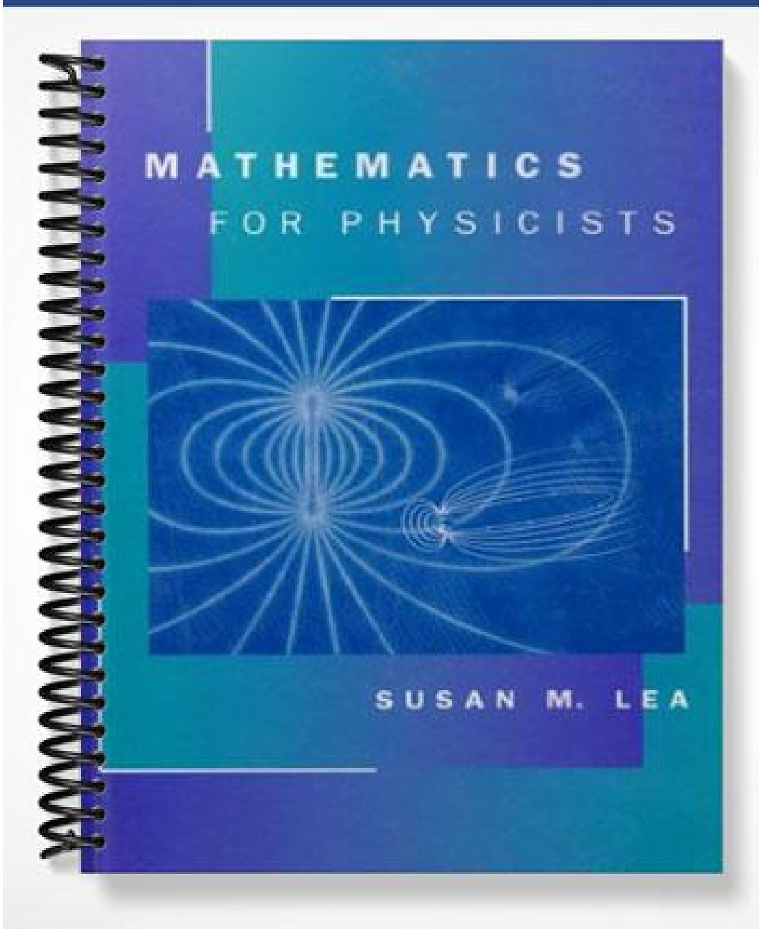
SOLUTIONS MANUAL



Chapter 2: Complex variables

1. If $z_1 = 5 + 2i$ and $z_2 = 3 - 4i$, find z_1/z_2 and $z_1 \times z_2$.

$$\frac{z_1}{z_2} = \frac{5+2i}{3-4i} = \left(\frac{5+2i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{15+26i+8i^2}{9+16} = \frac{7}{25} + \frac{26}{25}i$$

$$z_1 z_2 = (5+2i)(3-4i) = 15 - 14i - 8i^2 = 23 - 14i$$

2. Use the polar representation of z to write an expression for z^3 in terms of r and θ. Use your result to express $\cos 3\theta$ and $\sin 3\theta$ in terms of cos θ and $\sin\theta$.

$$z^3 = \left(re^{i\theta}\right)^3 = r^3 e^{3i\theta}$$
$$r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta)$$
$$\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

The real part gives:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos\theta\left(\cos^2\theta - 3\sin^2\theta\right) = \cos\theta\left(4\cos^2\theta - 3\right)$$

and from the imaginary part:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) = \sin\theta\left(3 - 4\sin^2\theta\right)$$

3. Prove De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

4. The equation $(y - y_0)^2 = 4a(x - x_0)$ describes a parabola. Write this equation in terms of $z = x + iy$. Hint: use the geometric definition of the parabola.

The parabola is a curve such that for any point on the curve the distance from a point is equal to the distance to a line. In this case the point is at $\left(x_0 + a,\ y_0\right)$. The distance from the point is d where:

$$d = \sqrt{(x - x_0 - a)^2 + (y - y_0)^2}$$
$$d^2 = (x - x_0)^2 - 2a(x - x_0) + a^2 + (y - y_0)^2$$

Using the equation of the parabola:

$$d^2 = (x - x_0)^2 + 2a(x - x_0) + a^2 = (x - x_0 + a)^2 = s^2$$

where

$$s = x - (x_0 - a)$$

is the distance from the vertical line at $x = x_0 - a$.

Now we can express these ideas using complex numbers. The distance from the point $z_0 = (x_0 + a,\ y_0)$ is $|z - z_0|$ and the distance from the line is $\mathrm{Re}(z - (x_0 - a))$. Thus the equation we want is:

$$|z - z_0| = \mathrm{Re}(z - (x_0 - a))$$

5. Show that the equation

$$|z - c| + |z - d| = \alpha$$

represents an ellipse in the complex plane, where c and d are complex constants, and α is a real constant. Use geometrical arguments to determine the position of the center of the ellipse and its semi-major and semi-minor axes.

The absolute value $|z - c|$ is the distance between a point P in the Argand diagram described by $z = x + iy$ and the point C described by the number c. Thus the equation describes a curve such that the sum of the distances of P from the points C and D is a constant (α). This is the definition of an ellipse. The points C and D are the foci of the ellipse, so its center is half way between them, at $z = \frac{1}{2}(c + d)$.

When P is at the end of the semi-major axis, then $|z - c| = a(1 - e)$ and $|z - d| = a(1 + e)$ so $\alpha = 2a$, and the semi-major axis is

$$a = \alpha/2$$

Then also

$$|d - c| = 2ae \Rightarrow e = \frac{|d - c|}{\alpha}$$

and so

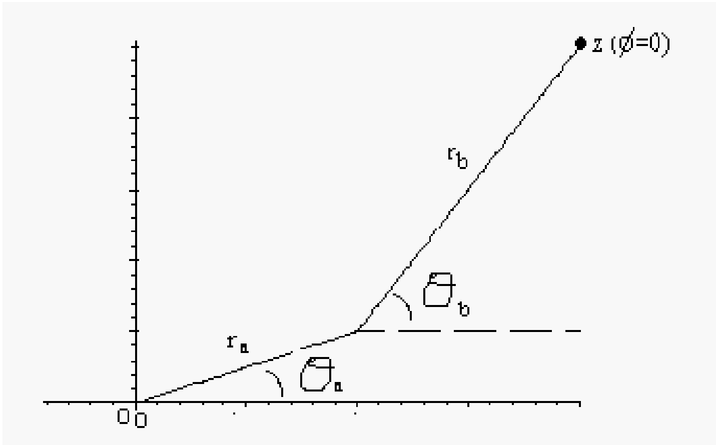
$$b = a\sqrt{1 - e^2} = \frac{\alpha}{2}\sqrt{1 - \frac{|d - c|^2}{\alpha^2}} = \frac{1}{2}\sqrt{\alpha^2 - |d - c|^2}$$

6. Show that the equation

$$z = ae^{i\phi} + be^{-i\phi}$$

represents an ellipse in the complex plane, where a and b are complex constants and ϕ is a real variable. Determine the position of the center of the ellipse and its semi-major and semi-minor axes.

First recall that multiplication by $e^{i\phi}$ corresponds to rotation counter-clockwise by an angle ϕ (Figure 2.3c) . Thus if $a = r_a e^{i\theta_a}$ and $b = r_b e^{i\theta_b}$, then z is represented as follows:



Now as ϕ increases, the lower line rotates counterclockwise, while the upper line rotates clockwise. The two lines align when:

$$\theta_a + \phi = \theta_b - \phi$$

or

$$\phi = \frac{\theta_b - \theta_a}{2}$$

which is the direction of the major axis. The length of the major axis is $r_a + r_b = |a| + |b|.$ The smallest value of $|z|$ occurs when the two ``vectors'' are in opposite directions, i.e.

$$\theta_a + \phi = \theta_b - \phi + \pi$$

or

$$\phi = \frac{\theta_b - \theta_a}{2} + \frac{\pi}{2}$$

Thus the minor axis, of length $||a| - |b||$ is perpendicular to the major axis, as expected.

The angle that the major axis makes with the $x-$ axis is $\theta_a + \left(\frac{\theta_b - \theta_a}{2}\right) = \frac{\theta_b + \theta_a}{2}.$ Let x' and y' be the coordinates of z with axes coincident with the major and minor axes of the ellipse. Then :

$$\begin{aligned} z &= Ae^{i\phi} + Be^{-i\phi} = r_a \exp(i(\theta_a + \phi)) + r_b \exp(i(\theta_b - \phi)) \\ &= \exp\left(i\frac{(\theta_a + \theta_b)}{2}\right)\left[r_a \exp\left(i\left(\frac{(\theta_a - \theta_b)}{2} + \phi\right)\right) + r_b \exp\left(-i\left(\frac{(\theta_a - \theta_b)}{2} + \phi\right)\right)\right] \\ &= \exp\left(i\frac{(\theta_a + \theta_b)}{2}\right)z' \end{aligned}$$

Again we note that the factor $\exp\left(i\frac{(\theta_a + \theta_b)}{2}\right)$ rotates the number in square brackets (z') by an angle $\frac{(\theta_a + \theta_b)}{2}$ counter-clockwise. Thus:

$$z' = r_a e^{i\alpha} + r_b e^{-i\alpha} = (r_a + r_b)\cos\alpha + i(r_a - r_b)\sin\alpha$$

where $\alpha = \frac{(\theta_a - \theta_b)}{2} + \phi.$ Thus we have

$$\frac{(x')^2}{(r_a + r_b)^2} + \frac{(y')^2}{(r_a - r_b)^2} = 1$$

which is the equation of an ellipse with semi-major axis $|a| + |b|$ and semi-minor axis $||a| - |b||.$ The center of the ellipse is at the origin.

7. Find all solutions of the equations (**a**) $z^5 = -1.$

Write z in polar form:

$$r^5 e^{5i\theta} = -1 = 1\exp(i\pi + 2n\pi i)$$

for $0 \le n \le 4.$ Thus the solutions are

$$z = 1\exp\left(i\frac{\pi}{5} + i\frac{2n\pi}{5}\right)$$

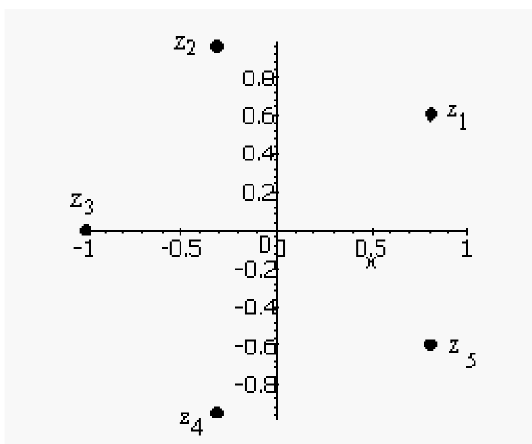
$$z_1 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) = 0.80902 + 0.58779i$$

$$z_2 = \cos\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) = \cos\left(\frac{3}{5}\pi\right) + i\sin\left(\frac{3\pi}{5}\right)$$
$$= -0.30902 + 0.95106i$$

$$z_3 = \cos(\pi) + i\sin(\pi) = -1$$

$$z_4 = \cos\left(\frac{7}{5}\pi\right) + i\sin\left(\frac{7\pi}{5}\right) = -0.30902 - 0.95106i$$

$$z_5 = \cos\left(\frac{9}{5}\pi\right) + i\sin\left(\frac{9\pi}{5}\right) = 0.80902 - 0.58779i$$



(**b**) $z^4 = 16.$ The roots are $2(1)^{1/4} = 2\left(e^{2\pi i n/4}\right),$ $n = 0, 1, 2, 3$

$$z_n = 2e^{in\pi/2}$$

These points are at the corners of a square: $z_0 = 2$ (on the real axis) $z_1 = 2i$ (on the imaginary axis), $z_2 = -2,$ $z_3 = -2i.$

8. Find all solutions of the equation (**a)** $\cos z = 100.$

Write $z = x + iy,$ where x and y are real, and expand the cosine:

$$\cos x \cos iy - \sin x \sin iy = 100$$
$$\cos x \cosh y - i \sin x \sinh y = 100$$

Writing the real and imaginary parts separately, we have:

$$\cos x \cosh y = 100 \quad \text{and} \quad \sin x \sinh y = 0$$

We can solve the second equation with either $y = 0,$ or $x = 0,$ $n\pi.$ But with $y = 0$ the first equation becomes $\cos x = 100$, which has no solutions. (Remember that x is real.) So we must choose $x = n\pi,$ where n is any positive or negative integer, or zero. Then:

$$\cos n\pi \cosh y = 100$$

Now the hyperbolic cosine is always positive if y is real, so we must choose n to be even, or zero. Then $\cos n\pi = +1$ and:

$$e^y + e^{-y} = 200$$
$$(e^y)^2 - 200e^y + 1 = 0$$
$$e^y = \frac{200 \pm \sqrt{200^2 - 4}}{2}$$
$$= 100 \pm \sqrt{9999} = 199.99,\ 5.0001 \times 10^{-3}$$

and thus

$$y = \ln(199.99) = 5.2983$$

or

$$y = \ln\left(5.0001 \times 10^{-3}\right) = -5.2983$$

Both values give the same value for the cosh. Then

$$\boxed{z = 2m\pi \pm 5.2983i}$$

(**b)** $\sin z = 6$

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y = 6$$

Equating real and imaginary parts:

$$\sin x \cosh y = 6$$
$$\cos x \sinh y = 0$$

Clearly $y = 0$ is not a viable solution, so we need

$$\cos x = 0 \Rightarrow x = \left(n + \frac{1}{2}\right)\pi$$

Then

$$\sin\left(n + \frac{1}{2}\right)\pi \cosh y = (-1)^n \cosh y = 6$$

Since cosh y is always positive (y is real) then n must be even, and

$$\begin{aligned} e^y + e^{-y} &= 12 \\ e^{2y} - 12e^y + 1 &= 0 \\ e^y &= \frac{12 \pm \sqrt{144-4}}{2} = 6 \pm \sqrt{35} \end{aligned}$$

Thus

$$y = \ln\left(6 \pm \sqrt{35}\right) = 2.4779 \text{ or } -2.4779$$

Thus

$$z = \left(2n + \frac{1}{2}\right)\pi \pm 2.4779i$$

9. Find all solutions of the equation $\cosh z = -5.$

$\cosh z = \cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy = \cosh x \cos y + i \sinh x \sin y = -5.$ The imaginary part must be zero, so we must have $x = 0$ or $y = n\pi.$ The real part would be $\cos y$ or $(-1)^n \cosh x$ in the two cases. Since $\cos y$ can never equal $-5,$ we must choose $y = n\pi$ with n odd, and then setting the real part equal to -5 we need

$$\cosh x = 5$$

and the solution is : $x = \pm 2.2924.$ Thus $z = \pm 2.2924 + (2n+1)\pi,$ where n is any positive or negative integer.

10. Find all numbers z such that $z = \ln(-5).$

$$z = \ln\left(5e^{i\pi + 2\pi i n}\right) = \ln 5 + i\pi(2n+1) = 1.6094 + i\pi(2n+1)$$

11. Investigate the function $w = 1/\sqrt{z}.$ Find the functions $u(r,\theta)$ and $v(r,\theta)$ where $w = u + iv.$ How many branches does this function have? Find the image of the unit circle under this mapping.

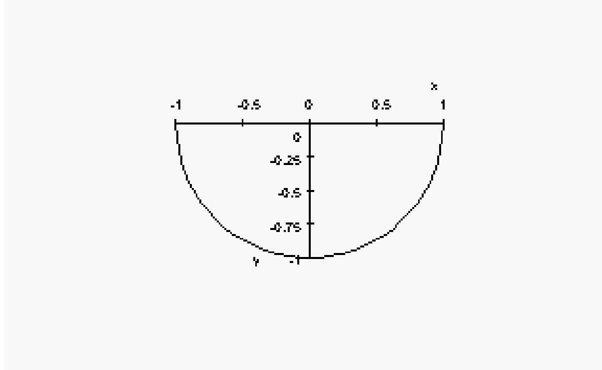
$$\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{r}} e^{-i\theta/2} = \frac{1}{\sqrt{r}}\left(\cos\frac{\theta}{2} - i \sin\frac{\theta}{2}\right)$$

Thus

$$u = \rho\cos\phi = \frac{1}{\sqrt{r}}\cos\frac{\theta}{2}; \quad v = \rho\sin\phi = -\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}$$

The function has a branch point at $z = 0,$ and it has two branches. Two circuits of the $z-$ plane give the whole $w-$ plane.

The unit circle is defined by $|z| = r = 1, \ \theta \le \theta < \pi.$ Then in the $w-$ plane we get a piece of the unit circle: $|w| = \sqrt{r} = 1,$ and, for the principal branch, $\phi = -\theta/2.$ So $0 \ge \phi > -\pi.$



12. The function $w(z) = z^{1/4}$. Find the functions $u(r,\theta)$ and $v(r,\theta)$ where $w = u + iv$. How many branches does this function have? Find the image under this mapping of a square of side 1 centered at the origin .

$$w = \left(re^{i\theta}\right)^{1/4} = r^{1/4}e^{i\theta/4} = r^{1/4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)$$

Thus

$$u = r^{1/4}\cos\frac{\theta}{4} \text{ and } v = r^{1/4}\sin\frac{\theta}{4}$$

The function has four branches since we have to go around the original plane four times to get the whole $w-$ plane.

