What are the stresses σ_M and σ_O in the middle and outer cables, respectively? (*Note:* See Table 2-1 in Section 2.2 for properties of cables.)



Solution 2.4-5 **Three cables in tension**



AREAS OF CABLES (from Table 2-1)

Middle cable: $A_M = 0.268 \text{ in.}^2$

Outer cables: $A_O = 0.119 \text{ in.}^2$

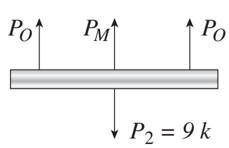
(for each cable)

FIRST LOADING

$$P_1 = 12 \text{ k} \left(\text{Each cable carries } \frac{P_1}{3} \text{ or } 4 \text{ k.}\right)$$

SECOND LOADING

$P_2 = 9$ k (additional load)



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \qquad 2P_O + P_M - P_2 = 0 \tag{1}$$

EQUATION OF COMPATIBILITY

$$\delta_M = \delta_O \tag{2}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_M = \frac{P_M L}{EA_M} \qquad \delta_O = \frac{P_o L}{EA_o} \tag{3, 4}$$

SUBSTITUTE INTO COMPATIBILITY EQUATION:

$$\frac{P_M L}{EA_M} = \frac{P_O L}{EA_O} \qquad \frac{P_M}{A_M} = \frac{P_O}{A_O} \tag{5}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$P_M = P_2\left(\frac{A_M}{A_M + 2A_O}\right) = (9 \text{ k})\left(\frac{0.268 \text{ in.}^2}{0.506 \text{ in.}^2}\right)$$

$$= 4.767 \text{ k}$$

$$P_o = P_2\left(\frac{A_o}{A_M + 2A_O}\right) = (9 \text{ k})\left(\frac{0.119 \text{ in.}^2}{0.506 \text{ in.}^2}\right)$$

$$= 2.117 \text{ k}$$

FORCES IN CABLES

Middle cable: Force = 4 k + 4.767 k = 8.767 k

Outer cables: Force = 4 k + 2.117 k = 6.117 k

(for each cable)

(a) PERCENT OF TOTAL LOAD CARRIED BY MIDDLE CABLE

$$\text{Percent} = \frac{8.767 \text{ k}}{21 \text{ k}}(100\%) = 41.7\% \quad \longleftarrow$$

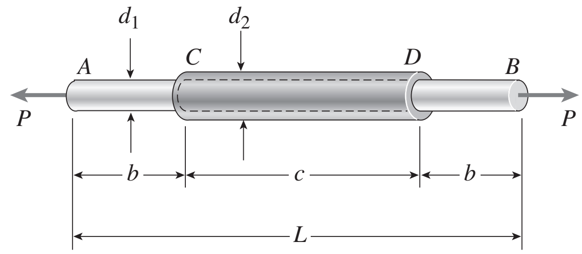
(b) STRESSES IN CABLES ($\sigma = P/A$)

$$\text{Middle cable: } \sigma_M = \frac{8.767 \text{ k}}{0.268 \text{ in.}^2} = 32.7 \text{ ksi} \quad \longleftarrow$$

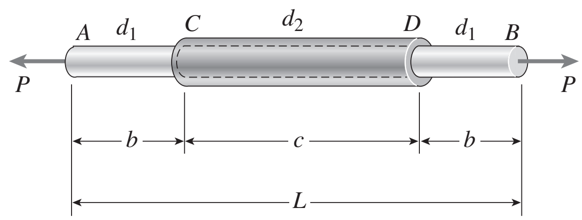
$$\text{Outer cables: } \sigma_O = \frac{6.117 \text{ k}}{0.119 \text{ in.}^2} = 51.4 \text{ ksi} \quad \longleftarrow$$

Problem 2.4-6 A plastic rod AB of length $L = 0.5$ m has a diameter $d_1 = 30$ mm (see figure). A plastic sleeve CD of length $c = 0.3$ m and outer diameter $d_2 = 45$ mm is securely bonded to the rod so that no slippage can occur between the rod and the sleeve. The rod is made of an acrylic with modulus of elasticity $E_1 = 3.1$ GPa and the sleeve is made of a polyamide with $E_2 = 2.5$ GPa.

(a) Calculate the elongation δ of the rod when it is pulled by axial forces $P = 12$ kN.
(b) If the sleeve is extended for the full length of the rod, what is the elongation?
(c) If the sleeve is removed, what is the elongation?



Solution 2.4-6 **Plastic rod with sleeve**



$P = 12$ kN $\quad d_1 = 30$ mm $\quad b = 100$ mm

$L = 500$ mm $\quad d_2 = 45$ mm $\quad c = 300$ mm

Rod: $E_1 = 3.1$ GPa

Sleeve: $E_2 = 2.5$ GPa

$$\text{Rod: } A_1 = \frac{\pi d_1^2}{4} = 706.86 \text{ mm}^2$$

$$\text{Sleeve: } A_2 = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$$E_1A_1 + E_2A_2 = 4.400 \text{ MN}$$

(a) Elongation of rod

$$\text{Part } AC\text{: } \delta_{AC} = \frac{Pb}{E_1A_1} = 0.5476 \text{ mm}$$

$$\text{Part } CD\text{: } \delta_{CD} = \frac{Pc}{E_1A_1E_2A_2}$$

$$= 0.81815 \text{ mm}$$

(From Eq. 2-13 of Example 2-5)

$$\delta = 2\delta_{AC} + \delta_{CD} = 1.91 \text{ mm} \quad \longleftarrow$$

(b) Sleeve at full length

$$\delta = \delta_{CD}\left(\frac{L}{c}\right) = (0.81815 \text{ mm})\left(\frac{500 \text{ mm}}{300 \text{ mm}}\right)$$

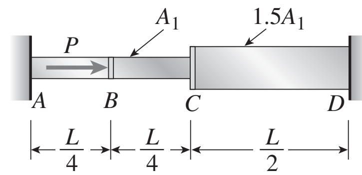
$$= 1.36 \text{ mm} \quad \longleftarrow$$

(c) Sleeve removed

$$\delta = \frac{PL}{E_1A_1} = 2.74 \text{ mm} \quad \longleftarrow$$

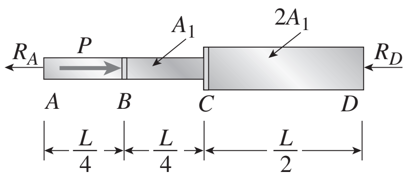
Problem 2.4-7 The axially loaded bar *ABCD* shown in the figure is held between rigid supports. The bar has cross-sectional area A_1 from *A* to *C* and $2A_1$ from *C* to *D*.

(a) Derive formulas for the reactions R_A and R_D at the ends of the bar.
(b) Determine the displacements δ_B and δ_C at points *B* and *C*, respectively.
(c) Draw a diagram in which the abscissa is the distance from the left-hand support to any point in the bar and the ordinate is the horizontal displacement δ at that point.



Solution 2.4-7 Bar with fixed ends

FREE-BODY DIAGRAM OF BAR



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \qquad R_A + R_D = P$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} + \delta_{BC} + \delta_{CD} = 0$ (Eq. 2)

Positive means elongation.

FORCE-DISPLACEMENT EQUATIONS

$$\delta_{AB} = \frac{R_A(L/4)}{EA_1} \qquad \delta_{BC} = \frac{(R_A - P)(L/4)}{EA_1} \qquad \text{(Eqs. 3, 4)}$$

$$\delta_{CD} = -\frac{R_D(L/2)}{E(2A_1)} \qquad \text{(Eq. 5)}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_A L}{4EA_1} + \frac{(R_A - P)(L)}{4EA_1} - \frac{R_D L}{4EA_1} = 0 \quad \text{(Eq. 6)}$$

(a) REACTIONS

Solve simultaneously Eqs. (1) and (6):

$$R_A = \frac{2P}{3} \qquad R_D = \frac{P}{3} \quad \longleftarrow$$

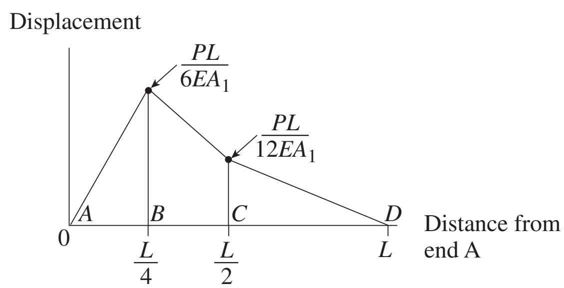
(b) DISPLACEMENTS AT POINTS *B* AND *C*

$$\delta_B = \delta_{AB} = \frac{R_A L}{4EA_1} = \frac{PL}{6EA_1} \text{ (To the right)} \quad \longleftarrow$$

$$\delta_C = |\delta_{CD}| = \frac{R_D L}{4EA_1}$$

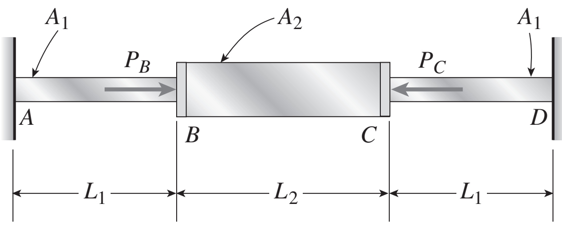
$$= \frac{PL}{12EA_1} \text{ (To the right)} \quad \longleftarrow$$

(c) DISPLACEMENT DIAGRAM

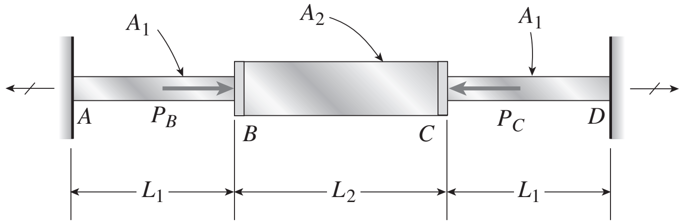


Problem 2.4-8 The fixed-end bar *ABCD* consists of three prismatic segments, as shown in the figure. The end segments have cross-sectional area $A_1 = 840\ \text{mm}^2$ and length $L_1 = 200$ mm. The middle segment has cross-sectional area $A_2 = 1260\ \text{mm}^2$ and length $L_2 = 250$ mm. Loads P_B and P_C are equal to 25.5 kN and 17.0 kN, respectively.

(a) Determine the reactions R_A and R_D at the fixed supports.
(b) Determine the compressive axial force F_{BC} in the middle segment of the bar.



Solution 2.4-8 Bar with three segments



$P_B = 25.5$ kN $\quad P_C = 17.0$ kN

$L_1 = 200$ mm $\quad L_2 = 250$ mm

$A_1 = 840\ \text{mm}^2 \quad A_2 = 1260\ \text{mm}^2$

m = meter

FREE-BODY DIAGRAM

EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{horiz}} = 0 \ \underset{+}{\rightarrow}\ \underset{-}{\leftarrow}$

$P_B + R_D - P_C - R_A = 0$ or

$$R_A - R_D = P_B - P_C = 8.5\ \text{kN} \qquad \text{(Eq. 1)}$$

EQUATION OF COMPATIBILITY

δ_{AD} = elongation of entire bar

$$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0 \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{R_A}{E}\left(238.095\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 3)}$$

$$\delta_{BC} = \frac{(R_A - P_B)L_2}{EA_2}$$

$$= \frac{R_A}{E}\left(198.413\ \frac{1}{\text{m}}\right) - \frac{P_B}{E}\left(198.413\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 4)}$$

$$\delta_{CD} = \frac{R_D L_1}{EA_1} = \frac{R_D}{E}\left(238.095\ \frac{1}{\text{m}}\right) \qquad \text{(Eq. 5)}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\frac{R_A}{E}\left(238.095\ \frac{1}{\text{m}}\right) + \frac{R_A}{E}\left(198.413\ \frac{1}{\text{m}}\right)$$

$$-\frac{P_B}{E}\left(198.413\ \frac{1}{\text{m}}\right) + \frac{R_D}{E}\left(238.095\ \frac{1}{\text{m}}\right) = 0$$

Simplify and substitute $P_B = 25.5$ kN:

$$R_A\left(436.508\ \frac{1}{\text{m}}\right) + R_D\left(238.095\ \frac{1}{\text{m}}\right)$$

$$= 5{,}059.53\ \frac{\text{kN}}{\text{m}} \qquad \text{(Eq. 6)}$$

(a) REACTIONS R_A AND R_D

Solve simultaneously Eqs. (1) and (6).

From (1): $R_D = R_A - 8.5$ kN

Substitute into (6) and solve for R_A:

$$R_A\left(674.603\ \frac{1}{\text{m}}\right) = 7083.34\ \frac{\text{kN}}{\text{m}}$$

$R_A = 10.5$ kN $\longleftarrow$

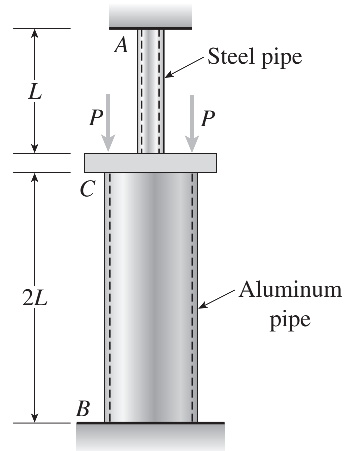
$R_D = R_A - 8.5\ \text{kN} = 2.0$ kN $\longleftarrow$

(b) COMPRESSIVE AXIAL FORCE F_{BC}

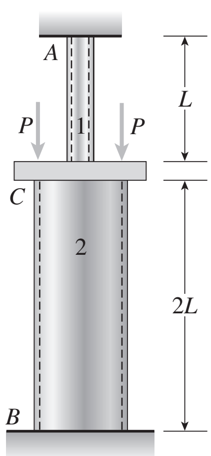
$F_{BC} = P_B - R_A = P_C - R_D = 15.0$ kN $\longleftarrow$

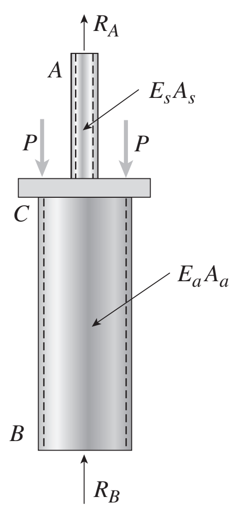
Problem 2.4-9 The aluminum and steel pipes shown in the figure are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. The aluminum pipe is twice as long as the steel pipe. Two equal and symmetrically placed loads P act on the plate at C.

(a) Obtain formulas for the axial stresses σ_a and σ_s in the aluminum and steel pipes, respectively.
(b) Calculate the stresses for the following data: $P = 12$ k, cross-sectional area of aluminum pipe $A_a = 8.92$ in.^{2}, cross-sectional area of steel pipe $A_s = 1.03$ in.^{2}, modulus of elasticity of aluminum $E_a = 10 \times 10^6$ psi, and modulus of elasticity of steel $E_s = 29 \times 10^6$ psi.



Solution 2.4-9 **Pipes with intermediate loads**





Pipe 1 is steel.
Pipe 2 is aluminum.

EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{vert}} = 0 \qquad R_A + R_B = 2P \qquad$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0 \qquad$ (Eq. 2)

(A positive value of δ means elongation.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A L}{E_s A_s} \qquad \delta_{BC} = -\frac{R_B(2L)}{E_a A_a} \qquad \text{(Eqs. 3, 4))}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3) and (4) into Eq. (2):

$$\frac{R_A L}{E_s A_s} - \frac{R_B(2L)}{E_a A_a} = 0 \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$R_A = \frac{4E_s A_s P}{E_a A_a + 2E_s A_s} \qquad R_B = \frac{2E_a A_a P}{E_a A_a + 2E_s A_s} \qquad \text{(Eqs. 6, 7)}$$

(a) AXIAL STRESSES

$$\text{Aluminum: } \sigma_a = \frac{R_B}{A_a} = \frac{2E_a P}{E_a A_a + 2E_s A_s} \longleftarrow \qquad \text{(Eq. 8)}$$

(compression)

$$\text{Steel: } \sigma_s = \frac{R_A}{A_s} = \frac{4E_s P}{E_a A_a + 2E_s A_s} \longleftarrow \qquad \text{(Eq. 9)}$$

(tension)

(b) NUMERICAL RESULTS

$P = 12$ k $\qquad A_a = 8.92$ in.^{2} $\qquad A_s = 1.03$ in.^{2}

$E_a = 10 \times 10^6$ psi $\qquad E_s = 29 \times 10^6$ psi

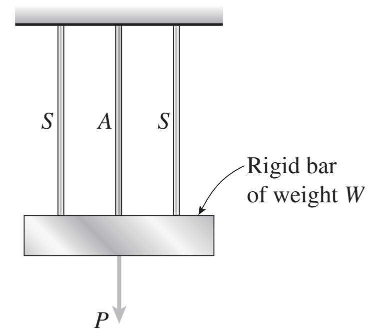
Substitute into Eqs. (8) and (9):

$\sigma_a = 1{,}610$ psi (compression) $\longleftarrow$

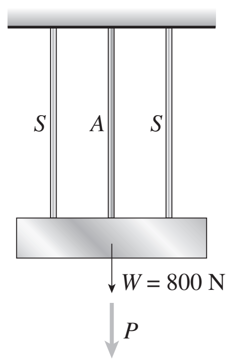
$\sigma_s = 9{,}350$ psi (tension) $\longleftarrow$

Problem 2.4-10 A rigid bar of weight $W = 800$ N hangs from three equally spaced vertical wires, two of steel and one of aluminum (see figure). The wires also support a load P acting at the midpoint of the bar. The diameter of the steel wires is 2 mm, and the diameter of the aluminum wire is 4 mm.

What load P_{allow} can be supported if the allowable stress in the steel wires is 220 MPa and in the aluminum wire is 80 MPa? (Assume $E_s = 210$ GPa and $E_a = 70$ GPa.)



Solution 2.4-10 **Rigid bar hanging from three wires**



STEEL WIRES

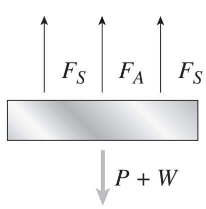
$d_s = 2$ mm $\quad \sigma_s = 220$ MPa $\quad E_s = 210$ GPa

ALUMINUM WIRES

$d_A = 4$ mm $\quad \sigma_A = 80$ MPa

$E_A = 70$ GPa

FREE-BODY DIAGRAM OF RIGID BAR



EQUATION OF EQUILIBRIUM

$\Sigma F_{vert} = 0$

$2F_s + F_A - P - W = 0$ (Eq. 1)

EQUATION OF COMPATIBILITY

$\delta_s = \delta_A$ (Eq. 2)

FORCE DISPLACEMENT RELATIONS

$$\delta_s = \frac{F_s L}{E_s A_s} \qquad \delta_A = \frac{F_A L}{E_A A_A} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_s L}{E_s A_s} = \frac{F_A L}{E_A A_A} \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$F_A = (P + W)\left(\frac{E_A A_A}{E_A A_A + 2E_s A_s}\right) \qquad \text{(Eq. 6)}$$

$$F_s = (P + W)\left(\frac{E_s A_s}{E_A A_A + 2E_s A_s}\right) \qquad \text{(Eq. 7)}$$

STRESSES IN THE WIRES

$$\sigma_A = \frac{F_A}{A_A} = \frac{(P + W)E_A}{E_A A_A + 2E_s A_s} \qquad \text{(Eq. 8)}$$

$$\sigma_s = \frac{F_s}{A_s} = \frac{(P + W)E_s}{E_A A_A + 2E_s A_s} \qquad \text{(Eq. 9)}$$

ALLOWABLE LOADS (FROM EQS. (8) AND (9))

$$P_A = \frac{\sigma_A}{E_A}(E_A A_A + 2E_s A_s) - W \qquad \text{(Eq. 10)}$$

$$P_s = \frac{\sigma_s}{E_s}(E_A A_A + 2E_s A_s) - W \qquad \text{(Eq. 11)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQS. (10) AND (11):

$$A_s = \frac{\pi}{4}(2 \text{ mm})^2 = 3.1416 \text{ mm}^2$$

$$A_A = \frac{\pi}{4}(4 \text{ mm})^2 = 12.5664 \text{ mm}^2$$

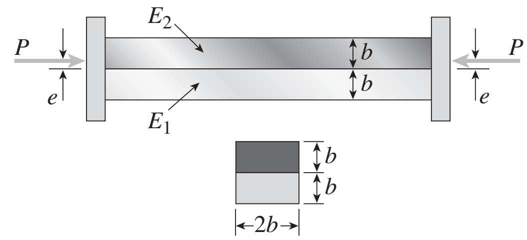
$P_A = 1713$ N

$P_s = 1504$ N

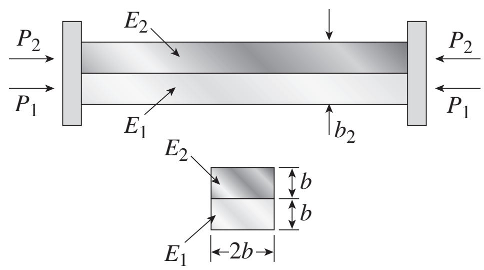
Steel governs. $\quad P_{allow} = 1500$ N $\longleftarrow$

Problem 2.4-11 A *bimetallic* bar (or composite bar) of square cross section with dimensions $2b \times 2b$ is constructed of two different metals having moduli of elasticity E_1 and E_2 (see figure). The two parts of the bar have the same cross-sectional dimensions. The bar is compressed by forces P acting through rigid end plates. The line of action of the loads has an eccentricity e of such magnitude that each part of the bar is stressed uniformly in compression.

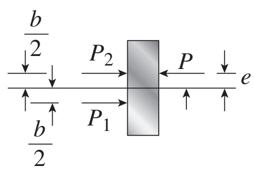
(a) Determine the axial forces P_1 and P_2 in the two parts of the bar.
(b) Determine the eccentricity e of the loads.
(c) Determine the ratio σ_1/σ_2 of the stresses in the two parts of the bar.



Solution 2.4-11 **Bimetallic bar in compression**



FREE-BODY DIAGRAM
(Plate at right-hand end)



EQUATIONS OF EQUILIBRIUM

$$\Sigma F = 0 \qquad P_1 + P_2 = P \qquad \text{(Eq. 1)}$$

$$\Sigma M = 0 \qquad Pe + P_1\left(\frac{b}{2}\right) - P_2\left(\frac{b}{2}\right) = 0 \qquad \text{(Eq. 2)}$$

EQUATION OF COMPATIBILITY

$$\delta_2 = \delta_1$$

$$\frac{P_2L}{E_2A} = \frac{P_1L}{E_1A} \quad \text{or} \quad \frac{P_2}{E_2} = \frac{P_1}{E_1} \qquad \text{(Eq. 3)}$$

(a) AXIAL FORCES

Solve simultaneously Eqs. (1) and (3):

$$P_1 = \frac{PE_1}{E_1 + E_2} \qquad P_2 = \frac{PE_2}{E_1 + E_2} \longleftarrow$$

(b ECCENTRICITY OF LOAD P

Substitute P_1 and P_2 into Eq. (2) and solve for e:

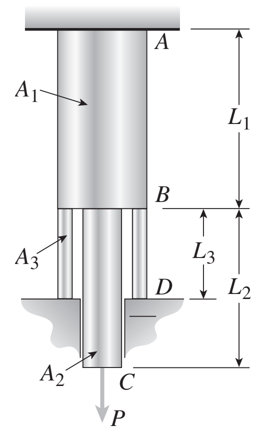
$$e = \frac{b(E_2 - E_1)}{2(E_2 + E_1)} \longleftarrow$$

(c) RATIO OF STRESSES

$$\sigma_1 = \frac{P_1}{A} \qquad \sigma_2 = \frac{P_2}{A} \qquad \frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2} \longleftarrow$$

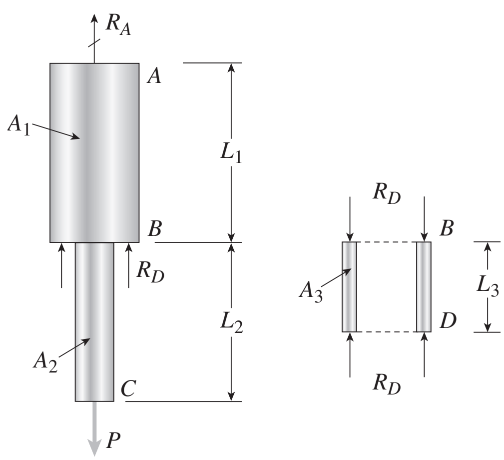
Problem 2.4-12 A circular steel bar *ABC* ($E = 200$ GPa) has cross-sectional area A_1 from *A* to *B* and cross-sectional area A_2 from *B* to *C* (see figure). The bar is supported rigidly at end *A* and is subjected to a load *P* equal to 40 kN at end *C*. A circular steel collar *BD* having cross-sectional area A_3 supports the bar at *B*. The collar fits snugly at *B* and *D* when there is no load.

Determine the elongation δ_{AC} of the bar due to the load *P*. (Assume $L_1 = 2L_3 = 250$ mm, $L_2 = 225$ mm, $A_1 = 2A_3 = 960$ mm^2, and $A_2 = 300$ mm^2.)



Solution 2.4-12 Bar supported by a collar

FREE-BODY DIAGRAM OF BAR *ABC* AND COLLAR *BD*



EQUILIBRIUM OF BAR *ABC*

$\Sigma F_{\text{vert}} = 0 \qquad R_A + R_D - P = 0$ (Eq. 1)

COMPATIBILITY (distance *AD* does not change)

$\delta_{AB}(\text{bar}) + \delta_{BD}(\text{collar}) = 0$ (Eq. 2)

(Elongation is positive.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} \qquad \delta_{BD} = -\frac{R_D L_3}{EA_3}$$

Substitute into Eq. (2):

$$\frac{R_A L_1}{EA_1} - \frac{R_D L_3}{EA_3} = 0 \qquad \text{(Eq. 3)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (3):

$$R_A = \frac{PL_3A_1}{L_1A_3 + L_3A_1} \qquad R_D = \frac{PL_1A_3}{L_1A_3 + L_3A_1}$$

CHANGES IN LENGTHS (Elongation is positive)

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{PL_1L_3}{E(L_1A_3 + L_3A_1)} \qquad \delta_{BC} = \frac{PL_2}{EA_2}$$

ELONGATION OF BAR *ABC*

$$\delta_{AC} = \delta_{AB} + \delta_{AC}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 40$ kN $\qquad E = 200$ GPa

$L_1 = 250$ mm

$L_2 = 225$ mm

$L_3 = 125$ mm

$A_1 = 960$ mm^2

$A_2 = 300$ mm^2

$A_3 = 480$ mm^2

RESULTS:

$R_A = R_D = 20$ kN

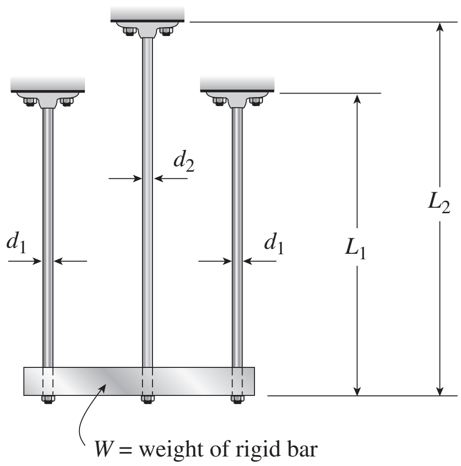
$\delta_{AB} = 0.02604$ mm

$\delta_{BC} = 0.15000$ mm

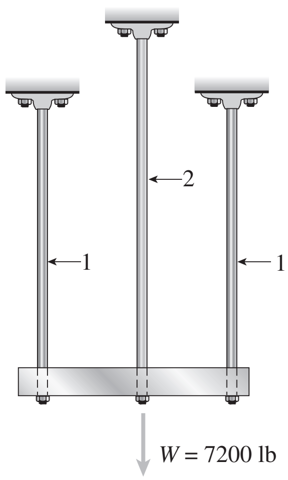
$\delta_{AC} = \delta_{AB} + \delta_{AC} = 0.176$ mm ←

Problem 2.4-13 A horizontal rigid bar of weight $W = 7200$ lb is supported by three slender circular rods that are equally spaced (see figure). The two outer rods are made of aluminum ($E_1 = 10 \times 10^6$ psi) with diameter $d_1 = 0.4$ in. and length $L_1 = 40$ in. The inner rod is magnesium ($E_2 = 6.5 \times 10^6$ psi) with diameter d_2 and length L_2. The allowable stresses in the aluminum and magnesium are 24,000 psi and 13,000 psi, respectively.

If it is desired to have all three rods loaded to their maximum allowable values, what should be the diameter d_2 and length L_2 of the middle rod?



Solution 2.4-13 **Bar supported by three rods**



BAR 1 ALUMINUM

$E_1 = 10 \times 10^6$ psi

$d_1 = 0.4$ in.

$L_1 = 40$ in.

$\sigma_1 = 24{,}000$ psi

BAR 2 MAGNESIUM

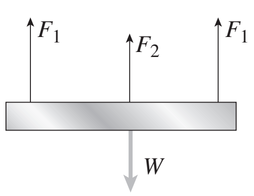
$E_2 = 6.5 \times 10^6$ psi

$d_2 = ?$ $\quad L_2 = ?$

$\sigma_2 = 13{,}000$ psi

FREE-BODY DIAGRAM OF RIGID BAR

EQUATION OF EQUILIBRIUM



$$\Sigma F_{\text{vert}} = 0$$

$$2F_1 + F_2 - W = 0 \qquad \text{(Eq. 1)}$$

FULLY STRESSED RODS

$F_1 = \sigma_1 A_1 \qquad F_2 = \sigma_2 A_2$

$$A_1 = \frac{\pi d_1^2}{4} \qquad A_2 = \frac{\pi d_2^2}{4}$$

Substitute into Eq. (1):

$$2\sigma_1\left(\frac{\pi d_1^2}{4}\right) + \sigma_2\left(\frac{\pi d_2^2}{4}\right) = W$$

Diameter d_1 is known; solve for d_2:

$$d_2^2 = \frac{4W}{\pi\sigma_2} - \frac{2\sigma_1 d_1^2}{\sigma_2} \longleftarrow \qquad \text{(Eq. 2)}$$

SUBSTITUTE NUMERICAL VALUES:

$$d_2^2 = \frac{4(7200\text{ lb})}{\pi(13{,}000\text{ psi})} - \frac{2(24{,}000\text{ psi})\,(0.4\text{ in.})^2}{13{,}000\text{ psi}}$$

$$= 0.70518\text{ in.}^2 - 0.59077\text{ in.}^2 = 0.11441\text{ in.}^2$$

$$d_2 = 0.338\text{ in.} \longleftarrow$$

EQUATION OF COMPATIBILITY

$$\delta_1 = \delta_2 \qquad \text{(Eq. 3)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} = \sigma_1\left(\frac{L_1}{E_1}\right) \qquad \text{(Eq. 4)}$$

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} = \sigma_2\left(\frac{L_2}{E_2}\right) \qquad \text{(Eq. 5)}$$

Substitute (4) and (5) into Eq. (3):

$$\sigma_1\left(\frac{L_1}{E_1}\right) = \sigma_2\left(\frac{L_2}{E_2}\right)$$

Length L_1 is known; solve for L_2:

$$L_2 = L_1\left(\frac{\sigma_1 E_2}{\sigma_2 E_1}\right) \longleftarrow \qquad \text{(Eq. 6)}$$

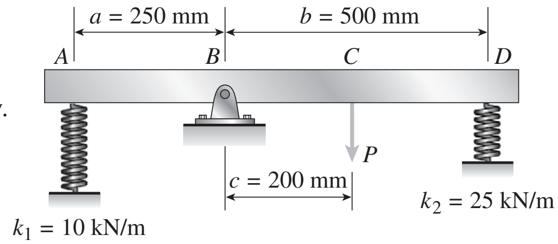
SUBSTITUTE NUMERICAL VALUES:

$$L_2 = (40\text{ in.})\left(\frac{24{,}000\text{ psi}}{13{,}000\text{ psi}}\right)\left(\frac{6.5 \times 10^6\text{ psi}}{10 \times 10^6\text{ psi}}\right)$$

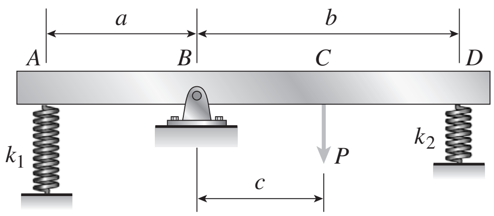
$$= 48.0\text{ in.} \longleftarrow$$

Problem 2.4-14 A rigid bar *ABCD* is pinned at point *B* and supported by springs at *A* and *D* (see figure). The springs at *A* and *D* have stiffnesses $k_1 = 10$ kN/m and $k_2 = 25$ kN/m, respectively, and the dimensions *a*, *b*, and *c* are 250 mm, 500 mm, and 200 mm, respectively. A load *P* acts at point *C*.

If the angle of rotation of the bar due to the action of the load *P* is limited to 3°, what is the maximum permissible load P_{max}?



Solution 2.4-14 **Rigid bar supported by springs**



NUMERICAL DATA

$a = 250$ mm

$b = 500$ mm

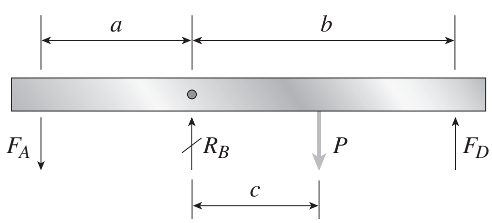
$c = 200$ mm

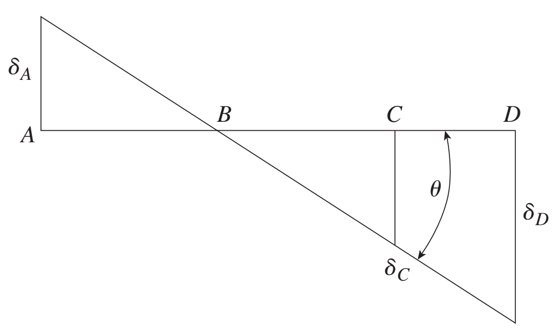
$k_1 = 10$ kN/m

$k_2 = 25$ kN/m

$$\theta_{max} = 3° = \frac{\pi}{60} \text{ rad}$$

FREE-BODY DIAGRAM AND DISPLACEMENT DIAGRAM





EQUATION OF EQUILIBRIUM

$$\Sigma M_B = 0 \qquad F_A(a) - P(c) + F_D(b) = 0 \qquad \text{(Eq. 1)}$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_A}{a} = \frac{\delta_D}{b} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_A = \frac{F_A}{k_1} \qquad \delta_D = \frac{F_D}{k_2} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_A}{ak_1} = \frac{F_D}{bk_2} \qquad \text{(Eq. 5)}$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$F_A = \frac{ack_1P}{a^2k_1 + b^2k_2} \qquad F_D = \frac{bck_2P}{a^2k_1 + b^2k_2}$$

ANGLE OF ROTATION

$$\delta_D = \frac{F_D}{k_2} = \frac{bcP}{a^2k_1 + b^2k_2} \qquad \theta = \frac{\delta_D}{b} = \frac{cP}{a^2k_1 + b^2k_2}$$

MAXIMUM LOAD

$$P = \frac{\theta}{c}(a^2k_1 + b^2k_2)$$

$$P_{max} = \frac{\theta_{max}}{c}(a^2k_1 + b^2k_2) \quad \longleftarrow$$

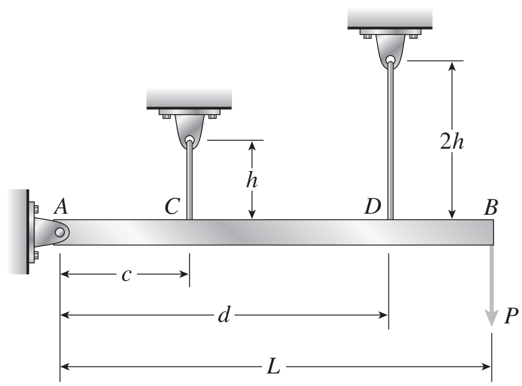
SUBSTITUTE NUMERICAL VALUES:

$$P_{max} = \frac{\pi/60 \text{ rad}}{200 \text{ mm}}[(250 \text{ mm})^2(10 \text{ kN/m}) + (500 \text{ mm})^2(25 \text{ kN/m})]$$

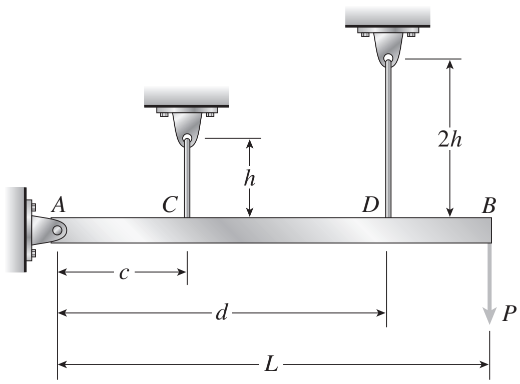
$$= 1800 \text{ N} \quad \longleftarrow$$

Problem 2.4-15 A rigid bar AB of length $L = 66$ in. is hinged to a support at A and supported by two vertical wires attached at points C and D (see figure). Both wires have the same cross-sectional area ($A = 0.0272$ in.^{2}) and are made of the same material (modulus $E = 30 \times 10^6$ psi). The wire at C has length $h = 18$ in. and the wire at D has length twice that amount. The horizontal distances are $c = 20$ in. and $d = 50$ in.

