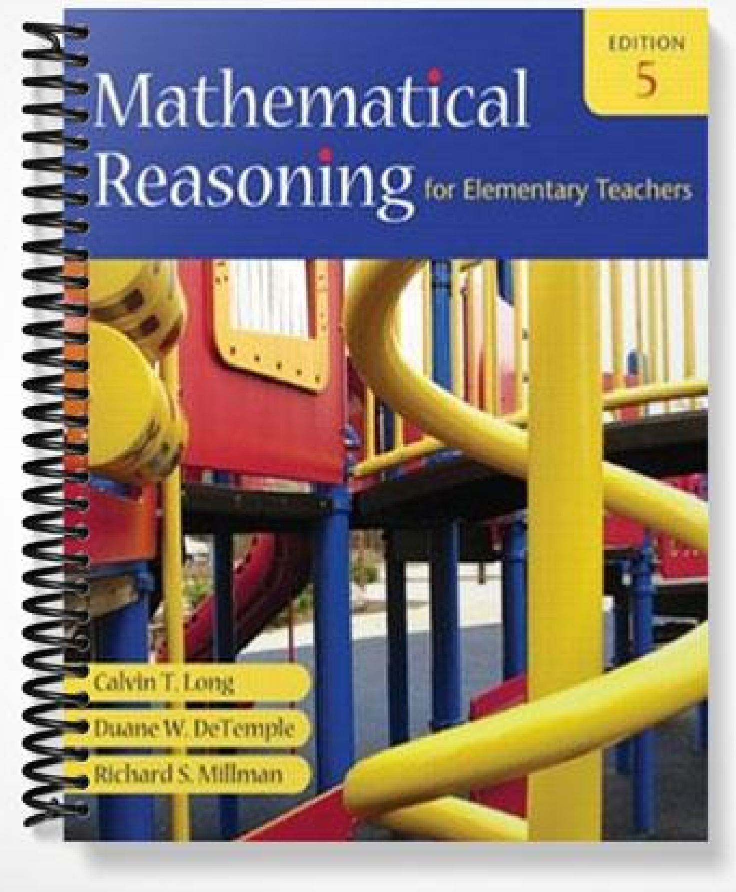
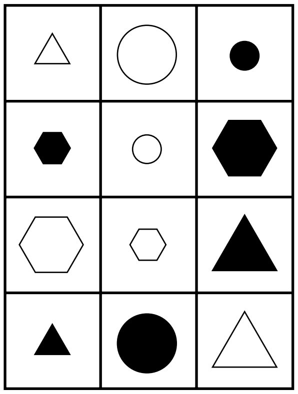
SOLUTIONS MANUAL



Chapter 2 Sets and Whole Numbers

HANDS ON (page 78)
Attribute Tic-Tac-Toe

In variation 2, a tie is possible, as shown in the example.



Problem Set 2.1 (page 87)

1. **(a)** {Arizona, California, Idaho, Oregon, Utah}

   (b) {Maine, Maryland, Massachusetts, Michigan, Minnesota, Mississippi, Missouri, Montana}

   (c) {Arizona}

2. **(a)** {l, i, s, t, h, e, m, n, a, o, y, c}

   (b) {a, e, m, t}

3. **(a)** {7, 8, 9, 10, 11, 12,13}

   (b) {9, 11, 13}

   (c) {4, 8, 12, 16, 20}

   (d) Since $2 = 2 \cdot 1$, $4 = 2 \cdot 2$, $6 = 2 \cdot 3$, $8 = 2 \cdot 4$, $10 = 2 \cdot 5$, etc., the set is {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}.

   (e) Since $3 = 2 \cdot 1 + 1$, $5 = 2 \cdot 2 + 1$, $7 = 2 \cdot 3 + 1$, $9 = 2 \cdot 4 + 1$, $11 = 2 \cdot 5 + 1$, $13 = 2 \cdot 6 + 1$, $15 = 2 \cdot 7 + 1$, $17 = 2 \cdot 8 + 1$,and $19 = 2 \cdot 9 + 1$, the set is {3, 5, 7, 9, 11, 13, 15, 17, 19}.

   (f) Since $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, and $16 = 4^2$, the set is {1, 4, 9, 16}.