The line $x = \frac{1}{2}$, $y = 0$ to $+\frac{1}{2}$ ($r = \frac{1}{2}\sec\theta, 0 \leq \theta \leq \pi/4)$ is mapped to

$$w = \left(\frac{1}{2}\sec\theta\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The top side at $y = 1/2$ ($r = \frac{1}{2\sin\theta}$ $\pi/4 < \theta < 3\pi/4)$ maps to

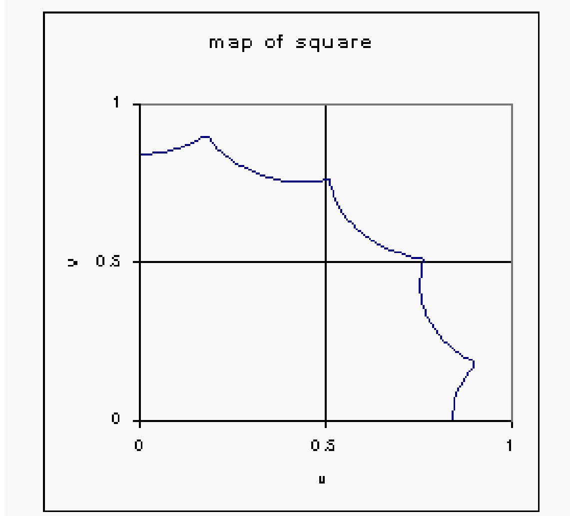
$$w = \left(\frac{1}{2\sin\theta}\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The left side at $x = -1/2$ ($r = \frac{1}{2}\sec\theta, 3\pi/4 \leq \theta \leq 5\pi/4)$ is mapped to:

$$\left(\frac{1}{2}\sec(\theta - \pi)\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The bottom at $y = -1/2$ ($r = \frac{1}{2\sin(\theta-\pi)}$ $5\pi/4 < \theta < 7\pi/4)$ maps to

$$w = \left(\frac{1}{2\sin(\theta - \pi)}\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$



The entire square has mapped into the first quadrant and has been deformed into a curvy polygon. The other four branches of the function would close the polygon by completing the other three quadrants.

13. Oblate spheroidal coordinates $u,\ v,\ w$ are defined in terms of cylindrical coordinates $\rho,\ \phi,\ z$ by the relations:

$$\rho + iz = c\cosh(u + iv),\ \ w = \phi$$

Show that the surfaces of constant u and constant v are ellipsoids and hyperboloids, respectively. What values of u and v correspond to the $z-$ axis and the $z = 0$ plane?

$$\rho + iz = c(\cosh u \cos v + i \sinh u \sin v)$$

Equating real and imaginary parts, we have:

$$\rho = c\cosh u \cos v \text{ and } z = c \sinh u \sin v$$

We want to find the shape of the constant u and constant v surfaces. First eliminate v :

$$\cos v = \frac{\rho}{c\cosh u} \qquad \text{and} \qquad \sin v = \frac{z}{c\sinh u}$$

Thus

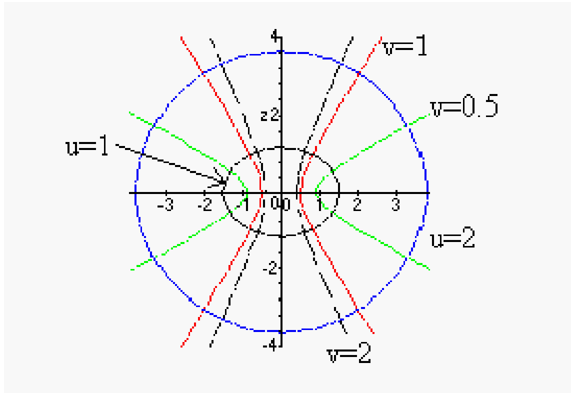
$$1 = \cos^2 v + \sin^2 v = \left(\frac{\rho}{c\cosh u}\right)^2 + \left(\frac{z}{c\sinh u}\right)^2$$

Thus the surfaces of constant u are ellipsoids with semi-major axis c cosh u and semi-minor axis $c\sinh u.$ Similarly, by solving for $\cosh u$ and $\sinh u,$ squaring and subtracting, we find:

$$1 = \cosh^2 u - \sinh^2 u = \left(\frac{\rho}{c\cos v}\right)^2 - \left(\frac{z}{c\sin v}\right)^2$$

so the constant v surfaces are hyperboloids.

The $z-$ axis is described by $\cos v = 0,$ i.e. $v = \pm\frac{\pi}{2}.$ Then $z = \pm c\sinh u$ which ranges from $-\infty$ to $+\infty$ as u does. The $z = 0$ plane is described by $u = 0$ or $v = 0$ or $v = \pi.$ These choices correspond to different regions for $\rho.$ But ρ is always positive, so we don't need $v = \pi.$ Thus $-\pi/2 \le v \le +\pi/2$, $0 \le u \le \infty$ and $0 \le w < 2\pi$ describes all of space.



This plot shows surfaces of constant u and constant v for $c = 1.$

14. An AC circuit contains a capacitor C in series with a coil with resistance R and inductance $L.$ The circuit is driven by an AC power supply with emf $\varepsilon = \varepsilon_0 \cos \omega t.$

(a) Use Kirchhoff's rules to write equations for the steady-state current in the circuit.

Loop rule:

$$\varepsilon_0 \cos \omega t = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

Charge conservation:

$$I = \frac{dQ}{dt}$$

(b) Using the fact that cos $\omega t = \mathrm{Re}\left(e^{i\omega t}\right)$, find the current through the power supply in the form:

$$I = \mathrm{Re}\left(\frac{\varepsilon_0}{Z}e^{i\omega t}\right)$$

where Z is the complex impedance of the circuit.

First write cos $\omega t = \mathrm{Re}\left(e^{i\omega t}\right)$ so the first equation becomes:

$$\mathrm{Re}\, \varepsilon_0 e^{i\omega t} = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

Now let $I = \mathrm{Re}\left(Ie^{i\omega t}\right)$. Then diffferentiate the loop equation with respect to time:

$$\begin{aligned} i\omega\varepsilon_0 &= R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} \\ i\omega\varepsilon_0 &= i\omega IR - \omega^2 LI + \frac{I}{C} \\ &= I\left(i\omega R - \omega^2 L + 1/C\right) \end{aligned}$$

Thus

$$I = \frac{\varepsilon_0}{R + i\omega L + \frac{1}{i\omega C}} = \frac{\varepsilon_0}{Z}$$

The complex impedance is:

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

(c) Use the result of (b) to find the amplitude and phase shift of the current. How much power is provided by the power supply? (Your answer should be the time-averaged power.)

Multiply top and bottom by the complex conjugate:

$$\begin{aligned}
I &= \mathrm{Re}\left(\varepsilon_0 \frac{R - i\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} e^{i\omega t} \right) \\
&= \frac{\varepsilon_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \left(R\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t \right) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \left(\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\omega t + \frac{\left(\omega L - \frac{1}{\omega C}\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\omega t \right) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} (\cos\phi\cos\omega t + \sin\phi\sin\omega t) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi)
\end{aligned}$$

Thus the amplitude is

$$I_0 = \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and the phase shift is:

$$\phi = \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

The time-averaged power is:

$$\begin{aligned}
P &= \langle I\varepsilon \rangle = \left\langle \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} (\cos\phi\cos\omega t + \sin\phi\sin\omega t)\varepsilon_0\cos\omega t \right\rangle \\
&= \frac{\varepsilon_0^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \frac{\cos\phi}{2} \\
&= \frac{1}{2} \frac{R\varepsilon_0^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}
\end{aligned}$$

(d) Show that the power is given by

$$P = \frac{1}{2}\mathrm{Re}(I\varepsilon^*)$$

$$P = \frac{1}{2}\mathrm{Re}(Z)\frac{\varepsilon_0^2}{|Z|^2} = \frac{1}{2}\mathrm{Re}\frac{Z|\varepsilon|^2}{|Z|^2} = \frac{1}{2}\mathrm{Re}\frac{Z\varepsilon\varepsilon^*}{ZZ^*} = \frac{1}{2}\mathrm{Re}\frac{Z^*\varepsilon\varepsilon^*}{ZZ^*}$$

since $\mathrm{Re}\,Z = \mathrm{Re}\,Z^*$. Then

$$P = \frac{1}{2}\mathrm{Re}\frac{\varepsilon^*}{Z^*}\varepsilon = \frac{1}{2}\mathrm{Re}\frac{\varepsilon}{Z}\varepsilon^*$$
$$= \frac{1}{2}\mathrm{Re}\,I^*\varepsilon = \frac{1}{2}\mathrm{Re}\,I\varepsilon^*$$

15. Small amplitude waves in a plasma are described by the relations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n_0 v) = 0$$
$$\varepsilon_0\frac{\partial E}{\partial x} = -en$$
$$\text{and } m\frac{\partial v}{\partial t} = -eE - m\nu v$$

where n_0, e, m, ν and ε_0 are constants. The constant ν is the collision frequency. Assume that n, E and v are all proportional to $\exp(ikx - i\omega t)$. Solve the equations for non-zero n, E and v to show that ω satisfies the equation:

$$\omega^2 + i\nu\omega = \frac{n_0 e^2}{m\varepsilon_0} \equiv \omega_p^2$$

and hence show that collisions damp the waves.

Putting in the exponential form, the equations become:

$$-i\omega n + ikn_0 v = 0$$
$$ik\varepsilon_o E = -en$$
$$\text{and } -i\omega m v = -eE - m\nu v$$

Use the second equation to eliminate E from the last:

$$-i\omega m v = -e\left(\frac{-en}{ik\varepsilon_0}\right) - m\nu v$$

and then use the first equation to eliminate n:

$$-i\omega m v = \left(\frac{e^2}{ik\varepsilon_0}\right)\left(\frac{kn_0 v}{\omega}\right) - m\nu v$$

Now we have an equation with v in every term. Either $v = 0$, a solution we are told to discard, or else:

$$-i\omega m = \frac{e^2}{i\varepsilon_0}\frac{n_0}{\omega} - m\nu$$
$$\omega^2 + i\omega\nu = \frac{n_0 e^2}{m\varepsilon_0} = \omega_p^2$$

which is the desired result. Now we solve this quadratic for ω

$$\omega = \frac{-i\nu \pm \sqrt{-\nu^2 + 4\omega_p^2}}{2}$$

With no collisions, $\nu = 0$, the solution is $\omega = \pm\omega_p$. With collisions, the real part of the frequency is slightly altered, but the important difference is the addition of the imaginary part $-i\nu/2$. The wave then has the form

$$\exp\left(ikx - it\left[\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}} - i\frac{\nu}{2}\right]\right) = \exp\left(ikx - it\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}}\right)\exp\left(i^2\frac{\nu}{2}t\right)$$
$$= \exp\left(ikx - it\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}}\right)\exp\left(-\frac{\nu}{2}t\right)$$

The real exponential shows that the wave amplitude decreases in time.

16. Write the real and imaginary parts u and v of the complex functions(a) $f = z^2 \sin z$ and (b) $f = \frac{1}{1+z}$. In each case, show that u and v obey the Cauchy-Riemann relations. Find the derivative df/dz first in terms of x and y, and then express the answer in terms of z. Is the result what you expected?

$$\begin{aligned} f &= (x+iy)^2 \sin(x+iy) \\ &= \left(x^2 + 2ixy + (iy)^2\right)(\sin x \cos iy + \cos x \sin iy) \\ &= \left(x^2 - y^2 + 2ixy\right)(\sin x \cosh y + i \cos x \sinh y) \end{aligned}$$

since

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = -\frac{\sinh y}{i} = i \sinh y$$

Thus

$$f = \left(x^2 - y^2\right) \sin x \cosh y - 2xy \cos x \sinh y + i\left(2xy \sin x \cosh y + \left(x^2 - y^2\right) \cos x \sinh y\right)$$

Thus

$$u = \left(x^2 - y^2\right) \sin x \cosh y - 2xy \cos x \sinh y$$

and

$$v = 2xy \sin x \cosh y + \left(x^2 - y^2\right) \cos x \sinh y$$

Then

$$\frac{\partial u}{\partial x} = 2x \sin x \cosh y + \left(x^2 - y^2\right) \cos x \cosh y - 2y \cos x \sinh y + 2xy \sin x \sinh y$$

while

$$\begin{aligned} \frac{\partial v}{\partial y} &= 2x \sin x \cosh y + 2xy \sin x \sinh y - 2y \cos x \sinh y + \left(x^2 - y^2\right) \cos x \cosh y \\ &= \frac{\partial u}{\partial x} \end{aligned}$$

So the first relation is satisfied.

Then

$$\frac{\partial u}{\partial y} = -2y \sin x \cosh y + \left(x^2 - y^2\right) \sin x \sinh y - 2x \cos x \sinh y - 2xy \cos x \cosh y$$

while

$$\begin{aligned} \frac{\partial v}{\partial x} &= 2y \sin x \cosh y + 2xy \cos x \cosh y + 2x \cos x \sinh y - \left(x^2 - y^2\right) \sin x \sinh y \\ &= -\frac{\partial u}{\partial y} \end{aligned}$$

and the second relation is also satisfied.

The derivative is

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= 2x \sin x \cosh y + (x^2 - y^2) \cos x \cosh y - 2y \cos x \sinh y + 2xy \sin x \sinh y \\ &\quad + i(2y \sin x \cosh y + 2xy \cos x \cosh y + 2x \cos x \sinh y - (x^2 - y^2) \sin x \sinh y) \\ &= 2(x+iy)(\sin x \cosh y + i \cos x \sinh y) + (x^2 - y^2 + 2ixy)(\cos x \cosh y - i \sin x \sinh y) \\ &= 2z \sin z + z^2 \cos z \end{aligned}$$

as expected.

(b) $f = \frac{1}{1+z} = \frac{1+x-iy}{(x+1)^2+y^2}$. Thus

$$u = \frac{1+x}{(x+1)^2+y^2};\ v = \frac{-y}{(x+1)^2+y^2}$$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{(x+1)^2+y^2} - \frac{2(1+x)^2}{\left((x+1)^2+y^2\right)^2} = \frac{(x+1)^2+y^2-2(1+x)^2}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2y(1+x)}{\left((x+1)^2+y^2\right)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2(1+x)y}{\left((x+1)^2+y^2\right)^2} = -\frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial y} = \frac{-1}{(x+1)^2+y^2} + \frac{2y^2}{\left((x+1)^2+y^2\right)^2} = \frac{2y^2-\left((x+1)^2+y^2\right)}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2} = \frac{\partial u}{\partial x}$$

So the CR relations are satisfied. Then the derivative is:

$$\frac{df}{dz} = \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2} + i\frac{2y(1+x)}{\left((x+1)^2+y^2\right)^2} = \frac{(y+i(1+x))^2}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{(iz^*+i)^2}{|z+1|^4} = -\frac{(z^*+1)^2}{(z+1)^2(z^*+1)^2} = -\frac{1}{(z+1)^2}$$

which is the expected result.

17. The variables x and y in a complex number $z = x+iy$ may be expressed in terms of z and its complex conjugate z^*:

$$x = \frac{1}{2}(z+z^*)$$
$$y = \frac{1}{2i}(z-z^*)$$

Show that the Cauchy-Riemann relations are equivalent to the condition

$$\frac{\partial f}{\partial z^*} \equiv 0.$$

We rewrite the derivatives using the chain rule. Suppose that $f = f(z,z^*)$. Then:

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right)$$
$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

If the Cauchy-Riemann relations are satisfied, both terms in square brackets are zero, and hence

$\frac{\partial f}{\partial z^*} \equiv 0,$ as required. This means that the function $f = f(z),$ and z^* does not appear.