(a) Determine the tensile stresses σ_C and σ_D in the wires due to the load $P = 340$ lb acting at end B of the bar.
(b) Find the downward displacement δ_B at end B of the bar.



Solution 2.4-15 **Bar supported by two wires**



$h = 18$ in.

$2h = 36$ in.

$c = 20$ in.

$d = 50$ in.

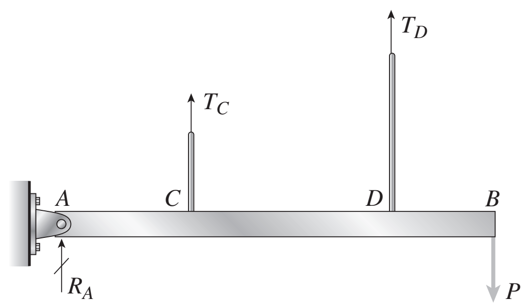
$L = 66$ in.

$E = 30 \times 10^6$ psi

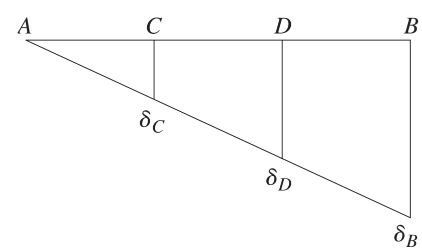
$A = 0.0272$ in.^{2}

$P = 340$ lb

FREE-BODY DIAGRAM



DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0$ ⟲ ⟳ $T_C(c) + T_D(d) = PL$ (Eq. 1)

EQUATION OF COMPATIBILITY

$$\frac{\delta_C}{c} = \frac{\delta_D}{d} \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_C = \frac{T_C h}{EA} \qquad \delta_D = \frac{T_D(2h)}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{T_C h}{cEA} = \frac{T_D(2h)}{dEA} \quad \text{or} \quad \frac{T_C}{c} = \frac{2T_D}{d} \qquad \text{(Eq. 5)}$$

TENSILE FORCES IN THE WIRES

Solve simultaneously Eqs. (1) and (5):

$$T_C = \frac{2cPL}{2c^2 + d^2} \qquad T_D = \frac{dPL}{2c^2 + d^2} \qquad \text{(Eqs. 6, 7)}$$

TENSILE STRESSES IN THE WIRES

$$\sigma_C = \frac{T_C}{A} = \frac{2cPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 8)}$$

$$\sigma_D = \frac{T_D}{A} = \frac{dPL}{A(2c^2 + d^2)} \qquad \text{(Eq. 9)}$$

DISPLACEMENT AT END OF BAR

$$\delta_B = \delta_D\left(\frac{L}{d}\right) = \frac{2hT_D}{EA}\left(\frac{L}{d}\right) = \frac{2hPL^2}{EA(2c^2 + d^2)} \qquad \text{(Eq. 10)}$$

SUBSTITUTE NUMERICAL VALUES

$$2c^2 + d^2 = 2(20 \text{ in.})^2 + (50 \text{ in.})^2 = 3300 \text{ in.}^2$$

(a) $$\sigma_C = \frac{2cPL}{A(2c^2 + d^2)} = \frac{2(20 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 10{,}000 \text{ psi} \longleftarrow$$

$$\sigma_D = \frac{dPL}{A(2c^2 + d^2)} = \frac{(50 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

$$= 12{,}500 \text{ psi} \longleftarrow$$

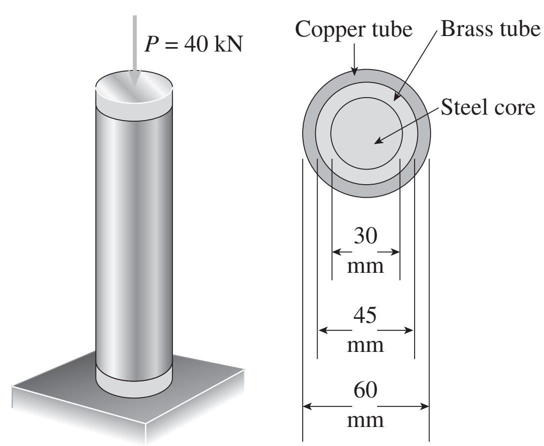
(b) $$\delta_B = \frac{2hPL^2}{EA(2c^2 + d^2)}$$

$$= \frac{2(18 \text{ in.})(340 \text{ lb})(66 \text{ in.})^2}{(30 \times 10^6 \text{ psi})(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)}$$

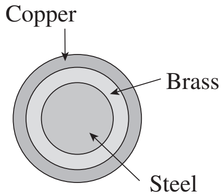
$$= 0.0198 \text{ in.} \longleftarrow$$

Problem 2.4-16 A trimetallic bar is uniformly compressed by an axial force $P = 40$ kN applied through a rigid end plate (see figure). The bar consists of a circular steel core surrounded by brass and copper tubes. The steel core has diameter 30 mm, the brass tube has outer diameter 45 mm, and the copper tube has outer diameter 60 mm. The corresponding moduli of elasticity are $E_s = 210$ GPa, $E_b = 100$ GPa, and $E_c = 120$ GPa.

Calculate the compressive stresses σ_s, σ_b, and σ_c in the steel, brass, and copper, respectively, due to the force P.



Solution 2.4-16 Trimetallic bar in compression

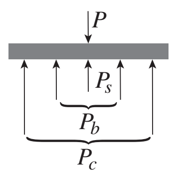


P_s = compressive force in steel core

P_b = compressive force in brass tube

P_c = compressive force in copper tube

FREE-BODY DIAGRAM OF RIGID END PLATE



EQUATION OF EQUILIBRIUM

$\Sigma F_{\text{vert}} = 0 \qquad P_s + P_b + P_c = P \qquad$ (Eq. 1)

EQUATIONS OF COMPATIBILITY

$\delta_s = \delta_b \qquad \delta_c = \delta_s \qquad$ (Eqs. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \qquad \delta_b = \frac{P_b L}{E_b A_b} \qquad \delta_c = \frac{P_c L}{E_c A_c} \qquad \text{(Eqs. 3, 4, 5)}$$

SOLUTION OF EQUATIONS

Substitute (3), (4), and (5) into Eqs. (2):

$$P_b = P_s \frac{E_b A_b}{E_s A_s} \qquad P_c = P_s \frac{E_c A_c}{E_s A_s} \qquad \text{(Eqs. 6, 7)}$$

SOLVE SIMULTANEOUSLY EQS. (1), (6), AND (7):

$$P_s = P \frac{E_s A_s}{E_s A_s + E_b A_b + E_c A_c}$$

$$P_b = P \frac{E_b A_b}{E_s A_s + E_b A_b + E_c A_c}$$

$$P_c = P \frac{E_c A_c}{E_s A_s + E_b A_b + E_c + A_c}$$

COMPRESSIVE STRESSES

Let $\Sigma EA = E_s A_s + E_b A_b + E_c A_c$

$$\sigma_s = \frac{P_s}{A_s} = \frac{PE_s}{\Sigma EA} \qquad \sigma_b = \frac{P_b}{A_b} = \frac{PE_b}{\Sigma EA}$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{PE_c}{\Sigma EA}$$

SUBSTITUTE NUMERICAL VALUES:

$P = 40$ kN $\qquad E_s = 210$ GPa

$E_b = 100$ GPa $\qquad E_c = 120$ GPa

$d_1 = 30$ mm $\qquad d_2 = 45$ mm $\qquad d_3 = 60$ mm

$$A_s = \frac{\pi}{4} d_1^2 = 706.86 \text{ mm}^2$$

$$A_b = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$$A_c = \frac{\pi}{4}(d_3^2 - d_2^2) = 1237.00 \text{ mm}^2$$

$\Sigma EA = 385.238 \times 10^6$ N

$$\sigma_s = \frac{PE_s}{\Sigma EA} = 21.8 \text{ MPa} \quad \longleftarrow$$

$$\sigma_b = \frac{PE_b}{\Sigma EA} = 10.4 \text{ MPa} \quad \longleftarrow$$

$$\sigma_c = \frac{PE_c}{\Sigma EA} = 12.5 \text{ MPa} \quad \longleftarrow$$

Thermal Effects

Problem 2.5-1 The rails of a railroad track are welded together at their ends (to form continuous rails and thus eliminate the clacking sound of the wheels) when the temperature is 60°F.

What compressive stress σ is produced in the rails when they are heated by the sun to 120°F if the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}$/°F and the modulus of elasticity $E = 30 \times 10^6$ psi?

Solution 2.5-1 **Expansion of railroad rails**

The rails are prevented from expanding because of their great length and lack of expansion joints.

Therefore, each rail is in the same condition as a bar with fixed ends (see Example 2-7).

The compressive stress in the rails may be calculated from Eq. (2-18).

$\Delta T = 120°\text{F} - 60°\text{F} = 60°\text{F}$

$$\sigma = E\alpha(\Delta T)$$
$$= (30 \times 10^6 \text{ psi})(6.5 \times 10^{-6}/°\text{F})(60°\text{F})$$
$$\sigma = 11{,}700 \text{ psi} \quad \longleftarrow$$

Problem 2.5-2 An aluminum pipe has a length of 60 m at a temperature of 10°C. An adjacent steel pipe at the same temperature is 5 mm longer than the aluminum pipe.

At what temperature (degrees Celsius) will the aluminum pipe be 15 mm longer than the steel pipe? (Assume that the coefficients of thermal expansion of aluminum and steel are $\alpha_a = 23 \times 10^{-6}$/°C and $\alpha_s = 12 \times 10^{-6}$/°C, respectively.)

Solution 2.5-2 **Aluminum and steel pipes**

INITIAL CONDITIONS

$L_a = 60$ m $\qquad T_0 = 10°$C

$L_s = 60.005$ m $\qquad T_0 = 10°$C

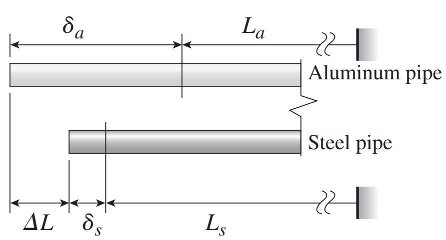
$\alpha_a = 23 \times 10^{-6}$/°C $\qquad \alpha_s = 12 \times 10^{-6}$/°C

FINAL CONDITIONS

Aluminum pipe is longer than the steel pipe by the amount $\Delta L = 15$ mm.

ΔT = increase in temperature

$\delta_a = \alpha_a(\Delta T)L_a \qquad \delta_s = \alpha_s(\Delta T)L_s$



From the figure above:

$\delta_a + L_a = \Delta L + \delta_s + L_s$

or, $\quad \alpha_a(\Delta T)L_a + L_a = \Delta L + \alpha_s(\Delta T)L_s + L_s$

Solve for ΔT:

$$\Delta T = \frac{\Delta L + (L_s - L_a)}{\alpha_a L_a - \alpha_s L_s} \quad \longleftarrow$$

Substitute numerical values:

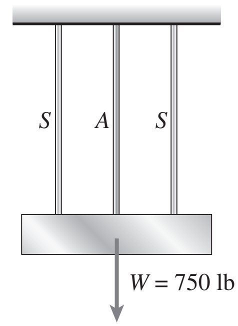
$\alpha_a L_a - \alpha_s L_s = 659.9 \times 10^{-6}$ m/°C

$$\Delta T = \frac{15 \text{ mm} + 5 \text{ mm}}{659.9 \times 10^{-6} \text{ m/°C}} = 30.31°\text{C}$$

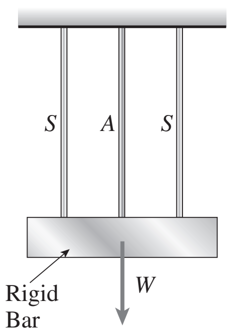
$$T = T_0 + \Delta T = 10°\text{C} + 30.31°\text{C}$$
$$= 40.3°\text{C} \quad \longleftarrow$$

Problem 2.5-3 A rigid bar of weight $W = 750$ lb hangs from three equally spaced wires, two of steel and one of aluminum (see figure). The diameter of the wires is ⅛ in. Before they were loaded, all three wires had the same length.

What temperature increase ΔT in all three wires will result in the entire load being carried by the steel wires? (Assume $E_s = 30 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}$/°F, and $\alpha_a = 12 \times 10^{-6}$/°F.)



Solution 2.5-3 Bar supported by three wires



S = steel $\quad A$ = aluminum

$W = 750$ lb

$$d = \frac{1}{8} \text{ in.}$$

$$A_s = \frac{\pi d^2}{4} = 0.012272 \text{ in.}^2$$

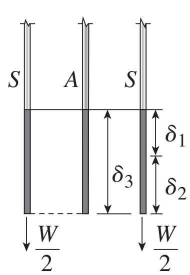
$E_s = 30 \times 10^6$ psi

$E_s A_s = 368{,}155$ lb

$\alpha_s = 6.5 \times 10^{-6}$/°F

$\alpha_a = 12 \times 10^{-6}$/°F

L = Initial length of wires



δ_1 = increase in length of a steel wire due to temperature increase ΔT

$= \alpha_s (\Delta T)L$

δ_2 = increase in length of a steel wire due to load $W/2$

$$= \frac{WL}{2E_s A_s}$$

δ_3 = increase in length of aluminum wire due to temperature increase ΔT

$= \alpha_a(\Delta T)L$

For no load in the aluminum wire:

$\delta_1 + \delta_2 = \delta_3$

$$\alpha_s(\Delta T)L + \frac{WL}{2E_s A_s} = \alpha_a(\Delta T)L$$

or

$$\Delta T = \frac{W}{2E_s A_s(\alpha_a - \alpha_s)} \longleftarrow$$

Substitute numerical values:

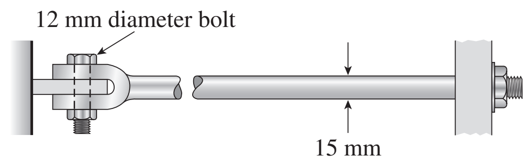
$$\Delta T = \frac{750 \text{ lb}}{(2)(368{,}155 \text{ lb})(5.5 \times 10^{-6}/°\text{F})}$$

$$= 185°\text{F} \longleftarrow$$

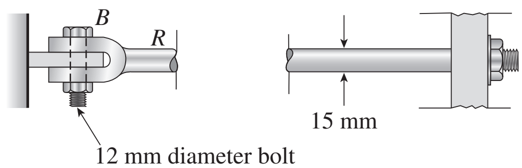
NOTE: If the temperature increase is larger than ΔT, the aluminum wire would be in compression, which is not possible. Therefore, the steel wires continue to carry all of the load. If the temperature increase is less than ΔT, the aluminum wire will be in tension and carry part of the load.

Problem 2.5-4 A steel rod of diameter 15 mm is held snugly (but without any initial stresses) between rigid walls by the arrangement shown in the figure.

Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. (For the steel rod, use $\alpha = 12 \times 10^{-6}/°C$ and $E = 200$ GPa.)



Solution 2.5-4 Steel rod with bolted connection



R = rod

B = bolt

P = tensile force in steel rod due to temperature drop ΔT

A_R = cross-sectional area of steel rod

From Eq. (2-17) of Example 2-7: $P = EA_R\alpha(\Delta T)$

Bolt is in double shear.

V = shear force acting over one cross section of the bolt

$$V = P/2 = \frac{1}{2}EA_R\alpha(\Delta T)$$

τ = average shear stress on cross section of the bolt

A_B = cross-sectional area of bolt

$$\tau = \frac{V}{A_B} = \frac{EA_R\,\alpha(\Delta T)}{2A_B}$$

Solve for ΔT: $\Delta T = \dfrac{2\tau A_B}{EA_R\alpha}$

$A_B = \dfrac{\pi d_B^2}{4}$ where d_B = diameter of bolt

$A_R = \dfrac{\pi d_R^2}{4}$ where d_R = diameter of steel rod

$$\Delta T = \frac{2\tau d_B^2}{E\alpha d_R^2} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\tau = 45$ MPa $\quad d_B = 12$ mm $\quad d_R = 15$ mm

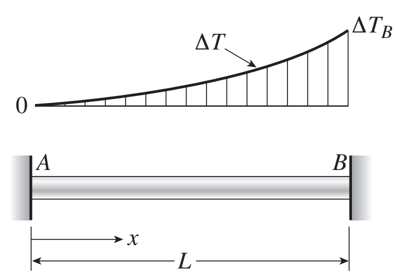
$\alpha = 12 \times 10^{-6}/°C \quad E = 200$ GPa

$$\Delta T = \frac{2(45\text{ MPa})(12\text{ mm})^2}{(200\text{ GPa})(12 \times 10^{-6}/°C)(15\text{ mm})^2}$$

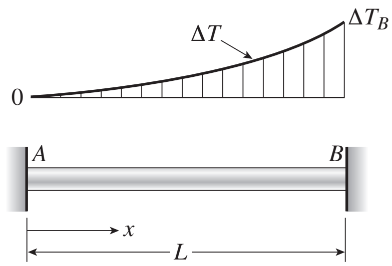
$$\Delta T = 24°C \longleftarrow$$

Problem 2.5-5 A bar AB of length L is held between rigid supports and heated nonuniformly in such a manner that the temperature increase ΔT at distance x from end A is given by the expression $\Delta T = \Delta T_B x^3/L^3$, where ΔT_B is the increase in temperature at end B of the bar (see figure).

Derive a formula for the compressive stress σ_c in the bar. (Assume that the material has modulus of elasticity E and coefficient of thermal expansion α.)



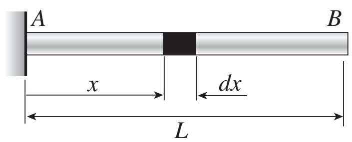
Solution 2.5-5 Bar with nonuniform temperature change



At distance x:

$$\Delta T = \Delta T_B\left(\frac{x^3}{L^3}\right)$$

Remove the support at end B of the bar:



Consider an element dx at a distance x from end A.

$d\delta$ = Elongation of element dx

$$d\delta = \alpha(\Delta T)dx = \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx$$

δ = elongation of bar

$$\delta = \int_0^L d\delta = \int_0^L \alpha(\Delta T_B)\left(\frac{x^3}{L^3}\right)dx = \frac{1}{4}\alpha(\Delta T_B)L$$

Compressive force P required to shorten the bar by the amount δ

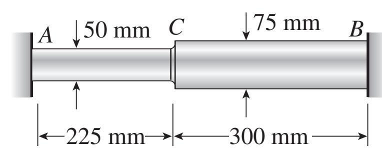
$$P = \frac{EA\delta}{L} = \frac{1}{4}EA\alpha(\Delta T_B)$$

Compressive stress in the bar

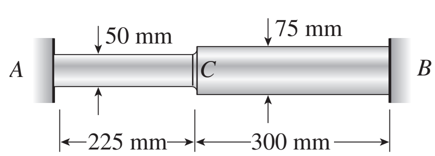
$$\sigma_c = \frac{P}{A} = \frac{E\alpha(\Delta T_B)}{4} \quad \longleftarrow$$

Problem 2.5-6 A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left- and right-hand parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion α is 100×10^{-6}/°C. The bar is subjected to a uniform temperature increase of 30°C.

Calculate the following quantities: (a) the compressive force P in the bar; (b) the maximum compressive stress σ_c; and (c) the displacement δ_C of point C.



Solution 2.5-6 Bar with rigid supports



$E = 6.0$ GPa $\qquad \alpha = 100 \times 10^{-6}$/°C

Left-hand part:

$L_1 = 225$ mm $\qquad d_1 = 50$ mm

$$A_1 = \frac{\pi}{4}d_1^2 = \frac{\pi}{4}(50\text{ mm})^2$$

$$= 1963.5\text{ mm}^2$$

$\Delta T = 30$°C

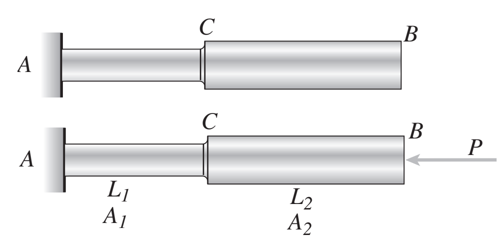
Right-hand part:

$L_2 = 300$ mm $\qquad d_2 = 75$ mm

$$A_2 = \frac{\pi}{4}d_2^2 = \frac{\pi}{4}(75\text{ mm})^2 = 4417.9\text{ mm}^2$$

(a) Compressive force P

Remove the support at end B.



δ_T = elongation due to temperature

$P = \alpha(\Delta T)(L_1 + L_2)$

$= 1.5750$ mm

δ_P = shortening due to P

$= \frac{PL_1}{EA_1} + \frac{PL_2}{EA_2}$

$= P(19.0986 \times 10^{-9}\text{ m/N} + 11.3177 \times 10^{-9}\text{ m/N})$

$= (30.4163 \times 10^{-9}\text{ m/N})P$

(P = newtons)

Compatibility: $\delta_T = \delta_P$

$1.5750 \times 10^{-3}\text{ m} = (30.4163 \times 10^{-9}\text{ m/N})P$

$P = 51{,}781$ N or $P = 51.8$ kN ←

(b) Maximum compressive stress

$$\sigma_c = \frac{P}{A_1} = \frac{51.78\text{ kN}}{1963.5\text{ mm}^2} = 26.4\text{ MPa} \quad \leftarrow$$

(c) Displacement of point C

δ_C = Shortening of AC

$$\delta_C = \frac{PL_1}{EA_1} - \alpha(\Delta T)L_1$$

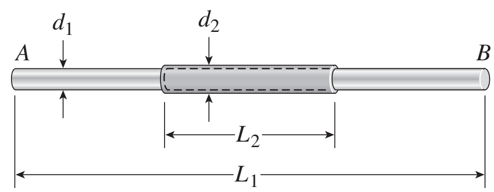
$= 0.9890$ mm $- 0.6750$ mm

$\delta_C = 0.314$ mm ←

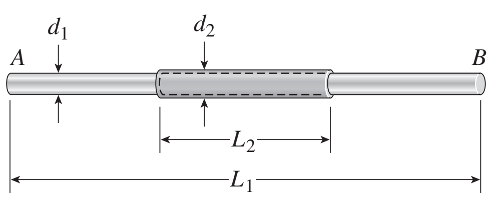
(Positive means AC shortens and point C displaces to the left.)

Problem 2.5-7 A circular steel rod AB (diameter $d_1 = 1.0$ in., length $L_1 = 3.0$ ft) has a bronze sleeve (outer diameter $d_2 = 1.25$ in., length $L_2 = 1.0$ ft) shrunk onto it so that the two parts are securely bonded (see figure).

Calculate the total elongation δ of the steel bar due to a temperature rise $\Delta T = 500°$F. (Material properties are as follows: for steel, $E_s = 30 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}$/°F; for bronze, $E_b = 15 \times 10^6$ psi and $\alpha_b = 11 \times 10^{-6}$/°F.)



Solution 2.5-7 **Steel rod with bronze sleeve**



$L_1 = 36$ in. $\quad L_2 = 12$ in.

Elongation of the two outer parts of the bar

$\delta_1 = \alpha_s(\Delta T)(L_1 - L_2)$

$= (6.5 \times 10^{-6}/°\text{F})(500°\text{F})(36\text{ in.} - 12\text{ in.})$

$= 0.07800$ in.

Elongation of the middle part of the bar

The steel rod and bronze sleeve lengthen the same amount, so they are in the same condition as the bolt and sleeve of Example 2-8. Thus, we can calculate the elongation from Eq. (2-21):

$$\delta_2 = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L_2}{E_s A_s + E_b A_b}$$

Substitute numerical values:

$\alpha_s = 6.5 \times 10^{-6}$/°F $\quad \alpha_b = 11 \times 10^{-6}$/°F

$E_s = 30 \times 10^6$ psi $\quad E_b = 15 \times 10^6$ psi

$d_1 = 1.0$ in.

$$A_s = \frac{\pi}{4}d_1^2 = 0.78540\text{ in.}^2$$

$d_2 = 1.25$ in.

$$A_b = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.44179\text{ in.}^2$$

$\Delta T = 500°$F $\quad L_2 = 12.0$ in.

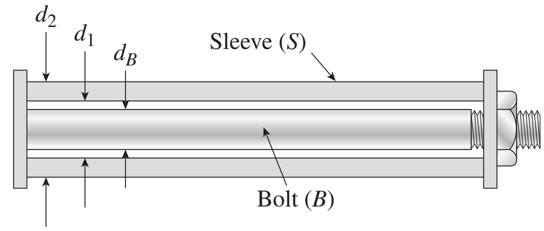
$\delta_2 = 0.04493$ in.

Total elongation

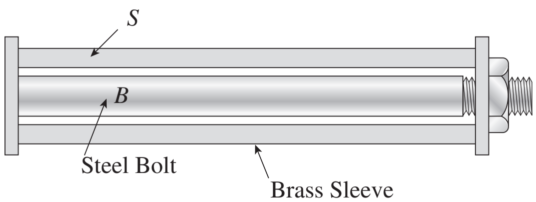
$\delta = \delta_1 + \delta_2 = 0.123$ in. ←

Problem 2.5-8 A brass sleeve S is fitted over a steel bolt B (see figure), and the nut is tightened until it is just snug. The bolt has a diameter $d_B = 25$ mm, and the sleeve has inside and outside diameters $d_1 = 26$ mm and $d_2 = 36$ mm, respectively.

Calculate the temperature rise ΔT that is required to produce a compressive stress of 25 MPa in the sleeve. (Use material properties as follows: for the sleeve, $\alpha_S = 21 \times 10^{-6}$/°C and $E_S = 100$ GPa; for the bolt, $\alpha_B = 10 \times 10^{-6}$/°C and $E_B = 200$ GPa.) (*Suggestion:* Use the results of Example 2-8.)



Solution 2.5-8 **Brass sleeve fitted over a Steel bolt**



Subscript S means "sleeve".

Subscript B means "bolt".

Use the results of Example 2-8.

σ_S = compressive force in sleeve

EQUATION (2-20a):

$$\sigma_S = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B} \text{ (Compression)}$$

SOLVE FOR ΔT:

$$\Delta T = \frac{\sigma_S(E_S A_S + E_B A_B)}{(\alpha_S - \alpha_B)E_S E_B A_B}$$

or

$$\Delta T = \frac{\sigma_S}{E_S(\alpha_S - \alpha_B)}\left(1 + \frac{E_S A_S}{E_B A_B}\right) \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\sigma_S = 25$ MPa

$d_2 = 36$ mm $\quad d_1 = 26$ mm $\quad d_B = 25$ mm

$E_S = 100$ GPa $\quad E_B = 200$ GPa

$\alpha_S = 21 \times 10^{-6}$/°C $\quad \alpha_B = 10 \times 10^{-6}$/°C

$$A_S = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}(620 \text{ mm}^2)$$

$$A_B = \frac{\pi}{4}(d_B)^2 = \frac{\pi}{4}(625 \text{ mm}^2)$$

$$1 + \frac{E_S A_S}{E_B A_B} = 1.496$$

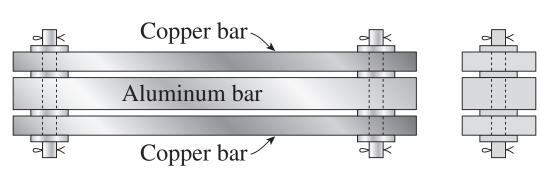
$$\Delta T = \frac{25 \text{ MPa } (1.496)}{(100 \text{ GPa})(11 \times 10^{-6}/°\text{C})}$$

$\Delta T = 34°\text{C} \longleftarrow$

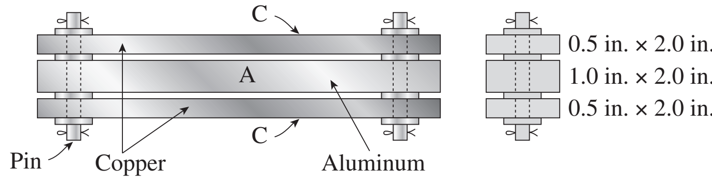
(Increase in temperature)

Problem 2.5-9 Rectangular bars of copper and aluminum are held by pins at their ends, as shown in the figure. Thin spacers provide a separation between the bars. The copper bars have cross-sectional dimensions 0.5 in. × 2.0 in., and the aluminum bar has dimensions 1.0 in. × 2.0 in.

Determine the shear stress in the 7/16 in. diameter pins if the temperature is raised by 100°F. (For copper, $E_c = 18{,}000$ ksi and $\alpha_c = 9.5 \times 10^{-6}$/°F; for aluminum, $E_a = 10{,}000$ ksi and $\alpha_a = 13 \times 10^{-6}$/°F.) *Suggestion:* Use the results of Example 2-8.



Solution 2.5-9 Rectangular bars held by pins



Diameter of pin: $d_P = \frac{7}{16}$ in. $= 0.4375$ in.

Area of pin: $A_P = \frac{\pi}{4} d_P^2 = 0.15033 \text{ in.}^2$

Area of two copper bars: $A_c = 2.0 \text{ in.}^2$

Area of aluminum bar: $A_a = 2.0 \text{ in.}^2$

$\Delta T = 100°\text{F}$

Copper: $E_c = 18{,}000$ ksi $\alpha_c = 9.5 \times 10^{-6}/°\text{F}$

Aluminum: $E_a = 10{,}000$ ksi $\alpha_a = 13 \times 10^{-6}/°\text{F}$

Use the results of Example 2-8.

Find the forces P_a and P_c in the aluminum bar and copper bar, respectively, from Eq. (2-19).

Replace the subscript "S" in that equation by "a" (for aluminum) and replace the subscript "B" by "c" (for copper), because α for aluminum is larger than α for copper.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_a A_a E_c A_c}{E_a A_a + E_c A_c}$$

Note that P_a is the compressive force in the aluminum bar and P_c is the combined tensile force in the two copper bars.

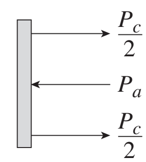
$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_c A_c}{1 + \frac{E_c A_c}{E_a A_a}}$$

SUBSTITUTE NUMERICAL VALUES:

$$P_a = P_c = \frac{(3.5 \times 10^{-6}/°\text{F})(100°\text{F})(18{,}000 \text{ ksi})(2 \text{ in.}^2)}{1 + \left(\frac{18}{10}\right)\left(\frac{2.0}{2.0}\right)}$$

$= 4{,}500$ lb

FREE-BODY DIAGRAM OF PIN AT THE LEFT END



V = shear force in pin

$= P_c/2$

$= 2{,}250$ lb

τ = average shear stress on cross section of pin

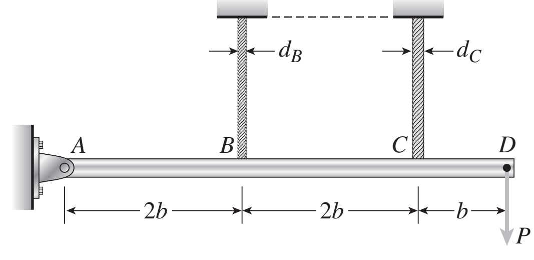
$$\tau = \frac{V}{A_P} = \frac{2{,}250 \text{ lb}}{0.15033 \text{ in.}^2}$$

$\tau = 15.0$ ksi ←

Problem 2.5-10 A rigid bar $ABCD$ is pinned at end A and supported by two cables at points B and C (see figure). The cable at B has nominal diameter $d_B = 12$ mm and the cable at C has nominal diameter $d_C = 20$ mm. A load P acts at end D of the bar.

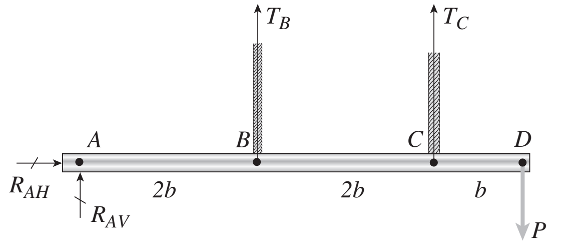
What is the allowable load P if the temperature rises by 60°C and each cable is required to have a factor of safety of at least 5 against its ultimate load?

(*Note:* The cables have effective modulus of elasticity $E = 140$ GPa and coefficient of thermal expansion $\alpha = 12 \times 10^{-6}/°\text{C}$. Other properties of the cables can be found in Table 2-1, Section 2.2.)



Solution 2.5-10 Rigid bar supported by two cables

FREE-BODY DIAGRAM OF BAR *ABCD*



T_B = force in cable B $\quad$ T_C = force in cable C

$d_B = 12$ mm $\quad$ $d_C = 20$ mm

From Table 2-1:

$A_B = 76.7 \text{ mm}^2$ $\quad$ $E = 140$ GPa

$\Delta T = 60°\text{C}$ $\quad$ $A_C = 173 \text{ mm}^2$

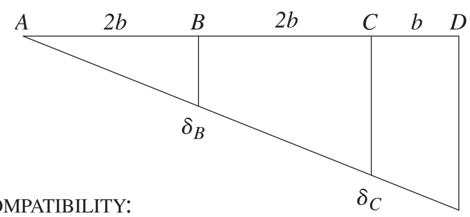
$\alpha = 12 \times 10^{-6}/°\text{C}$

EQUATION OF EQUILIBRIUM

$\Sigma M_A = 0 \quad T_B(2b) + T_C(4b) - P(5b) = 0$

or $\quad 2T_B + 4T_C = 5P$ (Eq. 1)

DISPLACEMENT DIAGRAM



COMPATIBILITY:

$\delta_C = 2\delta_B$ (Eq. 2)

FORCE-DISPLACEMENT AND TEMPERATURE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA_B} + \alpha(\Delta T)L \qquad \text{(Eq. 3)}$$

$$\delta_C = \frac{T_C L}{EA_C} + \alpha(\Delta T)L \qquad \text{(Eq. 4)}$$

SUBSTITUTE EQS. (3) AND (4) INTO EQ. (2):

$$\frac{T_C L}{EA_C} + \alpha(\Delta T)L = \frac{2T_B L}{EA_B} + 2\alpha(\Delta T)L$$

or

$$2T_B A_C - T_C A_B = -E\alpha(\Delta T)A_B A_C \qquad \text{(Eq. 5)}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (5):

$$T_B(346) - T_C(76.7) = -1{,}338{,}000 \qquad \text{(Eq. 6)}$$

in which T_B and T_C have units of newtons.