4. Answers will vary.

   (a) $\{x \in U \mid 11 \le x \le 14\}$ or $\{x \in U \mid 10 < x < 15\}$

   (b) $\left\{x \in U \,\middle|\, \begin{array}{r} x \text{ is even and } 6 \le x \le 16 \\ \text{and } x \ne 14 \end{array}\right\}$

   (c) $\{x \in U \mid x = 4n \text{ and } 1 \le n \le 5\}$

   (d) $\{x \in U \mid x = n^2 + 1 \text{ and } 1 \le n \le 4\}$

5. Answers will vary.

   (a) $\{x \in N \mid x \text{ is even and } x > 12\}$ or $\{x \in N \mid x = 2n \text{ for } n \in N \text{ and } n > 6\}$

   (b) $\left\{x \in N \,\middle|\, \begin{array}{r} x = n^2 \text{ for } n \in N \text{ with } n \text{ odd} \\ \text{and } n \ge 5 \end{array}\right\}$

   (c) $\{x \in N \mid x \text{ is divisible by } 3\}$

6. **(a)** True. The sets contain the same elements.

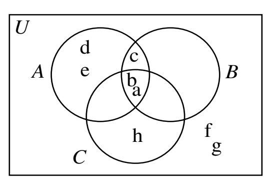
   (b) True. Every element in {6}—namely, 6—is in {6, 7, 8}.

   (c) True. The sets contain the same elements. Order doesn't matter.

   (d) True. 7 is in {6, 7, 8}, but the two sets are not the same.

   (e) False. The sets are equal.

   (f) False. 6 is not a member of {7}.

7. 

   (a) $B \cup C = \{a, b, c, h\}$

   (b) $A \cap B = \{a, b, c\}$

   (c) $B \cap C = \{a, b\}$

(d) $A \cup B = \{a, b, c, d, e\}$

(e) $\overline{A} = \{f, g, h\}$

(f) $A \cap C = \{a, b\}$

(g) $A \cup (B \cap C) = \{a, b, c, d, e\}$

8. (a) $M = \{45, 90, 135, 180, 225, 270, 315, ...\}$

(b) $L \cap M = \{90, 180, 270, ...\}$
This can be described as the set of natural numbers that are divisible by both 6 and 45, or as the set of natural numbers that are divisible by 90.

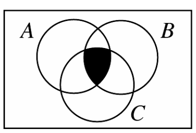
(c) 90

9. (a) $D = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144\}$

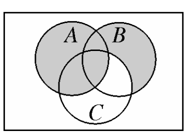
(b) $G \cap D = \{1, 2, 3, 6, 9, 18\}$

(c) 18

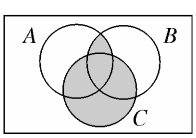
10. (a) $A \cap B \cap C$ is the set of elements common to all three sets.



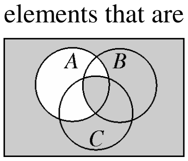
(b) $A \cup (B \cap \overline{C})$ is the set of elements in B but not in C, together with the elements of A.



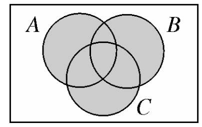
(c) $(A \cap B) \cup C$ is the set of elements in C or in both A and B.



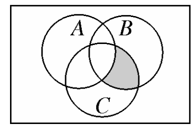
(d) $\overline{A} \cup (B \cap C)$ is the set of all elements that are not in A, together with additional elements that are in both B and C.



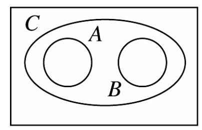
(e) $A \cup B \cup C$ is the set of all elements in A, B, and/or C.



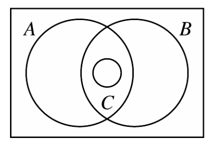
(f) $\overline{A} \cap B \cap C$ is the set of elements that are in both B and C, but not in A.



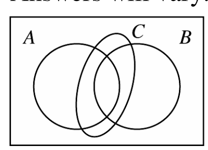
11. (a) A and B must be disjoint sets contained in C.



(b) Answers will vary.



(c) Answers will vary.



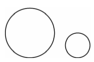
12. No. It is possible that there are elements of A that are also elements of B but not C, or C but not B. For example, let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3\}$. $A \cup B = A \cup C = \{1, 2, 3\}$, but $B \neq C$.

13. (a) Since
$A \cap B = \{6, 12, 18\}$,
$\overline{A \cap B} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20\}$
Since
$\overline{A} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ and
$\overline{B} = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$,
$\overline{A} \cup \overline{B} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20\}$

Since
$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$
$\overline{A \cup B} = \{1, 5, 7, 11, 13, 17, 19\}$.
Since
$\overline{A} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ and
$\overline{B} = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$,
$\overline{A} \cap \overline{B} = \{1, 5, 7, 11, 13, 17, 19\}$.