18. One of the functions $u_1 = 2(x-y)^2$ and $u_2 = \frac{x^3}{3} - xy^2$ is the real part of an analytic function $w(z) = u + iv.$ Which is it? Find the function $v(x,y)$ and write w as a function of $z.$

Both the real and imaginary parts of an analytic fucntion satisfy the equation

$$\nabla^2 u = 0$$

so let's test the two functions:

$$\nabla^2 u_1 = 4\frac{\partial}{\partial x}(x-y) - 4\frac{\partial}{\partial y}(x-y) = 4 + 4 = 8 \neq 0$$

and

$$\nabla^2 u_2 = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2yx = 2x - 2x = 0$$

So the correct function is $u_2.$

Then from the C-R relations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = x^2 - y^2 \Rightarrow v = x^2 y - \frac{y^3}{3} + f(x)$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2xy \Rightarrow v = x^2 y + f(y)$$

Thus

$$v = x^2 y - \frac{y^3}{3}$$

Then

$$f = \frac{x^3}{3} + ix^2 y - xy^2 - i\frac{y^3}{3} = \frac{1}{3}(x+iy)^3 = \frac{z^3}{3}$$

19. A cylinder of radius a has potential V on one half and $-V$ on the other half. The potential inside the cylinder may be written as a series:

$$\Phi(r,\theta) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{r^2}{a^2}\right)^{2n+1}\frac{\sin[2(2n+1)\theta]}{2n+1}$$

Express each term in the sum as the imaginary part of a complex number, and hence sum the series. Show that the result may be expressed in terms of an inverse tangent.

$$\begin{aligned}\Phi(r,\theta) &= \frac{4V}{\pi}\sum_{n=0}^{\infty}\mathrm{Im}\left(\frac{z^2}{a^2}\right)^{2n+1}\frac{1}{2n+1} \\ &= \frac{4V}{\pi a^2}\sum_{n=0}^{\infty}\mathrm{Im}\int_0^{z^2}\left(\frac{w}{a^2}\right)^{2n}dw \\ &= \frac{4V}{\pi a^2}\mathrm{Im}\int_0^{z^2}\sum_{n=0}^{\infty}\left(\frac{w}{a^2}\right)^{2n}dw\end{aligned}$$

The sum may be recognized as the geometric series (2.43)

$$\Phi(r,\theta) = \frac{4V}{\pi a^2}\mathrm{Im}\int_0^{z^2}\frac{1}{1-\left(w/a^2\right)^2}dw$$

To do the integral, let $w/a^2 = \sin\phi;\ dw/a^2 = \cos\phi\, d\phi.$

$$\begin{aligned}\Phi(r,\theta) &= \frac{4V}{\pi}\operatorname{Im}\int_0^{\sin^{-1}z^2/a^2}\frac{\cos\phi}{1-\sin^2\phi}d\phi\\ &= \frac{4V}{\pi}\operatorname{Im}\int_0^{\sin^{-1}z^2/a^2}\sec\phi\, d\phi\\ &= \frac{4V}{\pi}\operatorname{Im}\left[\ln(\sec\phi+\tan\phi)\Big|_0^{\sin^{-1}z^2/a^2}\right]\\ &= \frac{4V}{\pi}\operatorname{Im}\left\{\ln\left(\frac{1+z^2/a^2}{\sqrt{1-z^4/a^4}}\right)\right\}\\ &= \frac{4V}{\pi}\operatorname{Im}\left\{\ln\sqrt{\frac{1+z^2/a^2}{1-z^2/a^2}}\right\}\end{aligned}$$

Now the logarithm is

$$\ln\frac{1+z^2/a^2}{1-z^2/a^2} = \ln\left|\frac{1+z^2/a^2}{1-z^2/a^2}\right| + i\arg\left(\frac{1+z^2/a^2}{1-z^2/a^2}\right)$$

and thus

$$\Phi(r,\theta) = \frac{2V}{\pi}\arg\frac{1+z^2/a^2}{1-z^2/a^2}$$

Next we find the argument:

$$\begin{aligned}\frac{1+z^2/a^2}{1-z^2/a^2} &= \frac{a^2+r^2\cos 2\theta+ir^2\sin 2\theta}{a^2-r^2\cos 2\theta-ir^2\sin 2\theta}\\ &= \frac{\left(a^2+r^2\cos 2\theta+ir^2\sin 2\theta\right)\left(a^2-r^2\cos 2\theta+ir^2\sin 2\theta\right)}{\left(a^2-r^2\cos 2\theta\right)^2+r^4\sin^2 2\theta}\\ &= \frac{a^4-r^4+2ia^2r^2\sin 2\theta}{a^4-2a^2r^2\cos 2\theta+r^4} = Ae^{i\alpha}\end{aligned}$$

where

$$\tan\alpha = \frac{2a^2r^2\sin 2\theta}{a^4-r^4}$$

and thus

$$\Phi(r,\theta) = \frac{2V}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{a^4-r^4}$$

20. The function $f = \sin\left(z^2\right)$ (cf Example 2.10) also has a zero at $z = \sqrt{\pi}$. What is its order?

To find the order of the zero, we write the Taylor series centered at $\sqrt{\pi}$.

$$\left.\frac{df}{dz}\right|_{\sqrt{\pi}} = 2z\cos z^2\Big|_{\sqrt{\pi}} = 2\sqrt{\pi}\cos\pi = -2\sqrt{\pi}$$

Thus the series is

$$f(z) = -2\sqrt{\pi}\left(z-\sqrt{\pi}\right)+\cdots$$

and the zero is of order 1.

21. Find the Taylor series for the following functions about the point specified:

(a) $z\cos z$ about $z = 0$

The series is z times the cosine series, i.e.

$$\begin{aligned} z\cos z &= z\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right) \\ &= z - \frac{z^3}{2} + \frac{z^5}{5!} + \cdots \end{aligned}$$

(b) $\ln(1+z)$ about $z = 0$

At $z = 0,\ f(z) = \ln(1) = 0$

The derivative is

$$\frac{df}{dz} = \frac{1}{1+z} = 1 \text{ at } z = 0$$

The 2nd derivative is

$$\frac{d^2f}{dz^2} = -\frac{1}{(1+z)^2} = -1 \text{ at } z = 0$$

The 3nd derivative is

$$\frac{d^3f}{dz^3} = \frac{2}{(1+z)^3} = 2 \text{ at } z = 0$$

So the series is:

$$\begin{aligned} \ln(1+z) &= z - \frac{z^2}{2!} + \frac{2}{3!}z^3 + \cdots \\ &= z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots \end{aligned}$$

The radius of convergence is $1,$ since $\ln(1+z)$ has a branch point at $z = -1.$

(c) $\frac{\sin z}{z}$ about $z = \pi/2$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi/2} = \frac{2}{\pi}$$

The derivative is

$$\frac{df}{dz} = \frac{\cos z}{z} - \frac{\sin z}{z^2} = -\left(\frac{2}{\pi}\right)^2 \text{at } z = \frac{\pi}{2}$$

The 2nd derivative is

$$\frac{d^2f}{dz^2} = \frac{-\sin z}{z} - 2\frac{\cos z}{z^2} + 2\frac{\sin z}{z^3} = -\left(\frac{2}{\pi}\right) + 2\left(\frac{2}{\pi}\right)^3 \text{at } z = \frac{\pi}{2}$$

So the series is:

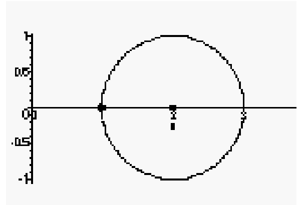
$$\frac{\sin z}{z} = \frac{2}{\pi} - \left(\frac{2}{\pi}\right)^2\left(z - \frac{\pi}{2}\right) + \frac{1}{\pi}\left[2\left(\frac{2}{\pi}\right)^2 - 1\right]\left(z - \frac{\pi}{2}\right)^2 + \cdots$$

The radius of convergence is $\infty,$ since the function has no singularities (other than the removable singularity at $z = 0.$)

(d) $\frac{1}{z^2-1}$ about $z = 2.$

First factor the denominator:

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$



There are poles at $z = \pm 1$. Now let $w = z - 2$:

$$f(z) = \frac{1}{2}\left(\frac{1}{w+1} - \frac{1}{w+3}\right)$$

Expand each term in a geometric series:

$$\begin{aligned}
f(z) &= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3}\left(\frac{1}{1+\frac{w}{3}}\right)\right) \\
&= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3}\left(1 - \frac{w}{3} + \left(\frac{w}{3}\right)^2 + \cdots\right)\right) \\
&= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3} + \frac{w}{9} - \frac{w^2}{3^3} + \cdots\right) \\
&= \frac{1}{2}\left(\frac{2}{3} - \frac{8w}{9} + w^2\left(1 - \frac{1}{3^3}\right) - w^3\left(1 - \frac{1}{3^4}\right) + \cdots\right) \\
&= \frac{1}{3} - \frac{4}{9}(z-2) + \frac{1}{2}\left(1 - \frac{1}{3^3}\right)(z-2)^2 - \frac{1}{2}\left(1 - \frac{1}{3^4}\right)(z-2)^3 + \cdots
\end{aligned}$$

The radius of convergence is 1, since $f(z)$ has a pole at $z = 1$.

22. Determine the Taylor or Laurent series for each of the following functions about the point specified:

(a) $\frac{\cos z}{z-1}$ about $z = 1$

The function has a pole at $z = 1$, so the series is a Laurent series.

First find the Taylor series for $\cos z$:

$$\cos z = \cos 1 - \sin 1(z-1) - \frac{\cos 1}{2}(z-1)^2 + \cdots$$

The general term is

$$\begin{aligned}
\frac{d^m}{dz^m}\cos z\Big|_{z=1}\frac{(z-1)^m}{m!} &= (-1)^{m/2}\frac{(z-1)^m}{m!}\cos 1 \text{ for } m \text{ even} \\
&= (-1)^{(m+1)/2}\frac{(z-1)^m}{m!}\sin 1 \text{ for } m \text{ odd}
\end{aligned}$$

and thus

$$\frac{\cos z}{z-1} = \cos 1\sum_{m=0}^{\infty}(-1)^m\frac{(z-1)^{2m}}{2m!} + \sin 1\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(z-1)^{2m+1}}{(2m+1)!}$$

The radius of convergence is infinite, since the function has no other poles or singularities.

(b) $\frac{\sin z^2}{z}$ about $z = 0$

The function is analytic at

$z = 0$ (there is a removable singularity) so the series is a Taylor series. We start with the series for $\sin z^2$:

$$\begin{aligned}\frac{\sin z^2}{z} &= \frac{1}{z}\left(z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots\right)\\ &= z - \frac{z^5}{3!} + \frac{z^9}{5!} + \cdots\end{aligned}$$

The radius of convergence is infinite, since the function has no poles or other singularities

(c) $\frac{e^z}{z-i\pi}$ about $z = i\pi$

There is a simple pole at $z = i\pi$: the series is a Laurent series:

$$\begin{aligned}\frac{e^z}{z-i\pi} &= e^{i\pi}\frac{e^{z-i\pi}}{z-i\pi}\\ &= -\frac{e^w}{w} = -\frac{1}{w}\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right)\\ &= -\frac{1}{w} - 1 - \frac{w}{2} - \frac{w^2}{6} - \cdots \text{where } w = z - i\pi\end{aligned}$$

The radius of convergence is infinite, since the function has no other poles or singularities.

(d) $\frac{\ln z}{z-1}$ about $z = 1$

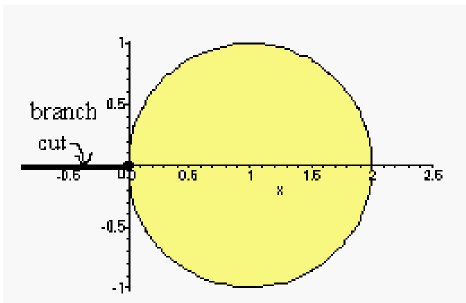
Th function has a branch point at $z = 0$. The singularity at $z = 1$ is removable, since $\ln z$ has a zero at $z = 1$. We should be able to find a Taylor series valid for $0 < |z-1| < 1$.

First find the Taylor series for $\ln z$. Let $w = z - 1$

$$\ln(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} + \cdots$$

So

$$\begin{aligned}\frac{\ln z}{z-1} &= \frac{1}{z-1}\left(z - 1 - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \cdots\right)\\ &= 1 - \frac{z-1}{2} + \frac{(z-1)^3}{3} + \cdots\end{aligned}$$



(e) $\tan^{-1}(z) = w$

$$z = \tan w = \frac{e^{iw} - e^{-iw}}{i\left(e^{iw} + e^{-iw}\right)}$$

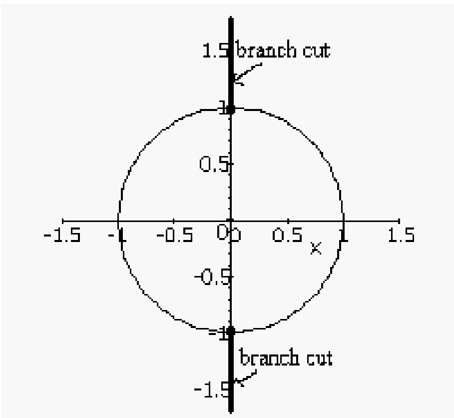
$$\begin{aligned}iz\left(e^{iw} + e^{-iw}\right) &= e^{iw} - e^{-iw}\\ e^{2iw}(iz-1) + iz + 1 &= 0\end{aligned}$$

So

$$w = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right)$$

There are branch points at $z = \pm i$. There is a Taylor series valid for $|z| < 1$.

$$\begin{aligned} w &= \frac{1}{2i}\left[iz - \frac{1}{2}\left(-z^2\right) + \cdots - (z \to -z)\right] \\ &= z + \frac{2}{3i}(iz)^3 + \frac{2}{5i}(iz)^5 + \cdots \\ &= z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots \end{aligned}$$



Problem 22

23. Determine *all* Taylor or Laurent series about the specified point for each of the following functions.

(a) $\frac{e^z}{z^2+1}$ about the origin.

The function is analytic about the origin, so there is a Taylor series. The function has poles at

$z = \pm i$, so the Taylor series is valid for $|z| < 1$. There is a Laurent series valid for $|z| > 1$.

Taylor series:

$$\begin{aligned} \frac{e^z}{z^2+1} &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right) \\ &= 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \frac{13}{24}z^4 + \frac{101}{120}z^5 - \frac{389}{720}z^6 + \cdots \text{ for } |z| < 1 \\ &= = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m \frac{z^{n+2m}}{n!} \end{aligned}$$

Laurent series:

$$\begin{aligned} \frac{e^z}{z^2+1} &= \frac{e^z}{z^2\left(1 + \frac{1}{z^2}\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m=0}^{\infty}(-1)^m\frac{1}{z^{2m}} \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^{n-2m}}{n!}(-1)^m \end{aligned}$$

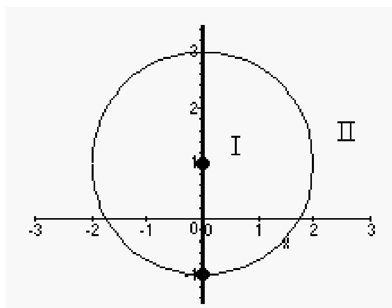
We may simplify the negative powers as follows:

$$\begin{aligned}\frac{e^z}{z^2+1} &= \frac{e^z}{z^2\left(1+\frac{1}{z^2}\right)} = \frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}\cdots\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)\\ &= \cdots\frac{1}{z^5}\left(1-\frac{1}{3!}+\cdots\right)+\frac{1}{z^4}\left(-1+\frac{1}{2}+-\frac{1}{4!}\cdots\right)+\frac{1}{z^3}\left(1-\frac{1}{3!}+\frac{1}{5!}+\cdots\right)\\ &\quad -\frac{1}{z^2}\left(1-\frac{1}{2}+\frac{1}{4!}+\cdots\right)\\ &\quad -\frac{1}{z}\left(1-\frac{1}{3!}+\cdots\right)+\frac{1}{2}-\frac{1}{4!}+\cdots+z\left(\frac{1}{3!}-\frac{1}{5!}+\right)+z^2\left(\frac{1}{4!}+\cdots\right)+z^3\left(\frac{1}{5!}+\cdots\right)+\cdots\\ &= \cos 1\sum_{n=1}^{\infty}\frac{(-1)^n}{z^{2n}}+\sin 1\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+1}}+\sum_{n=2m+2}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{z^{n-2m-2}}{n!}\end{aligned}$$

valid for $|z| > 1.$

(b) $\frac{1}{z^2+1}$ about $z = i$

The function has simple poles at $z = \pm i,$ so we can find a Laurent series valid for $0 < |z-i| < 2$ and another valid for $|z-i| > 2$.



$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)$$

Let $w = z - i$

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{w}-\frac{1}{w+2i}\right)$$

In Region I, expand the second term in a geometric series:

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{2i}\frac{1}{w}-\frac{1}{(2i)^2}\frac{1}{(1+w/2i)}\\ &= \frac{1}{2iw}+\frac{1}{4}\left(1-\frac{w}{2i}+\left(\frac{w}{2i}\right)^2-\left(\frac{w}{2i}\right)^3+\cdots\right)\\ &= -\frac{i}{2w}+\frac{1}{4}\left(1+\frac{iw}{2}-\frac{w^2}{4}-\frac{w^3}{8}i+\cdots\right)\\ &= \frac{1}{4}\left(1-\frac{(z-i)^2}{4}+\cdots\right)-\frac{i}{2}\left(\frac{1}{z-i}+\frac{z-i}{4}-\frac{(z-i)^3}{16}+\cdots\right)\end{aligned}$$

which is valid for $0 < |z-i| < 2.$

In the outer region (II) we expand the other way:

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{2i}\left(\frac{1}{w} - \frac{1}{w}\frac{1}{(1+2i/w)}\right)\\ &= \frac{1}{2iw}\left[1-\left(1-\frac{2i}{w}+\left(\frac{2i}{w}\right)^2-\left(\frac{2i}{w}\right)^3+\cdots\right)\right]\\ &= \frac{1}{2iw}\left[\frac{2i}{w}-\left(\frac{2i}{w}\right)^2+\left(\frac{2i}{w}\right)^3+\cdots\right]\\ &= \frac{1}{w^2}-\frac{2i}{w^3}-\frac{4}{w^4}+\cdots\\ &= \frac{1}{(z-i)^2}-\frac{2i}{(z-i)^3}-\frac{4}{(z-i)^4}+\cdots\end{aligned}$$

which is valid for $|z-i|>2.$

(c) $\frac{z}{z^2-9}$ about $z=3$

The function has poles at $z=\pm 3.$ We should be able to find a Laurent series valid for $0<|z-3|<6$ and another for $|z-3|>6.$

$$\frac{z}{z^2-9} = \frac{z}{(z-3)(z+3)} = \frac{w+3}{w(w+6)}$$

where $w=z-3.$ Then for $|w|<6$ we have:

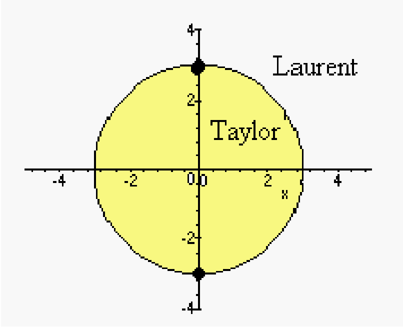
$$\begin{aligned}\frac{z}{z^2-9} &= \frac{w+3}{w}\frac{1}{6\left(1+\frac{w}{6}\right)}\\ &= \frac{w+3}{6w}\left(1-\frac{w}{6}+\frac{w^2}{6^2}-\frac{w^3}{6^3}+\cdots\right)\\ &= \left(\frac{1}{6}+\frac{1}{2w}\right)\left(1-\frac{w}{6}+\frac{w^2}{6^2}-\frac{w^3}{6^3}+\cdots\right)\\ &= \frac{1}{2w}+\frac{1}{12}-\frac{1}{72}w+\frac{1}{432}w^2-\frac{1}{1296}w^3+\cdots\\ &= \frac{1}{2(z-3)}+\frac{1}{12}-\frac{(z-3)}{72}+\frac{(z-3)^2}{432}+\cdots\end{aligned}$$

while for $|w|>6$

$$\begin{aligned}\frac{z}{z^2-9} &= \frac{w+3}{w}\frac{1}{w\left(1+\frac{6}{w}\right)}\\ &= \frac{w+3}{w^2}\left(1-\frac{6}{w}+\frac{6^2}{w^2}-\frac{6^3}{w^3}+\cdots\right)\\ &= \frac{1}{w}-\frac{3}{w^2}+\frac{18}{w^3}-\frac{108}{w^4}+\cdots\\ &= \frac{1}{z-3}-\frac{3}{(z-3)^2}+\frac{18}{(z-3)^3}-\frac{108}{(z-3)^4}+\cdots\end{aligned}$$

(d) $\frac{1}{z^2+9}$ about the origin.

