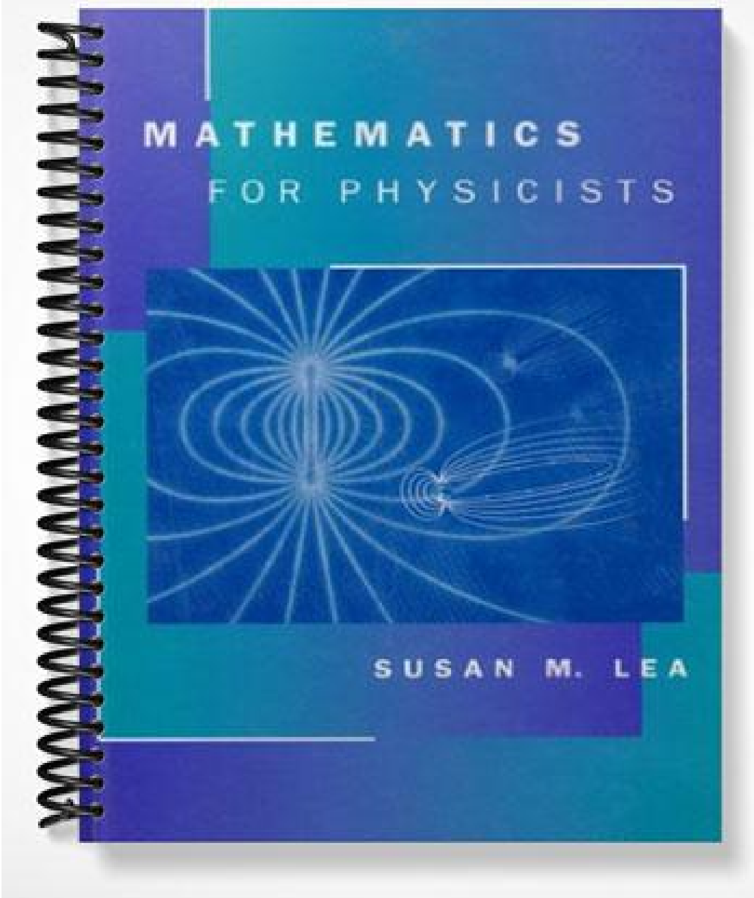
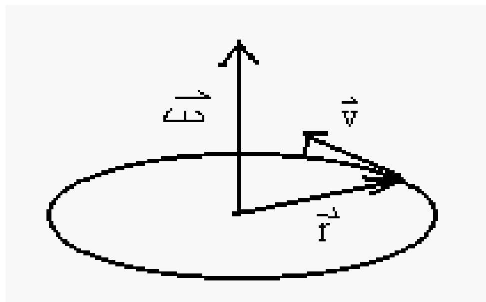
SOLUTIONS MANUAL



Chapter 1: Describing the universe

1. **Circular motion**. A particle is moving around a circle with angular velocity $\vec{\omega}$. Write its velocity vector $\vec{\mathbf{v}}$ as a vector product of $\vec{\omega}$ and the position vector $\mathbf{r}$ with respect to the center of the circle. Justify your expression. Differentiate your relation, and hence derive the angular form of Newton's second law ($\vec{\tau} = I\vec{\alpha}$) from the standard form (equation 1.8).



The direction of the velocity is perpendicular to $\vec{\omega}$ and also to the radius vector $\mathbf{r}$, and is given by putting your right thumb along the vector $\vec{\omega}$: your fingers then curl in the direction of the velocity. The speed is $v = \omega r$. Thus the vector relation we want is:

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

Differentiating, we get:

$$\begin{aligned}
\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} &= \frac{d\vec{\omega}}{dt} \times \vec{\mathbf{r}} + \vec{\omega} \times \frac{d\vec{\mathbf{r}}}{dt} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times \vec{\mathbf{v}} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{\mathbf{r}}\right) \\
&= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega}\left(\vec{\omega} \cdot \vec{\mathbf{r}}\right) - \omega^2 \vec{\mathbf{r}} \\
&= \vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}
\end{aligned}$$

since $\vec{\omega}$ is perpendicular to $\mathbf{r}$. The second term is the usual centripetal term. Then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

and

$$\begin{aligned}
\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} &= \vec{\mathbf{r}} \times m\left(\vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}\right) \\
&= m\left(\vec{\alpha} r^2 - \vec{\mathbf{r}}\left(\vec{\alpha} \cdot \vec{\mathbf{r}}\right)\right) \\
&= mr^2 \vec{\alpha} = I\vec{\alpha}
\end{aligned}$$

since $\vec{\alpha}$ is perpendicular to $\mathbf{r}$, and for a particle $I = mr^2$.

2. Find two vectors, each perpendicular to the vector $\vec{u} = (1, 2, 2)$ and perpendicular to each other. *Hint:* Use dot and cross products. Determine the transformation matrix a that allows you to transform to a new coordinate system with $x'-$axis along $\vec{u}$ and $y'-$ and $z'-$axes along your other two vectors.

We can find a vector $\vec{v}$ perpendicular to $\vec{u}$ by requiring that $\vec{u} \cdot \vec{v} = 0$. A vector satsifying this is:

$$\vec{v} = (0, 1, -1)$$

Now to find the third vector we choose

$$\vec{w} = \vec{u} \times \vec{v} = \left(\ 1,\ \ 2,\ \ 2\ \right) \times \left(\ 0,\ \ 1,\ \ -1\ \right) = \left(\ -4,\ \ 1,\ \ 1\ \right)$$

To find the transformation matrix, first we find the magnitude of each vector and the corresponding unit vectors:

$$\hat{u} = \frac{\left(\ 1,\ \ 2,\ \ 2\ \right)}{\sqrt{1+4+4}} = \frac{\left(\ 1,\ \ 2,\ \ 2\ \right)}{3}$$

$$\hat{v} = \frac{\left(\ 0,\ \ 1,\ \ -1\ \right)}{\sqrt{2}}$$

and

$$\hat{w} = \frac{\left(\ -4,\ \ 1,\ \ 1\ \right)}{\sqrt{16+1+1}} = \frac{\left(\ -4,\ \ 1,\ \ 1\ \right)}{3\sqrt{2}}$$

The elements of the transformation matrix are given by the dot products of the unit vectors along the old and new axes (equation 1.21)

$$A = \frac{1}{3\sqrt{2}} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}$$

To check, we evaluate:

$$A\hat{u} = \frac{1}{9\sqrt{2}} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{9\sqrt{2}} \begin{pmatrix} 9\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{u}'$$

as required. Similarly

$$A\hat{v} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and finally:

$$A\hat{w} = \frac{1}{18}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. Show that the vectors $\vec{u} =$ (15, 12, 16), $\vec{v} =$ (-20, 9, 12) and $\vec{w} =$ (0,-4, 3) are mutually orthogonal and right handed. Determine the transformation matrix that transforms from the original (x, y, z) cordinate system, to a system with x' $-$axis along $\vec{u}$, y' $-$axis along $\vec{v}$ and z' $-$axis along $\vec{w}$. Apply the transformation to find components of the vectors $\vec{a} = (1, 1, 1)$, $\vec{b} = (3, 2, 1)$ and $\vec{c} = (-2, 1, -2)$ in the prime system. Discuss the result for vector $\vec{c}$.

Two vectors are orthogonal if their dot product is zero.

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = -300 + 108 + 192 = 0$$

and

$$\vec{v} \cdot \vec{w} = \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -36 + 36 = 0$$

Finally

$$\vec{u} \cdot \vec{w} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -48 + 48 = 0$$

So the vectors are mutually orthogonal. In addition

$$\vec{u} \times \vec{v} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \times \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = \begin{pmatrix} 0 & -500 & 375 \end{pmatrix} = 125\vec{w}$$

So the vectors form a right-handed set.