(b) The results of part (a) suggest that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

14. (a) $R \cap C$ is the set of red circles. Draw two red circles—one large and one small.

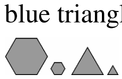


(b) $L \cap H$ is the set of large hexagons. Draw two large hexagons—one red and one blue.

(c) $T \cup H$ is the set of shapes that are either triangles or hexagons. Draw two large triangles, two large hexagons, two small triangles, and two small hexagons—a red and a blue of each.

(d) $L \cap T$ is the set of large triangles. Draw two large triangles—one red and one blue.

(e) $B \cap \overline{C}$ is the set of blue shapes that are not circles. That is, the set of four elements consisting of the large and small blue hexagons and the large and small blue triangles.



(f) $H \cap S \cap R$ is the set of hexagons that are both small and red. Draw one small red hexagon.

15. (a) $L \cap T$

(b) $B \cap (T \cup H)$ or $B \cap \overline{C}$

(c) $S \cup T$

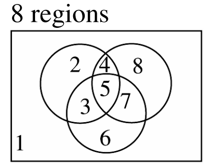
(d) $(B \cap C) \cup R$

16. (a) Answers will vary. One possibility is B = set of students taking piano lessons, C = set of students learning a musical instrument.

(b) Answers will vary.

(c) Answers will vary.

17. (a) 8 regions

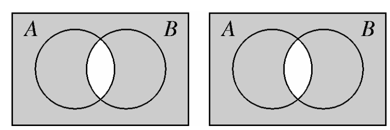


(b) By counting, there are 14 regions.

(c) Verify that $A \cap \overline{B} \cap \overline{C} \cap D$ has no region by observing that the region for $A \cap D$ is entirely contained in $B \cup C$. Likewise, the region for $B \cap C$ is entirely contained in $A \cup D$. The other missing region is $\overline{A} \cap B \cap C \cap \overline{D}$.

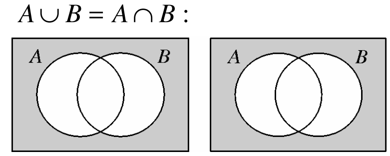
(d) Yes. Each loop contains 8 different regions, and there are 16 regions all together.

18. (a) These Venn diagrams show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$:



$\overline{A \cap B}$ $\overline{A} \cup \overline{B}$

(b) These Venn diagrams show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$:



$\overline{A \cup B}$ $\overline{A} \cap \overline{B}$

19. (a) There are 6 choices for shape, 2 choices for size, and 3 choices for color, so the number of pieces is $6 \times 2 \times 3 = 36$.

(b) $6 \times 2 \times 3 \times 2 = 72$

20. Children often interpret "or" as the "exclusive or" in which either one of two possibilities holds, but not both. Have the child consider the statement "Tomorrow, we will have music class or we will go to the gym", and point out that this remains true even if both music class and gym class take place.

21. **(a)** There appear to be 3 choices for shape, 3 choices for color, 2 choices for shading (shaded or not), and 3 choices for number of shapes used. The number of cards is probably $3 \times 3 \times 2 \times 3 = 54$. We cannot be sure of this result without knowing for sure that the deck does not contain any different shapes, colors, etc.

(b) Answers will vary.

(c) Answers will vary.

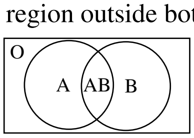
22. **(a)** There are eight subsets: ∅, {P}, {N}, {D}, {P, N}, {P, D}, {N, D}, {P, N, D}.

(b) There are 16 subsets: ∅, {P}, {N}, {D}, {P, N}, {P, D}, {N, D}, {P, N, D}, {Q}, {P, Q}, {N, Q}, {D, Q}, {P, N, Q}, {P, D, Q}, {N, D, Q}, {P, N, D, Q}.

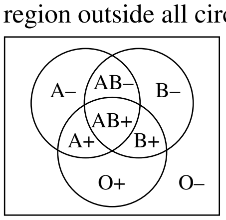
(c) Half of the subsets of {P, N, D, Q} contain Q.

(d) The number of subsets doubles with each additional element, so a set with n elements has 2^n subsets.

23. Note that in this diagram, O refers to the region outside both circles.



24. Answers will vary. (The loop shown on the bottom could have represented negative instead of positive, reversing all signs.) Note that in this diagram, O– refers to the region outside all circles.