The function has poles at $z=\pm 3i,$ so there is a Taylor series valid for $|z|<3$ and a Laurent series valid for $|z|>3.$



$$\begin{aligned}\frac{1}{z^2+9} &= \frac{1}{6i}\left(\frac{1}{z-3i} - \frac{1}{z+3i}\right)\\ &= -\frac{1}{6i}\frac{1}{3i}\left(\frac{1}{1-z/3i} + \frac{1}{1+z/3i}\right)\\ &= \frac{1}{18}\left(1 + \frac{z}{3i} + \left(\frac{z}{3i}\right)^2 + \left(\frac{z}{3i}\right)^3 + \cdots + 1 - \frac{z}{3i} + \left(\frac{z}{3i}\right)^2 - \left(\frac{z}{3i}\right)^3 + \cdots\right)\\ &= \frac{1}{9}\left(1 - \frac{z^2}{9} + \frac{z^4}{3^4} - \cdots\right)\end{aligned}$$

while for $|z| > 3$

$$\begin{aligned}\frac{1}{z^2+9} &= \frac{1}{6iz}\left(\frac{1}{1-3i/z} - \frac{1}{1+3i/z}\right)\\ &= \frac{1}{6iz}\left(1 + \frac{3i}{z} + \left(\frac{3i}{z}\right)^2 + \left(\frac{3i}{z}\right)^3 + \cdots - \left(1 - \frac{3i}{z} + \left(\frac{3i}{z}\right)^2 - \left(\frac{3i}{z}\right)^3 + \cdots\right)\right)\\ &= \frac{1}{3iz}\left(\frac{3i}{z} + \left(\frac{3i}{z}\right)^3 + \left(\frac{3i}{z}\right)^5 + \cdots\right)\\ &= \frac{1}{z^2} - \frac{9}{z^4} + \frac{3^4}{z^6} + \cdots\end{aligned}$$

24. Find all the singularities of each of the following functions, and describe each of them completely.

(a) $\frac{e^z}{z} - \sin\frac{1}{z}$

Expand out each term in a series:

$$\begin{aligned}\frac{e^z}{z} - \sin\frac{1}{z} &= \frac{1 + z + z^2/2 + \cdots}{z} - \left(\frac{1}{z} - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots\right)\\ &= 1 + \frac{z}{2} + \frac{z^2}{3!} + \cdots + \frac{1}{3!z^3} - \frac{1}{5!z^5} + \cdots\end{aligned}$$

This is a Laurent series with infinitely many negative powers, and it is valid up to the singularity at $z = 0$, so the function has an essential singularity at $z = 0$.

(b) $\frac{\cos z}{z} - \frac{\sin z}{z^2}$

Let's look at the series for this function about the origin:

$$\begin{aligned}\frac{\cos z}{z} - \frac{\sin z}{z^2} &= \frac{1 - z^2/2 + z^4/4! + \cdots}{z} - \frac{z - z^3/3! + z^5/5! - \cdots}{z^2}\\ &= -\frac{z}{2} - \frac{z}{6} + \frac{z^3}{4!} - \frac{z^3}{5!} + \cdots\\ &= -\frac{2}{3}z + \frac{z^3}{30} + \cdots\end{aligned}$$

This is a Taylor series valid for all z. Thus the function has a removable singularity at $z = 0$.

(c) $\frac{\tanh z}{z}$

The function has a removable singularity at $z = 0$:

$$\lim_{z\to 0}\frac{\tanh z}{z} = \lim_{z\to 0}\frac{\operatorname{sech}^2 z}{1} = 1$$

But the tanh function also has singularities regularly spaced along the imaginary axis.

$$\tanh iy = \frac{e^{-y} - e^{y}}{i(e^{-y} + e^{y})} = i\tan y$$

and $\tan y$ has singularities at $y = (2n+1)\pi/2.$ The singularities are all simple poles. For example

$$\begin{aligned}\cosh iy &= \cos y = -\sin(y - \pi/2)\\ &= -\left((y-\pi/2) - \frac{1}{3!}(y-\pi/2)^3 + \cdots\right)\end{aligned}$$

$$\begin{aligned}\lim_{z\to i\pi/2}\left(z - i\frac{\pi}{2}\right)\frac{\tanh z}{z} &= \lim_{z\to i\pi/2}\frac{\left(z - i\frac{\pi}{2}\right)\sinh z}{z\cosh z}\\ &= \lim_{z\to i\pi/2}\frac{\left(z - i\frac{\pi}{2}\right)\sinh z}{-z\left(-i(z - i\pi/2) - \frac{i}{3!}(z - i\pi/2)^3 + \cdots\right)}\\ &= \lim_{z\to i\pi/2}\frac{\sinh z}{iz\left(1 + \frac{1}{3!}(z - i\pi/2)^2 + \cdots\right)}\\ &= \frac{i}{i(i\pi/2)} = -\frac{2}{\pi}i\end{aligned}$$

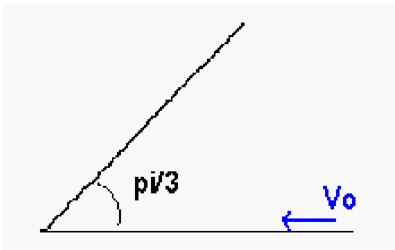
Since the limit exists, the pole is simple.

(d) $\ln\left(1 + z^2\right)$

The function has a branch point where $1 + z^2 = 0,$ or $z = \pm i.$

25. Incompressible fluid flows over a thin sheet from a distance

X_0 into a corner as shown in the diagram. The angle between the barriers is $\pi/3,$ and at $x = X_0,$ $\vec{v} = V_0\hat{x}.$ Assuming that the flow is as simple as possible, determine the streamlines of the flow. What is the velocity at $r = X_0/3, \theta = \pi/6?$



The velocity potential satsifies

$$\nabla^2\phi = 0$$

and thus we may look for a complex potential $\Phi = \phi + i\psi.$ Φ must be an analytic function in the region $\leq \theta \leq \pi/3$, and at $x = X_0$ we need

$$-\vec{\nabla}\phi = -V_0\hat{x}$$

The streamline function must be a constant on the surfaces $\theta = 0$ and

$\theta = \pi/3.$ We may take this constant to be zero, and then the function $\sin 3\theta$ does the job. (The function $\sin 3n\theta$ would also work, but would lead to more complicated flow.) This suggests that we look at the analytic function

$kz^3 = kr^3 e^{3i\theta} = kr^3(\cos 3\theta + i \sin 3\theta).$ The imaginary part of this function satisfies the boundary conditions at the two surfaces. Thus the streamlines are given by

$$\psi = kr^3 \sin 3\theta = \text{ constant}$$

and the velocity is given by

$$\begin{aligned}
\vec{\mathbf{v}} &= -\vec{\nabla}\phi = -\vec{\nabla}\left(kr^3 \cos 3\theta\right) \\
&= -3kr^2 \cos 3\theta \hat{\mathbf{r}} + 3kr^2 \sin 3\theta \hat{\boldsymbol{\theta}} \\
&= -3kr^2[\cos 3\theta(\hat{\mathbf{x}} \cos\theta + \hat{\mathbf{y}} \sin\theta) + \sin 3\theta(-\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta)] \\
&= 3kr^2[-\hat{\mathbf{x}}(\sin 3\theta \sin\theta + \cos 3\theta \cos\theta) - \hat{\mathbf{y}}(\cos 3\theta \sin\theta - \sin 3\theta \cos\theta)] \\
&= 3kr^2[-\hat{\mathbf{x}}(\cos 2\theta) + \hat{\mathbf{y}}(\sin 2\theta)]
\end{aligned}$$

Thus at $\theta = 0$ we have

$$\vec{\mathbf{v}} = -3kr^2 \hat{\mathbf{x}}$$

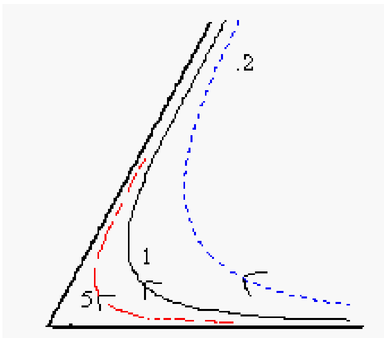
and so

$$3kX_0^2 = V_0$$

Thus the streamlines are given by

$$\begin{aligned}
\psi &= \frac{V_0 r^3}{3X_0^2} \sin 3\theta = \text{ constant } = C \\
r &= \left(\frac{3X_0^2 C}{V_0 \sin 3\theta}\right)^{1/3}
\end{aligned}$$

See Figure. $\frac{3X_0^2 C}{V_0} = 1$ (solid line), 5 (dashes), and 1/5 (dots).



The velocity is

$$\frac{\vec{\mathbf{v}}}{V_0} = \frac{r^2}{3X_0^2}(\hat{\mathbf{x}} \cos 2\theta + \hat{\mathbf{y}} \sin 2\theta)$$

and so at $r = X_0/2,$ $\theta = \pi/6$ we have

$$\begin{aligned}
\frac{\vec{\mathbf{v}}}{V_0} &= \hat{\mathbf{x}}\frac{1}{12} \sin\frac{\pi}{3} + \hat{\mathbf{y}}\frac{1}{12} \cos\frac{\pi}{3} \\
\vec{\mathbf{v}} &= \frac{V_0}{24}\left(\sqrt{3}\,\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)
\end{aligned}$$

26. Prove the Schwarz reflection principle: If a function $f(z)$ is analytic in a region including the real axis, and $f(x)$ is real when x is real,

$$f^*(z) = f(z^*)$$

Show that the result may be extended to functions that posess a Laurent series about the origin with real coefficients.

Verify the result for the functions (a) $f(z) = \cos z$ and (b) $f(z) = \tan^{-1}(z)$.

(c) Show that the result does not hold for all z if $f(z) = \ln(z)$ (the principal branch is assumed).

If the function is analytic, it may be expanded in a Taylor series about a point x_0 on the real axis:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$$

and since

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is real, then each of the a_n must be real. Then

$$\begin{aligned} f^*(z) &= \sum_n a_n^* (z - x_0)^n = \sum_n a_n \left(r^n e^{in\theta}\right)^* \\ &= \sum_n a_n r^n e^{-in\theta} = \sum_n a_n \left(re^{-i\theta}\right)^n = f(z^*) \end{aligned}$$

The proof extends trivially to the case where the series is a Laurent series with real coefficients.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}$$

So

$$(\cos z)^* = \frac{e^{-ix}e^{-y} + e^{ix}e^{y}}{2} = \frac{e^{-i(x-iy)} + e^{i(x-iy)}}{2} = \cos(z^*)$$

The function $w = \tan^{-1}(z)$ is trickier.

$$\tan w = z = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1}$$

Thus

$$\begin{aligned} iz\left(e^{2iw} + 1\right) &= e^{2iw} - 1 \\ e^{2iw} &= \frac{1 + iz}{1 - iz} \end{aligned}$$

and, choosing the principal branch of the logarithm,

$$w = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}[\ln(1+iz) - \ln(1-iz)]$$
$$= \frac{1}{2i}\left(\ln\sqrt{(1-y)^2+x^2} + i\tan^{-1}\frac{x}{1-y} - \ln\sqrt{(1+y)^2+x^2} - i\tan^{-1}\frac{-x}{1+y}\right)$$
$$= \frac{1}{2}\left(-i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y} + i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y}\right)$$

Then

$$f^*(z) = \frac{1}{2}\left(i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y} - i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y}\right)$$

and

$$f(z^*) = \frac{1}{2}\left(-i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y} + i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y}\right)$$

and the two expressions are the same.

Note that this function has branch points at $z = \pm i$, but it is analytic on the real axis.

(c)

$$\ln z = \ln r + i\theta$$

We proceed by showing that the relation fails at one point, $z = -1$. At $z = -1$, on the real axis,

$$\ln z = i\pi$$

Then

$$(\ln z)^* = -i\pi$$

but

$$\ln(z^*) = \ln(z) = i\pi$$

27. Find the residues of each of the following functions at the point specified.

(a) $\frac{z-2}{z^2-1}$ at $z = 1$

First factor the function:

$$\frac{z-2}{z^2-1} = \frac{z-2}{(z+1)(z-1)}$$

The function has a simple pole at $z = 1$ and the residue is:

$$\lim_{z\to 1}(z-1)\frac{z-2}{(z+1)(z-1)} = \lim_{z\to 1}\frac{z-2}{z+1} = \boxed{-\frac{1}{2}}$$

(b) $\exp\left(\frac{1}{z} - 1\right)$ at $z = 0$

First rewrite the function:

$$\exp\left(\frac{1}{z} - 1\right) = e^{-1}\exp\left(\frac{1}{z}\right)$$

and then expand in a Laurent series:

$$f(z) = \frac{1}{e}\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \cdots\right)$$

Now we can pick out the residue: it is the coefficient of $1/z.$ The residue is

$1/e$

(c) $\frac{\sin z}{z^2}$ at the origin

The easiest method here is to find the Laurent series:

$$\begin{aligned}\frac{\sin z}{z^2} &= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) \\ &= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots\end{aligned}$$

and thus the residue is

1
.

(d) $\frac{\cos z}{1/2 - \sin z}$ at $z = \pi/6$

Since the denominator is a function $h(z) = 1/2 - \sin z$ that has a simple zero at $z = \pi/6,$ we can use method 4. The derivative is

$$h'(z) = -\cos z$$

and so

$$\operatorname{Res} = \lim_{z\to\pi/6} \frac{\cos z}{-\cos z} = \boxed{-1}$$

28. Evaluate the following integrals:

(a) $\oint_C \frac{\cos z}{z} dz$ where C is a circle of radius 2 centered at the origin.

The integrand has a simple pole at $z = 0,$ which is inside the circle. The residue there is:

$$\lim_{z\to 0} \cos z = 1$$

and thus

$$\oint_C \frac{\cos z}{z} dz = 2\pi i$$

(b) $\oint_C \frac{\sinh z}{z-1} dz$ where C is a square of side 4 centered at the origin.

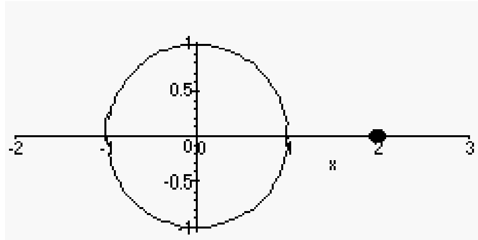
The integrand has a simple pole at $z = 1,$ which is inside the square. The residue there is:

$$\lim_{z\to 1} \sinh z = \sinh 1$$

and thus

$$\oint_C \frac{\sinh z}{z-1} dz = 2\pi i \sinh 1 = \pi i\left(e - \frac{1}{e}\right)$$

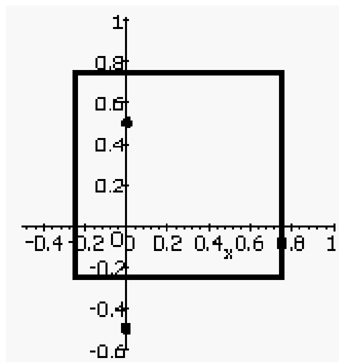
(c) $\oint_C \frac{z-1}{z+2}dz$ where C is a circle of radius 1 centered at the origin.



The integrand has a pole at $z = -2$, which is outside the circle. Thus:

$$\oint_C \frac{z-1}{z+2}dz = 0$$

(d) $\oint_C \frac{z}{4z^2+1}dz$ where C is a square of side 1 centered at the point $z = (1+i)/4$.