SOLVE SIMULTANEOUSLY EQS. (1) AND (6):

$$T_B = 0.2494\,P - 3{,}480 \qquad \text{(Eq. 7)}$$

$$T_C = 1.1253\,P + 1{,}740 \qquad \text{(Eq. 8)}$$

in which P has units of newtons.

SOLVE EQS. (7) AND (8) FOR THE LOAD P:

$$P_B = 4.0096\,T_B + 13{,}953 \qquad \text{(Eq. 9)}$$

$$P_C = 0.8887\,T_C - 1{,}546 \qquad \text{(Eq. 10)}$$

ALLOWABLE LOADS

From Table 2-1:

$(T_B)_{\text{ULT}} = 102{,}000$ N $\quad$ $(T_C)_{\text{ULT}} = 231{,}000$ N

Factor of safety = 5

$(T_B)_{\text{allow}} = 20{,}400$ N $\quad$ $(T_C)_{\text{allow}} = 46{,}200$ N

From Eq. (9): $P_B = (4.0096)(20{,}400 \text{ N}) + 13{,}953 \text{ N}$
$= 95{,}700$ N

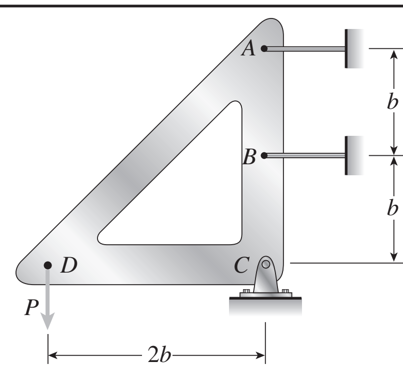
From Eq. (10): $P_C = (0.8887)(46{,}200 \text{ N}) - 1546 \text{ N}$
$= 39{,}500$ N

Cable C governs.

$P_{\text{allow}} = 39.5$ kN ←

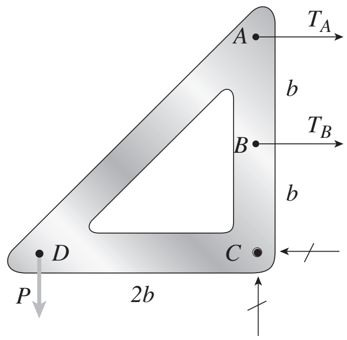
Problem 2.5-11 A rigid triangular frame is pivoted at C and held by two identical horizontal wires at points A and B (see figure). Each wire has axial rigidity $EA = 120$ k and coefficient of thermal expansion $\alpha = 12.5 \times 10^{-6}/°\text{F}$.

(a) If a vertical load $P = 500$ lb acts at point D, what are the tensile forces T_A and T_B in the wires at A and B, respectively?
(b) If, while the load P is acting, both wires have their temperatures raised by 180°F, what are the forces T_A and T_B?
(c) What further increase in temperature will cause the wire at B to become slack?



Solution 2.5-11 Triangular frame held by two wires

FREE-BODY DIAGRAM OF FRAME

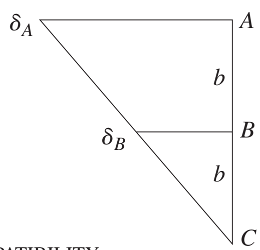


EQUATION OF EQUILIBRIUM

$\Sigma M_C = 0$

$P(2b) - T_A(2b) - T_B(b) = 0$ or $2T_A + T_B = 2P$ (Eq. 1)

DISPLACEMENT DIAGRAM



EQUATION OF COMPATIBILITY

$\delta_A = 2\delta_B$ (Eq. 2)

(a) LOAD P ONLY

Force-displacement relations:

$$\delta_A = \frac{T_A L}{EA} \qquad \delta_B = \frac{T_B L}{EA} \qquad \text{(Eq. 3, 4)}$$

(L = length of wires at A and B.)

Substitute (3) and (4) into Eq. (2):

$$\frac{T_A L}{EA} = \frac{2T_B L}{EA}$$

or $T_A = 2T_B$ (Eq. 5)

Solve simultaneously Eqs. (1) and (5):

$$T_A = \frac{4P}{5} \qquad T_B = \frac{2P}{5} \qquad \text{(Eqs. 6, 7)}$$

Numerical values:

$P = 500$ lb

$\therefore T_A = 400$ lb $\quad T_B = 200$ lb ←

(b) LOAD P AND TEMPERATURE INCREASE ΔT

Force-displacement and temperature-displacement relations:

$$\delta_A = \frac{T_A L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 8)}$$

$$\delta_B = \frac{T_B L}{EA} + \alpha(\Delta T)L \qquad \text{(Eq. 9)}$$

Substitute (8) and (9) into Eq. (2):

$$\frac{T_A L}{EA} + \alpha(\Delta T)L = \frac{2T_B L}{EA} + 2\alpha(\Delta T)L$$

or $T_A - 2T_B = EA\alpha(\Delta T)$ (Eq. 10)

Solve simultaneously Eqs. (1) and (10):

$$T_A = \frac{1}{5}[4P + EA\alpha(\Delta T)] \qquad \text{(Eq. 11)}$$

$$T_B = \frac{2}{5}[P - EA\alpha(\Delta T)] \qquad \text{(Eq. 12)}$$

Substitute numerical values:

$P = 500$ lb $\qquad EA = 120{,}000$ lb

$\Delta T = 180°F$

$\alpha = 12.5 \times 10^{-6}/°F$

$$T_A = \frac{1}{5}(2000 \text{ lb} + 270 \text{ lb}) = 454 \text{ lb} \quad \leftarrow$$

$$T_B = \frac{2}{5}(500 \text{ lb} - 270 \text{ lb}) = 92 \text{ lb} \quad \leftarrow$$

(c) WIRE B BECOMES SLACK

Set $T_B = 0$ in Eq. (12):

$P = EA\alpha(\Delta T)$

or

$$\Delta T = \frac{P}{EA\alpha} = \frac{500 \text{ lb}}{(120{,}000 \text{ lb})(12.5 \times 10^{-6}/°F)}$$

$= 333.3°F$

Further increase in temperature:

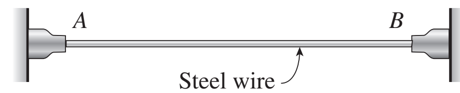
$\Delta T = 333.3°F - 180°F$

$= 153°F$ ←

Misfits and Prestrains

Problem 2.5-12 A steel wire AB is stretched between rigid supports (see figure). The initial prestress in the wire is 42 MPa when the temperature is 20°C.

(a) What is the stress σ in the wire when the temperature drops to 0°C?
(b) At what temperature T will the stress in the wire become zero? (Assume $\alpha = 14 \times 10^{-6}$/°C and $E = 200$ GPa.)



Solution 2.5-12 **Steel wire with initial prestress**

Initial prestress: $\sigma_1 = 42$ MPa

Initial temperature: $T_1 = 20°C$

$E = 200$ GPa

$\alpha = 14 \times 10^{-6}/°C$

(a) STRESS σ WHEN TEMPERATURE DROPS TO 0°C

$T_2 = 0°C \qquad \Delta T = 20°C$

Note: Positive ΔT means a *decrease* in temperature and an *increase* in the stress in the wire.

Negative ΔT means an *increase* in temperature and a *decrease* in the stress.

Stress σ equals the initial stress σ_1 plus the additional stress σ_2 due to the temperature drop.

From Eq. (2-18): $\sigma_2 = E\alpha(\Delta T)$

$$\sigma = \sigma_1 + \sigma_2 = \sigma_1 + E\alpha(\Delta T)$$
$$= 42 \text{ MPa} + (200 \text{ GPa})(14 \times 10^{-6}/°C)(20°C)$$
$$= 42 \text{ MPa} + 56 \text{ MPa} = 98 \text{ MPa} \longleftarrow$$

(b) TEMPERATURE WHEN STRESS EQUALS ZERO

$$\sigma = \sigma_1 + \sigma_2 = 0 \qquad \sigma_1 + E\alpha(\Delta T) = 0$$

$$\Delta T = -\frac{\sigma_1}{E\alpha}$$

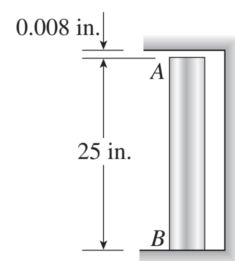
(Negative means increase in temp.)

$$\Delta T = -\frac{42 \text{ MPa}}{(200 \text{ GPa})(14 \times 10^{-6}/°C)} = -15°C$$

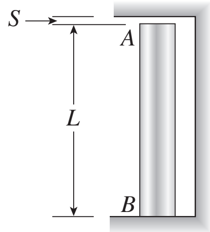
$$T = 20°C + 15°C = 35°C \longleftarrow$$

Problem 2.5-13 A copper bar AB of length 25 in. is placed in position at room temperature with a gap of 0.008 in. between end A and a rigid restraint (see figure).

Calculate the axial compressive stress σ_c in the bar if the temperature rises 50°F. (For copper, use $\alpha = 9.6 \times 10^{-6}$/°F and $E = 16 \times 10^6$ psi.)



Solution 2.5-13 Bar with a gap



$L = 25$ in.

$S = 0.008$ in.

$\Delta T = 50°F$ (increase)

$\alpha = 9.6 \times 10^{-6}/°F$

$E = 16 \times 10^6$ psi

δ = elongation of the bar if it is free to expand

$= \alpha(\Delta T)L$

δ_C = elongation that is prevented by the support

$= \alpha(\Delta T)L - S$

ε_C = strain in the bar due to the restraint

$= \delta_C/L$

σ_c = stress in the bar

$$= E\varepsilon_C = \frac{E\delta_C}{L} = \frac{E}{L}[\alpha(\Delta T)L - S] \quad \longleftarrow$$

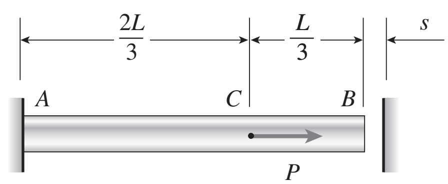
Note: This result is valid only if $\alpha(\Delta T)L \geq S$. (Otherwise, the gap is not closed).

Substitute numerical values:

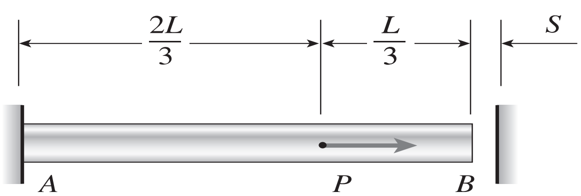
$$\sigma_c = \frac{16 \times 10^6 \text{ psi}}{25 \text{ in.}}[(9.6 \times 10^{-6}/°F)(50°F)(25 \text{ in.}) - 0.008 \text{ in.}] = 2{,}560 \text{ psi} \quad \longleftarrow$$

Problem 2.5-14 A bar AB having length L and axial rigidity EA is fixed at end A (see figure). At the other end a small gap of dimension s exists between the end of the bar and a rigid surface. A load P acts on the bar at point C, which is two-thirds of the length from the fixed end.

If the support reactions produced by the load P are to be equal in magnitude, what should be the size s of the gap?



Solution 2.5-14 Bar with a gap (load P)



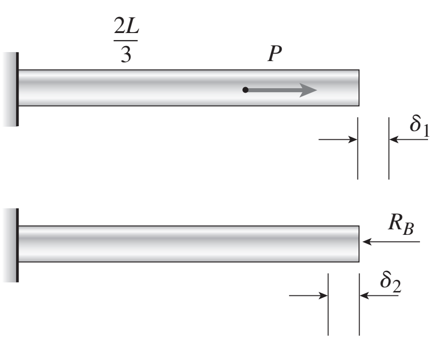
L = length of bar

S = size of gap

EA = axial rigidity

Reactions must be equal; find S.

FORCE-DISPLACEMENT RELATIONS



$$\delta_1 = \frac{P\left(\frac{2L}{3}\right)}{EA}$$

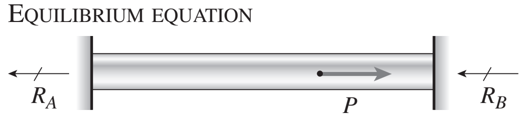
$$\delta_2 = \frac{R_B L}{EA}$$

COMPATIBILITY EQUATION

$\delta_1 - \delta_2 = S$ or

$$\frac{2PL}{3EA} - \frac{R_B L}{EA} = S \qquad \text{(Eq. 1)}$$

EQUILIBRIUM EQUATION



R_A = reaction at end A (to the left)

R_B = reaction at end B (to the left)

$P = R_A + R_B$

Reactions must be equal.

$\therefore R_A = R_B \qquad P = 2R_B \qquad R_B = \frac{P}{2}$

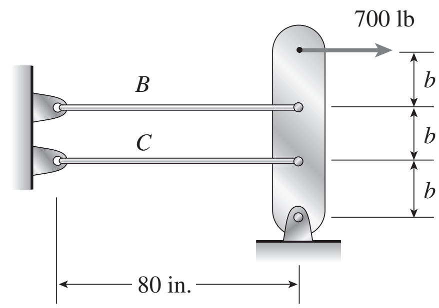
Substitute for R_B in Eq. (1):

$$\frac{2PL}{3EA} - \frac{PL}{2EA} = S \quad \text{or} \quad S = \frac{PL}{6EA} \quad \longleftarrow$$

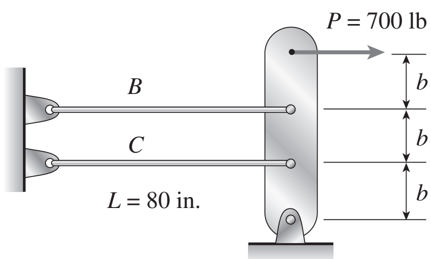
NOTE: The gap closes when the load reaches the value $P/4$. When the load reaches the value P, equal to $6EAs/L$, the reactions are equal ($R_A = R_B = P/2$). When the load is between $P/4$ and P, R_A is greater than R_B. If the load exceeds P, R_B is greater than R_A.

Problem 2.5-15 Wires B and C are attached to a support at the left-hand end and to a pin-supported rigid bar at the right-hand end (see figure). Each wire has cross-sectional area $A = 0.03$ in.^{2} and modulus of elasticity $E = 30 \times 10^6$ psi. When the bar is in a vertical position, the length of each wire is $L = 80$ in. However, before being attached to the bar, the length of wire B was 79.98 in. and of wire C was 79.95 in.

Find the tensile forces T_B and T_C in the wires under the action of a force $P = 700$ lb acting at the upper end of the bar.



Solution 2.5-15 Wires B and C attached to a bar



$P = 700$ lb

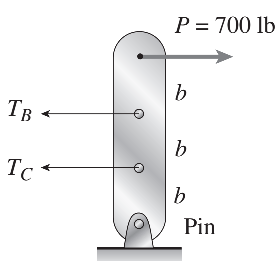
$A = 0.03$ in.^{2}

$E = 30\times10^6$ psi

$L_B = 79.98$ in.

$L_C = 79.95$ in.

Equilibrium Equation



$\Sigma M_{\text{pin}} = 0$

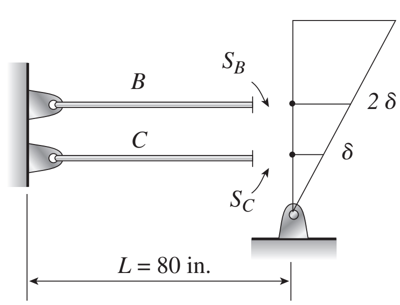
$T_C(b) + T_B(2b) = P(3b)$

$2T_B + T_C = 3P$ (Eq. 1)

Displacement diagram

$S_B = 80 \text{ in.} - L_B = 0.02$ in.

$S_C = 80 \text{ in.} - L_C = 0.05$ in.



Elongation of wires:

$\delta_B = S_B + 2\delta$ (Eq. 2)

$\delta_C = S_C + \delta$ (Eq. 3)

Force-displacement relations

$$\delta_B = \frac{T_B L}{EA} \qquad \delta_C = \frac{T_C L}{EA} \qquad \text{(Eqs. 4, 5)}$$

Solution of equations

Combine Eqs. (2) and (4):

$$\frac{T_B L}{EA} = S_B + 2\delta \qquad \text{(Eq. 6)}$$

Combine Eqs. (3) and (5):

$$\frac{T_C L}{EA} = S_C + \delta \qquad \text{(Eq. 7)}$$

Eliminate δ between Eqs. (6) and (7):

$$T_B - 2T_C = \frac{EAS_B}{L} - \frac{2EAS_C}{L} \qquad \text{(Eq. 8)}$$

Solve simultaneously Eqs. (1) and (8):

$$T_B = \frac{6P}{5} + \frac{EAS_B}{5L} - \frac{2EAS_C}{5L} \quad \longleftarrow$$

$$T_C = \frac{3P}{5} - \frac{2EAS_B}{5L} + \frac{4EAS_C}{5L} \quad \longleftarrow$$

Substitute numerical values:

$$\frac{EA}{5L} = 2250 \text{ lb/in.}$$

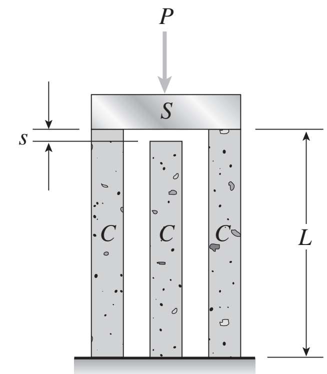
$T_B = 840 \text{ lb} + 45 \text{ lb} - 225 \text{ lb} = 660 \text{ lb}$ ⟵

$T_C = 420 \text{ lb} - 90 \text{ lb} + 450 \text{ lb} = 780 \text{ lb}$ ⟵

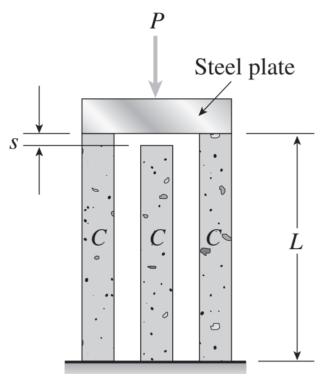
(Both forces are positive, which means tension, as required for wires.)

Problem 2.5-16 A rigid steel plate is supported by three posts of high-strength concrete each having an effective cross-sectional area $A = 40{,}000\ \text{mm}^2$ and length $L = 2$ m (see figure). Before the load P is applied, the middle post is shorter than the others by an amount $s = 1.0$ mm.

Determine the maximum allowable load P_{allow} if the allowable compressive stress in the concrete is $\sigma_{allow} = 20$ MPa. (Use $E = 30$ GPa for concrete.)



Solution 2.5-16 **Plate supported by three posts**



s = size of gap = 1.0 mm

L = length of posts = 2.0 m

$A = 40{,}000\ \text{mm}^2$

$\sigma_{allow} = 20$ MPa

$E = 30$ GPa

C = concrete post

DOES THE GAP CLOSE?

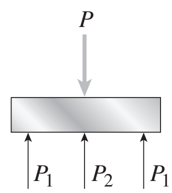
Stress in the two outer posts when the gap is just closed:

$$\sigma = E\varepsilon = E\left(\frac{s}{L}\right) = (30\ \text{GPa})\left(\frac{1.0\ \text{mm}}{2.0\ \text{m}}\right)$$

$$= 15\ \text{MPa}$$

Since this stress is less than the allowable stress, the allowable force P will close the gap.

EQUILIBRIUM EQUATION



$$2P_1 + P_2 = P \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION

δ_1 = shortening of outer posts

δ_2 = shortening of inner post

$$\delta_1 = \delta_2 + s \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{P_1 L}{EA} \qquad \delta_2 = \frac{P_2 L}{EA} \qquad \text{(Eqs. 3, 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_1 L}{EA} = \frac{P_2 L}{EA} + s \quad \text{or} \quad P_1 - P_2 = \frac{EAs}{L} \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P = 3P_1 - \frac{EAs}{L}$$

By inspection, we know that P_1 is larger than P_2. Therefore, P_1 will control and will be equal to $\sigma_{allow} A$.

$$P_{allow} = 3\sigma_{allow} A - \frac{EAs}{L}$$

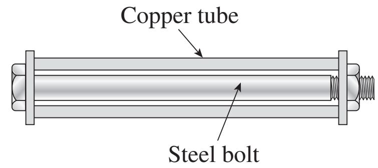
$$= 2400\ \text{kN} - 600\ \text{kN} = 1800\ \text{kN}$$

$$= 1.8\ \text{MN} \longleftarrow$$

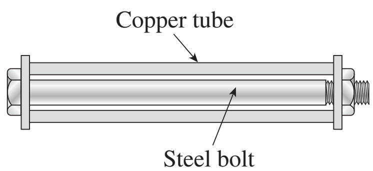
Problem 2.5-17 A copper tube is fitted around a steel bolt and the nut is turned until it is just snug (see figure). What stresses σ_s and σ_c will be produced in the steel and copper, respectively, if the bolt is now tightened by a quarter turn of the nut?

The copper tube has length $L = 16$ in. and cross-sectional area $A_c = 0.6$ in.^{2}, and the steel bolt has cross-sectional area $A_s = 0.2$ in.^{2} The pitch of the threads of the bolt is $p = 52$ mils (a mil is one-thousandth of an inch). Also, the moduli of elasticity of the steel and copper are $E_s = 30 \times 10^6$ psi and $E_c = 16 \times 10^6$ psi, respectively.

Note: The pitch of the threads is the distance advanced by the nut in one complete turn (see Eq. 2-22).



Solution 2.5-17 **Steel bolt and copper tube**



$L = 16$ in.

$p = 52$ mils $= 0.052$ in.

$n = \dfrac{1}{4}$ (See Eq. 2-22)

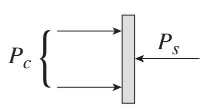
Steel bolt: $A_s = 0.2$ in.^{2}

$E_s = 30 \times 10^6$ psi

Copper tube: $A_c = 0.6$ in.^{2}

$E_c = 16 \times 10^6$ psi

Equilibrium equation

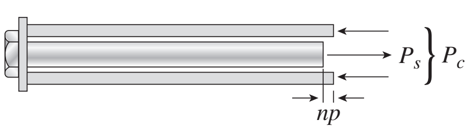


P_s = tensile force in steel bolt

P_c = compressive force in copper tube

$$P_c = P_s \qquad \text{(Eq. 1)}$$

Compatibility equation



δ_c = shortening of copper tube

δ_s = elongation of steel bolt

$$\delta_c + \delta_s = np \qquad \text{(Eq. 2)}$$

Force-displacement relations

$$\delta_c = \frac{P_c L}{E_c A_c} \qquad \delta_s = \frac{P_s L}{E_s A_s} \qquad \text{(Eq. 3, Eq. 4)}$$

Solution of equations

Substitute (3) and (4) into Eq. (2):

$$\frac{P_c L}{E_c A_c} + \frac{P_s L}{E_s A_s} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_s = P_c = \frac{npE_s A_s E_c A_c}{L(E_s A_s + E_c A_c)} \qquad \text{(Eq. 6)}$$

Substitute numerical values:

$P_s = P_c = 3{,}000$ lb

Stresses

Steel bolt:

$$\sigma_s = \frac{P_s}{A_s} = \frac{3{,}000 \text{ lb}}{0.2 \text{ in.}^2} = 15 \text{ ksi (tension)} \quad \longleftarrow$$

Copper tube:

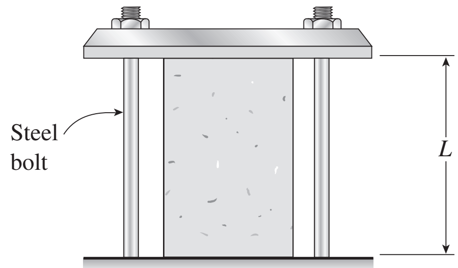
$$\sigma_c = \frac{P_c}{A_c} = \frac{3{,}000 \text{ lb}}{0.6 \text{ in.}^2}$$

$$= 5 \text{ ksi (compression)} \quad \longleftarrow$$

Problem 2.5-18 A plastic cylinder is held snugly between a rigid plate and a foundation by two steel bolts (see figure).

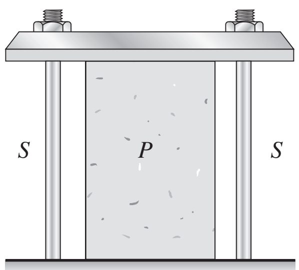
Determine the compressive stress σ_p in the plastic when the nuts on the steel bolts are tightened by one complete turn.

Data for the assembly are as follows: length $L = 200$ mm, pitch of the bolt threads $p = 1.0$ mm, modulus of elasticity for steel $E_s = 200$ GPa, modulus of elasticity for the plastic $E_p = 7.5$ GPa, cross-sectional area of one bolt $A_s = 36.0$ mm^2, and cross-sectional area of the plastic cylinder $A_p = 960$ mm^2.



Probs. 2.5-18 and 2.5-19

Solution 2.5-18 **Plastic cylinder and two steel bolts**



$L = 200$ mm

$P = 1.0$ mm

$E_s = 200$ GPa

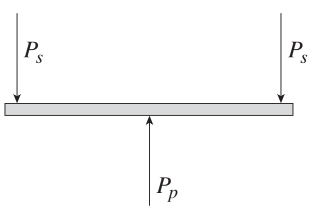
$A_s = 36.0$ mm^2 (for one bolt)

$E_p = 7.5$ GPa

$A_p = 960$ mm^2

$n = 1$ (See Eq. 2-22)

EQUILIBRIUM EQUATION

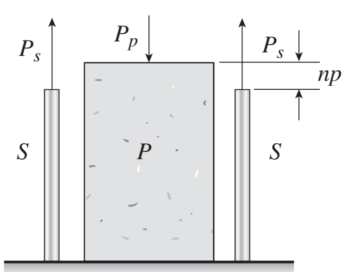


P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION



δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \qquad \text{(Eq. 2)}$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2\,np\,E_s A_s E_p}{L(E_p A_p + 2E_s A_s)} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 54.0 \times 10^{15}\ \text{N}^2/\text{m}^2$$

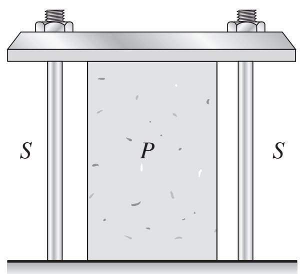
$$D = E_p A_p + 2E_s A_s = 21.6 \times 10^6\ \text{N}$$

$$\sigma_p = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(1.0\ \text{mm})}{200\ \text{mm}}\left(\frac{N}{D}\right)$$

$$= 25.0\ \text{MPa} \longleftarrow$$

Problem 2.5-19 Solve the preceding problem if the data for the assembly are as follows: length $L = 10$ in., pitch of the bolt threads $p = 0.058$ in., modulus of elasticity for steel $E_s = 30 \times 10^6$ psi, modulus of elasticity for the plastic $E_p = 500$ ksi, cross-sectional area of one bolt $A_s = 0.06$ in.^{2}, and cross-sectional area of the plastic cylinder $A_p = 1.5$ in.^{2}

Solution 2.5-19 Plastic cylinder and two steel bolts



$L = 10$ in.

$p = 0.058$ in.

$E_s = 30 \times 10^6$ psi

$A_s = 0.06$ in.^{2} (for one bolt)

$E_p = 500$ ksi

$A_p = 1.5$ in.^{2}

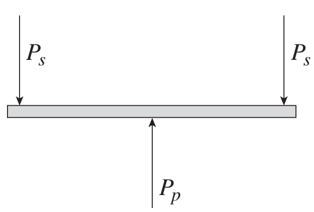
$n = 1$ (see Eq. 2-22)

Equilibrium equation

P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s \qquad \text{(Eq. 1)}$$

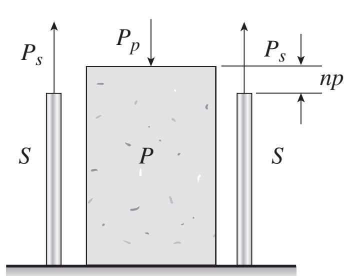


Compatibility equation

δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \qquad \text{(Eq. 2)}$$



Force-displacement relations

$$\delta_s = \frac{P_s L}{E_s A_s} \qquad \delta_p = \frac{P_p L}{E_p A_p} \qquad \text{(Eq. 3, Eq. 4)}$$

Solution of equations

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \qquad \text{(Eq. 5)}$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2\,np\,E_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

Stress in the plastic cylinder

$$\sigma_p = \frac{P_p}{A_p} = \frac{2\,np\,E_s A_s E_p}{L(E_p A_p + 2E_s A_s)} \longleftarrow$$

Substitute numerical values:

$$N = E_s A_s E_p = 900 \times 10^9 \text{ lb}^2/\text{in.}^2$$

$$D = E_p A_p + 2E_s A_s = 4350 \times 10^3 \text{ lb}$$

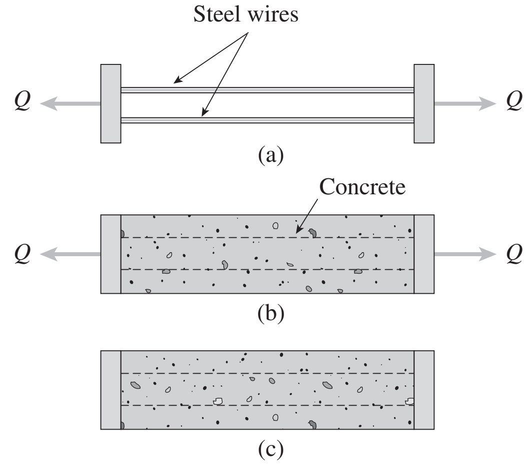
$$\sigma_P = \frac{2np}{L}\left(\frac{N}{D}\right) = \frac{2(1)(0.058 \text{ in.})}{10 \text{ in.}}\left(\frac{N}{D}\right)$$

$$= 2400 \text{ psi} \longleftarrow$$

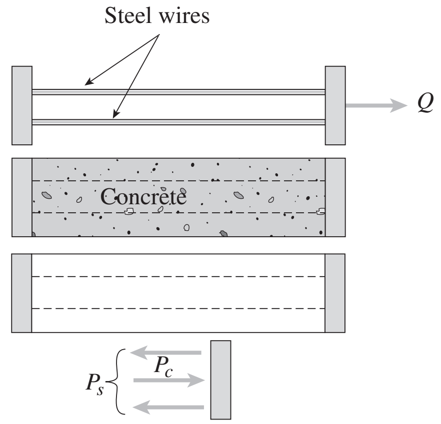
Problem 2.5-20 Prestressed concrete beams are sometimes manufactured in the following manner. High-strength steel wires are stretched by a jacking mechanism that applies a force Q, as represented schematically in part (a) of the figure. Concrete is then poured around the wires to form a beam, as shown in part (b).

After the concrete sets properly, the jacks are released and the force Q is removed [see part (c) of the figure]. Thus, the beam is left in a prestressed condition, with the wires in tension and the concrete in compression.

Let us assume that the prestressing force Q produces in the steel wires an initial stress $\sigma_0 = 620$ MPa. If the moduli of elasticity of the steel and concrete are in the ratio 12:1 and the cross-sectional areas are in the ratio 1:50, what are the final stresses σ_s and σ_c in the two materials?



Solution 2.5-20 **Prestressed concrete beam**



L = length

σ_0 = initial stress in wires

$$= \frac{Q}{A_s} = 620 \text{ MPa}$$

A_s = total area of steel wires

A_c = area of concrete

$= 50\ A_s$

$E_s = 12\ E_c$

P_s = final tensile force in steel wires

P_c = final compressive force in concrete

EQUILIBRIUM EQUATION

$$P_s = P_c \qquad \text{(Eq. 1)}$$

COMPATIBILITY EQUATION AND FORCE-DISPLACEMENT RELATIONS

δ_1 = initial elongation of steel wires

$$= \frac{QL}{E_s A_s} = \frac{\sigma_0 L}{E_s}$$

δ_2 = final elongation of steel wires

$$= \frac{P_s L}{E_s A_s}$$

δ_3 = shortening of concrete

$$= \frac{P_c L}{E_c A_c}$$

$$\delta_1 - \delta_2 = \delta_3 \quad \text{or} \quad \frac{\sigma_0 L}{E_s} - \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} \qquad \text{(Eq. 2, Eq. 3)}$$

Solve simultaneously Eqs. (1) and (3):

$$P_s = P_c = \frac{\sigma_0 A_s}{1 + \dfrac{E_s A_s}{E_c A_c}}$$

STRESSES

$$\sigma_s = \frac{P_s}{A_s} = \frac{\sigma_0}{1 + \dfrac{E_s A_s}{E_c A_c}} \quad \longleftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{\sigma_0}{\dfrac{A_c}{A_s} + \dfrac{E_s}{E_c}} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_0 = 620 \text{ MPa} \qquad \frac{E_s}{E_c} = 12 \qquad \frac{A_s}{A_c} = \frac{1}{50}$$

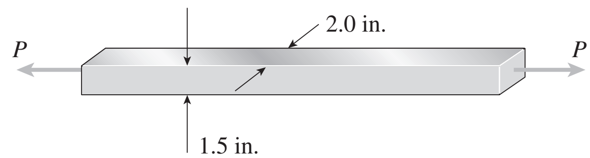
$$\sigma_s = \frac{620 \text{ MPa}}{1 + \dfrac{12}{50}} = 500 \text{ MPa (Tension)} \quad \longleftarrow$$

$$\sigma_c = \frac{620 \text{ MPa}}{50 + 12} = 10 \text{ MPa (Compression)} \quad \longleftarrow$$

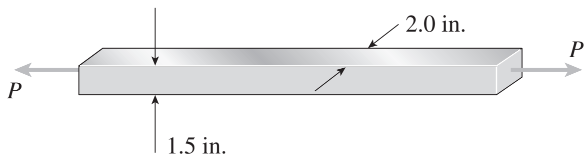
Stresses on Inclined Sections

Problem 2.6-1 A steel bar of rectangular cross section (1.5 in. × 2.0 in.) carries a tensile load P (see figure). The allowable stresses in tension and shear are 15,000 psi and 7,000 psi, respectively.

Determine the maximum permissible load P_{max}.



Solution 2.6-1 Rectangular bar in tension



$A = 1.5 \text{ in.} \times 2.0 \text{ in.}$

$= 3.0 \text{ in.}^2$

Maximum Normal Stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress: $\tau_{max} = \dfrac{\sigma_x}{2} = \dfrac{P}{2A}$

$\sigma_{allow} = 15{,}000 \text{ psi} \quad \tau_{allow} = 7{,}000 \text{ psi}$

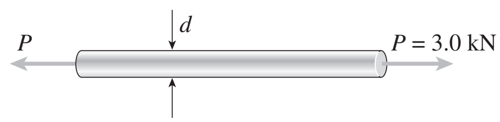
Because τ_{allow} is less than one-half of σ_{allow}, the shear stress governs.

$P_{max} = 2\tau_{allow} A = 2(7{,}000 \text{ psi})(3.0 \text{ in.}^2)$

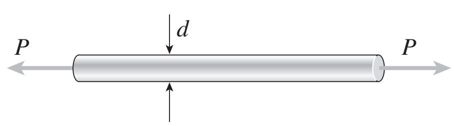
$= 42{,}000 \text{ lb}$ ←

Problem 2.6-2 A circular steel rod of diameter d is subjected to a tensile force $P = 3.0$ kN (see figure). The allowable stresses in tension and shear are 120 MPa and 50 MPa, respectively.

What is the minimum permissible diameter d_{min} of the rod?