To find the transformation matrix, first we find the magnitude of each vector and the corresponding unit vectors.

$$|\vec{u}|^2 = 15^2 + 12^2 + 16^2 = 625 \Rightarrow |\vec{u}| = \sqrt{625} = 25$$

So

$$\hat{u} = \frac{1}{25}\begin{pmatrix} 15, & 12, & 16 \end{pmatrix}$$

Similarly

$$\hat{\mathbf{v}} = \frac{\left(-20,\ \ 9,\ \ 12 \right)}{25}$$

and

$$\hat{\mathbf{w}} = \frac{\left(0,\ \ -4,\ \ 3 \right)}{5}$$

The elements of the transformation matrix are given by the dot products of the unit vectors along the old and new axes (equation 1.21)

$$A_{ij} = \hat{\mathbf{x}}'_i \cdot \hat{\mathbf{x}}_j$$

Thus the matrix is:

$$A = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}$$

Check:

$$A\vec{\mathbf{u}} = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 16 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 625 \\ 0 \\ 0 \end{pmatrix} = 25\hat{\mathbf{x}}'$$

as required.

Then:

$$\vec{\mathbf{a}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 43 \\ 1 \\ -5 \end{pmatrix}$$

and

$$\vec{\mathbf{b}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 85 \\ -30 \\ -25 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 17 \\ -6 \\ -5 \end{pmatrix}$$

$$\vec{\mathbf{c}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -50 \\ 25 \\ -50 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \vec{\mathbf{c}}$$

Since the components of the vector $\vec{c}$ remain unchanged, this vector must lie along the rotation axis.

4. A particle moves under the influence of electric and magnetic fields $\vec{E}$ and $\vec{B}$. Show that a particle moving with initial velocity $\vec{v}_0 = \frac{1}{B^2}\vec{E}\times\vec{B}$ is not accelerated if $\vec{E}$ is perpendicular to $\vec{B}$.

A particle reaches the origin with a velocity $\vec{v} = \vec{v}_0 + \varepsilon\hat{e}$, where $\hat{e}$ is a unit vector in the direction of $\vec{E}$ and $\varepsilon \ll v_0$. If $\vec{E} = E_0(1,1,1)$ and $\vec{B} = B_0(1,-2,1)$, set up a new coordinate system with x' $-$axis along $\vec{E}\times\vec{B}$ and y' $-$axis along $\vec{E}$. Determine the particle's position after a short time t. Determine the components of $\vec{v}(t)$ and $\vec{x}(t)$ in both the original and the new system. Give a criterion for ``short time''.

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) = q\left(\vec{E} + \left(\frac{1}{B^2}\vec{E}\times\vec{B}\right)\times\vec{B}\right)$$
$$= q\left(\vec{E} + \frac{1}{B^2}\left[\vec{B}\left(\vec{E}\cdot\vec{B}\right) - \vec{E}B^2\right]\right)$$

But if $\vec{E}$ is perpendicular to $\vec{B}$, then $\vec{E}\cdot\vec{B} = 0$, so:

$$\vec{F} = q\left(\vec{E} - \vec{E}\right) = 0$$

and if there is no force, then the particle does not accelerate.

With the given vectors for $\vec{E}$ and $\vec{B}$, then

$$\vec{E}\times\vec{B} = E_0B_0(1+2, 1-1, -2-1) = E_0B_0(3,0,-3)$$

Then , since $\left|\vec{B}\right| = \sqrt{6}B_0$

$$\vec{v}_0 = \frac{3}{6}\frac{E_0}{B_0}(1,0,-1) = \frac{1}{2}\frac{E_0}{B_0}(1,0,-1)$$

Now we want to create a new coordinate system with x' $-$ axis along the direction of $\vec{v}_0$. Then we can put the y'-axis along $\vec{E}_0$ and the z' $-$axis along $\vec{B}_0$. The components in the original system of unit vectors along the new axes are the rows of the transformation matrix. Thus the transformation matrix is:

$$\mathbb{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

and the new components of $\vec{\mathbf{v}}_0$ are

$$\vec{\mathbf{v}}_0' = \frac{\sqrt{2}}{2}\frac{E_0}{B_0}(1,0,0)$$

Let's check that the matrix we found actually does this:

$$\vec{\mathbf{v}}_0' = A\vec{\mathbf{v}}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\frac{E_0}{2B_0}$$

$$= \frac{E_0}{2B_0}\begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

as required.