The integrand has two simple poles, at $z = \pm i/2$. Only one, at $z = i/2$, is inside the square. The residue at $z = i/2$ is

$$\lim_{z\to i/2}\left(z-\frac{i}{2}\right)\frac{z}{(2z-i)(2z+i)} = \lim_{z\to i/2}\frac{z}{2(2z+i)} = \frac{i}{4(2i)} = \frac{1}{8}$$

and so

$$\oint_C \frac{z}{4z^2+1} = 2\pi i\frac{1}{8} = \boxed{\frac{\pi i}{4}}$$

29. Evaluate the following integrals:

(a) $\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}d\theta$

We evaluate as an ingtegral around the unit circle. Let $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

and

$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

and

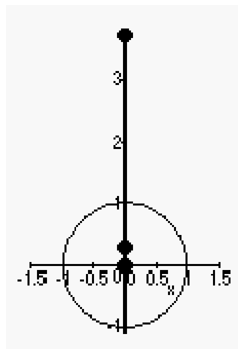
$$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Then

$$\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}d\theta = \oint_c \frac{1+\frac{1}{2}\left(z+\frac{1}{z}\right)}{2-\frac{1}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz}$$
$$= \oint_c \frac{2+\left(z+\frac{1}{z}\right)}{4i-\left(z-\frac{1}{z}\right)}\frac{dz}{z}$$
$$= \oint_c \frac{2z+\left(z^2+1\right)}{4iz-\left(z^2-1\right)}\frac{dz}{z}$$
$$= -\oint_c \frac{z^2+2z+1}{z^2-4iz-1}\frac{dz}{z}$$

The integrand has poles at $z = 0$ and

$$z = \frac{4i \pm \sqrt{-16+4}}{2} = 2i \pm i\sqrt{3}$$



Only the poles at $z = 0$ and $z = \left(2-\sqrt{3}\right)i$ are inside the circle. The residues at these poles are -1 and

$$\lim_{z\to\left(2-\sqrt{3}\right)i} \frac{(z+1)^2}{z}\frac{1}{z-\left(2+\sqrt{3}\right)i} = \frac{\left(\left(2-\sqrt{3}\right)i+1\right)^2}{\left(2-\sqrt{3}\right)i}\frac{1}{\left(2-\sqrt{3}\right)i-\left(2+\sqrt{3}\right)i}$$
$$= \frac{3-2\sqrt{3}-\left(2-\sqrt{3}\right)i}{\left(2-\sqrt{3}\right)i}\frac{1}{i\sqrt{3}}$$
$$= 1+i\frac{\sqrt{3}}{3}$$

So the integral is:

$$\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}d\theta = -2\pi i\left(-1+1+i\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}\pi$$

(b) $\int_0^{\pi} \frac{\sin^2\theta}{1+\cos^2\theta}d\theta$

Let $2\theta = \phi$. Then: $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, so

$$\int_0^{\pi} \frac{\frac{1}{2}(1-\cos 2\theta)}{1+\frac{1}{2}(\cos 2\theta + 1)}d\theta = \int_0^{2\pi} \frac{1-\cos\phi}{3+\cos\phi}\frac{d\phi}{2}$$

$$\frac{1}{2}\int_0^{2\pi}\frac{1-\cos\phi}{3+\cos\phi}d\phi = \frac{1}{2}\oint_{\text{unit circle}}\frac{1-\frac{1}{2}\left(z+\frac{1}{z}\right)}{3+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$
$$= \frac{1}{2i}\oint_{\text{unit circle}}\frac{2z-\left(z^2+1\right)}{6z+\left(z^2+1\right)}\frac{dz}{z}$$

The integrand has poles where

$$z = \frac{-6\pm\sqrt{36-4}}{2}$$
$$= -3\pm\sqrt{8} = -3\pm 2\sqrt{2} = -.17157, -5.8284$$

Only one of these poles is inside the unit circle. There is an additional pole at $z = 0.$ The residues are -1 and:

$$\lim_{z\to -3+2\sqrt{2}}\frac{-(z-1)^2}{z\left(z-\left(-3-2\sqrt{2}\right)\right)} = \frac{-\left(-3+2\sqrt{2}-1\right)^2}{\left(-3+2\sqrt{2}\right)\left(-3+2\sqrt{2}-\left(-3-2\sqrt{2}\right)\right)}$$
$$= \frac{-\left(-4+2\sqrt{2}\right)^2}{\left(-3+2\sqrt{2}\right)\left(4\sqrt{2}\right)} = \frac{-2\left(3-2\sqrt{2}\right)}{\left(-3+2\sqrt{2}\right)\sqrt{2}}$$
$$= \sqrt{2}$$

Thus the integral is:

$$\int_0^{\pi}\frac{\sin^2\theta}{1+\cos^2\theta}d\theta = \frac{1}{2i}2\pi i\left(-1+\sqrt{2}\right) = \pi\left(\sqrt{2}-1\right) = 1.3013$$

(c) $\int_0^{2\pi}\frac{1}{1+\sin^2\theta}d\theta$

$$\int_0^{2\pi}\frac{1}{1+\sin^2\theta}d\theta = \oint_{\text{unit circle}}\frac{1}{\left(1-\frac{1}{4}\left(z-\frac{1}{z}\right)^2\right)}\frac{dz}{iz}$$
$$= \oint_{\text{unit circle}}\frac{z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i}dz$$

The integrand has poles where

$$z = \pm\frac{1}{2}\left(z^2-1\right)$$
$$z^2-1\mp 2z = 0$$
$$z = \frac{\pm 2\pm\sqrt{4+4}}{2} = \pm 1\pm\sqrt{2}$$

Of these 4 poles only 2 are inside the circle, at $z = 1-\sqrt{2}$ and $z = -1+\sqrt{2}.$ The residues are:

$$\lim_{z\to 1-\sqrt{2}}\frac{\left(z-1+\sqrt{2}\right)z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i} = \lim_{z\to 1-\sqrt{2}}\frac{4iz}{\left(z-1-\sqrt{2}\right)\left(z^2+2z-1\right)}$$
$$= -\frac{1}{4}\sqrt{2}i$$

and

$$\lim_{z\to -1+\sqrt{2}} \frac{\left(z+1-\sqrt{2}\right)z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i} = \lim_{z\to -1+\sqrt{2}} \frac{4iz}{\left(z^2-2z-1\right)\left(z+1+\sqrt{2}\right)}$$
$$= -\frac{1}{4}i\sqrt{2}$$

Thus the integral is:

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}d\theta = 2\pi i\left(-\frac{1}{4}\sqrt{2}i\right)2 = \pi\sqrt{2}$$

(d) $\int_0^{\pi} \sin^{2n}\theta d\theta$

Since $\sin^2(-\theta) = \sin^2\theta$, we may rewrite the integral:

$$\int_0^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2}\oint_{\text{unit circle}} \left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^{2n}\frac{dz}{iz}$$
$$= \frac{1}{2^{2n+1}}\frac{(-1)^n}{i}\oint_{\text{unit circle}} \left(z^2-1\right)^{2n}\frac{dz}{z^{2n+1}}$$

The integrand has a pole of order $2n+1$ at $z=0$. The residue is:

$$\lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left(z^2-1\right)^{2n} = \lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\left(z^2\right)^{2n} - 2n\left(z^2\right)^{2n-1} + \frac{(2n)(2n-1)}{2}\left(z^2\right)^{2n-2} + \cdots\right]$$

All the terms in powers $> 2n$ are zero in the limit, and all the terms in powers $< 2n$ differentiate away. Thus

$$\lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}(z^2-1)^{2n} = \lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\frac{(2n)(2n-1)\ldots(2n-n+1)}{2.3.4\ldots n}(z^2)^n\right]$$
$$= \frac{(-1)^n}{(2n)!}\frac{(2n)(2n-1)\ldots(n+1)}{2.3.4\ldots n}(2n)!$$
$$= \frac{(-1)^n(2n)!}{(n)!(n)!}$$

Thus the integral is:

$$\int_0^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2^{2n+1}}\frac{(-1)^n}{i}2\pi i\frac{(-1)^n(2n)!}{[(n)!]^2}$$
$$= \frac{\pi}{2^{2n}}\frac{(2n)!}{[(n)!]^2}$$

30. Evaluate each of the following integrals:

(a) $\int_{-\infty}^{+\infty} \frac{1}{x^2+2}dx$

We close the contour with a big semicircle at infinity. The integral over the semicircle is:

$$\left|\int_{\text{semicircle}} \frac{1}{z^2+2}dz\right| \le \pi R\max\left|\frac{1}{z^2+2}\right|$$
$$= \pi R\max\frac{1}{R^2}\left|\frac{1}{1+2/z^2}\right| \le \frac{\pi}{R}\frac{1}{|1-2/|z|^2|}$$
$$\to 0 \text{ as } R = |z| \to \infty$$

The poles of the integrand are at $z = \pm i\sqrt{2}$. Only the pole at $z = +i\sqrt{2}$ is inside the contour. the residue is:

$$\lim_{z \to i\sqrt{2}} \frac{1}{z + i\sqrt{2}} = \frac{1}{2i\sqrt{2}}$$

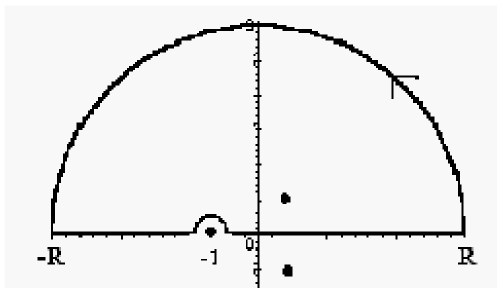
and the integral is:

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 2} dx = 2\pi i \left(\frac{1}{2i\sqrt{2}} \right) = \frac{\sqrt{2}}{2}\pi$$

(b) $P\int_{-\infty}^{+\infty} \frac{x}{x^3+1} dx$

We close the contour with a big semicircle at infinity. The integral over the semicircle is:

$$\begin{aligned} \left| \int_{\text{semicircle}} \frac{z}{z^3 + 1} dz \right| &\le \pi R \max \left| \frac{z}{z^3 + 1} \right| \\ &= \pi R \max \frac{R}{R^3} \left| \frac{1}{1 + 1/z^3} \right| \le \frac{\pi}{R} \frac{1}{|1 - 1/|z|^3|} \\ &\to 0 \text{ as } R = |z| \to \infty \end{aligned}$$



The integrand has poles at

$$\begin{aligned} z &= (-1)^{1/3} = e^{i\pi/3}, -1, e^{i5\pi/3} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

The first of these is inside the contour and the second is on it. We'll evaluate the principal value. The residue at $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is:

$$\begin{aligned} \lim_{z \to \frac{1}{2} + \frac{\sqrt{3}}{2}i} \left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \frac{z}{z^3 + 1} &= \lim_{z \to \frac{1}{2} + \frac{\sqrt{3}}{2}i} \frac{z}{(z + 1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} \\ &= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} \\ &= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{\left(\frac{3}{2} + \frac{1}{2}i\sqrt{3}\right)\left(i\sqrt{3}\right)} \\ &= -\frac{1}{6}i\left(\sqrt{3} + i\right) \end{aligned}$$

The integral around the little semicircle where $z = -1 + \varepsilon e^{i\theta}$ is:

$$\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{(-1+\varepsilon e^{i\theta})^3+1}\,i\varepsilon e^{i\theta}d\theta = \lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{-1+3\varepsilon e^{i\theta}-3\varepsilon^2 e^{2(i\theta)}+\varepsilon^3 e^{3(i\theta)}+1}\,i\varepsilon e^{i\theta}d\theta$$
$$= \lim_{\varepsilon\to 0} i\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{3e^{i\theta}-3\varepsilon e^{2(i\theta)}+\varepsilon^2 e^{3(i\theta)}}\,e^{i\theta}d\theta$$
$$= i\int_{\pi}^{0}\frac{-1}{3}d\theta = \frac{\pi}{3}i$$

Thus

$$P\int_{-\infty}^{+\infty}\frac{x}{x^3+1}dx + \frac{\pi}{3}i = 2\pi i\left(-\frac{1}{6}i\left(\sqrt{3}+i\right)\right) = \frac{1}{3}\pi\sqrt{3}+\frac{1}{3}i\pi$$

and so

$$P\int_{-\infty}^{+\infty}\frac{x}{x^3+1}dx = \frac{\sqrt{3}}{3}\pi$$

(c) $\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx$

There are no poles on the real axis, so we may assume that the integral is real. Then we may evaluate:

$$\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx = \mathrm{Re}\int_{-\infty}^{+\infty}\frac{\exp(i\omega x)}{x^2+9}dx$$

Close the contour with a big semicircle in the upper half plane. The integral along the semicircle is zero by Jordan's lemma. The poles are at $z = \pm 3i,$ but only the pole at $z = +3i$ is inside the contour. The residue is:

$$\lim_{z\to 3i}\frac{e^{i\omega z}}{z+3i} = \frac{e^{i\omega(3i)}}{6i} = \frac{e^{-3\omega}}{6i}$$

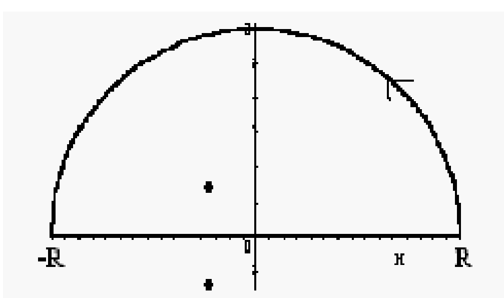
Then:

$$\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx = \mathrm{Re}\left(2\pi i\frac{e^{-3\omega}}{6i}\right) = \frac{\pi}{3}e^{-3\omega}$$

(d) $\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+2x+2}dx$

The poles of the integrand are where

$$z^2+2z+2 = 0$$
$$z = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$$



None are on the real axis. Thus we may take:

$$\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+2x+2}dx = \mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{ix}}{x^2+2x+2}dx$$

and close the contour with a big semicircle in the upper half plane. The integral along the semicircle is zero by Jordan's lemma. Only the pole at $z = -1+i$ is inside the contour. The residue is:

$$\lim_{z\to -1+i} \frac{ze^{iz}}{z-(-1-i)} = \frac{(-1+i)\exp(i(-1+i))}{2i} = \frac{(-1+i)\exp(-1-i)}{2i}$$

Thus the integral is:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+2x+2}dx &= \operatorname{Im}\left(2\pi i\frac{(-1+i)\exp(-1-i)}{2i}\right) \\ &= \operatorname{Im}\left(\pi e^{-1}(\sin 1-\cos 1)+i\pi e^{-1}(\cos 1+\sin 1)\right) \\ &= \pi e^{-1}(\cos 1+\sin 1) \\ &= 1.597\end{aligned}$$

31. Use a rectangular contour to evaluate the integrals:

(a) $\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$ $0 < \mathrm{Re}\, a < b$

The upper side of the rectangle should be at $y = 2\pi/b$ (for real b). Then on the upper side:

$$\int_{-\infty+i2\pi/b}^{+\infty+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz = \int_{-\infty}^{+\infty} \frac{e^{ax} e^{ia2\pi/b}}{1+e^{bx} e^{i2\pi}} dx = e^{ia2\pi/b} \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$$

Then around the whole rectangle:

$$\begin{aligned}\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}} dz &= \lim_{R\to\infty} \int_{-R}^{+R} \frac{e^{ax}}{1+e^{bx}} dx + \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \\ &\quad + \int_{R+i2\pi/b}^{-R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz + \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz \\ &= \lim_{R\to\infty} \left(1 - e^{ia2\pi/b}\right) \int_{-R}^{+R} \frac{e^{ax}}{1+e^{bx}} dx + \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \\ &\quad + \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz\end{aligned}$$

Along the end at $x = R$, with $a = \alpha + i\gamma$

$$\int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz = \int_{0}^{2\pi/b} \frac{e^{\alpha R - \gamma y} e^{i\alpha y + i\gamma R}}{1+e^{bR} e^{iby}} i dy = e^{(a-b)R} e^{i\gamma R} \int_{0}^{2\pi/b} \frac{e^{i\alpha y} e^{-\gamma y}}{e^{-bR}+e^{iby}} i dy$$

$$\left| \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \right| \le e^{\mathrm{Re}(a-b)R} \max \left| \frac{e^{i\alpha y} e^{-\gamma y}}{e^{-bR}+e^{iby}} \right| \frac{2\pi}{b} \le e^{\mathrm{Re}(a-b)R} \frac{1}{\left|e^{iby}\right| - e^{-bR}} \frac{2\pi}{b}$$

$$= e^{\mathrm{Re}(a-b)R} \frac{1}{1-e^{-bR}} \frac{2\pi}{b} \to 0 \text{ as } R \to \infty$$

provided that $\mathrm{Re}\, a < b$.