Solution 2.6-2 Steel rod in tension



$P = 3.0 \text{ kN} \quad A = \dfrac{\pi d^2}{4}$

Maximum normal stress: $\sigma_x = \dfrac{P}{A}$

Maximum shear stress: $\tau_{max} = \dfrac{\sigma_x}{2} = \dfrac{P}{2A}$

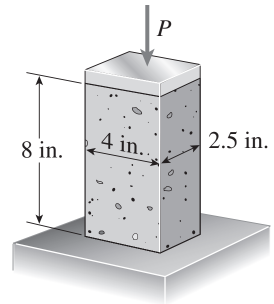
$\sigma_{allow} = 120 \text{ MPa} \qquad \tau_{allow} = 50 \text{ MPa}$

Because τ_{allow} is less than one-half of σ_{allow}, the shear stress governs.

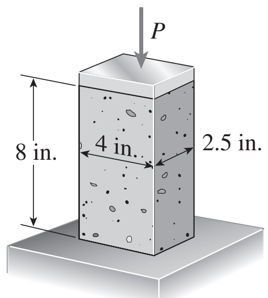
$$\tau_{max} = \frac{P}{2A} \quad \text{or} \quad 50 \text{ MPa} = \frac{3.0 \text{ kN}}{(2)\left(\dfrac{\pi d^2}{4}\right)}$$

Solve for d: $d_{min} = 6.18$ mm ←

Problem 2.6-3 A standard brick (dimensions 8 in. $\times$ 4 in. $\times$ 2.5 in.) is compressed lengthwise by a force P, as shown in the figure. If the ultimate shear stress for brick is 1200 psi and the ultimate compressive stress is 3600 psi, what force P_{max} is required to break the brick?



Solution 2.6-3 Standard brick in compression



$A = 2.5 \text{ in.} \times 4.0 \text{ in.} = 10.0 \text{ in.}^2$

Maximum normal stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress:

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{P}{2A}$$

$\sigma_{ult} = 3600 \text{ psi} \qquad \tau_{ult} = 1200 \text{ psi}$

Because τ_{ult} is less than one-half of σ_{ult}, the shear stress governs.

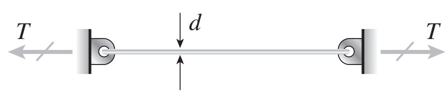
$$\tau_{max} = \frac{P}{2A} \quad \text{or} \quad P_{max} = 2A\tau_{ult}$$

$$P_{max} = 2(10.0 \text{ in.}^2)(1200 \text{ psi})$$

$$= 24{,}000 \text{ lb} \quad \longleftarrow$$

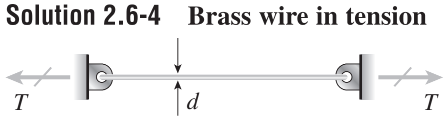
Problem 2.6-4 A brass wire of diameter $d = 2.42$ mm is stretched tightly between rigid supports so that the tensile force is $T = 92$ N (see figure).

What is the maximum permissible temperature drop ΔT if the allowable shear stress in the wire is 60 MPa? (The coefficient of thermal expansion for the wire is 20×10^{-6}/°C and the modulus of elasticity is 100 GPa.)



Probs. 2.6-4 and 2.6-5

Solution 2.6-4 Brass wire in tension



$d = 2.42$ mm

$$A = \frac{\pi d^2}{4} = 4.60 \text{ mm}^2$$

$\alpha = 20 \times 10^{-6}/°\text{C} \quad E = 100 \text{ GPa} \quad \tau_{allow} = 60 \text{ MPa}$

Initial tensile force: $T = 92$ N

Stress due to initial tension: $\sigma_x = \dfrac{T}{A}$

Stress due to temperature drop: $\sigma_x = E\alpha(\Delta T)$

(see Eq. 2-18 of Section 2.5)

Total stress: $\sigma_x = \dfrac{T}{A} + E\alpha(\Delta T)$

MAXIMUM SHEAR STRESS

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{1}{2}\left[\frac{T}{A} + E\alpha(\Delta T)\right]$$

Solve for temperature drop ΔT:

$$\Delta T = \frac{2\tau_{max} - T/A}{E\alpha} \qquad \tau_{max} = \tau_{allow}$$

SUBSTITUTE NUMERICAL VALUES:

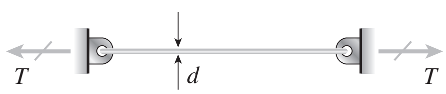
$$\Delta T = \frac{2(60 \text{ MPa}) - (92 \text{ N})/(4.60 \text{ mm}^2)}{(100 \text{ GPa})(20 \times 10^{-6}/°\text{C})}$$

$$= \frac{120 \text{ MPa} - 20 \text{ MPa}}{2 \text{ MPa}/°\text{C}} = 50°\text{C} \quad \longleftarrow$$

Problem 2.6-5 A brass wire of diameter $d = 1/16$ in. is stretched between rigid supports with an initial tension T of 32 lb (see figure).

(a) If the temperature is lowered by 50°F, what is the maximum shear stress τ_{max} in the wire?
(b) If the allowable shear stress is 10,000 psi, what is the maximum permissible temperature drop? (Assume that the coefficient of thermal expansion is 10.6×10^{-6}/°F and the modulus of elasticity is 15×10^6 psi.)

Solution 2.6-5 **Brass wire in tension**



$$d = \frac{1}{16} \text{ in.}$$

$$A = \frac{\pi d^2}{4}$$

$$= 0.003068 \text{ in.}^2$$

$$\alpha = 10.6 \times 10^{-6}/°\text{F}$$

$$E = 15 \times 10^6 \text{ psi}$$

Initial tensile force: $T = 32$ lb

Stress due to initial tension: $\sigma_x = \dfrac{T}{A}$

Stress due to temperature drop: $\sigma_x = E\alpha(\Delta T)$

(see Eq. 2-18 of Section 2.5)

Total stress: $\sigma_x = \dfrac{T}{A} + E\alpha(\Delta T)$

(a) Maximum shear stress when temperature drops 50°F

$$\tau_{max} = \frac{\sigma_x}{2} = \frac{1}{2}\left[\frac{T}{A} + E\alpha(\Delta T)\right] \qquad \text{(Eq. 1)}$$

Substitute numerical values:

$\tau_{max} = 9{,}190$ psi ←

(b) Maximum permissible temperature drop if $\tau_{allow} = 10{,}000$ psi ←

Solve Eq. (1) for ΔT:

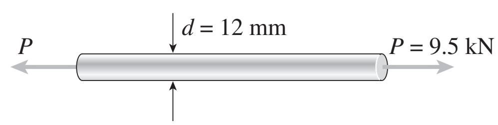
$$\Delta T = \frac{2\tau_{max} - T/A}{E\alpha} \qquad \tau_{max} = \tau_{allow}$$

Substitute numerical values:

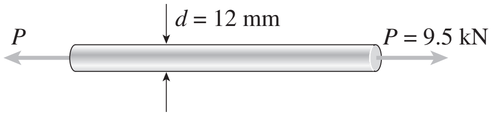
$\Delta T = 60.2°$F ←

Problem 2.6-6 A steel bar with diameter $d = 12$ mm is subjected to a tensile load $P = 9.5$ kN (see figure).

(a) What is the maximum normal stress σ_{max} in the bar?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-6 Steel bar in tension



$P = 9.5$ kN

(a) MAXIMUM NORMAL STRESS

$$\sigma_x = \frac{P}{A} = \frac{9.5\text{ kN}}{\frac{\pi}{4}(12\text{ mm})^2} = 84.0\text{ MPa}$$

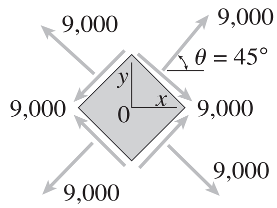
$\sigma_{max} = 84.0$ MPa ←

(b) MAXIMUM SHEAR STRESS

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$$\tau_{max} = \frac{\sigma_x}{2} = 42.0\text{ MPa} \quad \longleftarrow$$

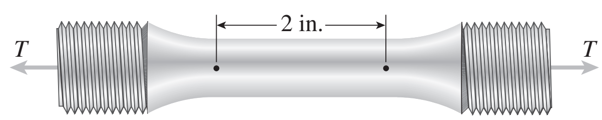
(c) STRESS ELEMENT AT $\theta = 45°$



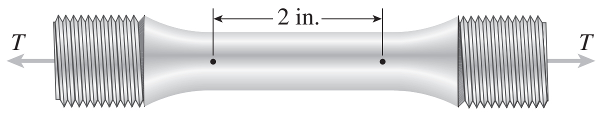
NOTE: All stresses have units of MPa.

Problem 2.6-7 During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.00120 in. with a gage length of 2 in. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity $E = 30 \times 10^6$ psi.

(a) What is the maximum normal stress σ_{max} in the specimen?
(b) What is the maximum shear stress τ_{max}?
(c) Draw a stress element oriented at an angle of 45° to the axis of the bar and show all stresses acting on the faces of this element.



Solution 2.6-7 Tension test



Elongation: $\delta = 0.00120$ in.

(2 in. gage length)

Strain: $\varepsilon = \dfrac{\delta}{L} = \dfrac{0.00120 \text{ in.}}{2 \text{ in.}} = 0.00060$

Hooke's law : $\sigma_x = E\varepsilon = (30 \times 10^6 \text{ psi})(0.00060)$

$= 18{,}000$ psi

(a) Maximum normal stress

σ_x is the maximum normal stress.

$\sigma_{\max} = 18{,}000$ psi ⟵

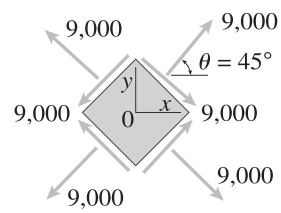
(b) Maximum shear stress

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$\tau_{\max} = \dfrac{\sigma_x}{2} = 9{,}000$ psi ⟵

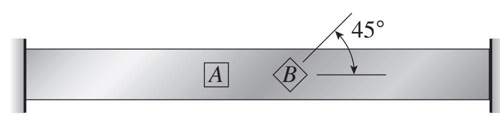
(c) Stress element at $\theta = 45°$

Note: All stresses have units of psi.

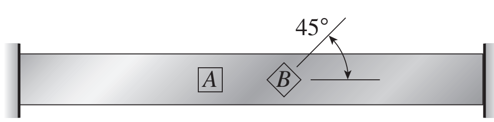


Problem 2.6-8 A copper bar with a rectangular cross section is held without stress between rigid supports (see figure). Subsequently, the temperature of the bar is raised 50°C.

Determine the stresses on all faces of the elements A and B, and show these stresses on sketches of the elements. (Assume $\alpha = 17.5 \times 10^{-6}$/°C and $E = 120$ GPa.)



Solution 2.6-8 Copper bar with rigid supports



$\Delta T = 50°\text{C}$ (Increase)

$\alpha = 17.5 \times 10^{-6}$/°C

$E = 120$ GPa

Stress due to temperature increase

$\sigma_x = E\alpha\,(\Delta T)$ (See Eq. 2-18 of Section 2.5)

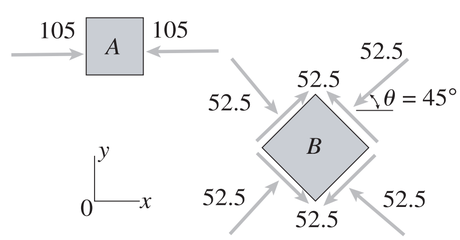
$= 105$ MPa (Compression)

Maximum shear stress

$\tau_{\max} = \dfrac{\sigma_x}{2}$

$= 52.5$ MPa

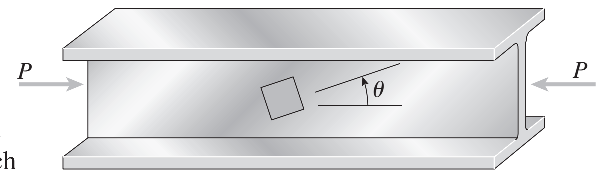
Stresses on elements A and B



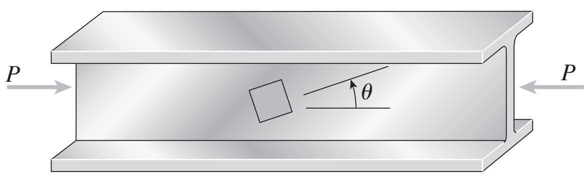
Note: All stresses have units of MPa.

Problem 2.6-9 A compression member in a bridge truss is fabricated from a wide-flange steel section (see figure). The cross-sectional area $A = 7.5$ in.^{2} and the axial load $P = 90$ k.

Determine the normal and shear stresses acting on all faces of stress elements located in the web of the beam and oriented at (a) an angle $\theta = 0°$, (b) an angle $\theta = 30°$, and (c) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element.



Solution 2.6-9 Truss member in compression



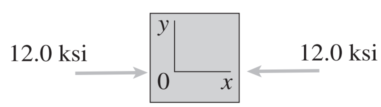
$P = 90$ k

$A = 7.5$ in.^{2}

$$\sigma_x = -\frac{P}{A} = -\frac{90\text{ k}}{7.5\text{ in.}^2}$$

$= -12.0$ ksi (Compression)

(a) $\theta = 0°$



(b) $\theta = 30°$

Use Eqs. (2-29a) and (2-29b):

$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0\text{ ksi})(\cos 30°)^2$

$= -9.0$ ksi

$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -(-12.0\text{ ksi})(\sin 30°)(\cos 30°)$

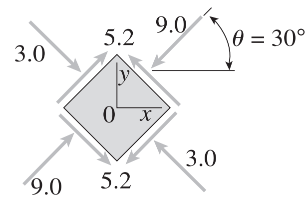
$= 5.2$ ksi

$\theta = 30° + 90° = 120°$

$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0\text{ ksi})(\cos 120°)^2 = -3.0$ ksi

$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -(-12.0\text{ ksi})(\sin 120°)(\cos 120°)$

$= -5.2$ ksi



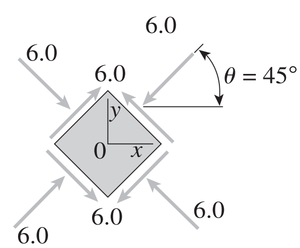
NOTE: All stresses have units of ksi.

(c) $\theta = 45°$

$\sigma_\theta = \sigma_x \cos^2\theta = (-12.0\text{ ksi})(\cos 45°)^2 = -6.0$ ksi

$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -(-12.0\text{ ksi})(\sin 45°)(\cos 45°)$

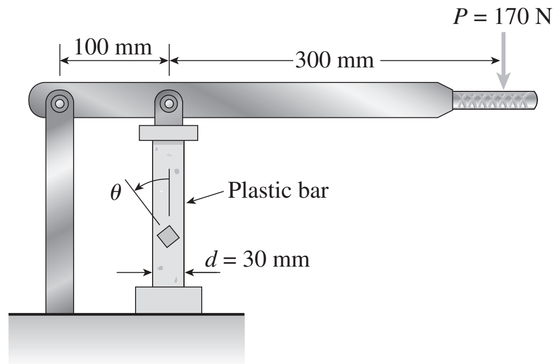
$= 6.0$ ksi



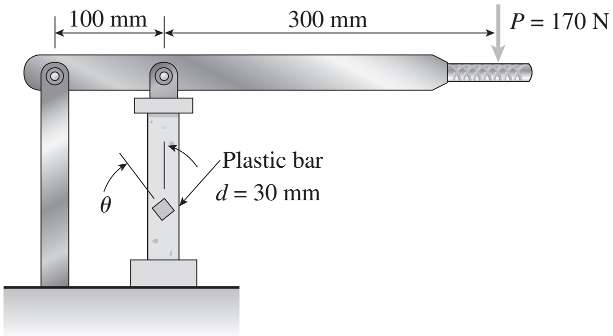
NOTE: All stresses have units of ksi.

Problem 2.6-10 A plastic bar of diameter $d = 30$ mm is compressed in a testing device by a force $P = 170$ N applied as shown in the figure.

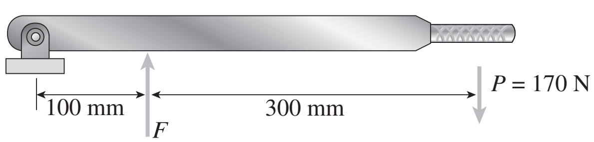
Determine the normal and shear stresses acting on all faces of stress elements oriented at (a) an angle $\theta = 0°$, (b) an angle $\theta = 22.5°$, and (c) an angle $\theta = 45°$. In each case, show the stresses on a sketch of a properly oriented element.



Solution 2.6-10 Plastic bar in compression



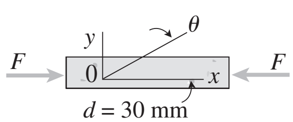
FREE-BODY DIAGRAM



F = Compressive force in plastic bar

$F = 4P = 4(170\text{ N}) = 680\text{ N}$

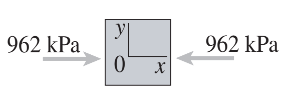
PLASTIC BAR (ROTATED TO THE HORIZONTAL)



$$\sigma_x = -\frac{F}{A} = -\frac{680\text{ N}}{\frac{\pi}{4}(30\text{ mm})^2}$$

$$= -962.0\text{ kPa (Compression)}$$

(a) $\theta = 0°$



(b) $\theta = 22.5°$

Use Eqs. (2-29a) and (2-29b)

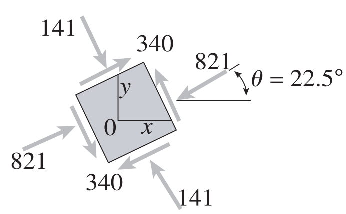
$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0\text{ kPa})(\cos 22.5°)^2$$
$$= -821\text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0\text{ kPa})(\sin 22.5°)(\cos 22.5°)$$
$$= 340\text{ kPa}$$

$$\theta = 22.5° + 90° = 112.5°$$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0\text{ kPa})(\cos 112.5°)^2$$
$$= -141\text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0\text{ kPa})(\sin 112.5°)(\cos 112.5°)$$
$$= -340\text{ kPa}$$

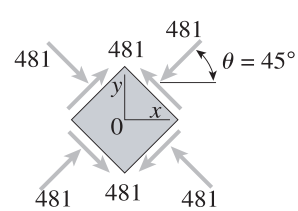


NOTE: All stresses have units of kPa.

(c) $\theta = 45°$

$$\sigma_\theta = \sigma_x \cos^2\theta = (-962.0\text{ kPa})(\cos 45°)^2$$
$$= -481\text{ kPa}$$

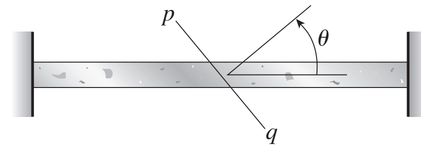
$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$
$$= -(-962.0\text{ kPa})(\sin 45°)(\cos 45°) = 481\text{ kPa}$$



NOTE: All stresses have units of kPa.

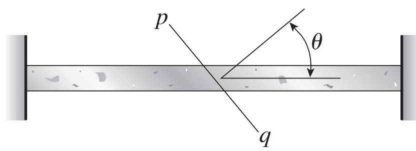
Problem 2.6-11 A plastic bar fits snugly between rigid supports at room temperature (68°F) but with no initial stress (see figure). When the temperature of the bar is raised to 160°F, the compressive stress on an inclined plane *pq* becomes 1700 psi.

(a) What is the shear stress on plane *pq*? (Assume $\alpha = 60 \times 10^{-6}$/°F and $E = 450 \times 10^3$ psi.)
(b) Draw a stress element oriented to plane *pq* and show the stresses acting on all faces of this element.



Probs. 2.6-11 and 2.6-12

Solution 2.6-11 **Plastic bar between rigid supports**



$\alpha = 60 \times 10^{-6}$/°F $\quad E = 450 \times 10^3$ psi

Temperature increase:

$\Delta T = 160°\text{F} - 68°\text{F} = 92°\text{F}$

NORMAL STRESS σ_x IN THE BAR

$\sigma_x = -E\alpha(\Delta T)$ (See Eq. 2-18 in Section 2.5)

$\sigma_x = -(450 \times 10^3 \text{ psi})(60 \times 10^{-6}/°\text{F})(92°\text{F})$

$= -2484$ psi (Compression)

ANGLE θ TO PLANE *pq*

$\sigma_\theta = \sigma_x \cos^2\theta$ For plane *pq*: $\sigma_\theta = -1700$ psi

Therefore, $-1700 \text{ psi} = (-2484 \text{ psi})(\cos^2\theta)$

$$\cos^2\theta = \frac{-1700 \text{ psi}}{-2484 \text{ psi}} = 0.6844$$

$\cos\theta = 0.8273 \quad \theta = 34.18°$

(a) SHEAR STRESS ON PLANE *pq*

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= -(-2484 \text{ psi})(\sin 34.18°)(\cos 34.18°)$

$= 1150$ psi (Counter clockwise) ⟵

(b) STRESS ELEMENT ORIENTED TO PLANE *pq*

$\theta = 34.18° \quad \sigma_\theta = -1700 \text{ psi} \quad \tau_\theta = 1150 \text{ psi}$

$\theta = 34.18° + 90° = 124.18°$

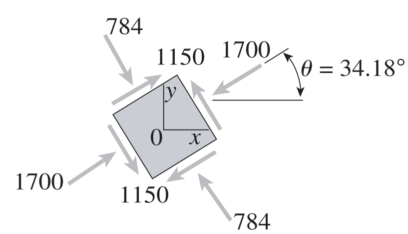
$\sigma_\theta = \sigma_x \cos^2\theta = (-2484 \text{ psi})(\cos 124.18°)^2$

$= -784$ psi

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= -(-2484 \text{ psi})(\sin 124.18°)(\cos 124.18°)$

$= -1150$ psi

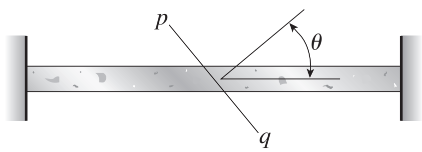


NOTE: All stresses have units of psi.

Problem 2.6-12 A copper bar is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane pq, for which $\theta = 55°$, are specified as 60 MPa in compression and 30 MPa in shear.

(a) What is the maximum permissible temperature rise ΔT if the allowable stresses on plane pq are not to be exceeded? (Assume $\alpha = 17 \times 10^{-6}$/°C and $E = 120$ GPa.)
(b) If the temperature increases by the maximum permissible amount, what are the stresses on plane pq?

Solution 2.6-12 **Copper bar between rigid supports**



$\alpha = 17 \times 10^{-6}$/°C

$E = 120$ GPa

Plane pq: $\theta = 55°$

Allowable stresses on plane pq:

$\sigma_{allow} = 60$ MPa (Compression)

$\tau_{allow} = 30$ MPa (Shear)

(a) Maximum permissible temperature rise ΔT

$\sigma_\theta = \sigma_x \cos^2\theta \quad -60 \text{ MPa} = \sigma_x (\cos 55°)^2$

$\sigma_x = -182.4$ MPa

$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$30 \text{ MPa} = -\sigma_x (\sin 55°)(\cos 55°)$

$\sigma_x = -63.85$ MPa

Shear stress governs. $\sigma_x = -63.85$ MPa

Due to temperature increase ΔT:

$\sigma_x = -E\alpha(\Delta T)$ (See Eq. 2-18 in Section 2.5)

$-63.85 \text{ MPa} = -(120 \text{ GPa})(17 \times 10^{-6}/°\text{C})(\Delta T)$

$\Delta T = 31.3°\text{C}$ ←

(b) Stresses on plane pq

$\sigma_x = -63.85$ MPa

$\sigma_\theta = \sigma_x \cos^2\theta = (-63.85 \text{ MPa})(\cos 55°)^2$

$= -21.0$ MPa (Compression) ←

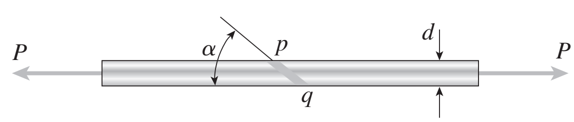
$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$= -(-63.85 \text{ MPa})(\sin 55°)(\cos 55°)$

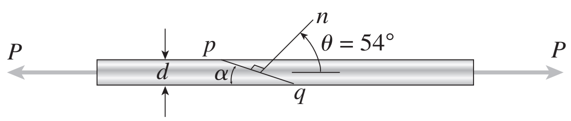
$= 30.0$ MPa (Counter clockwise) ←

Problem 2.6-13 A circular brass bar of diameter d is composed of two segments brazed together on a plane pq making an angle $\alpha = 36°$ with the axis of the bar (see figure). The allowable stresses in the brass are 13,500 psi in tension and 6500 psi in shear. On the brazed joint, the allowable stresses are 6000 psi in tension and 3000 psi in shear.

If the bar must resist a tensile force $P = 6000$ lb, what is the minimum required diameter d_{min} of the bar?



Solution 2.6-13 Brass bar in tension



$\alpha = 36°$

$\theta = 90° - \alpha = 54°$

$P = 6000 \text{ lb}$

$$A = \frac{\pi d^2}{4}$$

STRESS σ_x BASED UPON ALLOWABLE STRESSES IN THE BRASS

Tensile stress ($\theta = 0°$): $\sigma_{\text{allow}} = 13{,}500$ psi

$$\sigma_x = 13{,}500 \text{ psi} \qquad (1)$$

Shear stress ($\theta = 45°$): $\tau_{\text{allow}} = 6500$ psi

$$\tau_{\max} = \frac{\sigma_x}{2}$$

$$\sigma_x = 2\,\tau_{\text{allow}}$$
$$= 13{,}000 \text{ psi} \qquad (2)$$

STRESS σ_x BASED UPON ALLOWABLE STRESSES ON THE BRAZED JOINT ($\theta = 54°$)

$\sigma_{\text{allow}} = 6000$ psi (tension)

$\tau_{\text{allow}} = 3000$ psi (shear)

Tensile stress: $\sigma_\theta = \sigma_x \cos^2\theta$

$$\sigma_x = \frac{\sigma_{\text{allow}}}{\cos^2\theta} = \frac{6000 \text{ psi}}{(\cos 54°)^2}$$
$$= 17{,}370 \text{ psi} \qquad (3)$$

Shear stress: $\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$$\sigma_x = \left|\frac{\tau_{\text{allow}}}{\sin\theta\cos\theta}\right| = \frac{3{,}000 \text{ psi}}{(\sin 54°)(\cos 54°)}$$
$$= 6{,}310 \text{ psi} \qquad (4)$$

ALLOWABLE STRESS

Compare (1), (2), (3), and (4).

Shear stress on the brazed joint governs.

$\sigma_x = 6310$ psi

DIAMETER OF BAR

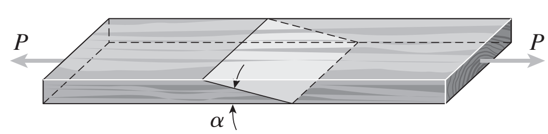
$$A = \frac{P}{\sigma_x} = \frac{6000 \text{ lb}}{6310 \text{ psi}} = 0.951 \text{ in.}^2$$

$$A = \frac{\pi d^2}{4} \qquad d^2 = \frac{4A}{\pi} \qquad d_{\min} = \sqrt{\frac{4A}{\pi}}$$

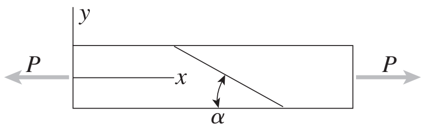
$$d_{\min} = 1.10 \text{ in.} \longleftarrow$$

Problem 2.6-14 Two boards are joined by gluing along a scarf joint, as shown in the figure. For purposes of cutting and gluing, the angle α between the plane of the joint and the faces of the boards must be between 10° and 40°. Under a tensile load P, the normal stress in the boards is 4.9 MPa.

(a) What are the normal and shear stresses acting on the glued joint if $\alpha = 20°$?
(b) If the allowable shear stress on the joint is 2.25 MPa, what is the largest permissible value of the angle α?
(c) For what angle α will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?



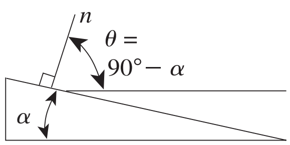
Solution 2.6-14 **Two boards joined by a scarf joint**



$10° \leq \alpha \leq 40°$

Due to load P: $\sigma_x = 4.9$ MPa

(a) Stresses on joint when $\alpha = 20°$



$\theta = 90° - \alpha = 70°$

$\sigma_\theta = \sigma_x \cos^2\theta = (4.9 \text{ MPa})(\cos 70°)^2$

$= 0.57$ MPa ⟵

$\tau_\theta = -\sigma_x \sin\theta \cos\theta$

$= (-4.9 \text{ MPa})(\sin 70°)(\cos 70°)$

$= -1.58$ MPa ⟵

(b) Largest angle α if $\tau_{allow} = 2.25$ MPa

$\tau_{allow} = -\sigma_x \sin\theta \cos\theta$

The shear stress on the joint has a negative sign. Its numerical value cannot exceed $\tau_{allow} = 2.25$ MPa. Therefore,

$-2.25 \text{ MPa} = -(4.9 \text{ MPa})(\sin\theta)(\cos\theta)$ or
$\sin\theta\cos\theta = 0.4592$

From trigonometry: $\sin\theta\cos\theta = \dfrac{1}{2}\sin 2\theta$

Therefore: $\sin 2\theta = 2(0.4592) = 0.9184$

Solving : $2\theta = 66.69°$ or $113.31°$

$\theta = 33.34°$ or $56.66°$

$\alpha = 90° - \theta \quad \therefore \alpha = 56.66°$ or $33.34°$

Since α must be between 10° and 40°, we select

$\alpha = 33.3°$ ⟵

Note: If α is between 10° and 33.3°,

$|\tau_\theta| < 2.25$ MPa.

If α is between 33.3° and 40°,

$|\tau_\theta| > 2.25$ MPa.

(c) What is α if $\tau_\theta = 2\sigma_\theta$?

Numerical values only:

$|\tau_\theta| = \sigma_x \sin\theta\cos\theta \qquad |\sigma_\theta| = \sigma_x \cos^2\theta$

$$\left|\frac{\tau_\theta}{\sigma_\theta}\right| = 2$$

$\sigma_x \sin\theta\cos\theta = 2\sigma_x\cos^2\theta$

$\sin\theta = 2\cos\theta$ or $\tan\theta = 2$

$\theta = 63.43° \quad \alpha = 90° - \theta$

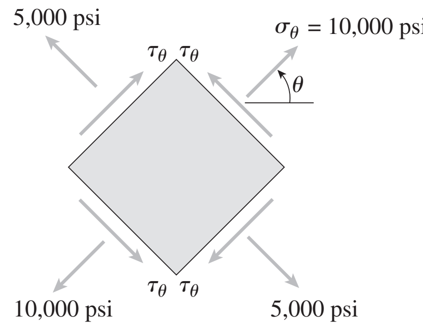
$\alpha = 26.6°$ ⟵

Note: For $\alpha = 26.6°$ and $\theta = 63.4°$, we find $\sigma_\theta = 0.98$ MPa and $\tau_\theta = -1.96$ MPa.

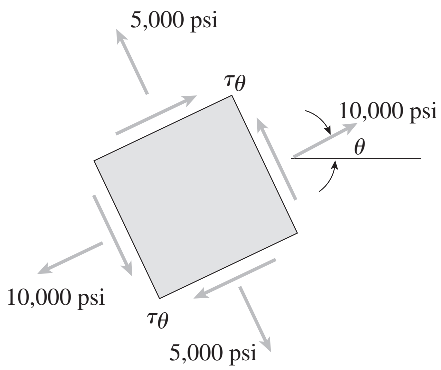
Thus, $\left|\dfrac{\tau_\theta}{\sigma_\theta}\right| = 2$ as required.

Problem 2.6-15 Acting on the sides of a stress element cut from a bar in uniaxial stress are tensile stresses of 10,000 psi and 5,000 psi, as shown in the figure.

(a) Determine the angle θ and the shear stress τ_θ and show all stresses on a sketch of the element.
(b) Determine the maximum normal stress σ_{max} and the maximum shear stress τ_{max} in the material.



Solution 2.6-15 **Bar in uniaxial stress**



(a) ANGLE θ AND SHEAR STRESS τ_θ

$$\sigma_\theta = \sigma_x \cos^2\theta$$

$$\sigma_\theta = 10{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{10{,}000 \text{ psi}}{\cos^2\theta} \qquad (1)$$

PLANE AT ANGLE $\theta + 90°$

$$\sigma_{\theta+90°} = \sigma_x[\cos(\theta + 90°)]^2 = \sigma_x[-\sin\theta]^2$$

$$= \sigma_x \sin^2\theta$$

$$\sigma_{\theta+90°} = 5{,}000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_{\theta+90°}}{\sin^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta} \qquad (2)$$

Equate (1) and (2):

$$\frac{10{,}000 \text{ psi}}{\cos^2\theta} = \frac{5{,}000 \text{ psi}}{\sin^2\theta}$$

$$\tan^2\theta = \frac{1}{2} \qquad \tan\theta = \frac{1}{\sqrt{2}} \qquad \theta = 35.26° \longleftarrow$$

From Eq. (1) or (2):

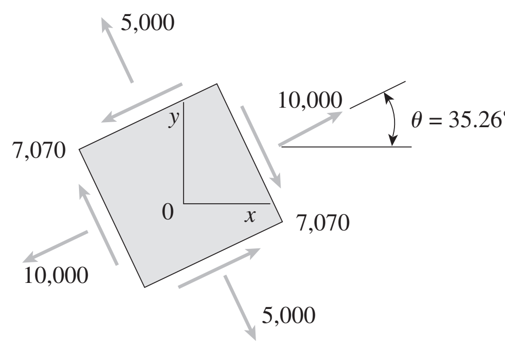
$$\sigma_x = 15{,}000 \text{ psi}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$

$$= (-15{,}000 \text{ psi})(\sin 35.26°)(\cos 35.26°)$$

$$= -7{,}070 \text{ psi} \longleftarrow$$

Minus sign means that τ_θ acts clockwise on the plane for which $\theta = 35.26°$.



NOTE: All stresses have units of psi.

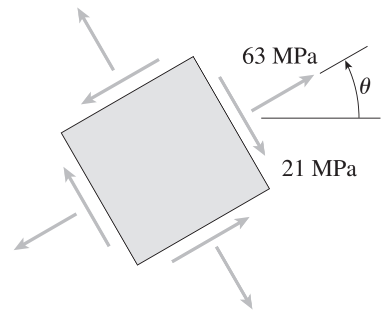
(b) MAXIMUM NORMAL AND SHEAR STRESSES

$$\sigma_{max} = \sigma_x = 15{,}000 \text{ psi} \longleftarrow$$

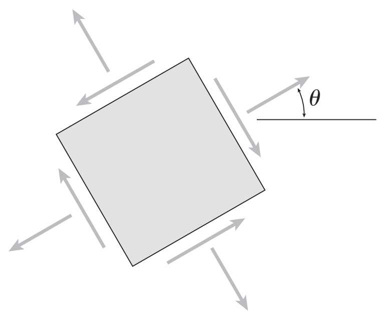
$$\tau_{max} = \frac{\sigma_x}{2} = 7{,}500 \text{ psi} \longleftarrow$$

Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 63$ MPa and a shear stress $\tau_\theta = -21$ MPa on a certain inclined plane (see figure).

Determine the stresses acting on all faces of a stress element oriented at $\theta = 30°$ and show the stresses on a sketch of the element.