Now let $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \varepsilon\hat{\mathbf{y}}'$. Then

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \left[\vec{\mathbf{v}}_0 + \varepsilon\hat{\mathbf{y}}'\right]\times\vec{\mathbf{B}}\right) = q\varepsilon\hat{\mathbf{y}}'\times\vec{\mathbf{B}}$$

in the new system, the components of $\vec{\mathbf{B}}$ are:

$$\vec{\mathbf{B}}' = B_0\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = B_0\begin{pmatrix} 0 \\ 0 \\ \sqrt{6} \end{pmatrix} = \sqrt{6}B_0\hat{\mathbf{z}}'$$

and so

$$\vec{\mathbf{F}}' = \sqrt{6}q\varepsilon B_0\hat{\mathbf{x}}' = m\vec{\mathbf{a}}$$

Since the initial velocity is $\vec{\mathbf{v}} = v_0\hat{\mathbf{x}}' + \varepsilon\hat{\mathbf{y}}'$ the particle's velocity at time t is:

$$\vec{\mathbf{v}}' = \left(v_0 + \frac{\sqrt{6}q\varepsilon B_0}{m}t\right)\hat{\mathbf{x}}' + \varepsilon\hat{\mathbf{y}}'$$

and the path is intially parabolic:

$$\vec{\mathbf{r}}' = \varepsilon t \hat{\mathbf{y}}' + \left(v_0 t + \frac{\sqrt{6}}{2} \frac{q\varepsilon B_0}{m} t^2 \right) \hat{\mathbf{x}}'$$

This result is valid so long as the initial velocity has not changed appreciably, so that the acceleration is approximately constant. That is:

$$t \ll \frac{v_0}{\varepsilon} \frac{2m}{\sqrt{6}\, q B_0} = \frac{v_0}{\varepsilon} \frac{2}{\omega_c}$$

or v_0/ε times (the cyclotron period divided by π). The time may be quite long if ε is small. Now we convert back to the original coordinates:

$$\begin{aligned}
\vec{\mathbf{r}} &= \mathbb{A}^{-1} \vec{\mathbf{r}}' = \mathbb{A}^{T} \vec{\mathbf{r}}' \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} v_0 t + \frac{\sqrt{6}}{2} \frac{q\varepsilon B_0}{m} t^2 \\ \varepsilon t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{v_0 t}{\sqrt{2}} + \sqrt{3}\, \varepsilon t \left(\frac{1}{2} \frac{q B_0}{m} t + \frac{1}{3} \right) \\ \frac{\sqrt{3}}{3} \varepsilon t \\ -\frac{v_0 t}{\sqrt{2}} - \sqrt{3}\, \varepsilon t \left(\frac{1}{2} \frac{q B_0}{m} t - \frac{1}{3} \right) \end{pmatrix}
\end{aligned}$$

5. A solid body rotates with angular velocity $\vec{\boldsymbol{\omega}}$. Using cylindrical coordinates with $z-$axis along the rotation axis, find the components of the velocity vector $\vec{\mathbf{v}}$ at an arbitrary point within the body. Use the expression for curl in cylindrical coordinates to evaluate $\vec{\nabla} \times \vec{\mathbf{v}}$. Comment on your answer.