Along the end at $x = -R$:

$$\int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz = \int_{2\pi/b}^{0} \frac{e^{-\alpha R - \gamma y} e^{i\alpha y - i\gamma R}}{1+e^{-bR} e^{iby}} i dy$$

$$\left| \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz \right| \le e^{-\mathrm{Re}\, a R} \max \left| \frac{e^{i\alpha y} e^{-\gamma y}}{1+e^{-bR} e^{iby}} \right| \frac{2\pi}{b}$$

$$= e^{-\mathrm{Re}\, a R} \frac{1}{1-e^{-bR}} \frac{2\pi}{b} \to 0 \text{ as } R \to \infty$$

provided that $\mathrm{Re}\, a > 0$. Thus we have:

$$\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}} dz = \left(1 - e^{ia2\pi/b}\right) \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$$

Now the integrand has a pole where

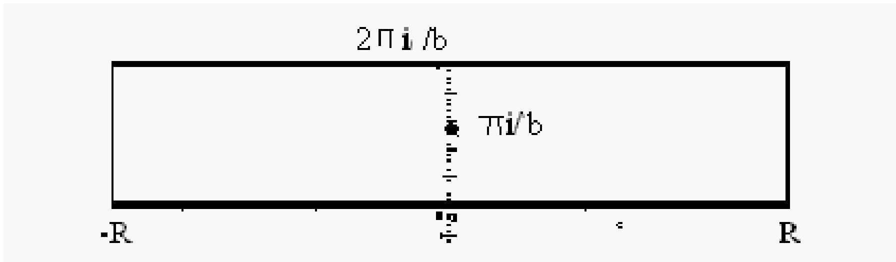
$$1 + e^{bz} = 0$$

or

$$z = i\pi/b$$

which is inside the contour. The residue there may be found from method 4:

$$\text{residue} = \lim_{z\to i\pi/b} \frac{e^{az}}{be^{bz}} = \frac{1}{b}\frac{e^{ai\pi/b}}{-1}$$



and so the integral is:

$$\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}}dz = 2\pi i\left(\frac{-e^{i\pi a/b}}{b}\right)$$

and thus

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}}dx = \frac{2\pi i}{b}\frac{-e^{i\pi a/b}}{1-e^{ia2\pi/b}} = \frac{2\pi i}{b}\frac{1}{\left(e^{i\pi a/b} - e^{-i\pi a/b}\right)}$$
$$= \frac{\pi}{b}\frac{1}{\sin \pi a/b}$$

The result is real, as expected.

(b) $\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx$

We put the top of the rectangle at $z = i\pi/a$. Then:

$$\oint_{\text{rectangle}} \frac{\sinh az}{\sinh 4az}dx = \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx + \int_{R}^{R+i\pi/a} \frac{\sinh az}{\sinh 4az}dz$$
$$+ \int_{R+i\pi/a}^{-R+i\pi/a} \frac{\sinh az}{\sinh 4az} + \int_{-R+i\pi/a}^{-R} \frac{\sinh az}{\sinh 4az}$$
$$= \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx + \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx = 2\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx$$

There are poles inside at $z = in\pi/4a$, $n = 1, 2,$ and 3. The residues are:

$$\lim_{z\to in\pi/4a} \frac{\sinh az}{4a\cosh 4az} = \frac{\sinh in\pi/4}{4a\cosh in\pi} = \frac{i\sin\frac{1}{4}n\pi}{4a\cos n\pi} = \frac{i}{4a}\frac{\sin\frac{n\pi}{4}}{(-1)^n}$$

Thus summing the 3 residues, we get:

$$\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax}dx = \pi i\frac{i}{4a}\left(-\sin\frac{\pi}{4} + \sin\frac{\pi}{2} - \sin\frac{3\pi}{4}\right) = -\frac{1}{4}\frac{\pi}{a}\left(-\sqrt{2}+1\right) = \frac{\pi}{a}\frac{\sqrt{2}-1}{4}$$

Note: the singularities on the top line at $z = i\pi/a$ and on the x-axis at $x = 0$ are removable.

(c) $\int_{-\infty}^{+\infty} \frac{x^2}{\cosh ax}dx$

Again we want the integral along the upper side of the contour to be a multiple of that along the lower. Here we find there is an additional integral that we have already evaluated. We can make use of the results

$$\cosh(az) = \cosh(a(x+iy)) = \cosh ax \cos ay + i \sinh ax \sin ay$$

So we can take $y = \pi/a$ on the upper side of the rectangle, so that $\cos ay = \cos\pi = -1$ and $\sin ay = \sin\pi = 0$. Then:

$$\oint_{\text{rectangle}} \frac{z^2}{\cosh az} dz = \lim_{R\to\infty}\left[\int_{-R}^{+R} \frac{x^2}{\cosh ax} dx + \int_{R}^{R+i\frac{\pi}{a}} \frac{z^2}{\cosh az} dz\right] + \lim_{R\to\infty}\left[\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}} \frac{z^2}{\cosh az} dz + \int_{-R+i\frac{\pi}{a}}^{-R} \frac{z^2}{\cosh az} dz\right]$$

On the top side:

$$\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}} \frac{z^2}{\cosh az} dz = \int_{R}^{-R} \frac{(x+i\pi/a)^2}{\cosh(ax+i\pi)} dx = \int_{-R}^{+R} \frac{x^2}{\cosh ax} dx + i\frac{2\pi}{a}\int_{-R}^{+R} \frac{x}{\cosh ax} dx - \frac{\pi^2}{a^2}\int_{-R}^{+R} \frac{1}{\cosh ax} dx$$

TThe second integral is zero because the integrand is odd and there are no poles on the real axis. The third integral was evaluated in § 2.7.3, Example 2.22. The result is π/a. Thus:

$$\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}} \frac{z^2}{\cosh az} dz = \int_{-R}^{+R} \frac{x^2}{\cosh ax} dx - \left(\frac{\pi}{a}\right)^3$$

Now at the two ends, we have:

$$\left|\int_{R}^{R+i\frac{\pi}{a}} \frac{z^2}{\cosh az} dz\right| = \left|\int_{0}^{i\frac{\pi}{a}} \frac{(R+iy)^2}{e^{aR}e^{iay} - e^{-aR}e^{-iay}} i\,dy\right| \le \frac{\pi}{a}e^{-aR}\left|\frac{2\left(R^2+\pi^2/a^2\right)}{1-e^{-4aR}}\right| \to 0 \text{ as } R\to\infty$$

for $a > 0$. A similar proof works for Re$a < 0$: just factor out e^{-aR} in the denominator.

Now we have:

$$\oint_{\text{rectangle}} \frac{z^2}{\cosh az} dz = 2\lim_{R\to\infty}\int_{-R}^{+R} \frac{x^2}{\cosh ax} dx - \left(\frac{\pi}{a}\right)^3 = 2\pi i \sum(\text{residues})$$

There is a pole where

$$\cosh az = 0$$

i.e. at

$$z = i\frac{\pi}{2a}$$

and the residue there is:

$$\lim_{z\to i\pi/2a} \frac{z^2}{a\sinh z} = \frac{(i\pi/2a)^2}{ia} = -\frac{\pi^2}{4ia^3}$$

and therefore

$$\int_{-\infty}^{+\infty} \frac{x^2}{\cosh ax} dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^3 - \pi i \frac{\pi^2}{4ia^3}$$
$$= \frac{1}{2}\left(\frac{\pi}{a}\right)^3 - \frac{1}{4}\frac{\pi^3}{a^3} = \frac{1}{4}\frac{\pi^3}{a^3}$$

32. Evaluate the integrals

(a)

$$\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx$$

The integrand has a branch point at the origin and a branch cut, which we may take along the positive real axis. Let's evaluate

$$\oint_C \frac{z^{3/2}}{1+z^3} dz$$

where C is the keyhole contour in Figure 2.36.

Along the bottom of the branch cut:

$$\int_{0,\text{bottom}}^{\infty} \frac{z^{3/2}}{1+z^3} dz = \int_{0,\text{bottom}}^{\infty} \frac{r^{3/2} e^{3\pi i}}{1+r^3} dr = -\int_0^{\infty} \frac{r^{3/2}}{1+r^3} dr$$

Now along the big circle, we have:

$$\left|\int_{\text{circle}} \frac{z^{3/2}}{1+z^3} dz\right| = \lim_{R\to\infty} \left|\int_0^{2\pi} \frac{R^{3/2}}{1+R^3 e^{3i\theta}} iR\, e^{i\theta} d\theta\right|$$
$$\le \frac{2\pi R^{5/2}}{R^3-1} \to \frac{2\pi}{R^{1/2}} \to 0 \text{ as } R \to \infty$$

The integrand has poles at $z = (-1)^{1/3} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $e^{3\pi i/3} = -1$, $e^{5\pi i/3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$. All three are inside the contour. the residues are:

$$\text{Res}\left(e^{i\pi/3}\right) = \lim_{z\to -1} \frac{z^{3/2}}{3z^2} = \lim_{z\to -1} \frac{1}{3z^{1/2}} = \frac{1}{3e^{i\pi/6}} = \frac{\sqrt{3}-i}{6}$$

$$\text{Res}(-1) = \frac{1}{3e^{i\pi/2}} = -\frac{i}{3}$$

and

$$\text{Res}\left(e^{5\pi i/3}\right) = \frac{1}{3e^{5\pi i/6}} = \frac{-\sqrt{3}-i}{6}$$

So

$$\oint_C \frac{z^{3/2}}{1+z^3} dz = 2\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx = 2\pi i\left(\frac{\sqrt{3}-i}{6} - \frac{i}{3} - \frac{\sqrt{3}+i}{6}\right)$$
$$\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx = \frac{2}{3}\pi$$

(b)

$$\int_0^\infty \frac{x^{1/3}}{x^2+1}dx$$

Use the keyhole contour. There are two poles inside, at $z = \pm i$, that is, $z = e^{i\pi/2}$ and $e^{i3\pi/2}$.

$$\oint_C \frac{z^{1/3}}{z^2+1}dz = \int_0^\infty \frac{r^{1/3}}{r^2+1}dr + \int_\infty^0 \frac{r^{1/3}e^{2\pi i/3}}{r^2+1}dr = 2\pi i\left(\frac{e^{i\pi/6}}{2i} + \frac{e^{i3\pi/6}}{-2i}\right)$$

$$\left(1 - e^{2\pi i/3}\right)I = \pi\left(e^{i\pi/6} - e^{i3\pi/6}\right)$$

$$I = \frac{\pi\left(e^{i\pi/6} - e^{i3\pi/6}\right)}{\left(1 - e^{2\pi i/3}\right)} = \pi\frac{e^{i\pi/6}\left(1 - e^{2\pi i/6}\right)}{\left(1 - e^{2\pi i/3}\right)}$$

$$= \pi\frac{e^{i\pi/6}\left(1 - e^{i\pi/3}\right)}{\left(1 - e^{-\pi i/3}\right)} = \frac{\pi}{2}\frac{\left(\sqrt{3}+i\right)\left(1 - i\sqrt{3}\right)}{\left(3 - i\sqrt{3}\right)}$$

$$= \frac{\pi}{6}\frac{\left(\sqrt{3}+i\right)\left(1 - i\sqrt{3}\right)}{\left(1 - i\sqrt{3}/3\right)}\frac{\left(1 + i\sqrt{3}/3\right)}{\left(1 + i\sqrt{3}/3\right)}$$

$$= \pi\frac{\sqrt{3}}{3}$$

Check that the integral around the small circle goes to zero:

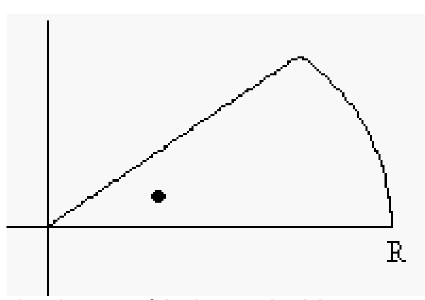
$$\int_0^{2\pi}\frac{\varepsilon^{1/3}e^{i\theta/3}\varepsilon e^{i\theta}}{1+\varepsilon^2 e^{2i\theta}}id\theta \to 0 \text{ as } \varepsilon \to 0$$

33. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^{2N}}$$

by integrating over a pie-slice contour with sides at $\phi = 0$ and at $\phi = \pi/N$, $0 \le r < \infty$.

We evaluate $\int \frac{dz}{1+z^{2N}}$ over the suggested contour.



On the curved part of the contour, the integral is bounded by

$$\left|\int\frac{dz}{1+z^{2N}}\right| \le \frac{\pi R}{N}\frac{2}{R^{2N}} \to 0 \text{ as } R \to \infty$$

On the straight line at $\phi = \pi/N$, $z = re^{i\pi/N}$ and we have

$$\int_{\infty}^{0} \frac{dre^{i\pi/N}}{1 + r^{2N}e^{i2\pi}} = -e^{i\pi/N}\int_{0}^{\infty} \frac{dr}{1 + r^{2N}}$$

Thus

$$\int_{0}^{\infty} \frac{dx}{1 + x^{2N}} = \frac{1}{1 - e^{i\pi/N}} \oint \frac{dz}{1 + z^{2N}}$$

The integrand has a pole where

$$1 + z^{2N} = 0$$

or

$$z_p = e^{i\frac{\pi}{2N}}$$

(the other roots are outside the contour) and the residue there is

$$\lim_{z \to z_p} \frac{1}{2Nz^{2N-1}} = \frac{1}{2N}\exp\left(-i\frac{\pi}{2N}(2N - 1)\right) = \frac{1}{2N}\exp\left(-i\pi + i\frac{\pi}{2N}\right)$$
$$= -\frac{1}{2N}\exp\left(i\frac{\pi}{2N}\right)$$

Thus

$$\int_{0}^{\infty} \frac{dx}{1 + x^{2N}} = \frac{1}{1 - e^{i\pi/N}} 2\pi i \left(-\frac{1}{2N}\exp\left(i\frac{\pi}{2N}\right)\right)$$
$$= -\frac{\pi}{2N}\frac{2i}{\exp\left(-i\frac{\pi}{2N}\right) - \exp\left(i\frac{\pi}{2N}\right)}$$
$$= \frac{\pi}{2N\sin(\pi/2N)}$$

34. Evaluate the integral

$$\int_{0}^{\infty} e^{ix^2} dx$$

along the positive real axis by making the change of variable $u^2 = -ix^2$. Take care to discuss the path of integration for the u −integral. Use the Cauchy theorem to show that the resulting u −integral may be reduced to a known integral along the real axis. Hence show that $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. (The result has numerous applications in physics, for example in signal propagation.)**.**

$$\sin x^2 = \operatorname{Im} e^{ix^2}$$

and letting $u = \sqrt{-i}\,x = e^{-i\pi/4}x$, then

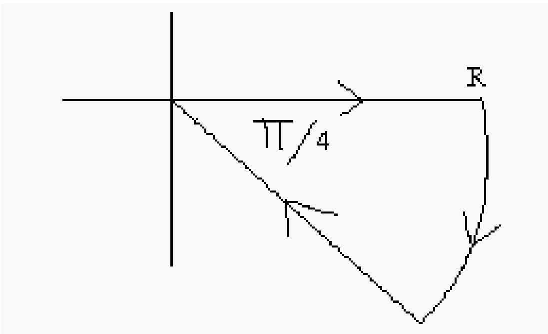
$$\int_{0}^{\infty} e^{ix^2} dx = \int_{0}^{\infty} e^{-u^2} e^{i\pi/4} du$$

The path of integration is moved off the real axis: when $x = R \to \infty$, then $u = e^{-i\pi/4}R \to \infty$ along a line making an angle $-\pi/4$ with the real axis. But the integral around the closed contour formed by the real axis, this line, and the arc at infinity is zero because there are no poles of the integrand inside, and the integral along the arc $\to 0$:

$$\int_{arc} = \int_0^{-\pi/4} e^{-R^2 e^{2i\theta}} Rie^{i\theta} d\theta = iR\int_0^{-\pi/4} e^{-R^2(\cos 2\theta + i\sin 2\theta)} e^{i\theta} d\theta$$

and cos2θ is positive throughout the range $-\pi/4 \le \theta \le 0,$ so the integral $\to 0.$ Thus the integral along the sloping line equals the integral along the real axis. Thus

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \int_0^\infty e^{-u^2} du = e^{i\pi/4} \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}(1+i)\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+i)$$



Problem 34.