Solution 2.6-16 **Bar in uniaxial stress**



$\sigma_\theta = 63$ MPa $\quad \tau_\theta = -21$ MPa

INCLINED PLANE AT ANGLE θ

$\sigma_\theta = \sigma_x \cos^2\theta$

$63 \text{ MPa} = \sigma_x \cos^2\theta$

$$\sigma_x = \frac{63 \text{ MPa}}{\cos^2\theta} \quad (1)$$

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$-21 \text{ MPa} = -\sigma_x \sin\theta\cos\theta$

$$\sigma_x = \frac{21 \text{ MPa}}{\sin\theta\cos\theta} \quad (2)$$

Equate (1) and (2):

$$\frac{63 \text{ MPa}}{\cos^2\theta} = \frac{21 \text{ MPa}}{\sin\theta\cos\theta}$$

or

$$\tan\theta = \frac{21}{63} = \frac{1}{3} \qquad \theta = 18.43°$$

From (1) or (2): $\sigma_x = 70.0$ MPa (tension)

STRESS ELEMENT AT $\theta = 30°$

$\sigma_\theta = \sigma_x \cos^2\theta = (70 \text{ MPa})(\cos 30°)^2$

$= 52.5$ MPa

$\tau_\theta = -\sigma_x \sin\theta\cos\theta$

$= (-70 \text{ MPa})(\sin 30°)(\cos 30°)$

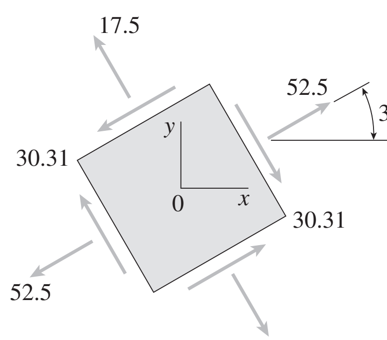
$= -30.31$ MPa

Plane at $\theta = 30° + 90° = 120°$

$\sigma_\theta = (70 \text{ MPa})(\cos 120°)^2 = 17.5$ MPa

$\tau_\theta = (-70 \text{ MPa})(\sin 120°)(\cos 120°)$

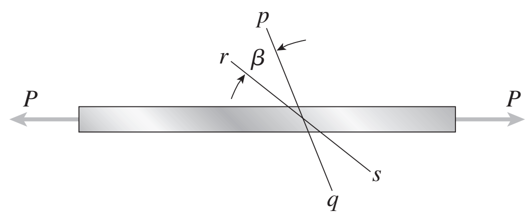
$= 30.31$ MPa



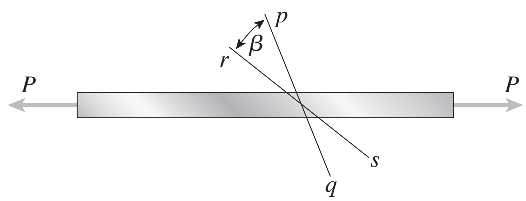
NOTE: All stresses have units of MPa.

Problem 2.6-17 The normal stress on plane *pq* of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane *rs*, which makes an angle $\beta = 30°$ with plane *pq*, the stress is found to be 2500 psi.

Determine the maximum normal stress σ_{max} and maximum shear stress τ_{max} in the bar.



Solution 2.6-17 **Bar in tension**



Eq. (2-29a):

$\sigma_\theta = \sigma_x \cos^2\theta$

$\beta = 30°$

Plane *pq*: $\sigma_1 = \sigma_x \cos^2\theta_1$ $\quad\sigma_1 = 7500$ psi

Plane *rs*: $\sigma_2 = \sigma_x \cos^2(\theta_1 + \beta)$ $\quad\sigma_2 = 2500$ psi

Equate σ_x from σ_1 and σ_2:

$$\sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \qquad \text{(Eq. 1)}$$

or

$$\frac{\cos^2\theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \qquad \frac{\cos\theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \qquad \text{(Eq. 2)}$$

Substitute numerical values into Eq. (2):

$$\frac{\cos\theta_1}{\cos(\theta_1 + 30°)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$\theta_1 = 30°$

Maximum normal stress (from Eq. 1)

$$\sigma_{max} = \sigma_x = \frac{\sigma_1}{\cos^2\theta_1} = \frac{7500 \text{ psi}}{\cos^2 30°}$$

$$= 10{,}000 \text{ psi} \quad \longleftarrow$$

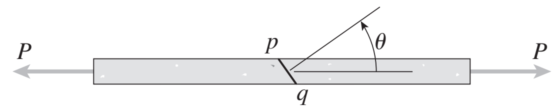
Maximum shear stress

$$\tau_{max} = \frac{\sigma_x}{2} = 5{,}000 \text{ psi} \quad \longleftarrow$$

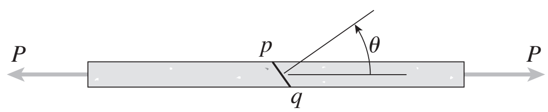
Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane *pq* (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45°. The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.

(a) Determine the angle θ so that the bar will carry the largest load P. (Assume that the strength of the glued joint controls the design.)

(b) Determine the maximum allowable load P_{max} if the cross-sectional area of the bar is 225 mm².



Solution 2.6-18 **Bar in tension with glued joint**



$25° < \theta < 45°$

$A = 225 \text{ mm}^2$

On glued joint: $\sigma_{allow} = 5.0 \text{ MPa}$

$\tau_{allow} = 3.0 \text{ MPa}$

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2\theta \qquad \sigma_x = \frac{\sigma_\theta}{\cos^2\theta} = \frac{5.0 \text{ MPa}}{\cos^2\theta} \qquad (1)$$

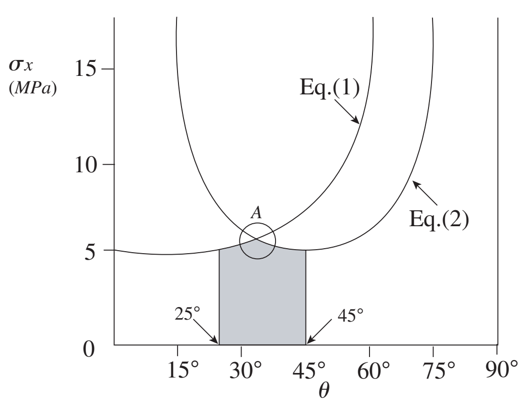
$$\tau_\theta = -\sigma_x \sin\theta\cos\theta$$

Since the direction of τ_θ is immaterial, we can write:
$|\tau_\theta| = \sigma_x \sin\theta\cos\theta$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin\theta\cos\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta} \qquad (2)$$

GRAPH OF EQS. (1) AND (2)



(a) DETERMINE ANGLE θ FOR LARGEST LOAD

Point A gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point A, we equate Eqs. (1) and (2).

$$\frac{5.0 \text{ MPa}}{\cos^2\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta}$$

$$\tan\theta = \frac{3.0}{5.0} \qquad \theta = 30.96° \quad \longleftarrow$$

(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

$$\sigma_x = \frac{5.0 \text{ MPa}}{\cos^2\theta} = \frac{3.0 \text{ MPa}}{\sin\theta\cos\theta} = 6.80 \text{ MPa}$$

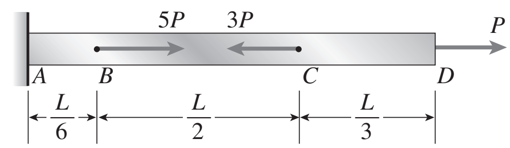
$$P_{max} = \sigma_x A = (6.80 \text{ MPa})(225 \text{ mm}^2)$$

$$= 1.53 \text{ kN} \quad \longleftarrow$$

Strain Energy

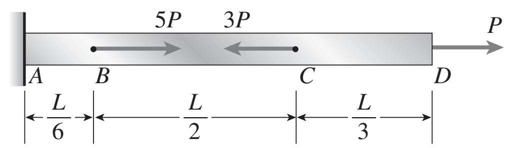
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar AD of length L, cross-sectional area A, and modulus of elasticity E is subjected to loads $5P$, $3P$, and P acting at points B, C, and D, respectively (see figure). Segments AB, BC, and CD have lengths $L/6$, $L/2$, and $L/3$, respectively.



(a) Obtain a formula for the strain energy U of the bar.
(b) Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.^{2}, and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 **Bar with three loads**



$P = 6$ k

$L = 52$ in.

$E = 10.4 \times 10^6$ psi

$A = 2.76$ in.^{2}

Internal axial forces

$N_{AB} = 3P \qquad N_{BC} = -2P \qquad N_{CD} = P$

Lengths

$$L_{AB} = \frac{L}{6} \qquad L_{BC} = \frac{L}{2} \qquad L_{CD} = \frac{L}{3}$$

(a) Strain energy of the bar (Eq. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{1}{2EA}\left[(3P)^2\left(\frac{L}{6}\right) + (-2P)^2\left(\frac{L}{2}\right) + (P)^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{P^2 L}{2EA}\left(\frac{23}{6}\right) = \frac{23P^2L}{12EA} \longleftarrow$$

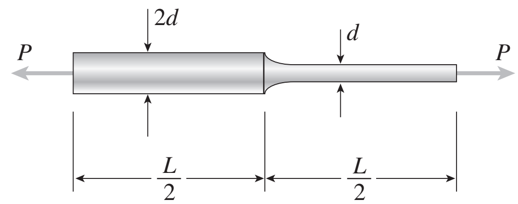
(b) Substitute numerical values:

$$U = \frac{23(6\text{ k})^2(52\text{ in.})}{12(10.4 \times 10^6\text{ psi})(2.76\text{ in.}^2)}$$

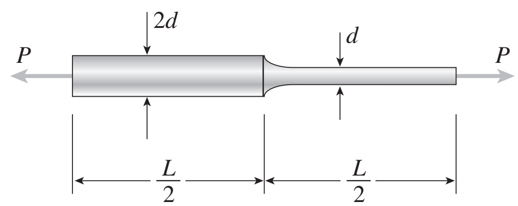
$$= 125\text{ in.-lb} \longleftarrow$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E.

(a) Obtain a formula for the strain energy U of the bar due to the load P.
(b) Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 **Bar with two segments**



(a) STRAIN ENERGY OF THE BAR

Add the strain energies of the two segments of the bar (see Eq. 2-40).

$$U = \sum_{i=1}^{2} \frac{N_i^2 L_i}{2 E_i A_i} = \frac{P^2(L/2)}{2E}\left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)}\right]$$

$$= \frac{P^2 L}{\pi E}\left(\frac{1}{4d^2} + \frac{1}{d^2}\right) = \frac{5P^2 L}{4\pi E d^2} \longleftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$P = 27$ kN $\qquad L = 600$ mm

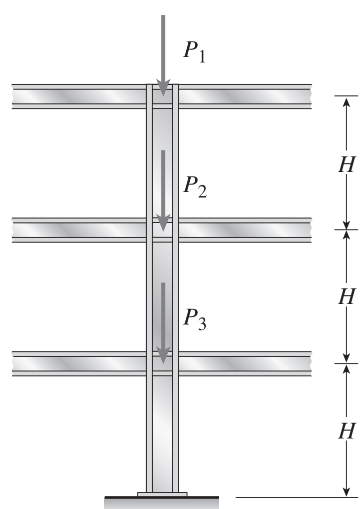
$d = 40$ mm $\qquad E = 105$ GPa

$$U = \frac{5(27\text{ kN})^2(600\text{ mm})}{4\pi(105\text{ GPa})(40\text{ mm})^2}$$

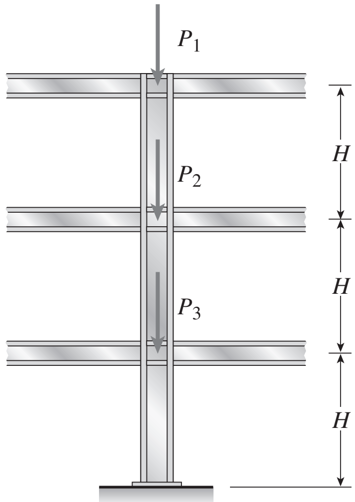
$$= 1.036\text{ N}\cdot\text{m} = 1.036\text{ J} \longleftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^{2}, and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 **Three-story column**



$H = 10.5$ ft $\quad E = 30 \times 10^6$ psi

$A = 15.5$ in.^{2} $\quad P_1 = 40$ k

$P_2 = P_3 = 60$ k

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-40).

Upper segment: $N_1 = -P_1$

Middle segment: $N_2 = -(P_1 + P_2)$

Lower segment: $N_3 = -(P_1 + P_2 + P_3)$

STRAIN ENERGY

$$U = \sum \frac{N_i^2 L_i}{2E_i A_i}$$

$$= \frac{H}{2EA}[P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2]$$

$$= \frac{H}{2EA}[Q]$$

$$[Q] = (40\text{ k})^2 + (100\text{ k})^2 + (160\text{ k})^2 = 37{,}200\text{ k}^2$$

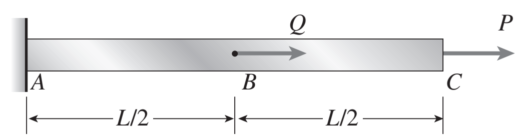
$$2EA = 2(30 \times 10^6\text{ psi})(15.5\text{ in.}^2) = 930 \times 10^6\text{ lb}$$

$$U = \frac{(10.5\text{ ft})(12\text{ in./ft})}{930 \times 10^6\text{ lb}}[37{,}200\text{ k}^2]$$

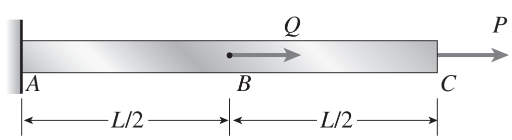
$$= 5040\text{ in.-lb} \quad \longleftarrow$$

Problem 2.7-4 The bar *ABC* shown in the figure is loaded by a force *P* acting at end *C* and by a force *Q* acting at the midpoint *B*. The bar has constant axial rigidity *EA*.

(a) Determine the strain energy U_1 of the bar when the force *P* acts alone ($Q = 0$).
(b) Determine the strain energy U_2 when the force *Q* acts alone ($P = 0$).
(c) Determine the strain energy U_3 when the forces *P* and *Q* act simultaneously upon the bar.



Solution 2.7-4 **Bar with two loads**



(a) FORCE *P* ACTS ALONE ($Q = 0$)

$$U_1 = \frac{P^2 L}{2EA} \quad \longleftarrow$$

(b) FORCE *Q* ACTS ALONE ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(c) FORCES *P* AND *Q* ACT SIMULTANEOUSLY

Segment *BC*: $U_{BC} = \dfrac{P^2(L/2)}{2EA} = \dfrac{P^2 L}{4EA}$

Segment *AB*: $U_{AB} = \dfrac{(P + Q)^2(L/2)}{2EA}$

$$= \frac{P^2 L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2 L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \quad \longleftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if *Q* is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 **Strain-energy density**

DATA:

Material	Weight density (lb/in.^{3})	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

STRAIN ENERGY PER UNIT VOLUME

$$U = \frac{P^2 L}{2EA} \qquad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$u = \frac{U}{V} = \frac{\sigma^2}{2E}$$

At the proportional limit:

$u = u_R$ = modulus of resistance

$$u_R = \frac{\sigma_{PL}^2}{2E} \qquad \text{(Eq. 1)}$$

STRAIN ENERGY PER UNIT WEIGHT

$$U = \frac{P^2 L}{2EA} \qquad \text{Weight } W = \gamma AL$$

γ = weight density

$$u_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

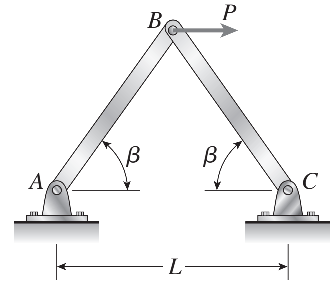
$$u_W = \frac{\sigma_{PL}^2}{2\gamma E} \qquad \text{(Eq. 2)}$$

RESULTS

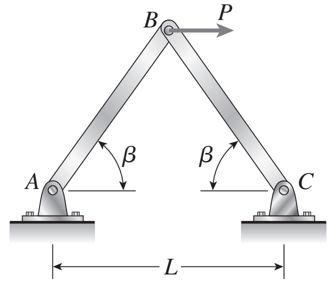
	u_R (psi)	u_W (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss ABC shown in the figure is subjected to a horizontal load P at joint B. The two bars are identical with cross-sectional area A and modulus of elasticity E.

(a) Determine the strain energy U of the truss if the angle $\beta = 60°$.
(b) Determine the horizontal displacement δ_B of joint B by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 **Truss subjected to a load P**



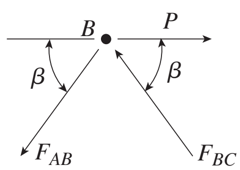
$\beta = 60°$

$L_{AB} = L_{BC} = L$

$\sin\beta = \sqrt{3}/2$

$\cos\beta = 1/2$

FREE-BODY DIAGRAM OF JOINT B



$\Sigma F_{vert} = 0 \quad \uparrow_{+} \quad \downarrow^{-}$

$-F_{AB}\sin\beta + F_{BC}\sin\beta = 0$

$F_{AB} = F_{BC}$ (Eq. 1)

$\Sigma F_{horiz} = 0 \quad \xrightarrow{+} \quad \xleftarrow{-}$

$-F_{AB}\cos\beta - F_{BC}\cos\beta + P = 0$

$$F_{AB} = F_{BC} = \frac{P}{2\cos\beta} = \frac{P}{2(1/2)} = P \qquad \text{(Eq. 2)}$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-40)

$$U = \sum \frac{N_i^2 L_i}{2E_iA_i} = \frac{(N_{AB})^2L}{2EA} + \frac{(N_{BC})^2L}{2EA}$$

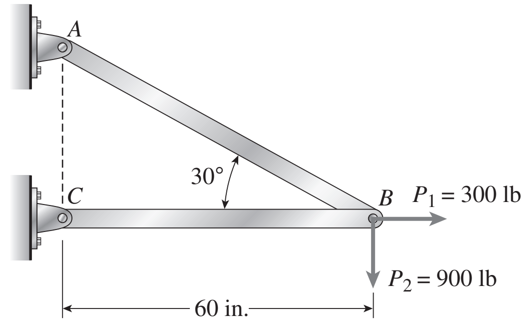
$$= \frac{P^2L}{EA} \quad \longleftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT B (EQ. 2-42)

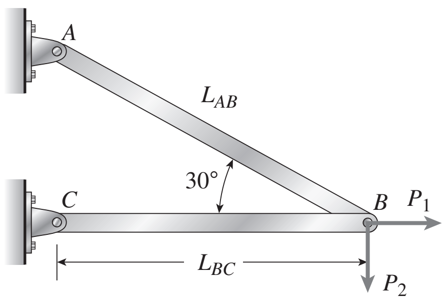
$$\delta_B = \frac{2U}{P} = \frac{2}{P}\left(\frac{P^2L}{EA}\right) = \frac{2PL}{EA} \quad \longleftarrow$$

Problem 2.7-7 The truss *ABC* shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.^{2} and are made of steel with $E = 30 \times 10^6$ psi.

(a) Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
(b) Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
(c) Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 **Truss with two loads**



$P_1 = 300$ lb

$P_2 = 900$ lb

$A = 2.4$ in.^{2}

$E = 30 \times 10^6$ psi

$L_{BC} = 60$ in.

$\beta = 30°$

$$\sin \beta = \sin 30° = \frac{1}{2}$$

$$\cos \beta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30°} = \frac{120}{\sqrt{3}} \text{ in.} = 69.282 \text{ in.}$$

$$2EA = 2(30 \times 10^6 \text{ psi})(2.4 \text{ in.}^2) = 144 \times 10^6 \text{ lb}$$

Forces F_{AB} and F_{BC} in the bars

From equilibrium of joint *B*:

$$F_{AB} = 2P_2 = 1800 \text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300 \text{ lb} - 1558.8 \text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	−1558.8 lb	−1258.8 lb

(a) Load P_1 acts alone

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300 \text{ lb})^2(60 \text{ in.})}{144 \times 10^6 \text{ lb}}$$

$$= 0.0375 \text{ in.-lb} \quad \longleftarrow$$

(b) Load P_2 acts alone

$$U_2 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\Big[(1800 \text{ lb})^2(69.282 \text{ in.})$$

$$+ (-1558.8 \text{ lb})^2(60 \text{ in.})\Big]$$

$$= \frac{370.265 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} = 2.57 \text{ in.-lb} \quad \longleftarrow$$

(c) Loads P_1 and P_2 act simultaneously

$$U_3 = \frac{1}{2EA}\left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC}\right]$$

$$= \frac{1}{2EA}\Big[(1800 \text{ lb})^2(69.282 \text{ in.})$$

$$+ (-1258.8 \text{ lb})^2(60 \text{ in.})\Big]$$

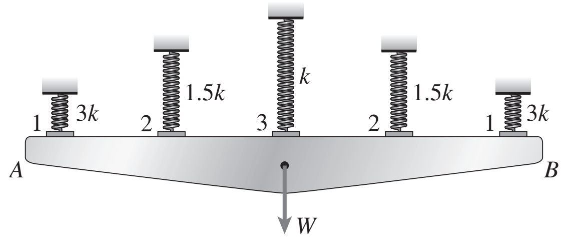
$$= \frac{319.548 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}}$$

$$= 2.22 \text{ in.-lb} \quad \longleftarrow$$

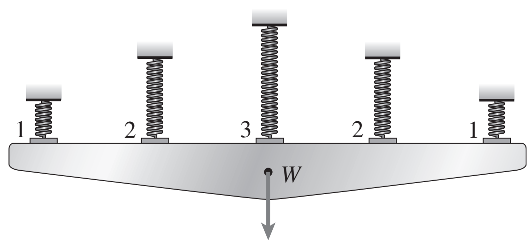
Note: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar AB supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k, respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar AB, which has weight W, causes the springs to elongate by an amount δ.

(a) Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
(b) Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W.
(c) Determine the forces F_1, F_2, and F_3 in the springs.
(d) Evaluate the strain energy U, the displacement δ, and the forces in the springs if $W = 600$ N and $k = 7.5$ N/mm.



Solution 2.7-8 **Rigid bar supported by springs**



$k_1 = 3k$

$k_2 = 1.5k$

$k_3 = k$

δ = downward displacement of rigid bar

For a spring: $U = \dfrac{k\delta^2}{2}$ Eq. (2-38b)

(a) Strain energy U of all springs

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2}$$

$$= 5k\delta^2 \longleftarrow$$

(b) Displacement δ

Work done by the weight W equals $\dfrac{W\delta}{2}$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \longleftarrow$$

(c) Forces in the springs

$$F_1 = 3k\delta = \frac{3W}{10} \quad F_2 = 1.5k\delta = \frac{3W}{20} \longleftarrow$$

$$F_3 = k\delta = \frac{W}{10} \longleftarrow$$

(d) Numerical values

$W = 600\,\text{N} \quad k = 7.5\,\text{N/mm} = 7500\,\text{N/mm}$

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4\ \text{N} \cdot \text{m} = 2.4\ \text{J} \longleftarrow$$

$$\delta = \frac{W}{10k} = 8.0\ \text{mm} \longleftarrow$$

$$F_1 = \frac{3W}{10} = 180\ \text{N} \longleftarrow$$

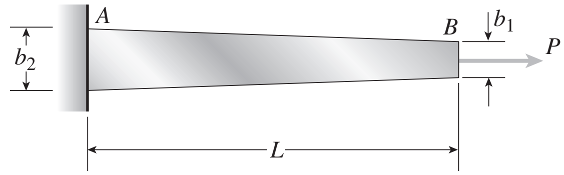
$$F_2 = \frac{3W}{20} = 90\ \text{N} \longleftarrow$$

$$F_3 = \frac{W}{10} = 60\ \text{N} \longleftarrow$$

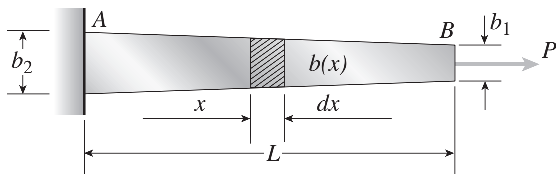
Note: $W = 2F_1 + 2F_2 + F_3 = 600\,\text{N}$ (Check)

Problem 2.7-9 A slightly tapered bar AB of rectangular cross section and length L is acted upon by a force P (see figure). The width of the bar varies uniformly from b_2 at end A to b_1 at end B. The thickness t is constant.

(a) Determine the strain energy U of the bar.
(b) Determine the elongation δ of the bar by equating the strain energy to the work done by the force P.



Solution 2.7-9 **Tapered bar of rectangular cross section**



$$b(x) = b_2 - \frac{(b_2 - b_1)x}{L}$$

$$A(x) = tb(x)$$

$$= t\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]$$

(a) Strain energy of the bar

$$U = \int \frac{[N(x)]^2 dx}{2EA(x)} \qquad \text{(Eq. 2-41)}$$

$$= \int_0^L \frac{P^2 dx}{2Etb(x)} = \frac{P^2}{2Et}\int_0^L \frac{dx}{b_2 - (b_2 - b_1)\frac{x}{L}} \qquad (1)$$

From Appendix C: $\displaystyle\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx)$

Apply this integration formula to Eq. (1):

$$U = \frac{P^2}{2Et}\left[\frac{1}{-(b_2 - b_1)(\frac{1}{L})}\ln\left[b_2 - \frac{(b_2 - b_1)x}{L}\right]\right]_0^L$$

$$= \frac{P^2}{2Et}\left[\frac{-L}{(b_2 - b_1)}\ln b_1 - \frac{-L}{(b_2 - b_1)}\ln b_2\right]$$

$$U = \frac{P^2 L}{2Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \longleftarrow$$

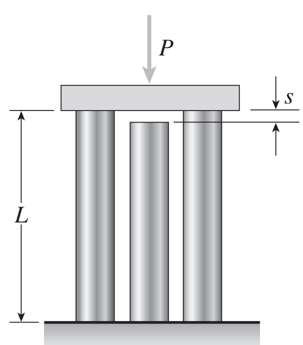
(b) Elongation of the bar (Eq. 2-42)

$$\delta = \frac{2U}{P} = \frac{PL}{Et(b_2 - b_1)}\ln\frac{b_2}{b_1} \quad \longleftarrow$$

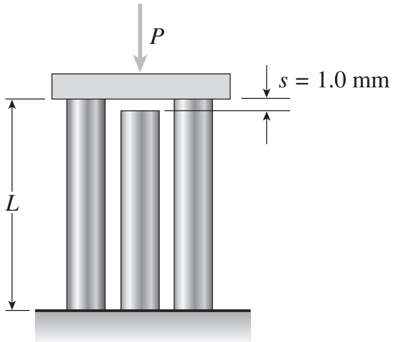
Note: This result agrees with the formula derived in Prob. 2.3-11.

Problem 2.7-10 A compressive load P is transmitted through a rigid plate to three magnesium-alloy bars that are identical except that initially the middle bar is slightly shorter than the other bars (see figure). The dimensions and properties of the assembly are as follows: length $L = 1.0$ m, cross-sectional area of each bar $A = 3000\ \text{mm}^2$, modulus of elasticity $E = 45$ GPa, and the gap $s = 1.0$ mm.

(a) Calculate the load P_1 required to close the gap.
(b) Calculate the downward displacement δ of the rigid plate when $P = 400$ kN.
(c) Calculate the total strain energy U of the three bars when $P = 400$ kN.
(d) Explain why the strain energy U is *not* equal to $P\delta/2$. (*Hint:* Draw a load-displacement diagram.)



Solution 2.7-10 Three bars in compression



$s = 1.0\,\text{mm}$

$L = 1.0\,\text{m}$

For each bar:

$A = 3000\,\text{mm}^2$

$E = 45\,\text{GPa}$

$$\frac{EA}{L} = 135 \times 10^6\ \text{N/m}$$

(a) LOAD P_1 REQUIRED TO CLOSE THE GAP

In general, $\delta = \dfrac{PL}{EA}$ and $P = \dfrac{EA\delta}{L}$

For two bars, we obtain:

$$P_1 = 2\left(\frac{EAs}{L}\right) = 2(135 \times 10^6\ \text{N/m})(1.0\ \text{mm})$$

$$P_1 = 270\ \text{kN} \longleftarrow$$

(b) DISPLACEMENT δ FOR $P = 400\,\text{kN}$

Since $P > P_1$, all three bars are compressed. The force P equals P_1 plus the additional force required to compress all three bars by the amount $\delta - s$.

$$P = P_1 + 3\left(\frac{EA}{L}\right)(\delta - s)$$

or $400\ \text{kN} = 270\ \text{kN} + 3(135 \times 10^6\ \text{N/m})(\delta - 0.001\,\text{m})$

Solving, we get $\delta = 1.321$ mm $\longleftarrow$

(c) STRAIN ENERGY U FOR $P = 400\,\text{kN}$

$$U = \sum \frac{EA\delta^2}{2L}$$

Outer bars: $\delta = 1.321\,\text{mm}$

Middle bar: $\delta = 1.321\,\text{mm} - s$

$= 0.321\,\text{mm}$

$$U = \frac{EA}{2L}[2(1.321\ \text{mm})^2 + (0.321\ \text{mm})^2]$$

$$= \frac{1}{2}(135 \times 10^6\ \text{N/m})(3.593\ \text{mm}^2)$$

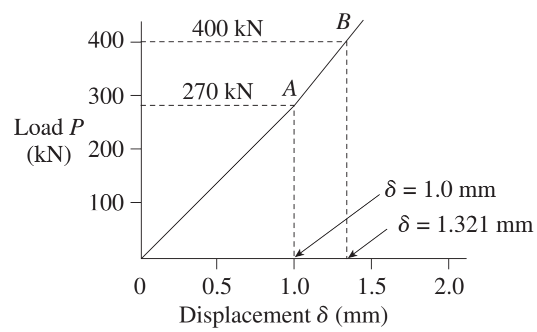
$$= 243\ \text{N} \cdot \text{m} = 243\ \text{J} \longleftarrow$$

(d) LOAD-DISPLACEMENT DIAGRAM

$U = 243\ \text{J} = 243\,\text{N} \cdot \text{m}$

$$\frac{P\delta}{2} = \frac{1}{2}(400\ \text{kN})(1.321\ \text{mm}) = 264\ \text{N} \cdot \text{m}$$

The strain energy U is *not* equal to $\dfrac{P\delta}{2}$ because the load-displacement relation is not linear.

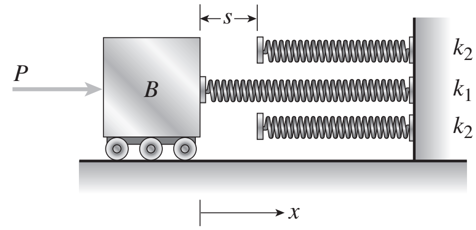


U = area under line OAB.

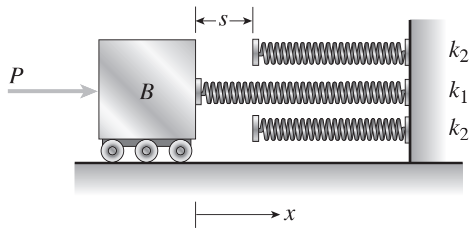
$\dfrac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U.

Problem 2.7-11 A block B is pushed against three springs by a force P (see figure). The middle spring has stiffness k_1 and the outer springs each have stiffness k_2. Initially, the springs are unstressed and the middle spring
is longer than the outer springs (the difference in length is denoted s).

(a) Draw a force-displacement diagram with the force P as ordinate and the displacement x of the block as abscissa.
(b) From the diagram, determine the strain energy U_1 of the springs when $x = 2s$.
(c) Explain why the strain energy U_1 is not equal to $P\delta/2$, where $\delta = 2s$.



Solution 2.7-11 Block pushed against three springs



Force P_0 required to close the gap:

$$P_0 = k_1 s \qquad (1)$$

FORCE-DISPLACEMENT RELATION BEFORE GAP IS CLOSED

$$P = k_1 x \qquad (0 \le x \le s)(0 \le P \le P_0) \qquad (2)$$

FORCE-DISPLACEMENT RELATION AFTER GAP IS CLOSED

All three springs are compressed. Total stiffness equals $k_1 + 2k_2$. Additional displacement equals $x - s$. Force P equals P_0 plus the force required to compress all three springs by the amount $x - s$.

$$P = P_0 + (k_1 + 2k_2)(x - s)$$

$$= k_1 s + (k_1 + 2k_2)x - k_1 s - 2k_2 s$$

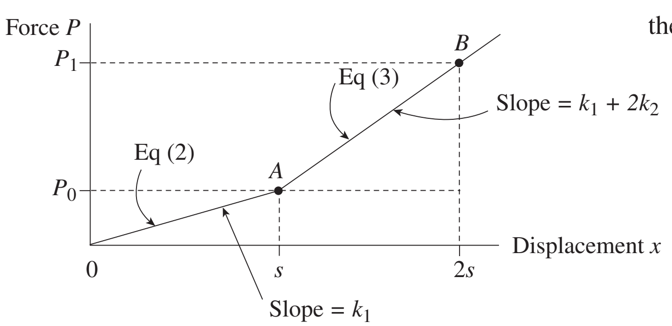
$$P = (k_1 + 2k_2)x - 2k_2 s \quad (x \ge s); (P \ge P_0) \qquad (3)$$

P_1 = force P when $x = 2s$

Substitute $x = 2s$ into Eq. (3):

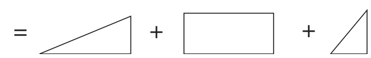
$$P_1 = 2(k_1 + k_2)s \qquad (4)$$

(a) FORCE-DISPLACEMENT DIAGRAM



(b) STRAIN ENERGY U_1 WHEN $x = 2s$

U_1 = Area below force-displacement curve



$$= \frac{1}{2}P_0 s + P_0 s + \frac{1}{2}(P_1 - P_0)s = P_0 s + \frac{1}{2}P_1 s$$

$$= k_1 s^2 + (k_1 + k_2)s^2$$

$$U_1 = (2k_1 + k_2)s^2 \quad \longleftarrow \qquad (5)$$

(c) STRAIN ENERGY U_1 IS NOT EQUAL TO $\dfrac{P\delta}{2}$

For $\delta = 2s$: $\dfrac{P\delta}{2} = \dfrac{1}{2}P_1(2s) = P_1 s = 2(k_1 + k_2)s^2$

(This quantity is greater than U_1.)

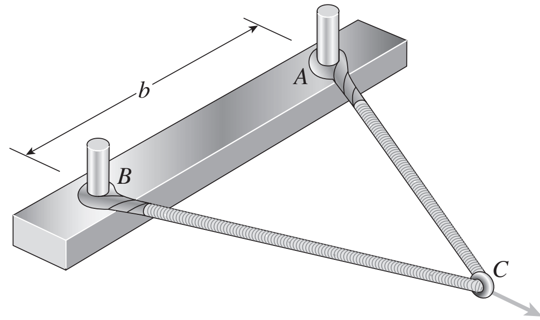
U_1 = area under line OAB.

$\dfrac{P\delta}{2}$ = area under a straight line from O to B, which is larger than U_1.

Thus, $\dfrac{P\delta}{2}$ is *not* equal to the strain energy because the force-displacement relation is not linear.

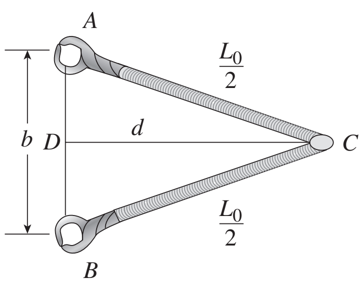
Problem 2.7-12 A bungee cord that behaves linearly elastically has an unstressed length $L_0 = 760$ mm and a stiffness $k = 140$ N/m. The cord is attached to two pegs, distance $b = 380$ mm apart, and pulled at its midpoint by a force $P = 80$ N (see figure).

(a) How much strain energy U is stored in the cord?
(b) What is the displacement δ_C of the point where the load is applied?
(c) Compare the strain energy U with the quantity $P\delta_C/2$.
(*Note:* The elongation of the cord is *not* small compared to its original length.)