The velocity has only a $\phi-$component.

$$\vec{\mathbf{v}} = (0, \rho\omega, 0)$$

Then the curl is given by:

$$\begin{aligned}
\vec{\nabla} \times \vec{\mathbf{v}} &= \hat{\boldsymbol{\rho}} \left(\frac{1}{\rho} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho} \right) + \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi} \right] \\
&= \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho^2 \omega \right) \right] = \hat{\mathbf{z}} \frac{1}{\rho} (2\rho\omega) = 2\omega \hat{\mathbf{z}} = 2\vec{\boldsymbol{\omega}}
\end{aligned}$$

Thus the curl of the velocity equals twice the angular velocity- this seems logical for an operator called curl.

6. Starting from conservation of mass in a fixed volume V, use the divergence theorem to derive the continuity equation for fluid flow:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

where ρ is the fluid density and $\vec{v}$ its velocity.

The mass inside the volume can change only if fluid flows in or out across the boundary. Thus:

$$\frac{dM}{dt} = -\int_S \rho \vec{v} \cdot \hat{n} dA$$

where flow outward ($\vec{v} \cdot \hat{n} > 0$) decreases the mass. Now if the volume is fixed, then:

$$\frac{dM}{dt} = \frac{d}{dt}\int \rho dV = \int \frac{\partial \rho}{\partial t} dV = -\int_S \rho \vec{v} \cdot \hat{n} dA$$

Then from the divergence theorem:

$$\int \frac{\partial \rho}{\partial t} dV = -\int \vec{\nabla} \cdot (\rho \vec{v}) dV$$

and since this must be true for *any* volume V, then

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

7. Find the matrix that represents the transformation obtained by (a) rotating about the $x-$axis by 45° counterclockwise, and then (b) rotating about the $y'-$axis by 30° clockwise. What are the components of a unit vector along the original $z-$axis in the new (double-prime) system?

The first rotation is represented by the matrix

$$\mathbb{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & \sin 45^\circ \\ 0 & -\sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

The second rotation is:

$$\mathbb{A}_2 = \begin{pmatrix} \cos(-30^\circ) & 0 & \sin(-30^\circ) \\ 0 & 1 & 0 \\ -\sin(-30^\circ) & 0 & \cos(-30^\circ) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

And the result of the two rotations is:

$$\mathbb{A}_2\mathbb{A}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}$$

The new components of the orignal $z-$axis are:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

8. Does the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represent a rotation of the coordinate axes? If not, what transformation does it represent? Draw a diagram showing the old and new coordinate axes, and comment.

The determinant of this matrix is:

$$\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\cos^2\theta - \sin^2\theta = -1$$

Thus this transformation cannot be a rotation since a rotation matrix has determinant $+1$.
Let's see where the axes go:

$$\mathbb{A}\hat{\mathbf{x}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

and

$$\mathbb{A}\hat{\mathbf{y}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

while

$$\mathbb{A}\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are the components of the original $x-$ and y axes in the new system. The new $x'-$ and y' axes have the following components in the original system:

$$\hat{\mathbf{u}} = \mathbb{A}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

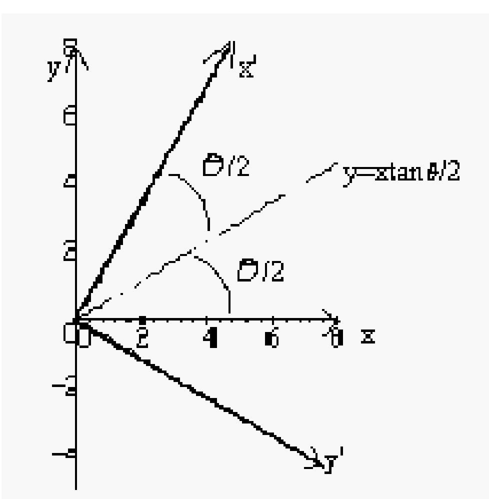
where

$$\mathbb{A}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus:

$$\hat{\mathbf{u}} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

The picture looks like this:



Problem 8: $\theta = \pi/3$