Thus

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

35. The power radiated per unit solid angle by a charge undergoing simple harmonic motion is

$$\frac{dP}{d\Omega} = K\sin^2\theta \frac{\cos^2 \omega t}{(1+\beta\cos\theta\sin\omega t)^5}$$

where the constant $K = e^2 c\beta^4/4\pi a^2$ and $\beta = a\omega/c$ is the speed amplitude$/c$ (see.e.g, Jackson p 701). Using methods from section 7.2.1, perform the time average over one period to show that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K}{8}\sin^2\theta \frac{4+\beta^2\cos^2\theta}{\left(1-\beta^2\cos^2\theta\right)^{7/2}}$$

Write $\phi = \omega t.$ Then the time average is:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{2\pi}\int_0^{2\pi} \frac{\cos^2\phi}{(1+\beta\cos\theta\sin\phi)^5} d\phi$$

We can simplify by doing one integration by parts:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{(-4\beta\cos\theta)2\pi}\left[\frac{\cos\phi}{(1+\beta\cos\theta\sin\phi)^4}\Bigg|_0^{2\pi} - \int_0^{2\pi} \frac{-\sin\phi}{(1+\beta\cos\theta\sin\phi)^4} d\phi \right]$$
$$= \frac{K\sin^2\theta}{8\pi\beta\cos\theta}\left[\int_0^{2\pi} \frac{-\sin\phi}{(1+\beta\cos\theta\sin\phi)^4} d\phi \right]$$

We convert to an integral over the unit circle in the z $-$plane and write $\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and

$\sin\phi = \frac{1}{2i}\left(z - \frac{1}{z}\right)$. Also $dz = e^{i\phi} i\,d\phi$, so $d\phi = dz/iz$. With $A = \beta\cos\theta$, the integral is

$$\int_0^{2\pi} \frac{-\sin\phi}{(1 + A\sin\phi)^4} d\phi = -\frac{1}{2i}\oint_{\text{unit circle}} \frac{\left(z - \frac{1}{z}\right)}{\left(1 + \frac{A}{2i}\left(z - \frac{1}{z}\right)\right)^4}\frac{dz}{iz}$$
$$= \frac{1}{2}\oint_{\text{unit circle}} \frac{z^2\left(z^2 - 1\right)}{\left(z - iB\left(z^2 - 1\right)\right)^4} dz$$

where $B = A/2$. The denominator is

$$\left(z - iBz^2 + iB\right)^4 = (-iB)^5\left(z^2 - \frac{z}{iB} - 1\right)^4 = (-iB)^4\left(z^2 + i\frac{z}{B} - 1\right)^4$$

and there are 2 fourth-order poles at:

$$z = \frac{-i/B \pm \sqrt{-1/B^2 + 4}}{2} = \pm\sqrt{1 - 1/\left(4B^2\right)} - \frac{i}{2B} = \frac{\pm\sqrt{A^2 - 1} - i}{A}$$

where $A < 1$ and so the square root is imaginary:

$$z = \frac{i}{A}\left[-1 \pm \sqrt{1 - A^2}\right]$$

Only one of the two poles is inside the unit circle:

$$z_p = \frac{i}{A}\left[-1 + \sqrt{1 - A^2}\right]$$

Now we find the residue using method 3:

$$\operatorname{Res} f(z_p) = \lim_{z \to z_p} \frac{1}{3!}\frac{d^3}{dz^3}(z - z_p)^4 \frac{z^2\left(z^2 - 1\right)}{((z - z_p)(z - z_0))^4}$$
$$= \lim_{z \to z_p} \frac{1}{3!}\frac{d^3}{dz^3}\frac{\left(z^4 - z^2\right)}{(z - z_0)^4}$$

where

$$z_0 = \frac{i}{A}\left[-1 - \sqrt{1 - A^2}\right]$$

Thus

$$\operatorname{Res} f(z_p) = \lim_{z \to z_p} \frac{1}{3!}\frac{d^2}{dz^2}\left[\frac{4z^3 - 2z}{(z - z_0)^4} - 4\frac{z^4 - z^2}{(z - z_0)^5}\right]$$
$$= \lim_{z \to z_p} \frac{1}{3}\frac{d}{dz}\left(\frac{6z^2 - 1}{(z - z_0)^4} - 8z\frac{2z^2 - 1}{(z - z_0)^5} + 10\frac{z^4 - z^2}{(z - z_0)^6}\right)$$
$$= \lim_{z \to z_p} 4\left(\frac{z}{(z - z_0)^4} - \frac{6z^2 - 1}{(z - z_0)^5} + 5z\frac{2z^2 - 1}{(z - z_0)^6} - 5\frac{z^2(z^2 - 1)}{(z - z_0)^7}\right)$$

Now

$$z_p - z_0 = 2i\frac{\sqrt{1 - A^2}}{A} = \frac{2iC}{A}$$

and

$$z_p^2 - 1 = \frac{z_p}{iB} = \frac{2}{iA}\frac{i}{A}\left[-1 + \sqrt{1 - A^2}\,\right] = \frac{2}{A^2}\left[-1 + \sqrt{1 - A^2}\,\right]$$
$$= \frac{2}{A^2}(C - 1) \text{ where } C = \sqrt{1 - A^2} = \sqrt{1 - \beta^2 \cos^2\theta}$$

So

$$z_p = i\left(\frac{C-1}{A}\right) \text{ and } z_p^2 = -\frac{(C-1)^2}{A^2}$$

Thus

$$\operatorname{Res} f(z_p) = 4\left(\frac{z}{(z-z_0)^4} - \frac{6z^2 - 1}{(z-z_0)^5} + 5z\frac{2z^2-1}{(z-z_0)^6} - 5\frac{z^2\left(z^2-1\right)}{(z-z_0)^7}\right)$$
$$= 4\left(\frac{A}{2iC}\right)^4\left(\begin{array}{c} i\left(\frac{C-1}{A}\right) - \frac{-6\frac{(C-1)^2}{A^2}-1}{\frac{2iC}{A}} + \\ 5i\left(\frac{C-1}{A}\right)\frac{-2\frac{(C-1)^2}{A^2}-1}{\left(\frac{2iC}{A}\right)^2} + 5\frac{(C-1)^2}{A^2}\frac{\frac{2}{A^2}(C-1)}{\left(\frac{2iC}{A}\right)^3}\end{array}\right)$$
$$= \frac{1}{4}\frac{A^4}{C^4}\left(\begin{array}{c} i\left(\frac{C-1}{A}\right) - i\frac{6(C-1)^2+A^2}{2AC} + \\ \frac{5}{4}i\left(\frac{C-1}{A}\right)\frac{2(C-1)^2+A^2}{C^2} + i\frac{5}{4}\frac{(C-1)^3}{AC^3}\end{array}\right)$$
$$= \frac{i}{16}\frac{A^3}{C^7}\left(\begin{array}{c} 4(C-1)C^3 - 2\left(6(C-1)^2 + A^2\right)C^2 \\ +5C(C-1)\left(2(C-1)^2 + A^2\right) + 5(C-1)^3\end{array}\right)$$
$$= \frac{i}{16}\frac{A^3}{C^7}\left(2C^4 - 5C^3 + 3C^2 + 3C^2A^2 + 5C - 5A^2C - 5\right)$$
$$= -\frac{i}{16}\frac{A^5}{C^7}\left(4 + A^2\right)$$
$$= \frac{-i\left(\beta^5\cos^5\theta\right)}{16}\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

and thus the integral is:

$$\frac{1}{2}\oint_{\text{unit circle}} \frac{z\left(z^2-1\right)}{\left(z - iB\left(z^2-1\right)\right)^4}dz = \frac{2\pi i}{(-iB)^4}\frac{K\sin^2\theta}{8\pi\beta\cos\theta}\left(\frac{\beta^4\cos^4\theta}{16}\right)\frac{1 + 4\beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$
$$= 2\pi\beta\cos\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

and finally

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{8\pi\beta\cos\theta}2\pi\beta\cos\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$
$$= \frac{1}{4}K\sin^2\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

as required.

36. Langmuir waves. The equation for the Langmuir wave dispersion relation takes the form:

$$0 = 1 + \frac{\omega_p^2}{k}\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv$$

where ω_p is the plasma frequency $ne^2/\varepsilon_0 m$ and $f(v)$ is the 1-dimensional Maxwellian

$$f(v) = \sqrt{\frac{m}{2\pi k_B T}}\exp\left(-\frac{mv^2}{2k_B T}\right)$$

Notice that the integrand has a singularity at $v = \omega/k.$ Landau showed that the integral is to be regarded as an integral along the real axis in the complex $v-$ plane, and that the correct integration path passes around and **under** the pole.

(a) Show that the integral may be expressed as:

$$\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv - \frac{i\pi}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k}$$

(cf Section 7.3.5)

The principal value is defined in the section referred to

$$P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\omega/k-\varepsilon} + \int_{\omega/k+\varepsilon}^{+\infty}\right)\frac{\partial f(v)/\partial v}{\omega - kv}dv$$

We need to add to this the integral around the small semicircle that passes beneath the pole. On this path, $v = \omega/k + \varepsilon e^{i\theta}$, and the integral is

$$\frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k}\int_{-\pi}^{0}\frac{\varepsilon i e^{i\theta}}{-\varepsilon e^{i\theta}}d\theta = \frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} i\left(-\theta|_{-\pi}^{0}\right) = \frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} i(-\pi)$$

which is the required result.

(b) Evaluate the principal value approximately, assuming $\omega/k \gg v_T = \sqrt{k_B T/m}$ and hence find the frequency ω as a function of $k.$ What is the effect of the pole at ω/k?

First we integrate by parts:

$$P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = P\left[\frac{f}{\omega - kv}\bigg|_{-\infty}^{+\infty} - k\int_{-\infty}^{+\infty}\frac{f}{(\omega - kv)^2}dv\right]$$
$$= 0 - kP\int_{-\infty}^{+\infty}\frac{f}{(\omega - kv)^2}dv$$

Because of the exponential in the Maxwellian, the numerator is very small except when $v \le v_t \ll \omega/k.$ Thus we expand the denominator:

$$\frac{1}{(\omega - kv)^2} = -\frac{1}{\omega^2(kv/\omega - 1)^2} = -\frac{1}{\omega^2}\left(1 + 2\frac{kv}{\omega} + 3\left(\frac{kv}{\omega}\right)^2 + \cdots\right)$$

and thus, integrating by parts

$$P\int_{-\infty}^{+\infty} \frac{\partial f(v)/\partial v}{\omega - kv} dv = -\frac{k}{\omega^2}\int_{-\infty}^{+\infty} f\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \cdots\right) dv$$
$$= -\frac{k}{\omega^2}\left(1 + 0 + 3\frac{k^2}{\omega^2}v_t^2 + \cdots\right)$$

Finally, the pole on the real axis contributes a term:

$$-\frac{i\pi}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} = -i\frac{\pi}{kv_t}\frac{1}{\sqrt{2\pi}}\left(-\frac{v}{v_t^2}\right)\exp\left(-\frac{v^2}{2v_t^2}\right)\bigg|_{v=\omega/k}$$
$$= i\sqrt{\frac{\pi}{2}}\left(\frac{\omega}{k^2v_t^3}\right)\exp\left(-\frac{\omega^2}{2k^2v_t^2}\right)$$

This term is small because the exponent is large, so let's neglect it for the moment. Then:

$$0 = 1 - \frac{\omega_p^2}{k}\frac{k}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_t^2\right) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_t^2\right)$$

To zeroth order the result is $\omega = \omega_p$. The first order correction gives:

$$\omega^2 = \omega_p^2 + 3k^2v_t^2$$

the Langmuir wave dispersion relation. Now we add in the small imaginary part:

$$\omega^2 = \omega_p^2 + 3k^2v_t^2 - i\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p^3}{k^2v_t^3}\right)\exp\left(-\frac{\omega_p^2}{2k^2v_t^2}\right)$$

Thus ω must have an imaginary part, $\omega = \omega_r + i\gamma$, and thus $\omega^2 \simeq \omega_r^2 + 2i\omega_r\gamma$, with

$$\gamma = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p^2}{k^2v_t^3}\right)\exp\left(-\frac{\omega_p^2}{2k^2v_t^2}\right)$$

The wave form

exp $(ikx - i\omega t) = \exp(ikx - i\omega_r t - i(i\gamma t)) = \exp(ikx - i\omega_r t)\exp\gamma t$ shows that with

anegative γ, the wave is damped.

(c) How would the result change if the path of integration passed over, rather than under, the pole? The contribution from the pole would change sign, and we would predict growth of the waves rather than damping. This is contradicted by experiment.

37. Is the mapping $w = z^2$ conformal? Find the image in the $w -$ plane of the circle $|z - i| = 1$ in the $z-$ plane, and plot it.

The function $w = z^2$ is analytic. The derivative

$$\frac{dw}{dz} = 2z$$

is not zero except at the origin. Thus the mapping is conformal except at the origin.

The circle is described by

$$(z - i)(z^* + i) = 1$$

or

$$zz^* + i(z - z^*) = 0$$

which maps to

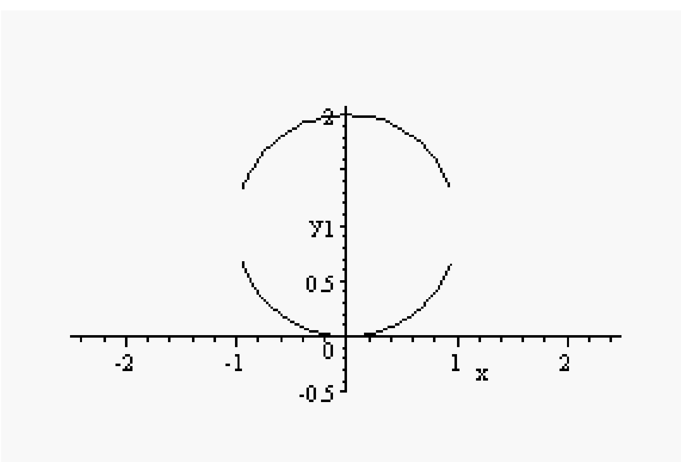
$$\sqrt{w}\left(\sqrt{w}\right)^* + i\left(\sqrt{w} - \sqrt{w}^*\right) = 0$$

and if $w = \rho e^{i\phi}$,

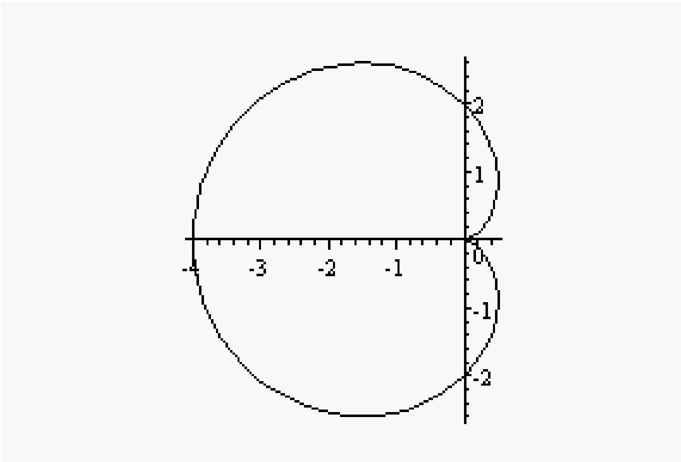
$$\rho + i\sqrt{\rho}\left(e^{i\phi/2} - e^{-i\phi/2}\right) = 0$$

$$\sqrt{\rho} = 2\sin\frac{\phi}{2}$$

Here's the plot:



$z-$ plane



$w-$ plane

Invariance of angles breaks down at $z = 0,$ where the mapping is not conformal.

38. Is the mapping $w = z + \frac{1}{z}$ conformal? Find the image in the $w-$ plane of (a) the $x-$ axis , (b) the $y-$ axis, and (c) the unit circle in the $z-$ plane.

The function $w = z + \frac{1}{z}$ is analytic except at $z = 0$ and at infinity. The derivative is

$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

which is zero at $z = \pm 1.$ Thus the mapping is not conformal at these two points.

(a) The real axis maps to

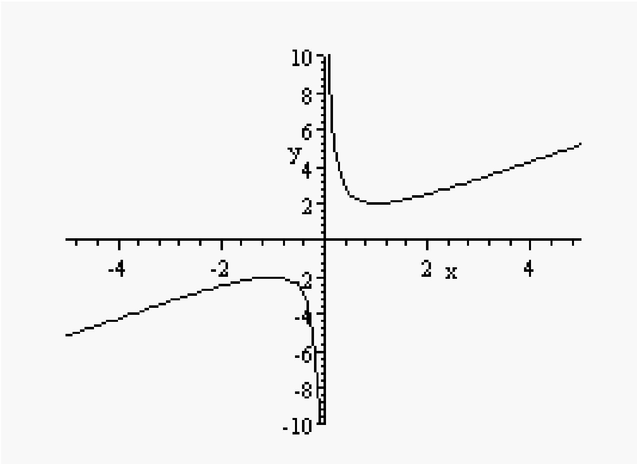
$$w = x + \frac{1}{x}$$

The origin maps to infinity, the positive $x-$ axis maps to the positive $u-$ axis with $u > 2$, and the negative $x-$ axis maps to the negative $u-$ axis with $u < -2.$

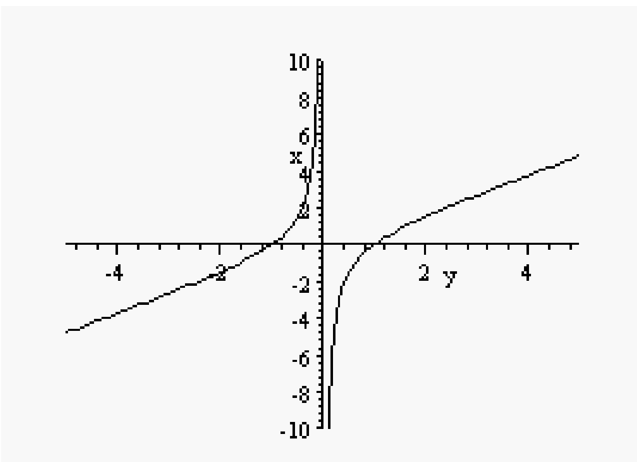
(b) The imaginary axis maps to

$$w = iy + \frac{1}{iy} = i\left(y - \frac{1}{y}\right)$$

Thus the points $y = \pm 1$ map to the origin. Points with $0 < y < 1$ map to negative $v,$ while points with $y > 1$ map to positive $v.$



u versus
x



v versus
y

(c)The unit circle $z = e^{i\theta}$ maps to

$$w = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

---a chunk of the real- w axis between $u = -2$ and $u = +2.$

A capacitor plate has a cylindrical bump of radius $a.$ on it. The second plate is a distance $d \gg a$ away. One plate is maintained at potential $V,$ and the other is grounded. Find the potential everywhere between the plates.