Solution 2.7-12 **Bungee cord subjected to a load *P*.**

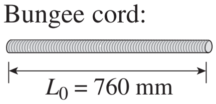
DIMENSIONS BEFORE THE LOAD P IS APPLIED



$L_0 = 760$ mm $\quad \frac{L_0}{2} = 380$ mm

$b = 380$ mm

Bungee cord:

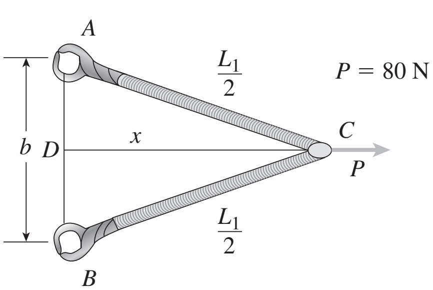


$k = 140$ N/m

From triangle ACD:

$$d = \frac{1}{2}\sqrt{L_0^2 - b^2} = 329.09 \text{ mm} \qquad (1)$$

DIMENSIONS AFTER THE LOAD P IS APPLIED



Let x = distance CD

Let L_1 = stretched length of bungee cord

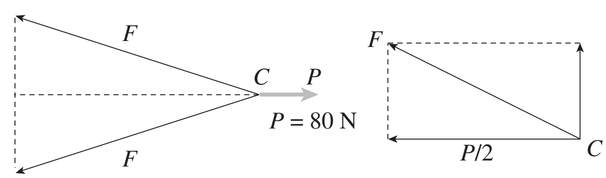
From triangle ACD:

$$\frac{L_1}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} \qquad (2)$$

$$L_1 = \sqrt{b^2 + 4x^2} \qquad (3)$$

EQUILIBRIUM AT POINT C

Let F = tensile force in bungee cord



$$\frac{F}{P/2} = \frac{L_1/2}{x} \qquad F = \left(\frac{P}{2}\right)\left(\frac{L_1}{2}\right)\left(\frac{1}{x}\right)$$

$$= \frac{P}{2}\sqrt{1 + \left(\frac{b}{2x}\right)^2} \qquad (4)$$

ELONGATION OF BUNGEE CORD

Let δ = elongation of the entire bungee cord

$$\delta = \frac{F}{k} = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (5)$$

Final length of bungee cord = original length + δ

$$L_1 = L_0 + \delta = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \qquad (6)$$

(Continued)

SOLUTION OF EQUATIONS

Combine Eqs. (6) and (3):

$$L_1 = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = \sqrt{b^2 + 4x^2}$$

$$\text{or} \quad L_1 = L_0 + \frac{P}{4kx}\sqrt{b^2 + 4x^2} = \sqrt{b^2 + 4x^2}$$

$$L_0 = \left(1 - \frac{P}{4kx}\right)\sqrt{b^2 + 4x^2} \qquad (7)$$

This equation can be solved for x.

SUBSTITUTE NUMERICAL VALUES INTO EQ. (7):

$$760 \text{ mm} = \left[1 - \frac{(80 \text{ N})(1000 \text{ mm/m})}{4(140 \text{ N/m})x}\right] \times \sqrt{(380 \text{ mm})^2 + 4x^2} \qquad (8)$$

$$760 = \left(1 - \frac{142.857}{x}\right)\sqrt{144{,}400 + 4x^2} \quad (9) \qquad (9)$$

Units: x is in millimeters

Solve for x (Use trial & error or a computer program):

$x = 497.88$ mm

(a) STRAIN ENERGY U OF THE BUNGEE CORD

$$U = \frac{k\delta^2}{2} \quad k = 140 \text{ N/m} \quad P = 80 \text{ N}$$

From Eq. (5):

$$\delta = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = 305.81 \text{ mm}$$

$$U = \frac{1}{2}(140 \text{ N/m})(305.81 \text{ mm})^2 = 6.55 \text{ N}\cdot\text{m}$$

$U = 6.55$ J ←

(b) DISPLACEMENT δ_C OF POINT C

$\delta_C = x - d = 497.88 \text{ mm} - 329.09 \text{ mm}$

$= 168.8$ mm ←

(c) COMPARISON OF STRAIN ENERGY U WITH THE QUANTITY $P\delta_C/2$

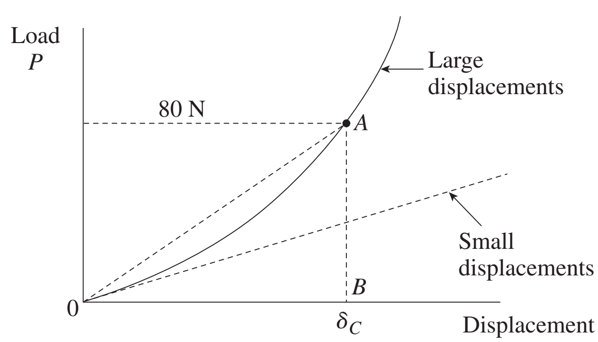
$U = 6.55$ J

$$\frac{P\delta_C}{2} = \frac{1}{2}(80 \text{ N})(168.8 \text{ mm}) = 6.75 \text{ J}$$

The two quantities are not the same. The work done by the load P is *not* equal to $P\delta_C/2$ because the load-displacement relation (see below) is non-linear when the displacements are large. (The *work* done by the load P *is* equal to the strain energy because the bungee cord behaves elastically and there are no energy losses.)

U = area OAB under the curve OA.

$\frac{P\delta_C}{2}$ = area of triangle OAB, which is greater than U.

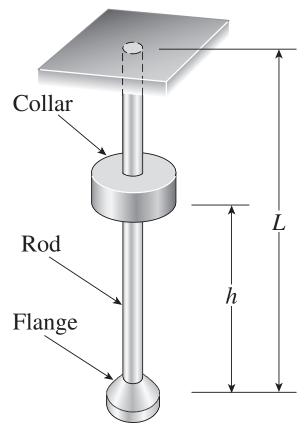


Impact Loading

The problems for Section 2.8 are to be solved on the basis of the assumptions and idealizations described in the text. In particular, assume that the material behaves linearly elastically and no energy is lost during the impact.

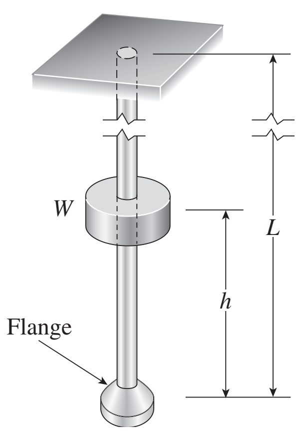
Problem 2.8-1 A sliding collar of weight $W = 150$ lb falls from a height $h = 2.0$ in. onto a flange at the bottom of a slender vertical rod (see figure). The rod has length $L = 4.0$ ft, cross-sectional area $A = 0.75$ in.^{2}, and modulus of elasticity $E = 30 \times 10^6$ psi.

Calculate the following quantities: (a) the maximum downward displacement of the flange, (b) the maximum tensile stress in the rod, and (c) the impact factor.



Probs. 2.8-1, 2.8-2, and 2.8-3

Solution 2.8-1 Collar falling onto a flange



$W = 150$ lb

$h = 2.0$ in. $\qquad L = 4.0$ ft $= 48$ in.

$E = 30 \times 10^6$ psi $\qquad A = 0.75$ in.^{2}

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.00032 \text{ in.}$$

Eq. of (2-53):

$$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 0.0361 \text{ in.} \longleftarrow$$

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 22{,}600 \text{ psi} \longleftarrow$$

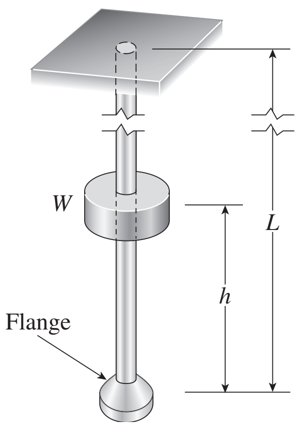
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0361 \text{ in.}}{0.00032 \text{ in.}}$$

$$= 113 \longleftarrow$$

Problem 2.8-2 Solve the preceding problem if the collar has mass $M = 80$ kg, the height $h = 0.5$ m, the length $L = 3.0$ m, the cross-sectional area $A = 350\ \text{mm}^2$, and the modulus of elasticity $E = 170$ GPa.

Solution 2.8-2 Collar falling onto a flange



$M = 80\,\text{kg}$

$W = Mg = (80\,\text{kg})(9.81\,\text{m/s}^2)$

$= 784.8\,\text{N}$

$h = 0.5\,\text{m}$ $\quad L = 3.0\,\text{m}$

$E = 170\,\text{GPa}$ $\quad A = 350\,\text{mm}^2$

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.03957\ \text{mm}$$

Eq. (2-53): $\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$

$= 6.33\ \text{mm}$ ←

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 359\ \text{MPa} \quad \leftarrow$$

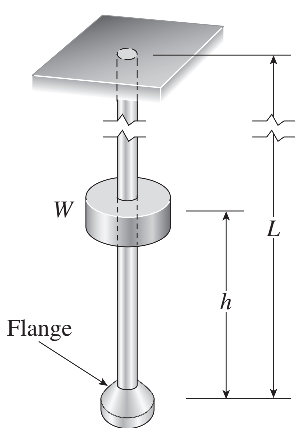
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{6.33\ \text{mm}}{0.03957\ \text{mm}}$$

$= 160$ ←

Problem 2.8-3 Solve Problem 2.8-1 if the collar has weight $W = 50$ lb, the height $h = 2.0$ in., the length $L = 3.0$ ft, the cross-sectional area $A = 0.25\ \text{in.}^2$, and the modulus of elasticity $E = 30{,}000$ ksi.

Solution 2.8-3 Collar falling onto a flange



$W = 50\,\text{lb}$ $\quad h = 2.0\,\text{in.}$

$L = 3.0\,\text{ft} = 36\,\text{in.}$

$E = 30{,}000\,\text{psi}$ $\quad A = 0.25\,\text{in.}^2$

(a) Downward displacement of flange

$$\delta_{st} = \frac{WL}{EA} = 0.00024\ \text{in.}$$

Eq. (2-53): $\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$

$= 0.0312$ in. ←

(b) Maximum tensile stress (Eq. 2-55)

$$\sigma_{max} = \frac{E\delta_{max}}{L} = 26{,}000\ \text{psi} \quad \leftarrow$$

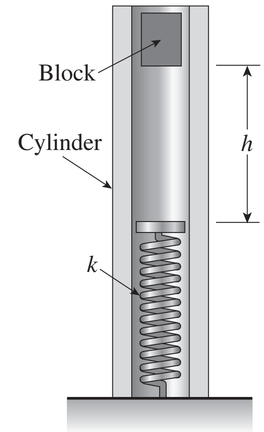
(c) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{0.0312\ \text{in.}}{0.00024\ \text{in.}}$$

$= 130$ ←

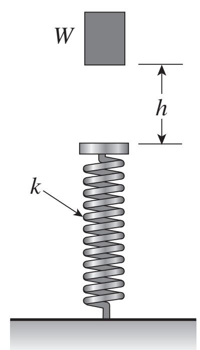
Problem 2.8-4 A block weighing $W = 5.0$ N drops inside a cylinder from a height $h = 200$ mm onto a spring having stiffness $k = 90$ N/m (see figure).

(a) Determine the maximum shortening of the spring due to the impact, and (b) determine the impact factor.



Prob. 2.8-4 and 2.8-5

Solution 2.8-4 **Block dropping onto a spring**



$W = 5.0\,\text{N} \qquad h = 200\,\text{mm} \qquad k = 90\,\text{N/m}$

(a) Maximum shortening of the spring

$$\delta_{st} = \frac{W}{k} = \frac{5.0\text{ N}}{90\text{ N/m}} = 55.56\text{ mm}$$

Eq. (2-53): $$\delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 215\text{ mm} \longleftarrow$$

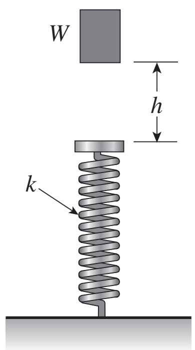
(b) Impact factor (Eq. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{215\text{ mm}}{55.56\text{ mm}}$$

$$= 3.9 \longleftarrow$$

Problem 2.8-5 Solve the preceding problem if the block weighs $W = 1.0$ lb, $h = 12$ in., and $k = 0.5$ lb/in.

Solution 2.8-5 Block dropping onto a spring



$W = 1.0\,\text{lb} \qquad h = 12\,\text{in.} \qquad k = 0.5\,\text{lb/in.}$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{1.0\text{ lb}}{0.5\text{ lb/in.}} = 2.0\text{ in.}$$

$$\text{Eq. (2-53):} \quad \delta_{max} = \delta_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right]$$

$$= 9.21\text{ in.} \longleftarrow$$

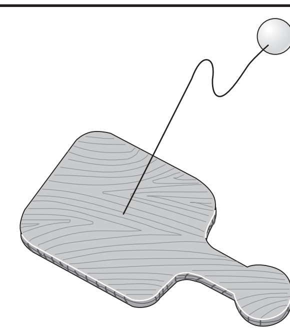
(b) IMPACT FACTOR (EQ. 2-61)

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} = \frac{9.21\text{ in.}}{2.0\text{ in.}}$$

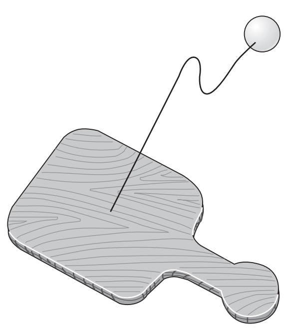
$$= 4.6 \longleftarrow$$

Problem 2.8-6 A small rubber ball (weight $W = 450$ mN) is attached by a rubber cord to a wood paddle (see figure). The natural length of the cord is $L_0 = 200$ mm, its cross-sectional area is $A = 1.6\text{ mm}^2$, and its modulus of elasticity is $E = 2.0$ MPa. After being struck by the paddle, the ball stretches the cord to a total length $L_1 = 900$ mm.

What was the velocity v of the ball when it left the paddle? (Assume linearly elastic behavior of the rubber cord, and disregard the potential energy due to any change in elevation of the ball.)



Solution 2.8-6 Rubber ball attached to a paddle



$g = 9.81\,\text{m/s}^2 \qquad E = 2.0\,\text{MPa}$

$A = 1.6\,\text{mm}^2 \qquad L_0 = 200\,\text{mm}$

$L_1 = 900\,\text{mm} \qquad W = 450\,\text{mN}$

WHEN THE BALL LEAVES THE PADDLE

$$KE = \frac{Wv^2}{2g}$$

WHEN THE RUBBER CORD IS FULLY STRETCHED:

$$U = \frac{EA\delta^2}{2L_0} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

$$v^2 = \frac{gEA}{WL_0}(L_1 - L_0)^2$$

$$v = (L_1 - L_0)\sqrt{\frac{gEA}{WL_0}} \longleftarrow$$

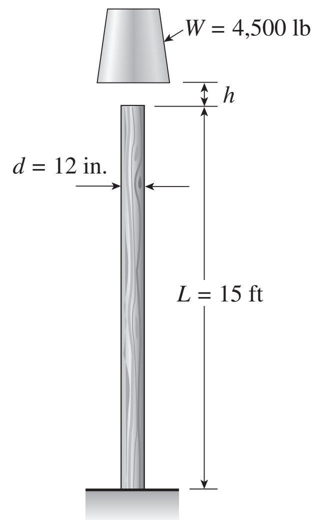
SUBSTITUTE NUMERICAL VALUES:

$$v = (700\text{ mm})\sqrt{\frac{(9.81\text{ m/s}^2)(2.0\text{ MPa})(1.6\text{ mm}^2)}{(450\text{ mN})(200\text{ mm})}}$$

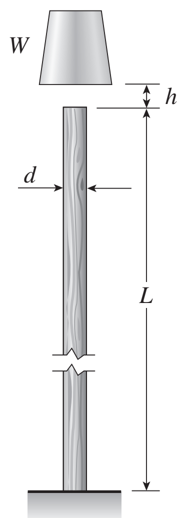
$$= 13.1\text{ m/s} \longleftarrow$$

Problem 2.8-7 A weight $W = 4500$ lb falls from a height h onto a vertical wood pole having length $L = 15$ ft, diameter $d = 12$ in., and modulus of elasticity $E = 1.6 \times 10^6$ psi (see figure).

If the allowable stress in the wood under an impact load is 2500 psi, what is the maximum permissible height h?



Solution 2.8-7 Weight falling on a wood pole



$W = 4500\,\text{lb} \qquad d = 12\,\text{in.}$

$L = 15\,\text{ft} = 180\,\text{in.}$

$$A = \frac{\pi d^2}{4} = 113.10\ \text{in.}^2$$

$E = 1.6 \times 10^6\,\text{psi}$

$\sigma_{allow} = 2500\,\text{psi}\ (= \sigma_{max})$

Find h_{max}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{4500\ \text{lb}}{113.10\ \text{in.}^2} = 39.79\ \text{psi}$$

MAXIMUM HEIGHT h_{max}

Eq. (2–59): $\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for h:

$$h = h_{max} = \frac{L\sigma_{max}}{2E}\left(\frac{\sigma_{max}}{\sigma_{st}} - 2\right) \longleftarrow$$

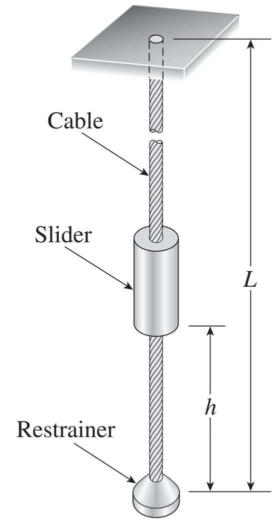
SUBSTITUTE NUMERICAL VALUES:

$$h_{max} = \frac{(180\ \text{in.})(2500\ \text{psi})}{2(1.6 \times 10^6\ \text{psi})}\left(\frac{2500\ \text{psi}}{39.79\ \text{psi}} - 2\right)$$

$$= 8.55\ \text{in.} \longleftarrow$$

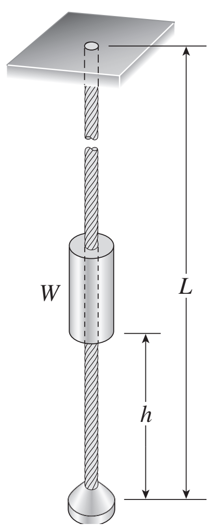
Problem 2.8-8 A cable with a restrainer at the bottom hangs vertically from its upper end (see figure). The cable has an effective cross-sectional area $A = 40\text{ mm}^2$ and an effective modulus of elasticity $E = 130$ GPa. A slider of mass $M = 35$ kg drops from a height $h = 1.0$ m onto the restrainer.

If the allowable stress in the cable under an impact load is 500 MPa, what is the minimum permissible length L of the cable?



Probs. 2.8-8 and 2.8-9

Solution 2.8-8 **Slider on a cable**



$W = Mg = (35\text{ kg})(9.81\text{ m/s}^2) = 343.4\text{ N}$

$A = 40\text{ mm}^2 \qquad E = 130\text{ GPa}$

$h = 1.0\text{ m} \qquad \sigma_{allow} = \sigma_{max} = 500\text{ MPa}$

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{343.4\text{ N}}{40\text{ mm}^2} = 8.585\text{ MPa}$$

MINIMUM LENGTH L_{min}

Eq. (2–59): $$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

$$L = L_{min} = \frac{2Eh\sigma_{st}}{\sigma_{max}(\sigma_{max} - 2\sigma_{st})} \quad \longleftarrow$$

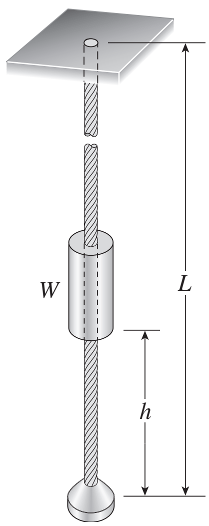
SUBSTITUTE NUMERICAL VALUES:

$$L_{min} = \frac{2(130\text{ GPa})(1.0\text{ m})(8.585\text{ MPa})}{(500\text{ MPa})[500\text{ MPa} - 2(8.585\text{ MPa})]}$$

$$= 9.25\text{ mm} \quad \longleftarrow$$

Problem 2.8-9 Solve the preceding problem if the slider has weight $W = 100$ lb, $h = 45$ in., $A = 0.080\text{ in.}^2$, $E = 21 \times 10^6$ psi, and the allowable stress is 70 ksi.

Solution 2.8-9 **Slider on a cable**



$W = 100\,\text{lb}$

$A = 0.080\text{ in.}^2 \quad E = 21 \times 10^6\text{ psi}$

$h = 45\text{ in} \quad \sigma_{allow} = \sigma_{max} = 70\,\text{ksi}$

Find minimum length L_{min}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{100\text{ lb}}{0.080\text{ in.}^2} = 1250\text{ psi}$$

MINIMUM LENGTH L_{min}

Eq. (2-59): $\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}\right]$

or

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}}\right)^{1/2}$$

Square both sides and solve for L:

$$L = L_{min} = \frac{2Eh\sigma_{st}}{\sigma_{max}(\sigma_{max} - 2\sigma_{st})} \longleftarrow$$

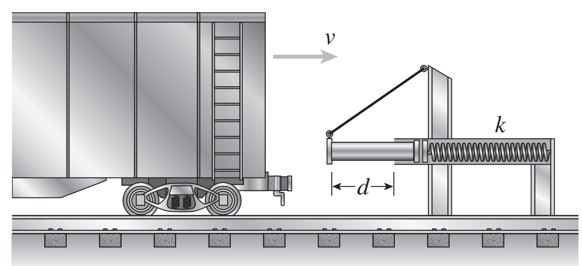
SUBSTITUTE NUMERICAL VALUES:

$$L_{min} = \frac{2(21 \times 10^6\text{ psi})(45\text{ in.})(1250\text{ psi})}{(70{,}000\text{ psi})[70{,}000\text{ psi} - 2(1250\text{ psi})]}$$

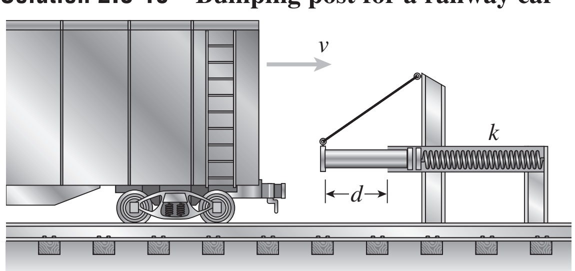
$$= 500\text{ in.} \longleftarrow$$

Problem 2.8-10 A bumping post at the end of a track in a railway yard has a spring constant $k = 8.0$ MN/m (see figure). The maximum possible displacement d of the end of the striking plate is 450 mm.

What is the maximum velocity v_{max} that a railway car of weight $W = 545$ kN can have without damaging the bumping post when it strikes it?



Solution 2.8-10 **Bumping post for a railway car**



$k = 8.0\,\text{MN/m} \qquad W = 545\,\text{kN}$

d = maximum displacement of spring

$d = \delta_{max} = 450\,\text{mm}$

Find v_{max}

KINETIC ENERGY BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS COMPRESSED TO THE MAXIMUM ALLOWABLE AMOUNT

$$U = \frac{k\delta_{max}^2}{2} = \frac{kd^2}{2}$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{kd^2}{2} \qquad v^2 = \frac{kd^2}{W/g}$$

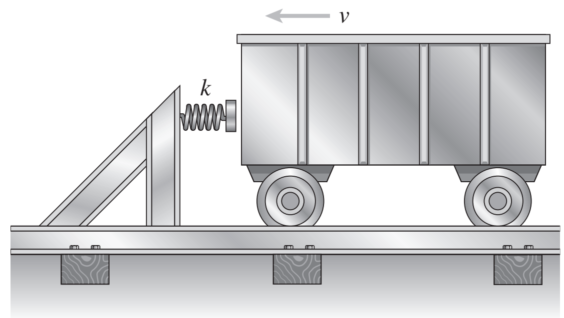
$$v = v_{max} = d\sqrt{\frac{k}{W/g}} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

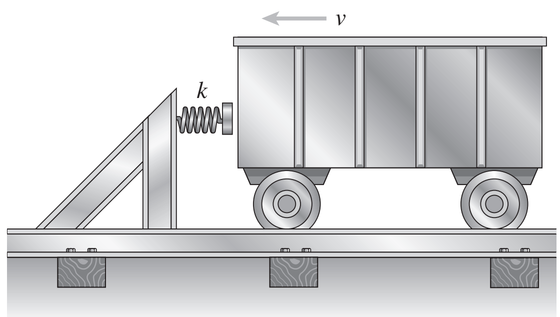
$$v_{max} = (450\text{ mm})\sqrt{\frac{8.0\text{ MN/m}}{(545\text{ kN})/(9.81\text{ m/s}^2)}}$$

$$= 5400\text{ mm/s} = 5.4\text{ m/s} \longleftarrow$$

Problem 2.8-11 A bumper for a mine car is constructed with a spring of stiffness $k = 1120$ lb/in. (see figure). If a car weighing 3450 lb is traveling at velocity $v = 7$ mph when it strikes the spring, what is the maximum shortening of the spring?



Solution 2.8-11 **Bumper for a mine car**



$k = 1120$ lb/in. $\quad W = 3450$ lb

$v = 7$ mph $= 123.2$ in./sec

$g = 32.2$ ft/sec$^2 = 386.4$ in./sec^2

Find the shortening δ_{max} of the spring.

KINETIC ENERGY JUST BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS FULLY COMPRESSED

$$U = \frac{k\delta_{max}^2}{2}$$

Conservation of energy

$$KE = U \qquad \frac{Wv^2}{2g} = \frac{k\delta_{max}^2}{2}$$

Solve for δ_{max}: $\quad \delta_{max} = \sqrt{\frac{Wv^2}{gk}}$ ←

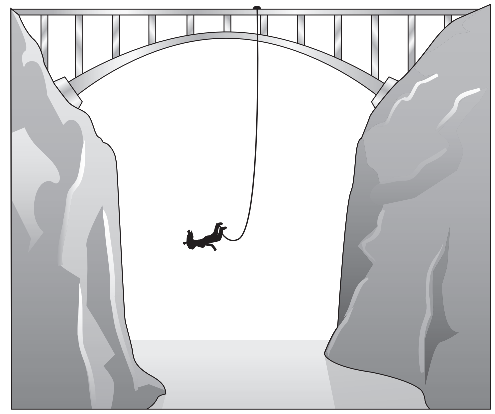
SUBSTITUTE NUMERICAL VALUES:

$$\delta_{max} = \sqrt{\frac{(3450 \text{ lb})(123.2 \text{ in./sec})^2}{(386.4 \text{ in./sec}^2)(1120 \text{ lb/in.})}}$$

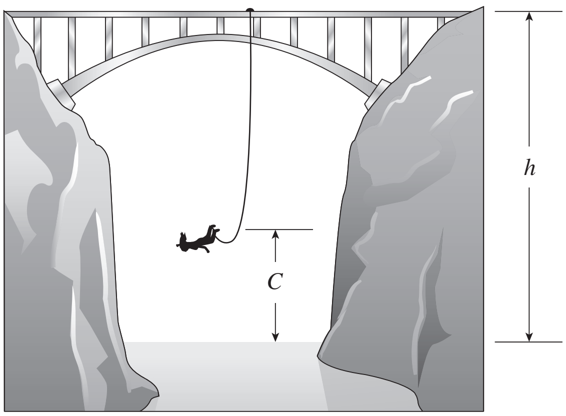
$$= 11.0 \text{ in.}$$ ←

Problem 2.8-12 A bungee jumper having a mass of 55 kg leaps from a bridge, braking her fall with a long elastic shock cord having axial rigidity $EA = 2.3$ kN (see figure).

If the jumpoff point is 60 m above the water, and if it is desired to maintain a clearance of 10 m between the jumper and the water, what length L of cord should be used?



Solution 2.8-12 Bungee jumper



$W = Mg = (55\ \text{kg})(9.81\ \text{m/s}^2)$

$= 539.55\ \text{N}$

$EA = 2.3\ \text{kN}$

Height: $h = 60\ \text{m}$

Clearance: $C = 10\ \text{m}$

Find length L of the bungee cord.

$P.E.$ = Potential energy of the jumper at the top of bridge (with respect to lowest position)

$= W(L + \delta_{max})$

U = strain energy of cord at lowest position

$$= \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2L}$$

$$\text{or} \quad \delta_{max}^2 - \frac{2WL}{EA}\delta_{max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$

$$= \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

VERTICAL HEIGHT

$$h = C + L + \delta_{max}$$

$$h - C = L + \frac{WL}{EA}\left[1 + \left(1 + \frac{2EA}{W}\right)^{1/2}\right]$$

SOLVE FOR L:

$$L = \frac{h - C}{1 + \dfrac{W}{EA}\left[1 + \left(1 + \dfrac{2EA}{W}\right)^{1/2}\right]} \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{W}{EA} = \frac{539.55\ \text{N}}{2.3\ \text{kN}} = 0.234587$$

Numerator $= h - C = 60\ \text{m} - 10\ \text{m} = 50\ \text{m}$

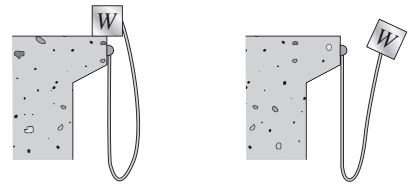
$$\text{Denominator} = 1 + (0.234587) \times \left[1 + \left(1 + \frac{2}{0.234587}\right)^{1/2}\right]$$

$$= 1.9586$$

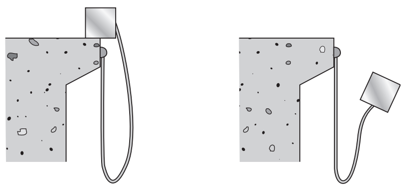
$$L = \frac{50\ \text{m}}{1.9586} = 25.5\ \text{m} \longleftarrow$$

Problem 2.8-13 A weight W rests on top of a wall and is attached to one end of a very flexible cord having cross-sectional area A and modulus of elasticity E (see figure). The other end of the cord is attached securely to the wall. The weight is then pushed off the wall and falls freely the full length of the cord.

(a) Derive a formula for the impact factor.
(b) Evaluate the impact factor if the weight, when hanging statically, elongates the band by 2.5% of its original length.



Solution 2.8-13 Weight falling off a wall



W = Weight

Properties of elastic cord:

E = modulus of elasticity

A = cross-sectional area

L = original length

δ_{max} = elongation of elastic cord

$P.E.$ = potential energy of weight before fall (with respect to lowest position)

$P.E. = W(L + \delta_{max})$

Let U = strain energy of cord at lowest position

$$U = \frac{EA\delta_{max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \qquad W(L + \delta_{max}) = \frac{EA\delta_{max}^2}{2\,L}$$

or $$\delta_{max}^2 - \frac{2WL}{EA}\delta_{max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR δ_{max}:

$$\delta_{max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA}\right)^2 + 2L\left(\frac{WL}{EA}\right)\right]^{1/2}$$

STATIC ELONGATION

$$\delta_{st} = \frac{WL}{EA}$$

IMPACT FACTOR

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left[1 + \frac{2EA}{W}\right]^{1/2} \quad \longleftarrow$$

NUMERICAL VALUES

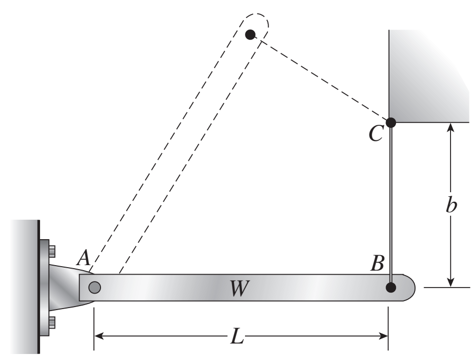
$\delta_{st} = (2.5\%)(L) = 0.025L$

$$\delta_{st} = \frac{WL}{EA} \qquad \frac{W}{EA} = 0.025 \qquad \frac{EA}{W} = 40$$

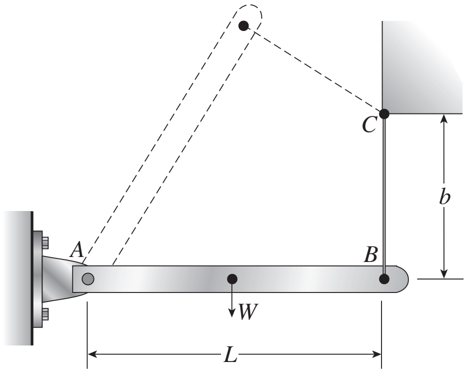
Impact factor $= 1 + [1 + 2(40)]^{1/2} = 10$ $\longleftarrow$

Problem 2.8-14 A rigid bar AB having mass M = 1.0 kg and length L = 0.5 m is hinged at end A and supported at end B by a nylon cord BC (see figure). The cord has cross-sectional area A = 30 mm^2, length b = 0.25 m, and modulus of elasticity E = 2.1 GPa.

If the bar is raised to its maximum height and then released, what is the maximum stress in the cord?



Solution 2.8-14 Falling bar *AB*



RIGID BAR:

$W = Mg = (1.0\,\text{kg})(9.81\,\text{m/s}^2)$

$= 9.81\text{ N}$

$L = 0.5\,\text{m}$

NYLON CORD:

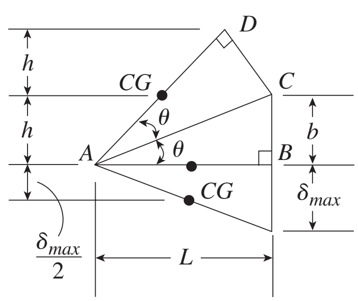
$A = 30\,\text{mm}^2$

$b = 0.25\,\text{m}$

$E = 2.1\,\text{GPa}$

Find maximum stress $\sigma_{\max}$ in cord BC.

GEOMETRY OF BAR AB AND CORD BC



$\overline{CD} = \overline{CB} = b$

$\overline{AD} = \overline{AB} = L$

h = height of center of gravity of raised bar AD

$\delta_{\max}$ = elongation of cord

From triangle ABC: $\sin\theta = \dfrac{b}{\sqrt{b^2 + L^2}}$

$$\cos\theta = \frac{L}{\sqrt{b^2 + L^2}}$$

From line AD : $\sin 2\theta = \dfrac{2h}{AD} = \dfrac{2h}{L}$

From Appendix C: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore\ \frac{2h}{L} = 2\left(\frac{b}{\sqrt{b^2 + L^2}}\right)\left(\frac{L}{\sqrt{b^2 + L^2}}\right) = \frac{2bL}{b^2 + L^2}$$

and $\quad h = \dfrac{bL^2}{b^2 + L^2} \qquad$ (Eq. 1)

CONSERVATION OF ENERGY

$P.E.$ = potential energy of raised bar AD

$$= W\left(h + \frac{\delta_{\max}}{2}\right)$$

U = strain energy of stretched cord $= \dfrac{EA\delta_{\max}^2}{2b}$

$$P.E. = U \quad W\left(h + \frac{\delta_{\max}}{2}\right) = \frac{EA\delta_{\max}^2}{2b} \qquad \text{(Eq. 2)}$$

For the cord: $\delta_{\max} = \dfrac{\sigma_{\max} b}{E}$

Substitute into Eq. (2) and rearrange:

$$\sigma_{\max}^2 - \frac{W}{A}\sigma_{\max} - \frac{2WhE}{bA} = 0 \qquad \text{(Eq. 3)}$$

Substitute from Eq. (1) into Eq. (3):

$$\sigma_{\max}^2 - \frac{W}{A}\sigma_{\max} - \frac{2WL^2E}{A(b^2 + L^2)} = 0 \qquad \text{(Eq. 4)}$$

SOLVE FOR $\sigma_{\max}$:

$$\sigma_{\max} = \frac{W}{2A}\left[1 + \sqrt{1 + \frac{8L^2EA}{W(b^2 + L^2)}}\right] \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

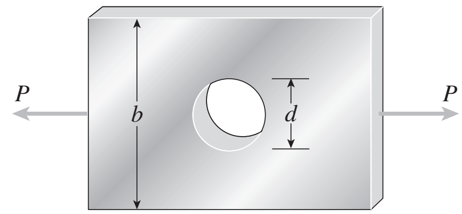
$\sigma_{\max} = 33.3\text{ MPa} \quad \leftarrow$

Stress Concentrations

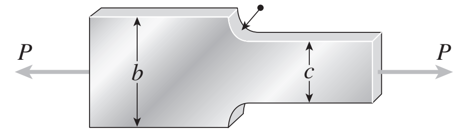
The problems for Section 2.10 are to be solved by considering the stress-concentration factors and assuming linearly elastic behavior.