We want to convert to a coordinate system with $a = 1,$ so let $x' = x/a,$ $y' = y/a$. Then the cylinder has radius $r' = 1.$ Now we map to the $w-$ plane using the mapping $w = z' + \frac{1}{z'}$. This maps the cylinder plus x' axis to the u' axis. The second plate has coordinate $y' = d/a \gg 1$. It maps to the line $v = y' - \frac{1}{y'} \simeq d/a$. In this plane the potential is

$$\phi = V\frac{a}{d}v$$

whic is zero for $v = 0$ and V for $v = d/a$. The complex potential is then:

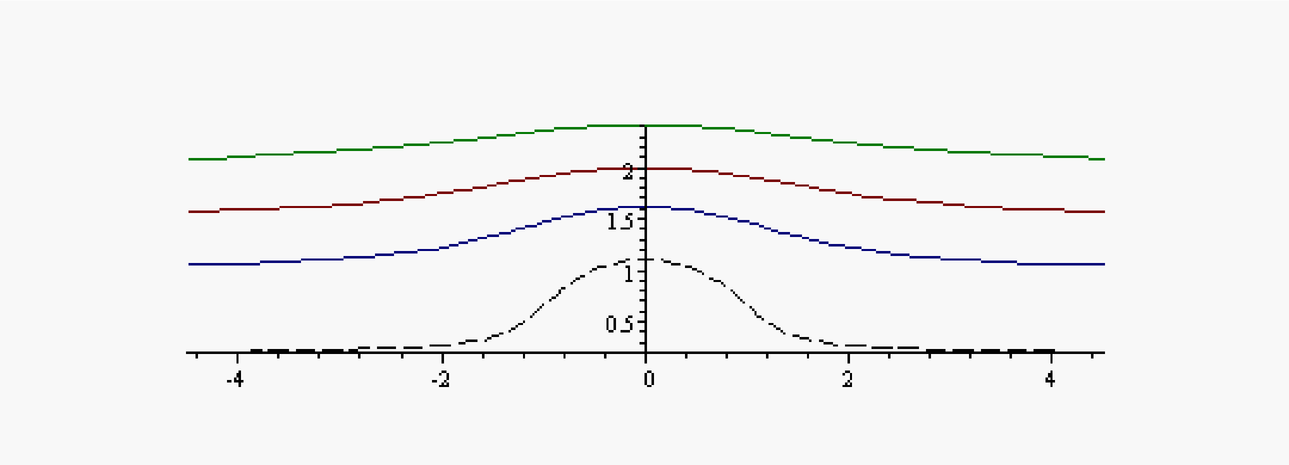
$$\Phi = V\frac{a}{d}w = \psi + i\phi$$

Mapping back, the potential in the z' − plane is:

$$\begin{aligned}\Phi &= V\frac{a}{d}\left(z' + \frac{1}{z'}\right) \\ &= V\frac{a}{d}\left(r'e^{i\theta} + \frac{1}{r'}e^{-i\theta}\right)\end{aligned}$$

so the electric potential is:

$$\begin{aligned}\phi &= V\frac{a}{d}\sin\theta\left(r' - \frac{1}{r'}\right) \\ &= V\frac{a}{d}\sin\theta\left(\frac{r}{a} - \frac{a}{r}\right)\end{aligned}$$



Equipotentials for $\phi/V = 0.1$ (dashed), 0.5 (solid blue), 0.75 (red) and 1 (green).

39. Show that the mapping $z = w + e^{w}$ is conformal except at a finite set of points.

A parallel plate capacitor has plates that extend from $x = -1$ to $x = -\infty$. Find an appropriate scaling that allows you to place the plates at $y = \pm\pi.$ Show that the given transformation maps the plates to the lines $v = \pm\pi.$ Solve for the potential between the plates in the $w-$ plane, map to the $z-$ plane, and hence find the equipotential surfaces at the ends of the capacitor. Sketch the field lines. This is the so-called fringing field.

Choose $y = 2s\pi/d$ where s is a coordinate measured perpendicular to the plates, and d is the plate separation. The function $w + e^w$ is analytic everywhere, and the derivative is

$$\frac{dz}{dw} = 1 + e^w$$

It is non-zero except at the points

$$e^w = -1$$
$$w = \pm i\pi, \quad \pm 3i\pi \text{ etc}$$

or, equivalently,

$$z = -1 \pm i(2n + 1)\pi.$$

The mapping takes the form:

$$x + iy = u + iv + e^u e^{iv} = u + e^u \cos v + i(v + e^u \sin v)$$

Then for $v = 0$ $(\text{the real axis in the } u-\text{plane})$ x ranges from $-\infty$ to $+\infty,$ i.e. we get the whole real axis in the $z-$ plane. The line $\omega = i\pi$ maps to $x = u - e^u,$ $y = \pi.$ x ranges from $-\infty$ at $u = \infty$ to -1 at $u = 0.$ This is the top plate of the capacitor. Similarly $w = -i\pi$ maps to the lower plate.

The mapping $w = f(z)$ has a branch point at each of the points $z = -1 \pm i(2n + 1)\pi.$ Each 2 $\pi-$ wide strip of the $w-$ plane maps to the whole $z-$ plane. For each branch there are two points in the $z-$ plane at which the mapping is not conformal.

In the $w-$ plane we can write the potential as $\phi = vV/2\pi,$ giving a complex potential $\Phi = wV/2\pi,$ with the complex part being the physical potential. Equipotentials correspond to $v =$ const $= v_0.$ The corresponding curves in the $z-$ plane are:

$$x = u + e^u \cos v_0$$
$$y = v_0 + e^u \sin v_0$$

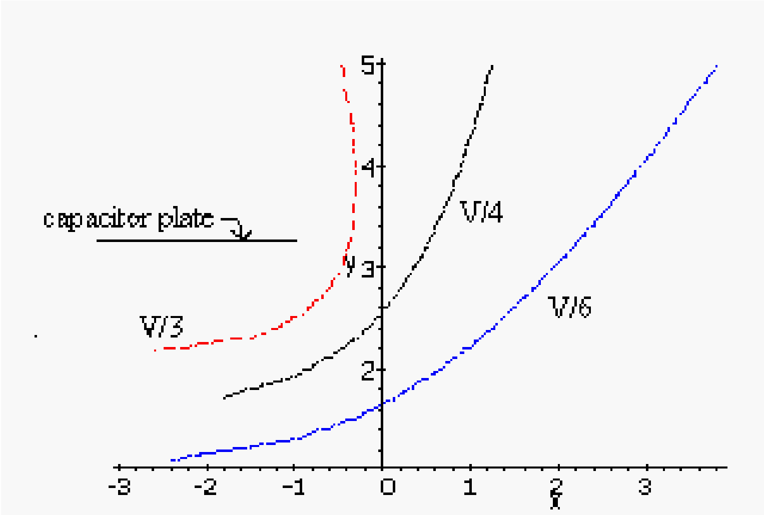
Thus

$$e^u = \frac{y - v_0}{\sin v_0}$$

and

$$x = (y - v_0)\cot v_0 + \ln\left(\frac{y - v_0}{\sin v_0}\right)$$

The equipotenials are shown in the figure.



40. Two conducting cylinders, each of radius $a,$ are touching. An insulating strip lies along the line at which they touch. One cylinder is grounded and the other is at potential $V.$ Use one of the mappings from the chapter to solve for the potential outside the cylinders.

. The transformation $z = 2a/w$ maps each of the cylinders to a straight line in the $w-$ plane. For a circle centered at $z = \pm ia$ with radius a we may write a point on the circle as

$$z = \pm ia + ae^{i\phi}$$

which maps to

$$w = \frac{2a}{\pm ia + ae^{i\phi}} = \frac{2a}{\left(\pm ia + ae^{i\phi}\right)\left(\mp ia + ae^{-i\phi}\right)}\left(\mp ia + ae^{-i\phi}\right)$$

$$= \frac{1}{(1 \pm \sin\phi)}(\mp i + \cos\phi - i\sin\phi)$$

$$= \frac{\cos\phi}{1 \pm \sin\phi} \mp i$$

As ϕ varies, u takes on all real values and w falls on the lines $\mp i$.

In the $w-$ plane, the potential is $\phi = (v+1)V/2$. So we can write a complex potential

$$\Phi = (w+i)V/2$$

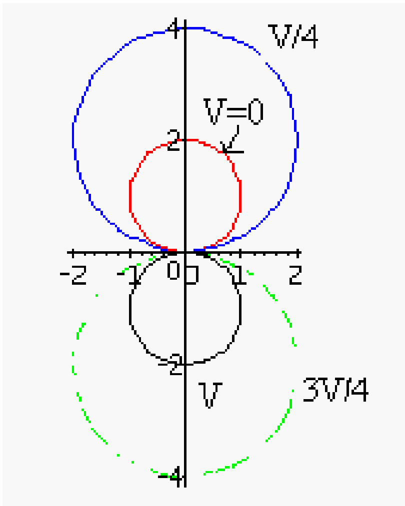
where the physical potential is the imaginary part. In the $z-$ plane we have:

$$\Phi = \left(\frac{2a}{z} + i\right)\frac{V}{2}$$

$$\Phi = \left(\frac{2a(x-iy)}{x^2+y^2} + i\right)\frac{V}{2} = \left(\frac{2a}{r}e^{-i\theta} + i\right)\frac{V}{2}$$

The imaginary part is:

$$\phi = -V\frac{a}{r}\sin\theta + \frac{V}{2} = \frac{-Vay}{x^2+y^2} + \frac{V}{2}$$



Problem 40. Equipotentials for $\Phi = 0, V/4,$ 3 V /4 and V.

The equipotentials are given by

$$r = a\frac{\sin\theta}{\left(\frac{1}{2}-\frac{\phi}{V}\right)}$$

Chapter 2: Complex variables.

41. Show that the mapping $w = 1/(z-2)$ maps the arcs (a) $|z-4| = 2$ with end points at $z = 3 \pm \sqrt{3}i$ and (b) $|z-(2+i)| = 1$ with end points at $z = 3+i$ and $z = 1+i$ to straight line segments.

(a) The arc is described by the equation

$$(z-4)(z^*-4) = 4$$
$$zz^* - 4(z+z^*) + 12 = 0$$

Using the transformation:

$$z = 2 + \frac{1}{w}$$

and thus

$$\left(2+\frac{1}{w}\right)\left(2+\frac{1}{w}\right)^* - 4\left(2+\frac{1}{w}+\left(2+\frac{1}{w}\right)^*\right) + 12 = 0$$
$$4 + 2\left(\frac{1}{w}+\frac{1}{w^*}\right) + \frac{1}{ww^*} - 16 - 4\left(\frac{1}{w}+\frac{1}{w^*}\right) + 12 = 0$$
$$\frac{1}{ww^*} - 2\left(\frac{1}{w}+\frac{1}{w^*}\right) = 0$$

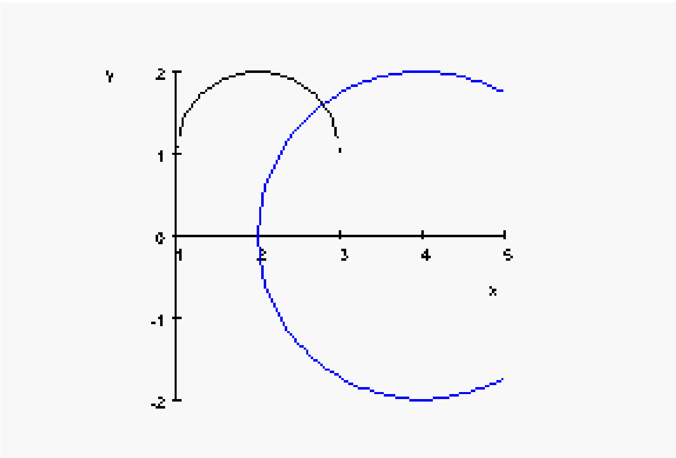
Thus

$$1 - 2(w+w^*) = 0$$

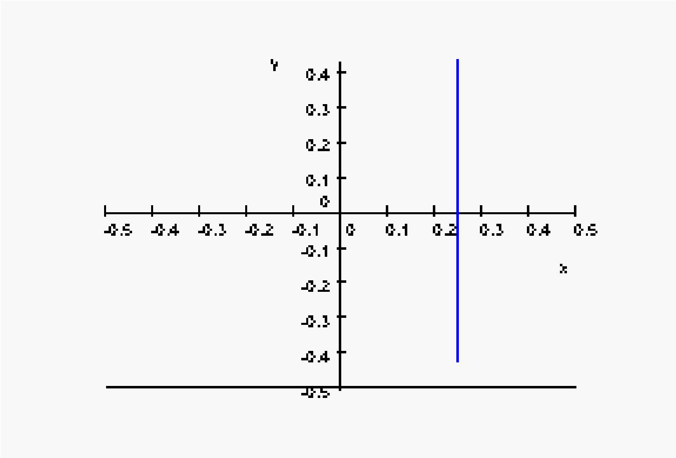
This a straight line parallel to the $v-$axis: $u = 1/4.$ With $z = 3 \pm \sqrt{3}i,$

$$w = \frac{1}{1 \pm \sqrt{3}i} = \frac{1 \mp \sqrt{3}i}{4}$$

The line extends from $v = \sqrt{3}/4$ to $v = -\sqrt{3}/4.$



The arc in the $z-$plane. (a) blue (b) black



The line in the $w-$plane

(b) The circle is

$$\begin{aligned} [z-(2+i)][z-(2+i)]^* &= 1 \\ \left(\frac{1}{w}-i\right)\left(\frac{1}{w^*}+i\right) &= 1 \\ (1-iw)(1+iw^*) &= ww^* \\ 1-i(w-w^*) &= 0 \\ 1 &= i(2iv) \\ v &= -\frac{1}{2} \end{aligned}$$

This is a straight line parallel to the $u-$axis. It extends from $w = \frac{1}{-1+i} = -\frac{1}{2} - \frac{i}{2}$ to $w = \frac{1}{1+i} = \frac{1}{2} - \frac{i}{2}$.

42. Show that $\Gamma(x) < 0$ for $-1 < x < 0.$

If $-1 < x < 0,$ then we can write

$$\Gamma(x) = \frac{\pi}{\sin \pi x \Gamma(1-x)} = \frac{\pi}{-|\sin \pi x| \Gamma(y)}$$

where $1 < y < 2.$ Then $\Gamma(y)$ is positive and hence $\Gamma(x)$ is negative.

43. Prove *Cauchy's inequality*: If $f(z)$ is analytic and bounded in a region $R:$ $|z-z_0| < R,$ and $|f(z)| < M$ on the circle $|z-z_0| = r < R,$ then the coefficients in the Taylor series expansion of f about z_0 (eqn 44) satisfy the inequality

$$|c_n| \le \frac{M}{r^n}$$

Hence prove *Liouville's theorem:*

If $f(z)$ is analytic and bounded in the entire complex plane, then it is a constant.

Using expression (45) with Γ equal to the circle of radius r,

$$|c_n| = \frac{1}{2\pi}\left|\oint_{\text{circle}} \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{1}{2\pi}M\oint_{\text{circle}} \frac{1}{r^{n+1}}r\,d\theta = \frac{M}{r^n}$$

as required.

To Prove Liouville's theorem, we let R and $r \to \infty$. Then $c_n \to 0$ for all $n > 0$. Thus $f(z) = c_0$, a constant.

44. A function $f(z)$ is analytic except for well-separated simple poles at $z = z_n$, $n = 1 - N$, $z_n \ne 0$. Show that the function may be expanded in a series

$$f(z) = f(0) + \sum_{n=1}^{N} a_n\left(\frac{1}{z_n} + \frac{1}{z - z_n}\right)$$

where a_n is the residue of f at z_n. Is the result valid for $N \to \infty$? Why or why not?

Hint: Evaluate the integral

$$I_N = \frac{1}{2\pi i}\int_{C_N} \frac{f(w)}{w(w-z)}dw$$

where C_N is a circle of radius R_N about the origin that contains the N poles. You may assume that $|f(z)| < \varepsilon R_N$ on C_N for ε a small positive constant.

The integrand has simple poles at the origin, at z, and at z_n, $n \le N$. Near one of the poles z_n, the integrand has the form

$$\frac{a_n}{w(w-z)(w-z_n)} + \sum_{k=0}^{\infty} \frac{c_k}{w(w-z)}(w-z_n)^k$$

The denominator of the first term has a simple zero at z_n and the sum is analytic at z_n, so the residue at z_n is

$$\frac{a_n}{\frac{d}{dw}[w(w-z)(w-z_n)]\Big|_{w=z_n}} = \frac{a_n}{[(w-z)(w-z_n)+w(w-z_n)+w(w-z)]|_{w=z_n}}$$
$$= \frac{a_n}{z_n(z_n-z)}$$

Thus

$$I_N = \frac{f(0)}{-z} + \frac{f(z)}{z} + \sum_{n=1}^{N} \frac{a_n}{z_n(z_n-z)}$$

But also

$$|I_N| \le \frac{1}{2\pi}\frac{\varepsilon R_N}{R_M(R_N-|z|)}2\pi R_N \le \frac{\varepsilon}{1-|z|/R_N}$$

Thus $I_n \to 0$ as $R_N \to \infty$ and so

$$f(z) = f(0) + \sum_{n=1}^{N}\frac{zb_n}{z_n(z-z_n)} = f(0) + \sum_{n=1}^{N} b_n\left(\frac{1}{z_n} + \frac{1}{z-z_n}\right)$$

as required.

The residue theorem holds when there are a finite number of poles inside the contour, so this proof is limited to finite N.

See also Jeffreys and Jeffreys 11.175.