Problem 2.10-1 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 3.0$ k. Each bar has thickness $t = 0.25$ in.

(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 1$ in. and $d = 2$ in. if the width $b = 6.0$ in.
(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 0.25$ in. and $R = 0.5$ in. if the bar widths are $b = 4.0$ in. and $c = 2.5$ in.

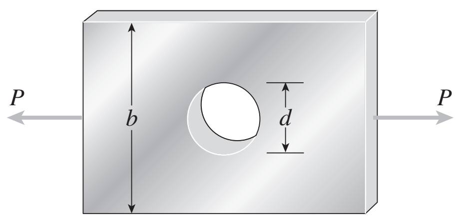


(a)

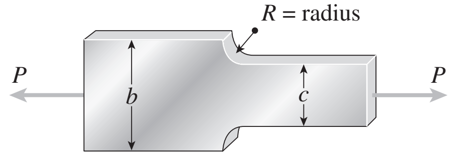


Probs. 2.10-1 and 2.10-2 (b)

Solution 2.10-1 Flat bars in tension



(a)



(b)

$P = 3.0\,\text{k} \quad t = 0.25\,\text{in.}$

(a) Bar with circular hole ($b = 6$ in.)

Obtain K from Fig. 2-63

For $d = 1$ in.: $c = b - d = 5$ in.

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(5\text{ in.})(0.25\text{ in.})} = 2.40\text{ ksi}$$

$$d/b = \frac{1}{6} \quad K \approx 2.60$$

$$\sigma_{max} = k\sigma_{nom} \approx 6.2\text{ ksi} \quad \longleftarrow$$

For $d = 2$ in.: $c = b - d = 4$ in.

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(4\text{ in.})(0.25\text{ in.})} = 3.00\text{ ksi}$$

$$d/b = \frac{1}{3} \quad K \approx 2.31$$

$$\sigma_{max} = K\sigma_{nom} \approx 6.9\text{ ksi} \quad \longleftarrow$$

(b) Stepped bar with shoulder fillets

$b = 4.0$ in. $\quad c = 2.5$ in.; Obtain k from Fig. 2-64

$$\sigma_{nom} = \frac{P}{ct} = \frac{3.0\text{ k}}{(2.5\text{ in.})(0.25\text{ in.})} = 4.80\text{ ksi}$$

For $R = 0.25$ in.: $R/c = 0.1 \qquad b/c = 1.60$

$$k \approx 2.30 \quad \sigma_{max} = K\sigma_{nom} \approx 11.0\text{ ksi} \quad \longleftarrow$$

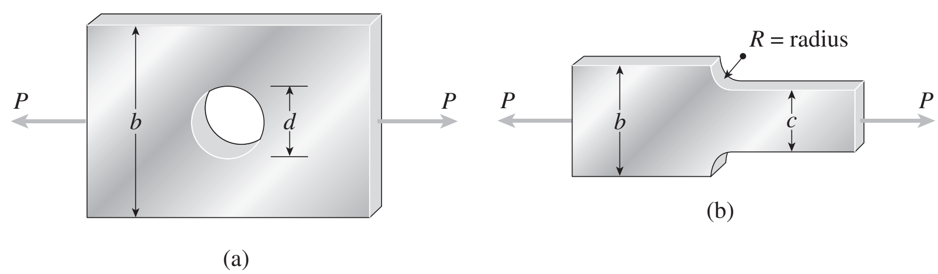
For $R = 0.5$ in.: $R/c = 0.2 \qquad b/c = 1.60$

$$K \approx 1.87 \quad \sigma_{max} = K\sigma_{nom} \approx 9.0\text{ ksi} \quad \longleftarrow$$

Problem 2.10-2 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 2.5$ kN. Each bar has thickness $t = 5.0$ mm.

(a) For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 12$ mm and $d = 20$ mm if the width $b = 60$ mm.
(b) For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 6$ mm and $R = 10$ mm if the bar widths are $b = 60$ mm and $c = 40$ mm.

Solution 2.10-2 **Flat bars in tension**



$P = 2.5\,\text{kN} \quad t = 5.0\,\text{mm}$

(a) Bar with circular hole ($b = 60$ mm)

Obtain K from Fig. 2-63

For $d = 12$ mm: $c = b - d = 48$ mm

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(48\text{ mm})(5\text{ mm})} = 10.42\text{ MPa}$$

$$d/b = \frac{1}{5} \qquad K \approx 2.51$$

$$\sigma_{max} = K\sigma_{nom} \approx 26\text{ MPa} \longleftarrow$$

For $d = 20$ mm: $c = b - d = 40$ mm

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(40\text{ mm})(5\text{ mm})} = 12.50\text{ MPa}$$

$$d/b = \frac{1}{3} \qquad K \approx 2.31$$

$$\sigma_{max} = K\sigma_{nom} \approx 29\text{ MPa} \longleftarrow$$

(b) Stepped bar with shoulder fillets

$b = 60$ mm $\qquad c = 40$ mm;

Obtain K from Fig. 2-64

$$\sigma_{nom} = \frac{P}{ct} = \frac{2.5\text{ kN}}{(40\text{ mm})(5\text{ mm})} = 12.50\text{ MPa}$$

For $R = 6$ mm: $R/c = 0.15 \qquad b/c = 1.5$

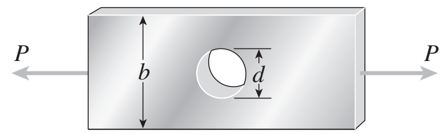
$$K \approx 2.00 \qquad \sigma_{max} = K\sigma_{nom} \approx 25\text{ MPa} \longleftarrow$$

For $R = 10$ mm: $R/c = 0.25 \qquad b/c = 1.5$

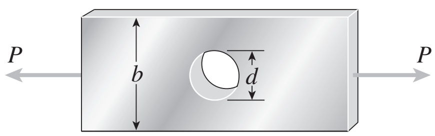
$$K \approx 1.75 \qquad \sigma_{max} = K\sigma_{nom} \approx 22\text{ MPa} \longleftarrow$$

Problem 2.10-3 A flat bar of width b and thickness t has a hole of diameter d drilled through it (see figure). The hole may have any diameter that will fit within the bar.

What is the maximum permissible tensile load P_{max} if the allowable tensile stress in the material is σ_t?



Solution 2.10-3 **Flat bar in tension**



t = thickness

σ_t = allowable tensile stress

Find P_{max}

Find K from Fig. 2-64

$$P_{max} = \sigma_{nom} ct = \frac{\sigma_{max}}{K} ct = \frac{\sigma_t}{K}(b - d)t$$

$$= \frac{\sigma_t}{K} bt\left(1 - \frac{d}{b}\right)$$

Because σ_t, b, and t are constants, we write:

$$P^* = \frac{P_{max}}{\sigma_t bt} = \frac{1}{K}\left(1 - \frac{d}{b}\right)$$

$\frac{d}{b}$	K	P^*
0	3.00	0.333
0.1	2.73	0.330
0.2	2.50	0.320
0.3	2.35	0.298
0.4	2.24	0.268

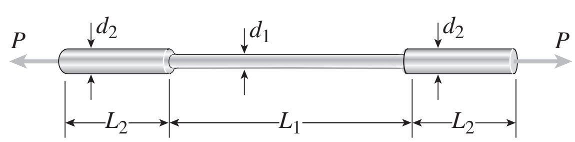
We observe that P_{max} decreases as d/b increases. Therefore, the maximum load occurs when the hole becomes very small.

$$\left(\frac{d}{b} \rightarrow 0 \quad \text{and} \quad K \rightarrow 3\right)$$

$$P_{max} = \frac{\sigma_t bt}{3} \quad \longleftarrow$$

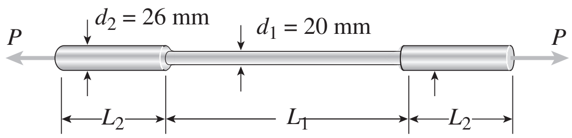
Problem 2.10-4 A round brass bar of diameter $d_1 = 20$ mm has upset ends of diameter $d_2 = 26$ mm (see figure). The lengths of the segments of the bar are $L_1 = 0.3$ m and $L_2 = 0.1$ m. Quarter-circular fillets are used at the shoulders of the bar, and the modulus of elasticity of the brass is $E = 100$ GPa.

If the bar lengthens by 0.12 mm under a tensile load P, what is the maximum stress σ_{max} in the bar?



Probs. 2.10-4 and 2.10-5

Solution 2.10-4 Round brass bar with upset ends



$E = 100\,\text{GPa}$

$\delta = 0.12\,\text{mm}$

$L_2 = 0.1\,\text{m}$

$L_1 = 0.3\,\text{m}$

$$R = \text{radius of fillets} = \frac{26\text{ mm} - 20\text{ mm}}{2} = 3\text{ mm}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $\quad P = \dfrac{\delta EA_1A_2}{2L_2A_1 + L_1A_2}$

Use Fig. 2-65 for the stress-concentration factor:

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta EA_2}{2L_2A_1 + L_1A_2} = \frac{\delta E}{2L_2\left(\frac{A_1}{A_2}\right) + L_1}$$

$$= \frac{\delta E}{2L_2\left(\frac{d_1}{d_2}\right)^2 + L_1}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.12\text{ mm})(100\text{ GPa})}{2(0.1\text{ m})\left(\frac{20}{26}\right)^2 + 0.3\text{ m}} = 28.68\text{ MPa}$$

$$\frac{R}{D_1} = \frac{3\text{ mm}}{20\text{ mm}} = 0.15$$

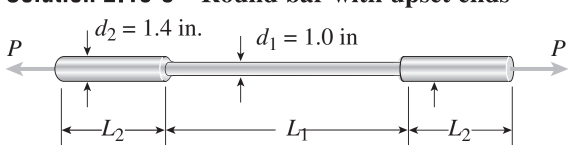
Use the dashed curve in Fig. 2-65. $K \approx 1.6$

$$\sigma_{\max} = K\sigma_{\text{nom}} \approx (1.6)(28.68\text{ MPa})$$

$$\approx 46\text{ MPa} \longleftarrow$$

Problem 2.10-5 Solve the preceding problem for a bar of monel metal having the following properties: $d_1 = 1.0$ in., $d_2 = 1.4$ in., $L_1 = 20.0$ in., $L_2 = 5.0$ in., and $E = 25 \times 10^6$ psi. Also, the bar lengthens by 0.0040 in. when the tensile load is applied.

Solution 2.10-5 Round bar with upset ends



$E = 25 \times 10^6$ psi

$\delta = 0.0040\,\text{in.}$

$L_1 = 20\,\text{in.}$

$L_2 = 5\,\text{in.}$

R = radius of fillets $\quad R = \dfrac{1.4\text{ in.} - 1.0\text{ in.}}{2}$

$= 0.2$ in.

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

Solve for P: $P = \dfrac{\delta EA_1A_2}{2L_2A_1 + L_1A_2}$

Use Fig. 2-65 for the stress-concentration factor.

$$\sigma_{\text{nom}} = \frac{P}{A_1} = \frac{\delta EA_2}{2L_2A_1 + L_1A_2} = \frac{\delta E}{2L_2\left(\frac{A_1}{A_2}\right) + L_1}$$

$$= \frac{\delta E}{2L_2\left(\frac{d_1}{d_2}\right)^2 + L_1}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.0040\text{ in.})(25 \times 10^6\text{ psi})}{2(5\text{ in.})\left(\frac{1.0}{1.4}\right)^2 + 20\text{ in.}} = 3{,}984\text{ psi}$$

$$\frac{R}{D_1} = \frac{0.2\text{ in.}}{1.0\text{ in.}} = 0.2$$

Use the dashed curve in Fig. 2-65. $K \approx 1.53$

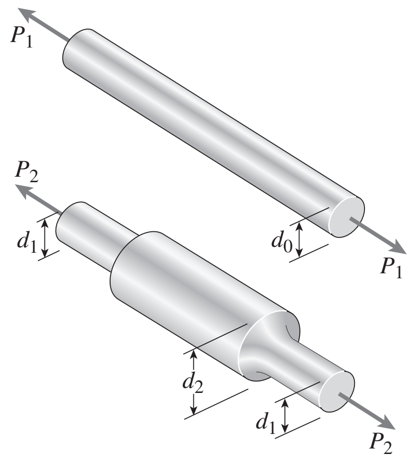
$$\sigma_{\max} = K\sigma_{\text{nom}} \approx (1.53)(3984\text{ psi})$$

$$\approx 6100\text{ psi} \longleftarrow$$

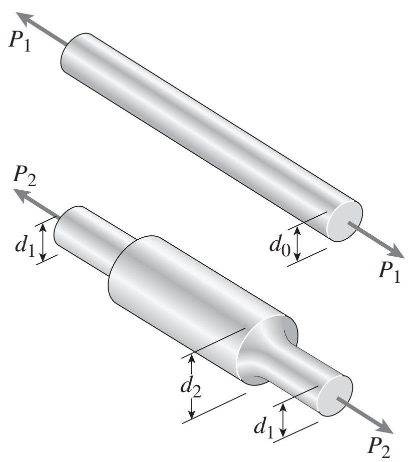
Problem 2.10-6 A prismatic bar of diameter $d_0 = 20$ mm is being compared with a stepped bar of the same diameter ($d_1 = 20$ mm) that is enlarged in the middle region to a diameter $d_2 = 25$ mm (see figure). The radius of the fillets in the stepped bar is 2.0 mm.

(a) Does enlarging the bar in the middle region make it stronger than the prismatic bar? Demonstrate your answer by determining the maximum permissible load P_1 for the prismatic bar and the maximum permissible load P_2 for the enlarged bar, assuming that the allowable stress for the material is 80 MPa.

(b) What should be the diameter d_0 of the prismatic bar if it is to have the same maximum permissible load as does the stepped bar?



Soluton 2.10-6 **Prismatic bar and stepped bar**



$d_0 = 20\,\text{mm}$

$d_1 = 20\,\text{mm}$

$d_2 = 25\,\text{mm}$

Fillet radius: $R = 2\,\text{mm}$

Allowable stress: $\sigma_t = 80\,\text{MPa}$

(a) Comparison of bars

Prismatic bar: $P_1 = \sigma_t A_0 = \sigma_t\left(\frac{\pi d_0^2}{4}\right)$

$$= (80\text{ MPa})\left(\frac{\pi}{4}\right)(20\text{ mm})^2 = 25.1\text{ kN} \longleftarrow$$

Stepped bar: See Fig. 2-65 for the stress-concentration factor.

$R = 2.0\,\text{mm} \quad D_1 = 20\,\text{mm} \quad D_2 = 25\,\text{mm}$

$R/D_1 = 0.10 \quad D_2/D_1 = 1.25 \quad K \approx 1.75$

$$\sigma_{\text{nom}} = \frac{P_2}{\frac{\pi}{4}d_1^2} = \frac{P_2}{A_1} \qquad \sigma_{\text{nom}} = \frac{\sigma_{\max}}{K}$$

$$P_2 = \sigma_{\text{nom}} A_1 = \frac{\sigma_{\max}}{K} A_1 = \frac{\sigma_t}{K} A_1$$

$$= \left(\frac{80\text{ MPa}}{1.75}\right)\left(\frac{\pi}{4}\right)(20\text{ mm})^2$$

$$\approx 14.4\text{ kN} \longleftarrow$$

Enlarging the bar makes it *weaker*, not stronger. The ratio of loads is $P_1/P_2 = K = 1.75$

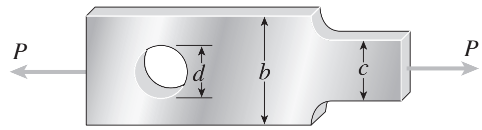
(b) Diameter of prismatic bar for the same allowable load

$$P_1 = P_2 \quad \sigma_t\left(\frac{\pi d_0^2}{4}\right) = \frac{\sigma_t}{K}\left(\frac{\pi d_1^2}{4}\right) \quad d_0^2 = \frac{d_1^2}{K}$$

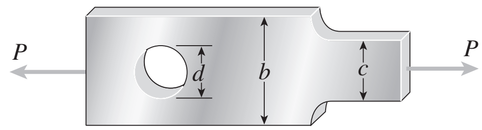
$$d_0 = \frac{d_1}{\sqrt{K}} \approx \frac{20\text{ mm}}{\sqrt{1.75}} \approx 15.1\text{ mm} \longleftarrow$$

Problem 2.10-7 A stepped bar with a hole (see figure) has widths $b = 2.4$ in. and $c = 1.6$ in. The fillets have radii equal to 0.2 in.

What is the diameter d_{max} of the largest hole that can be drilled through the bar without reducing the load-carrying capacity?



Solution 10-7 Stepped bar with a hole



$b = 2.4\,\text{in.}$

$c = 1.6\,\text{in.}$

Fillet radius: $R = 0.2\,\text{in.}$

Find d_{max}

BASED UPON FILLETS (Use Fig. 2-64)

$b = 2.4\,\text{in.} \quad c = 1.6\,\text{in.} \quad R = 0.2\text{ in.} \quad R/c = 0.125$

$b/c = 1.5 \quad K \approx 2.10$

$$P_{max} = \sigma_{nom} ct = \frac{\sigma_{max}}{K} ct = \frac{\sigma_{max}}{K}\left(\frac{c}{b}\right)(bt)$$

$$\approx 0.317\; bt\, \sigma_{max}$$

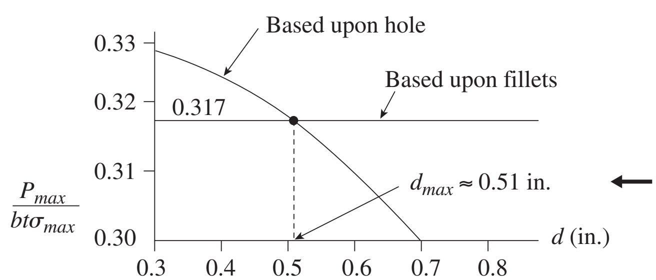
BASED UPON HOLE (Use Fig. 2-63)

$b = 2.4\,\text{in.} \quad d = \text{diameter of the hole (in.)} \quad c_1 = b - d$

$$P_{max} = \sigma_{nom}\, c_1 t = \frac{\sigma_{max}}{K}(b - d)t$$

$$= \frac{1}{K}\left(1 - \frac{d}{b}\right) bt\sigma_{max}$$

d (in.)	d/b	K	$P_{max}/bt\sigma_{max}$
0.3	0.125	2.66	0.329
0.4	0.167	2.57	0.324
0.5	0.208	2.49	0.318
0.6	0.250	2.41	0.311
0.7	0.292	2.37	0.299



Nonlinear Behavior (Changes in Lengths of Bars)

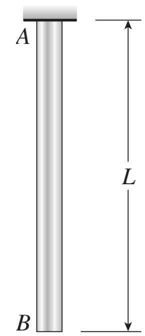
Problem 2.11-1 A bar AB of length L and weight density γ hangs vertically under its own weight (see figure). The stress-strain relation for the material is given by the Ramberg-Osgood equation (Eq. 2-71):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m$$

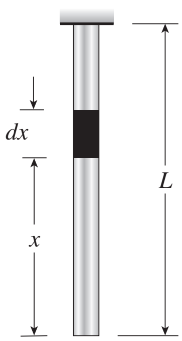
Derive the following formula

$$\delta = \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m$$

for the elongation of the bar.



Solution 2.11-1 **Bar hanging under its own weight**



Let A = cross-sectional area

Let N = axial force at distance x

$N = \gamma A x$

$$\sigma = \frac{N}{A} = \gamma x$$

STRAIN AT DISTANCE x

$$\varepsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\sigma}{\sigma_0}\right)^m = \frac{\gamma x}{E} + \frac{\sigma_0 \alpha}{E}\left(\frac{\gamma x}{\sigma_0}\right)^m$$

ELONGATION OF BAR

$$\delta = \int_0^L \varepsilon\, dx = \int_0^L \frac{\gamma x}{E}\, dx + \frac{\sigma_0 \alpha}{E}\int_0^L \left(\frac{\gamma x}{\sigma_0}\right)^m dx$$

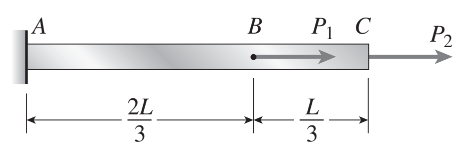
$$= \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E}\left(\frac{\gamma L}{\sigma_0}\right)^m \qquad \text{Q.E.D.} \leftarrow$$

Problem 2.11-2 A prismatic bar of length $L = 1.8$ m and cross-sectional area $A = 480\ \text{mm}^2$ is loaded by forces $P_1 = 30$ kN and $P_2 = 60$ kN (see figure). The bar is constructed of magnesium alloy having a stress-strain curve described by the following Ramberg-Osgood equation:

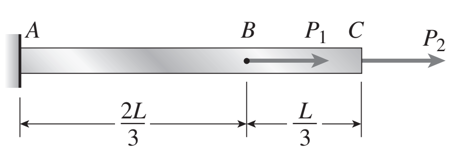
$$\epsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} \qquad (\sigma = \text{MPa})$$

in which σ has units of megapascals.

(a) Calculate the displacement δ_C of the end of the bar when the load P_1 acts alone.
(b) Calculate the displacement when the load P_2 acts alone.
(c) Calculate the displacement when both loads act simultaneously.



Solution 2.11-2 **Axially loaded bar**



$L = 1.8\ \text{m} \qquad A = 480\ \text{mm}^2$

$P_1 = 30\ \text{kN} \quad P_2 = 60\ \text{kN}$

Ramberg−Osgood Equation:

$$\varepsilon = \frac{\sigma}{45{,}000} + \frac{1}{618}\left(\frac{\sigma}{170}\right)^{10} (\sigma = \text{MPa})$$

Find displacement at end of bar.

(a) P_1 ACTS ALONE

$$AB: \ \sigma = \frac{P_1}{A} = \frac{30 \text{ kN}}{480 \text{ mm}^2} = 62.5 \text{ MPa}$$

$$\varepsilon = 0.001389$$

$$\delta_c = \varepsilon\left(\frac{2L}{3}\right) = 1.67 \text{ mm} \quad \longleftarrow$$

(b) P_2 ACTS ALONE

$$ABC: \ \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

$$\delta_c = \varepsilon L = 5.13 \text{ mm} \quad \longleftarrow$$

(c) BOTH P_1 AND P_2 ARE ACTING

$$AB: \ \sigma = \frac{P_1 + P_2}{A} = \frac{90 \text{ kN}}{480 \text{ mm}^2} = 187.5 \text{ MPa}$$

$$\varepsilon = 0.008477$$

$$\delta_{AB} = \varepsilon\left(\frac{2L}{3}\right) = 10.17 \text{ mm}$$

$$BC: \ \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\varepsilon = 0.002853$$

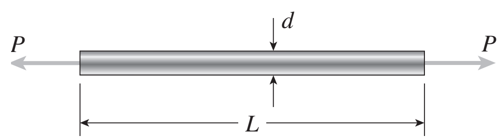
$$\delta_{BC} = \varepsilon\left(\frac{L}{3}\right) = 1.71 \text{ mm}$$

$$\delta_C = \delta_{AB} + \delta_{BC} = 11.88 \text{ mm} \quad \longleftarrow$$

(Note that the displacement when both loads act simultaneously is *not* equal to the sum of the displacements when the loads act separately.)

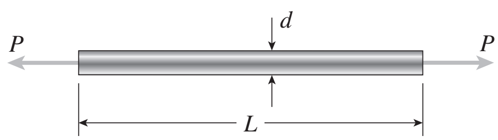
Problem 2.11-3 A circular bar of length $L = 32$ in. and diameter $d = 0.75$ in. is subjected to tension by forces P (see figure). The wire is made of a copper alloy having the following *hyperbolic stress-strain relationship*:

$$\sigma = \frac{18{,}000\epsilon}{1 + 300\epsilon} \qquad 0 \le \epsilon \le 0.03 \qquad (\sigma = \text{ksi})$$



(a) Draw a stress-strain diagram for the material.
(b) If the elongation of the wire is limited to 0.25 in. and the maximum stress is limited to 40 ksi, what is the allowable load P?

Solution 2.11-3 **Copper bar in tension**

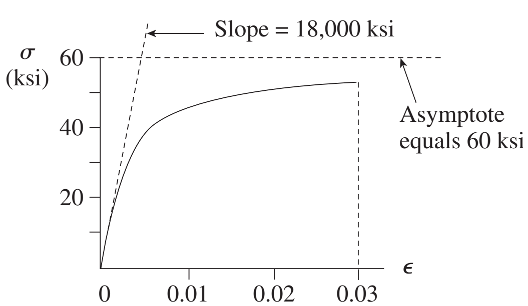


$L = 32$ in. $\qquad d = 0.75$ in.

$$A = \frac{\pi d^2}{4} = 0.4418 \text{ in.}^2$$

(a) STRESS-STRAIN DIAGRAM

$$\sigma = \frac{18{,}000\varepsilon}{1 + 300\varepsilon} \qquad 0 \le \varepsilon \le 0.03 \qquad (\sigma = \text{ksi})$$



(b) ALLOWABLE LOAD P

Max. elongation $\delta_{max} = 0.25$ in.

Max. stress $\sigma_{max} = 40$ ksi

Based upon elongation:

$$\varepsilon_{max} = \frac{\delta_{max}}{L} = \frac{0.25 \text{ in.}}{32 \text{ in.}} = 0.007813$$

$$\sigma_{max} = \frac{18{,}000\,\varepsilon_{max}}{1 + 300\,\varepsilon_{max}} = 42.06 \text{ ksi}$$

BASED UPON STRESS:

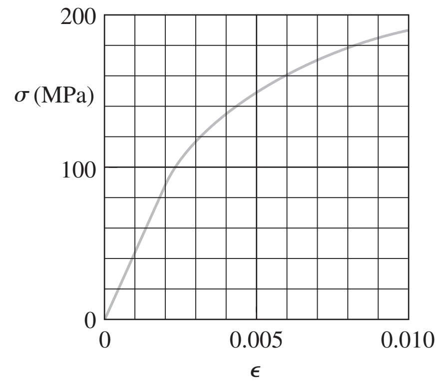
$\sigma_{max} = 40$ ksi

Stress governs. $P = \sigma_{max} A = (40 \text{ ksi})(0.4418 \text{ in.}^2)$

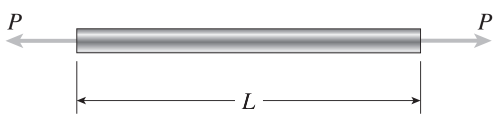
$$= 17.7 \text{ k} \quad \longleftarrow$$

Problem 2.11-4 A prismatic bar in tension has length $L = 2.0$ m and cross-sectional area $A = 249\ \text{mm}^2$. The material of the bar has the stress-strain curve shown in the figure.

Determine the elongation δ of the bar for each of the following axial loads: $P = 10$ kN, 20 kN, 30 kN, 40 kN, and 45 kN. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-4 **Bar in tension**



$L = 2.0\,\text{m}$

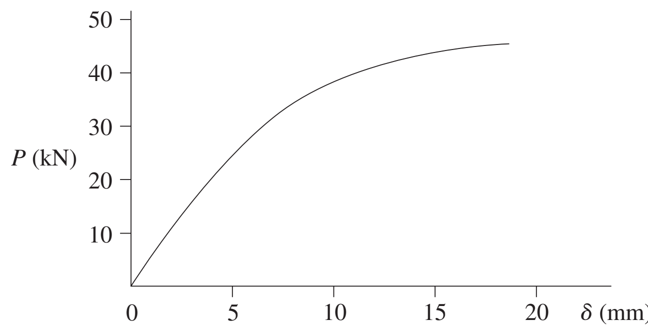
$A = 249\,\text{mm}^2$

STRESS-STRAIN DIAGRAM

(See the problem statement for the diagram)

LOAD-DISPLACEMENT DIAGRAM

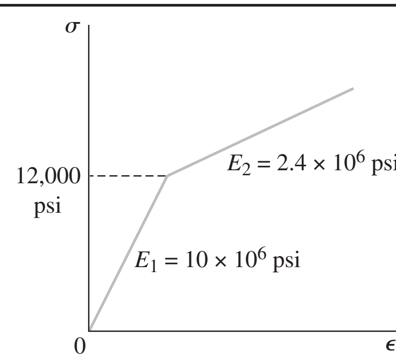
P (kN)	$\sigma = P/A$ (MPa)	ε (from diagram)	$\delta = \varepsilon L$ (mm)
10	40	0.0009	1.8
20	80	0.0018	3.6
30	120	0.0031	6.2
40	161	0.0060	12.0
45	181	0.0081	16.2



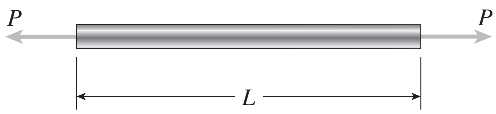
NOTE: The load-displacement curve has the same shape as the stress-strain curve.

Problem 2.11-5 An aluminum bar subjected to tensile forces P has length $L = 150$ in. and cross-sectional area $A = 2.0\ \text{in.}^2$ The stress-strain behavior of the aluminum may be represented approximately by the bilinear stress-strain diagram shown in the figure.

Calculate the elongation δ of the bar for each of the following axial loads: $P = 8$ k, 16 k, 24 k, 32 k, and 40 k. From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



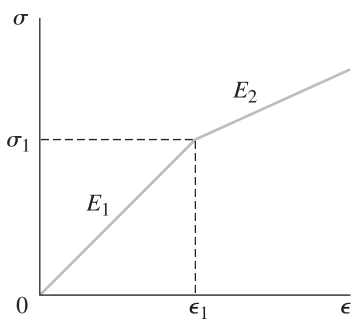
Solution 2.11-5 **Aluminum bar in tension**



$L = 150\text{ in.}$

$A = 2.0\text{ in.}^2$

STRESS-STRAIN DIAGRAM



$E_1 = 10 \times 10^6$ psi

$E_2 = 2.4 \times 10^6$ psi

$\sigma_1 = 12{,}000$ psi

$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{12{,}000\text{ psi}}{10 \times 10^6\text{ psi}}$$

$$= 0.0012$$

For $0 \leq \sigma \leq \sigma_1$:

$$\varepsilon = \frac{\sigma}{E_2} = \frac{\sigma}{10 \times 10^6\text{ psi}} \quad (\sigma = \text{psi}) \qquad \text{Eq. (1)}$$

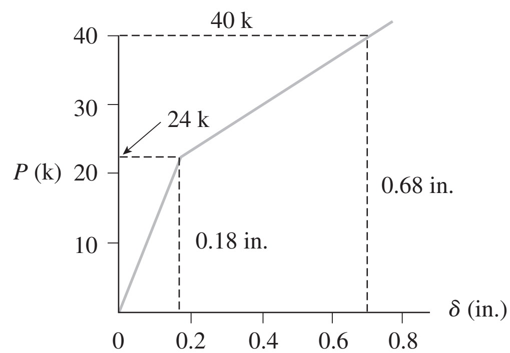
For $\sigma \geq \sigma_1$:

$$\varepsilon = \varepsilon_1 + \frac{\sigma - \sigma_1}{E_2} = 0.0012 + \frac{\sigma - 12{,}000}{2.4 \times 10^6}$$

$$= \frac{\sigma}{2.4 \times 10^6} - 0.0038 \quad (\sigma = \text{psi}) \qquad \text{Eq. (2)}$$

LOAD-DISPLACEMENT DIAGRAM

P (k)	$\sigma = P/A$ (psi)	ε (from Eq. 1 or Eq. 2)	$\delta = \varepsilon L$ (in.)
8	4,000	0.00040	0.060
16	8,000	0.00080	0.120
24	12,000	0.00120	0.180
32	16,000	0.00287	0.430
40	20,000	0.00453	0.680

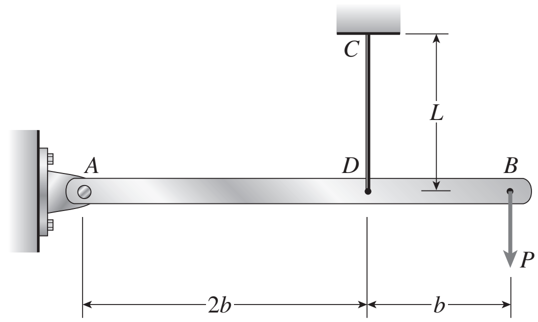


Problem 2.11-6 A rigid bar AB, pinned at end A, is supported by a wire CD and loaded by a force P at end B (see figure). The wire is made of high-strength steel having modulus of elasticity $E = 210$ GPa and yield stress $\sigma_Y = 820$ MPa. The length of the wire is $L = 1.0$ m and its diameter is $d = 3$ mm. The stress-strain diagram for the steel is defined by the *modified power law*, as follows:

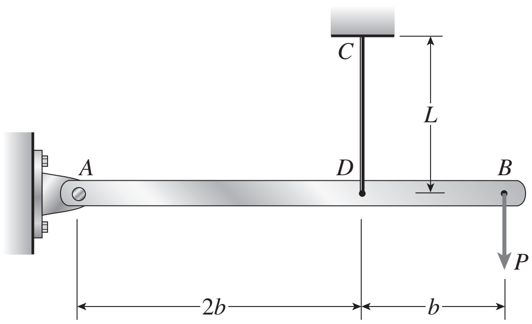
$$\sigma = E\epsilon \qquad 0 \le \sigma \le \sigma_Y$$

$$\sigma = \sigma_Y\left(\frac{E\epsilon}{\sigma_Y}\right)^n \qquad \sigma \ge \sigma_Y$$

(a) Assuming $n = 0.2$, calculate the displacement δ_B at the end of the bar due to the load P. Take values of P from 2.4 kN to 5.6 kN in increments of 0.8 kN.
(b) Plot a load-displacement diagram showing P versus δ_B.



Solution 2.11-6 **Rigid bar supported by a wire**



Wire: $E = 210\,\text{GPa}$

$\sigma_Y = 820\,\text{MPa}$

$L = 1.0\,\text{m}$

$d = 3\,\text{mm}$

$$A = \frac{\pi d^2}{4} = 7.0686\ \text{mm}^2$$

STRESS-STRAIN DIAGRAM

$$\sigma = E\varepsilon \quad (0 \le \sigma \le \sigma_Y) \qquad (1)$$

$$\sigma = \sigma_Y\left(\frac{E\varepsilon}{\sigma_Y}\right)^n \quad (\sigma \ge \sigma_Y) \quad (n = 0.2) \qquad (2)$$

(a) DISPLACEMENT δ_B AT END OF BAR

$$\delta = \text{elongation of wire} \quad \delta_B = \frac{3}{2}\delta = \frac{3}{2}\varepsilon L \qquad (3)$$

Obtain ε from stress-strain equations:

$$\text{From Eq. (1):} \ \ \varepsilon = \frac{\sigma}{E} \quad (0 \le \sigma \le \sigma_Y) \qquad (4)$$

$$\text{From Eq. (2):} \ \ \varepsilon = \frac{\sigma_Y}{E}\left(\frac{\sigma}{\sigma_Y}\right)^{1/n} \qquad (5)$$

$$\text{Axial force in wire: } F = \frac{3P}{2}$$

$$\text{Stress in wire: } \sigma = \frac{F}{A} = \frac{3P}{2A} \qquad (6)$$

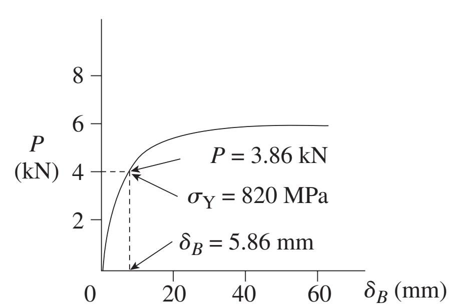
PROCEDURE: Assume a value of P
Calculate σ from Eq. (6)
Calculate ε from Eq. (4) or (5)
Calculate δ_B from Eq. (3)

P (kN)	σ (MPa) Eq. (6)	ε Eq. (4) or (5)	δ_B (mm) Eq. (3)
2.4	509.3	0.002425	3.64
3.2	679.1	0.003234	4.85
4.0	848.8	0.004640	6.96
4.8	1018.6	0.01155	17.3
5.6	1188.4	0.02497	37.5

For $\sigma = \sigma_Y = 820\,\text{MPa}$:

$\varepsilon = 0.0039048 \quad P = 3.864\,\text{kN} \quad \delta_B = 5.86\,\text{mm}$

(b) LOAD-DISPLACEMENT DIAGRAM

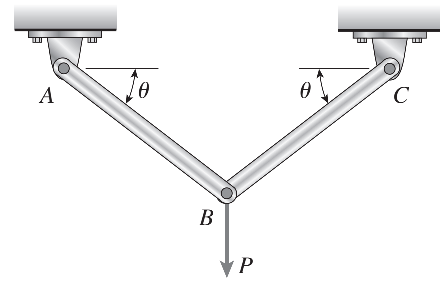


Elastoplastic Analysis

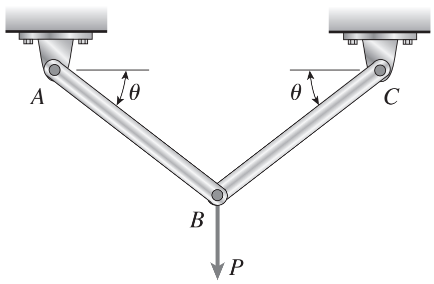
The problems for Section 2.12 are to be solved assuming that the material is elastoplastic with yield stress σ_Y, yield strain ϵ_Y, and modulus of elasticity E in the linearly elastic region (see Fig. 2-70).

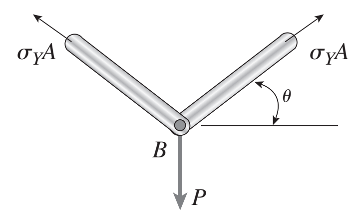
Problem 2.12-1 Two identical bars AB and BC support a vertical load P (see figure). The bars are made of steel having a stress-strain curve that may be idealized as elastoplastic with yield stress σ_Y. Each bar has cross-sectional area A.

Determine the yield load P_Y and the plastic load P_P.



Solution 2.12-1 **Two bars supporting a load P**





Structure is statically determinate. The yield load P_Y and the plastic lead P_P occur at the same time, namely, when both bars reach the yield stress.

JOINT B

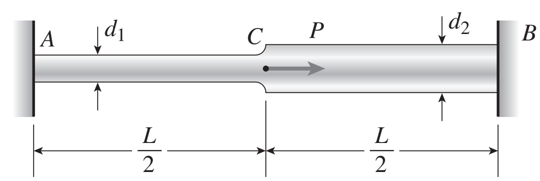
$\Sigma F_{\text{vert}} = 0$

$(2\sigma_Y A)\sin\theta = P$

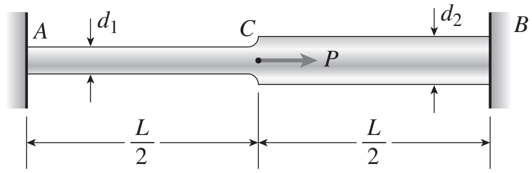
$P_Y = P_P = 2\sigma_Y A \sin\theta$ ←

Problem 2.12-2 A stepped bar ACB with circular cross sections is held between rigid supports and loaded by an axial force P at midlength (see figure). The diameters for the two parts of the bar are $d_1 = 20$ mm and $d_2 = 25$ mm, and the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.

Determine the plastic load P_P.



Solution 2.12-2 Bar between rigid supports

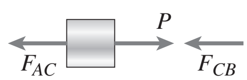


$d_1 = 20\,\text{mm}$ $\quad d_2 = 25\,\text{mm}$ $\quad \sigma_Y = 250\,\text{MPa}$

DETERMINE THE PLASTIC LOAD P_P:

At the plastic load, all parts of the bar are stressed to the yield stress.

Point C:



$F_{AC} = \sigma_Y A_1$ $\qquad F_{CB} = \sigma_Y A_2$

$P = F_{AC} + F_{CB}$

$P_P = \sigma_Y A_1 + \sigma_Y A_2 = \sigma_Y (A_1 + A_2)$ ←

SUBSTITUTE NUMERICAL VALUES:

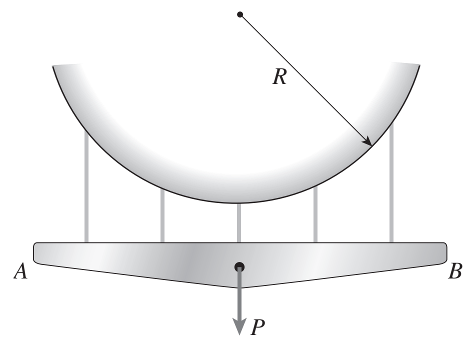
$$P_P = (250\text{ MPa})\left(\frac{\pi}{4}\right)(d_1^2 + d_2^2)$$

$$= (250\text{ MPa})\left(\frac{\pi}{4}\right)[(20\text{ mm})^2 + (25\text{ mm})^2]$$

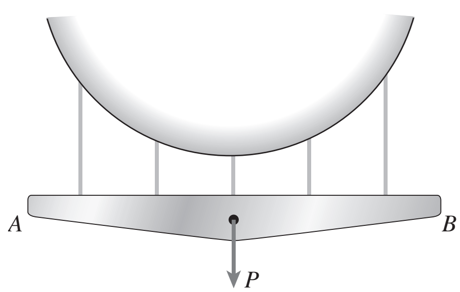
$$= 201\text{ kN}$$ ←

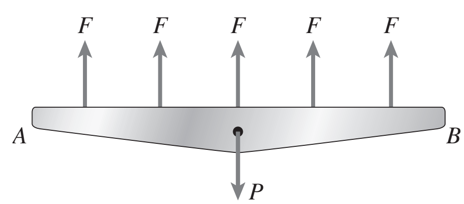
Problem 2.12-3 A horizontal rigid bar AB supporting a load P is hung from five symmetrically placed wires, each of cross-sectional area A (see figure). The wires are fastened to a curved surface of radius R.

(a) Determine the plastic load P_P if the material of the wires is elastoplastic with yield stress σ_Y.
(b) How is P_P changed if bar AB is flexible instead of rigid?
(c) How is P_P changed if the radius R is increased?



Solution 2.12-3 Rigid bar supported by five wires





$F = \sigma_Y A$

(a) PLASTIC LOAD P_P

At the plastic load, each wire is stressed to the yield stress. $\therefore$ $P_P = 5\sigma_Y A$ ←

(b) BAR AB IS FLEXIBLE

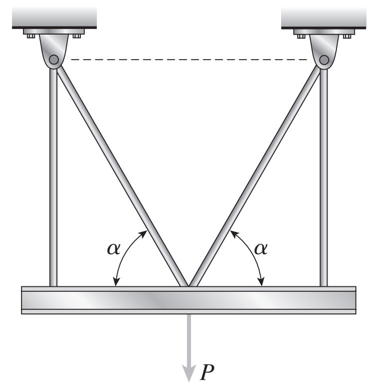
At the plastic load, each wire is stressed to the yield stress, so the plastic load is not changed. ←

(c) RADIUS R IS INCREASED

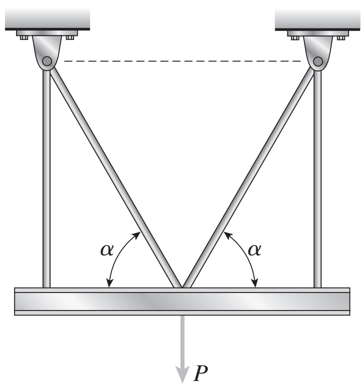
Again, the forces in the wires are not changed, so the plastic load is not changed. ←

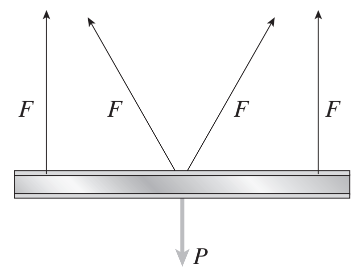
Problem 2.12-4 A load P acts on a horizontal beam that is supported by four rods arranged in the symmetrical pattern shown in the figure. Each rod has cross-sectional area A and the material is elastoplastic with yield stress σ_Y.

Determine the plastic load P_P.



Solution 2.12-4 **Beam supported by four rods**





At the plastic load, all four rods are stressed to the yield stress.

$F = \sigma_Y A$

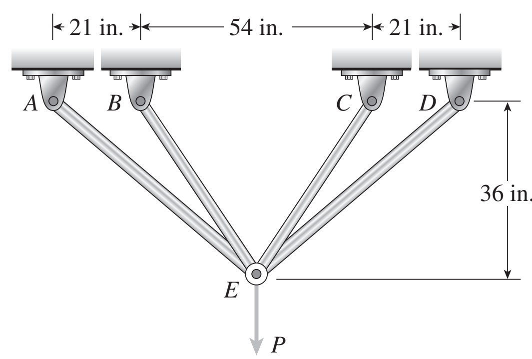
Sum forces in the vertical direction and solve for the load:

$P_P = 2F + 2F \sin \alpha$

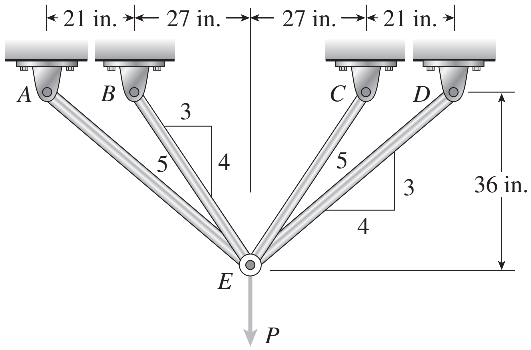
$P_P = 2\sigma_Y A\ (1 + \sin \alpha)$ ←

Problem 2.12-5 The symmetric truss $ABCDE$ shown in the figure is constructed of four bars and supports a load P at joint E. Each of the two outer bars has a cross-sectional area of 0.307 in.^{2}, and each of the two inner bars has an area of 0.601 in.^{2} The material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.

Determine the plastic load P_P.

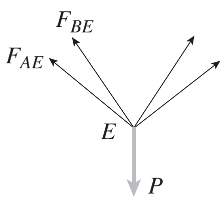


Solution 2.12-5 **Truss with four bars**



$L_{AE} = 60\,\text{in.}$ $\quad L_{BE} = 45\,\text{in.}$

JOINT E



Equilibrium:

$$2F_{AE}\left(\frac{3}{5}\right) + 2F_{BE}\left(\frac{4}{5}\right) = P$$

or

$$P = \frac{6}{5}F_{AE} + \frac{8}{5}F_{BE}$$

PLASTIC LOAD P_P

At the plastic load, all bars are stressed to the yield stress.

$F_{AE} = \sigma_Y A_{AE}$ $\qquad F_{BE} = \sigma_Y A_{BE}$

$$P_P = \frac{6}{5}\sigma_Y A_{AE} + \frac{8}{5}\sigma_Y A_{BE} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

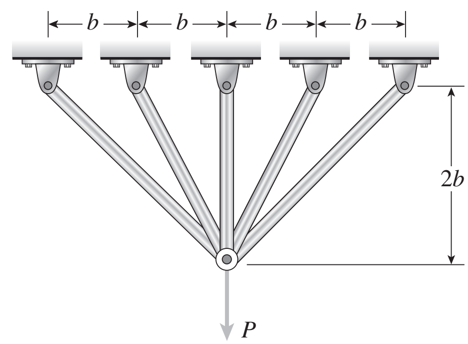
$A_{AE} = 0.307\,\text{in.}^2$ $\quad A_{BE} = 0.601\,\text{in.}^2$

$\sigma_Y = 36\,\text{ksi}$

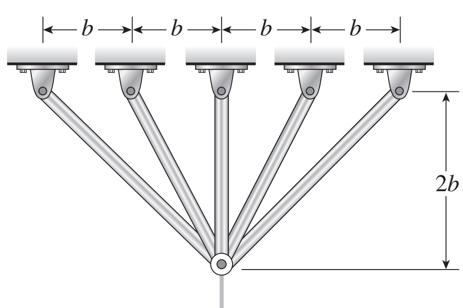
$$P_P = \frac{6}{5}(36\text{ ksi})(0.307\text{ in.}^2) + \frac{8}{5}(36\text{ ksi})(0.601\text{ in.}^2)$$

$$= 13.26\text{ k} + 34.62\text{ k} = 47.9\text{ k} \quad \longleftarrow$$

Problem 2.12-6 Five bars, each having a diameter of 10 mm, support a load P as shown in the figure. Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.



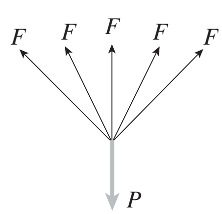
Solution 2.12-6 **Truss consisting of five bars**



$d = 10\,\text{mm}$ $\quad P$

$$A = \frac{\pi d^2}{4} = 78.54\text{ mm}^2$$

$\sigma_Y = 250\,\text{MPa}$



At the plastic load, all five bars are stressed to the yield stress

$F = \sigma_Y A$

Sum forces in the vertical direction and solve for the load:

$$P_P = 2F\left(\frac{1}{\sqrt{2}}\right) + 2F\left(\frac{2}{\sqrt{5}}\right) + F$$

$$= \frac{\sigma_Y A}{5}(5\sqrt{2} + 4\sqrt{5} + 5)$$

$$= 4.2031\,\sigma_Y A \quad \longleftarrow$$

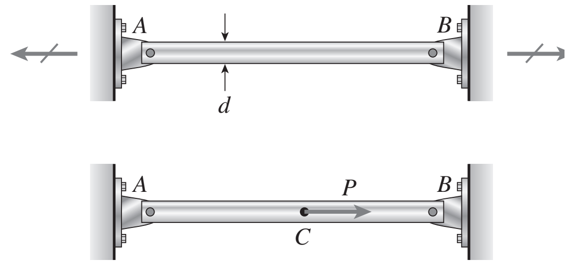
Substitute numerical values:

$$P_P = (4.2031)(250\,\text{MPa})(78.54\,\text{mm}^2)$$

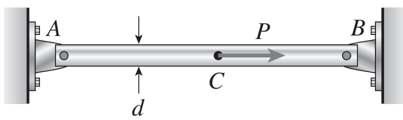
$$= 82.5\text{ kN} \quad \longleftarrow$$

Problem 2.12-7 A circular steel rod AB of diameter $d = 0.60$ in. is stretched tightly between two supports so that initially the tensile stress in the rod is 10 ksi (see figure). An axial force P is then applied to the rod at an intermediate location C.

(a) Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.
(b) How is P_P changed if the initial tensile stress is doubled to 20 ksi?



Solution 2.12-7 **Bar held between rigid supports**



$d = 0.6$ in.

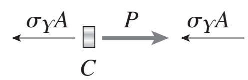
$\sigma_Y = 36$ ksi

Initial tensile stress = 10 ksi

(a) PLASTIC LOAD P_P

The presence of the initial tensile stress does not affect the plastic load. Both parts of the bar must yield in order to reach the plastic load.

POINT C:



$$P_P = 2\sigma_Y A = (2)(36 \text{ ksi})\left(\frac{\pi}{4}\right)(0.60 \text{ in.})^2$$

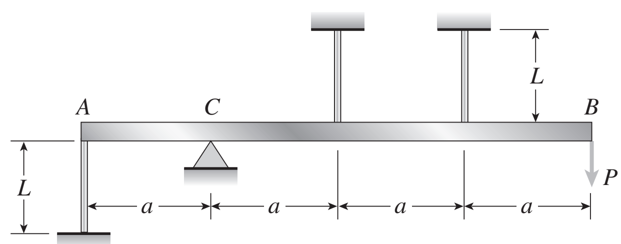
$$= 20.4 \text{ k} \quad \longleftarrow$$

(B) INITIAL TENSILE STRESS IS DOUBLED

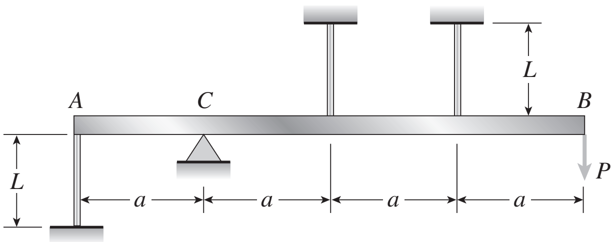
P_P is not changed. ⟵

Problem 2.12-8 A rigid bar ACB is supported on a fulcrum at C and loaded by a force P at end B (see figure). Three identical wires made of an elastoplastic material (yield stress σ_Y and modulus of elasticity E) resist the load P. Each wire has cross-sectional area A and length L.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point B.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point B when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load P as ordinate and the displacement δ_B of point B as abscissa.

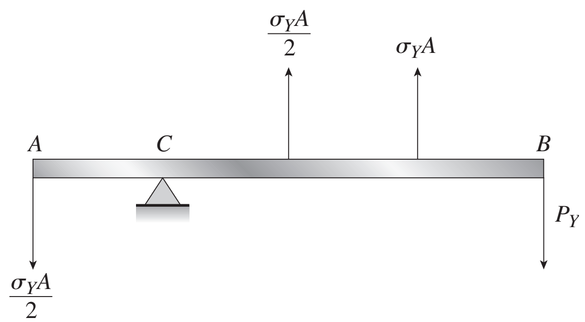


Solution 2.12-8 Rigid bar supported by wires



(a) YIELD LOAD P_Y

Yielding occurs when the most highly stressed wire reaches the yield stress σ_Y.



$\Sigma M_C = 0$

$$P_Y = \sigma_Y A \quad \longleftarrow$$

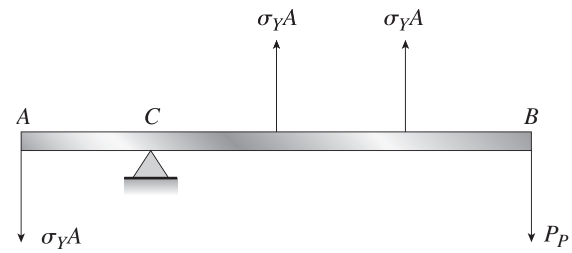
At point A:

$$\delta_A = \left(\frac{\sigma_Y A}{2}\right)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{2E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_Y = \frac{3\sigma_Y L}{2E} \quad \longleftarrow$$

(b) PLASTIC LOAD P_P



At the plastic load, all wires reach the yield stress.

$\Sigma M_C = 0$

$$P_P = \frac{4\sigma_Y A}{3} \quad \longleftarrow$$

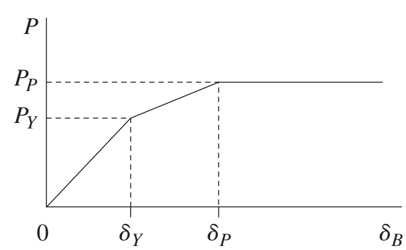
At point A:

$$\delta_A = (\sigma_Y A)\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_P = \frac{3\sigma_Y L}{E} \quad \longleftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

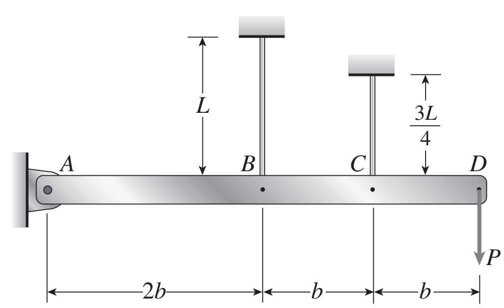


$$P_P = \frac{4}{3} P_Y$$

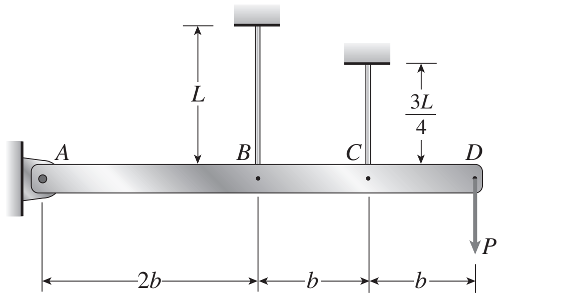
$$\delta_P = 2\delta_Y$$

Problem 2.12-9 The structure shown in the figure consists of a horizontal rigid bar $ABCD$ supported by two steel wires, one of length L and the other of length $3L/4$. Both wires have cross-sectional area A and are made of elastoplastic material with yield stress σ_Y and modulus of elasticity E. A vertical load P acts at end D of the bar.

(a) Determine the yield load P_Y and the corresponding yield displacement δ_Y at point D.
(b) Determine the plastic load P_P and the corresponding displacement δ_P at point D when the load just reaches the value P_P.
(c) Draw a load-displacement diagram with the load P as ordinate and the displacement δ_D of point D as abscissa.



Solution 2.12-9 Rigid bar supported by two wires

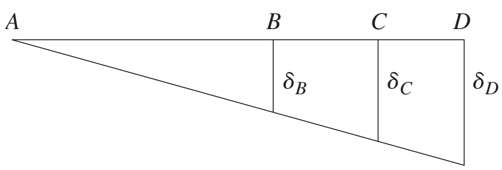


A = cross-sectional area

σ_Y = yield stress

E = modulus of elasticity

DISPLACEMENT DIAGRAM

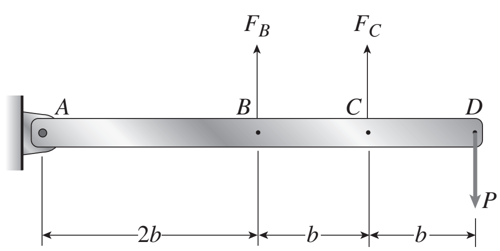


COMPATIBILITY:

$$\delta_C = \frac{3}{2}\delta_B \quad (1)$$

$$\delta_D = 2\delta_B \quad (2)$$

FREE-BODY DIAGRAM



EQUILIBRIUM:

$$\Sigma M_A = 0 \quad F_B(2b) + F_C(3b) = P(4b)$$

$$2F_B + 3F_C = 4P \quad (3)$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{F_B L}{EA} \qquad \delta_C = \frac{F_C\left(\frac{3}{4}L\right)}{EA} \quad (4, 5)$$

Substitute into Eq. (1):

$$\frac{3F_C L}{4EA} = \frac{3F_B L}{2EA}$$

$$F_C = 2F_B \quad (6)$$

STRESSES

$$\sigma_B = \frac{F_B}{A} \quad \sigma_C = \frac{F_C}{A} \quad \therefore \ \sigma_C = 2\sigma_B \quad (7)$$

Wire C has the larger stress. Therefore, it will yield first.

(a) YIELD LOAD

$$\sigma_C = \sigma_Y \quad \sigma_B = \frac{\sigma_C}{2} = \frac{\sigma_Y}{2} \quad \text{(From Eq. 7)}$$

$$F_C = \sigma_Y A \quad F_B = \frac{1}{2}\sigma_Y A$$

From Eq. (3):

$$2\left(\frac{1}{2}\sigma_Y A\right) + 3(\sigma_Y A) = 4P$$

$$P = P_Y = \sigma_Y A \longleftarrow$$

From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{2E}$$

From Eq. (2):

$$\delta_D = \delta_Y = 2\delta_B = \frac{\sigma_Y L}{E} \longleftarrow$$

(b) PLASTIC LOAD

At the plastic load, both wires yield.

$$\sigma_B = \sigma_Y = \sigma_C \qquad F_B = F_C = \sigma_Y A$$

From Eq. (3):

$$2(\sigma_Y A) + 3(\sigma_Y A) = 4P$$

$$P = P_P = \frac{5}{4}\sigma_Y A \longleftarrow$$

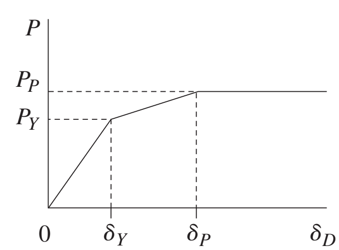
From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{E}$$

From Eq. (2):

$$\delta_D = \delta_P = 2\delta_B = \frac{2\sigma_Y L}{E} \longleftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

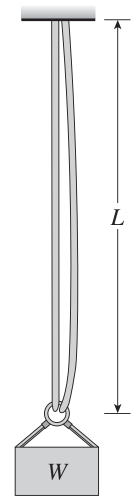


$$P_P = \frac{5}{4}P_Y$$

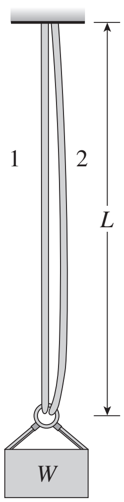
$$\delta_P = 2\delta_Y$$

Problem 2.12-10 Two cables, each having a length L of approximately 40 m, support a loaded container of weight W (see figure). The cables, which have effective cross-sectional area $A = 48.0\ \text{mm}^2$ and effective modulus of elasticity $E = 160$ GPa, are identical except that one cable is longer than the other when they are hanging separately and unloaded. The difference in lengths is $d = 100$ mm. The cables are made of steel having an elastoplastic stress-strain diagram with $\sigma_Y = 500$ MPa. Assume that the weight W is initially zero and is slowly increased by the addition of material to the container.

(a) Determine the weight W_Y that first produces yielding of the shorter cable. Also, determine the corresponding elongation δ_Y of the shorter cable.
(b) Determine the weight W_P that produces yielding of both cables. Also, determine the elongation δ_P of the shorter cable when the weight W just reaches the value W_P.
(c) Construct a load-displacement diagram showing the weight W as ordinate and the elongation δ of the shorter cable as abscissa. (*Hint*: The load displacement diagram is not a single straight line in the region $0 \le W \le W_Y$.)



Solution 2.12-10 Two cables supporting a load



$L = 40\,\text{m} \quad A = 48.0\,\text{mm}^2$

$E = 160\,\text{GPa}$

d = difference in length = 100 mm

$\sigma_Y = 500\,\text{MPa}$

INITIAL STRETCHING OF CABLE 1

Initially, cable 1 supports all of the load.

Let W_1 = load required to stretch cable 1 to the same length as cable 2

$$W_1 = \frac{EA}{L}d = 19.2\ \text{kN}$$

$\delta_1 = 100\,\text{mm}$ (elongation of cable 1)

$$\sigma_1 = \frac{W_1}{A} = \frac{Ed}{L} = 400\ \text{MPa}\ (\sigma_1 < \sigma_Y\ \therefore\ \text{OK})$$

(a) YIELD LOAD W_Y

Cable 1 yields first. $F_1 = \sigma_Y A = 24\,\text{kN}$

δ_{1Y} = total elongation of cable 1

$$\delta_{1Y} = \frac{F_1 L}{EA} = \frac{\sigma_Y L}{E} = 0.125\ \text{m} = 125\ \text{mm}$$

$\delta_Y = \delta_{1Y} = 125$ mm ⟵

δ_{2Y} = elongation of cable 2

$= \delta_{1Y} - d = 25\,\text{mm}$

$$F_2 = \frac{EA}{L}\delta_{2Y} = 4.8\ \text{kN}$$

$W_Y = F_1 + F_2 = 24\ \text{kN} + 4.8\ \text{kN}$

$= 28.8$ kN ⟵

(b) PLASTIC LOAD W_P

$F_1 = \sigma_Y A \quad F_2 = \sigma_Y A$

$W_P = 2\sigma_Y A = 48$ kN ⟵

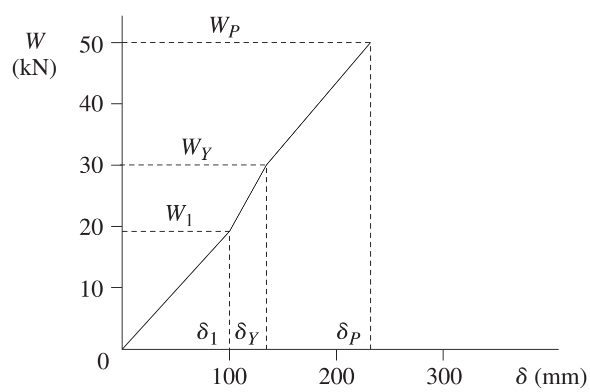
δ_{2P} = elongation of cable 2

$$= F_2\left(\frac{L}{EA}\right) = \frac{\sigma_Y L}{E} = 0.125\ \text{mm} = 125\ \text{mm}$$

$\delta_{1P} = \delta_{2P} + d = 225\,\text{mm}$

$\delta_P = \delta_{1P} = 225$ mm ⟵

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{W_Y}{W_1} = 1.5 \qquad \frac{\delta_Y}{\delta_1} = 1.25$$

$$\frac{W_P}{W_Y} = 1.667 \qquad \frac{\delta_P}{\delta_Y} = 1.8$$

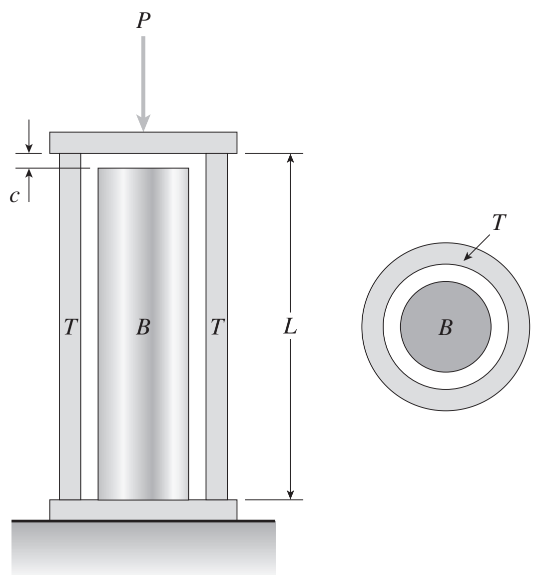
$0 < W < W_1$: slope = 192,000 N/m

$W_1 < W < W_Y$: slope = 384,000 N/m

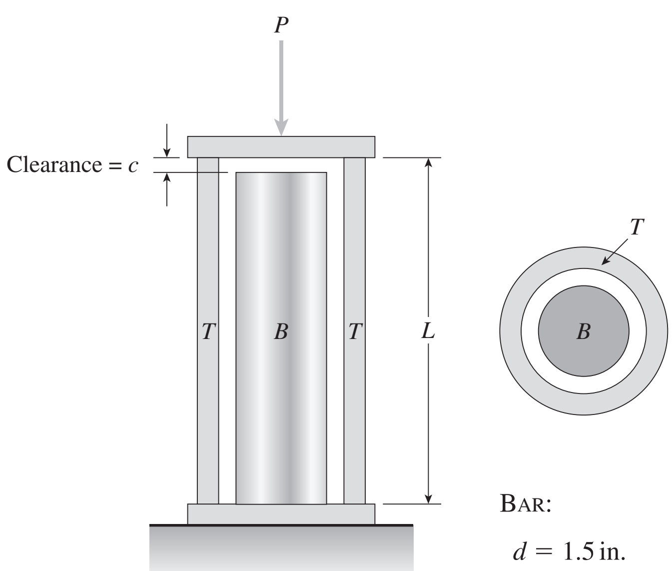
$W_Y < W < W_P$: slope = 192,000 N/m

Problem 2.12-11 A hollow circular tube T of length $L = 15$ in. is uniformly compressed by a force P acting through a rigid plate (see figure). The outside and inside diameters of the tube are 3.0 and 2.75 in., respectively. A concentric solid circular bar B of 1.5 in. diameter is mounted inside the tube. When no load is present, there is a clearance $c = 0.010$ in. between the bar B and the rigid plate. Both bar and tube are made of steel having an elastoplastic stress-strain diagram with $E = 29 \times 10^3$ ksi and $\sigma_Y = 36$ ksi.

(a) Determine the yield load P_Y and the corresponding shortening δ_Y of the tube.
(b) Determine the plastic load P_P and the corresponding shortening δ_P of the tube.
(c) Construct a load-displacement diagram showing the load P as ordinate and the shortening δ of the tube as abscissa. (*Hint:* The load-displacement diagram is not a single straight line in the region $0 \leq P \leq P_Y$.)



Solution 2.12-11 **Tube and bar supporting a load**



$L = 15$ in.

$c = 0.010$ in.

$E = 29 \times 10^3$ ksi

$\sigma_Y = 36$ ksi

TUBE:

$d_2 = 3.0$ in.

$d_1 = 2.75$ in.

$$A_T = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.1290 \text{ in.}^2$$

BAR:

$d = 1.5$ in.

$$A_B = \frac{\pi d^2}{4} = 1.7671 \text{ in.}^2$$

INITIAL SHORTENING OF TUBE T

Initially, the tube supports all of the load.

Let P_1 = load required to close the clearance

$$P_1 = \frac{EA_T}{L}c = 21{,}827 \text{ lb}$$

Let δ_1 = shortening of tube $\quad \delta_1 = c = 0.010$ in.

$$\sigma_1 = \frac{P_1}{A_T} = 19{,}330 \text{ psi} \qquad (\sigma_1 < \sigma_Y \ \therefore \ \text{OK})$$

(Continued)

(a) Yield load P_Y

Because the tube and bar are made of the same material, and because the strain in the tube is larger than the strain in the bar, the tube will yield first.

$F_T = \sigma_Y A_T = 40{,}644 \text{ lb}$

δ_{TY} = shortening of tube at the yield stress

$$\delta_{TY} = \frac{F_T L}{E A_T} = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$\delta_Y = \delta_{TY} = 0.01862 \text{ in.}$ ←

δ_{BY} = shortening of bar

$= \delta_{TY} - c = 0.008621 \text{ in.}$

$$F_B = \frac{E A_B}{L}\delta_{BY} = 29{,}453 \text{ lb}$$

$P_Y = F_T + F_B = 40{,}644 \text{ lb} + 29{,}453 \text{ lb}$
$= 70{,}097 \text{ lb}$

$P_Y = 70{,}100 \text{ lb}$ ←

(b) Plastic load P_P

$F_T = \sigma_Y A_T \qquad F_B = \sigma_Y A_B$

$P_P = F_T + F_B = \sigma_Y(A_T + A_B)$

$= 104{,}300 \text{ lb}$ ←

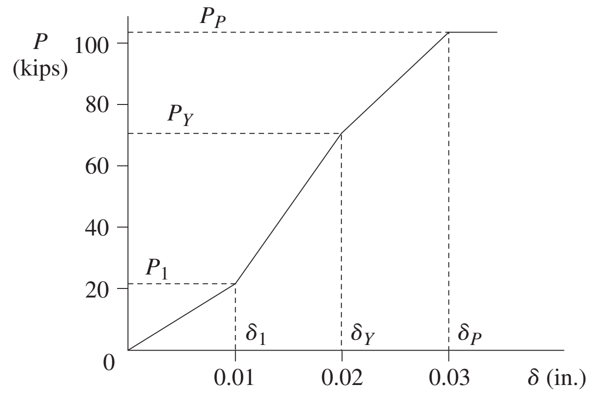
δ_{BP} = shortening of bar

$$= F_B\left(\frac{L}{E A_B}\right) = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$\delta_{TP} = \delta_{BP} + c = 0.028621 \text{ in.}$

$\delta_P = \delta_{TP} = 0.02862 \text{ in.}$ ←

(c) Load-displacement diagram



$$\frac{P_Y}{P_1} = 3.21 \qquad \frac{\delta_Y}{\delta_1} = 1.86$$

$$\frac{P_P}{P_Y} = 1.49 \qquad \frac{\delta_P}{\delta_Y} = 1.54$$

$0 < P < P_1$: slope = 2180 k/in.

$P_1 < P < P_Y$: slope = 5600 k/in.

$P_Y < P < P_P$: slope = 3420 k/in.