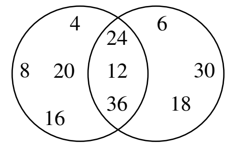


25. Answers will vary. Consult a thesaurus.

26. Answers will vary.

27. B. 4 students like all three foods, so there must be a region common to all three loops in the Venn diagram.

28. **(a)**



(b) The left hand loop seems to include multiples of 4 and the right hand loop multiples of 3 (or perhaps 6). The intersection would therefore contain multiples of 12. This gives the placement shown.

29. Choice H (21) only. It is both odd (as in set V) and divisible by 3 (as in set W).

30. Polya's Model
Understand the problem
A perfect square is a number which can be obtained by multiplying an integer by itself. For example, $25 = 5 \times 5$, $64 = 8 \times 8$, etc. The problem is to create a list using the integers from 1 to 15 inclusive so that any two adjacent numbers have a sum which is a perfect square.
Devise a plan
Many of the numbers can be added to more than one number to produce a square. For example, 3 can be added to 6 or 13, but the numbers 8 and 9 each have only one number to which they can be added to produce a perfect square. Therefore, 8 and 9 must go at the "ends" of the list.
Carry out the plan
If 8 is placed at the front of the list, it must be followed by 1, etc.: 8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9.
Look back
A key to solving the problem is the recognition that only two numbers 8 and 9 each have only one number from 1 to 15 inclusive to which they can be added to produce a perfect square.

31. Polya's Model
Understand the problem
Since all 10,000 poles must be numbered, we must use all possible 1-digit, 2-digit, 3-digit, and 4-digit numbers, as well as the number 10,000. How many twos will be required to write these numbers?
Devise a plan
If we start to write the list of numbers needed, we see that every 10th units digit will be a 2.

Also, all the 20s, 120s, ..., 9920s will require 2s in the tens position. Perhaps it will not be too difficult to determine how many are in the units, tens, hundreds, and thousands positions also. Of course, none will be needed in the ten thousands position since the highest number needed is 10,000.
Carry out the plan
Since the second number in each of the 1000 sets of ten consecutive numbers from 1 through 9999 end in 2, there are 1000 2s needed as units digits. Since there are ten consecutive 20s in each of the 100 sets of 100 consecutive numbers from 1 through 9999, there are $10 \times 100 = 1000$ 2s needed as tens digits. Since there are 100 consecutive 200s in each of the ten sets of 1000 consecutive integers from 1 through 9999, there are $100 \times 10 = 1000$ 2s needed as hundreds digits. Finally, all 1000 of the digits in the range $2000 \le n \le 2999$ begin with 2 so there are 1000 2s needed as thousands digits. Thus, in all, the power company should order 4000 2s.
Look back
This solution depended on counting the number of 2s in the units, tens, hundreds, and thousands digits to represent all the numbers from 1 through 9999. An alternate and perhaps simpler solution is to think of all possible arrangements of four digits. These include configurations such as 0037 which is the number 37. There are 10,000 of these arrangements, all are different and all digits appear equally often. Since there are 40,000 digits in all and $1/10$ of these are 2s, there must be 4000 2s.

Problem Set 2.2 (page 100)

1. **(a)** 13: ordinal
first: ordinal

(b) fourth: ordinal
second: ordinal
93: cardinal. In common usage, it represents the number 93 correct out of 100.

2. **(a)** Equivalent, since there are five letters in the set {A, B, M, N, P}

(b) Not equivalent since the sets have different numbers of elements.

(c) Equivalent, say by the correspondence
o ↔ t,n ↔ w,e ↔ o

(d) Not equivalent, since the set $\{0\}$ has one element and $\varnothing$ has no elements.

3. **(a)** $n(A)$ is 0, 1, 2, 3, or 4, since it has strictly fewer elements than of set B.

(b) $n(C)$ is 5, 6, 7, ..., any whole number greater than or equal to 5.

4. **(a)** $n(A) = 7$ because
$A = \{21, 22, 23, 24, 25, 26, 27\}$.

(b) $n(B) = 0$ because $B = \varnothing$. There is no natural number x such that $x + 1 = x$.

(c) The solutions for $(x-1)(x-9) = 0$ are $x = 1$ and $x = 9$, so $C = \{1, 9\}$ and $n(C) = 2$.

(d) $n(D) = 2$ because $D = \{40, 80\}$.

5. Yes, $A = \{1, 8, 27, 64\}$, $B =$ {California, Arizona, New Mexico, Texas}.
Since both sets have the same number of elements, $A \sim B$. We could have the correspondence 1 ↔ California,
8 ↔ Arizona, 27 ↔ New Mexico,
64 ↔ Texas.

6. **(a)** The correspondence $0 \leftrightarrow 1, 1 \leftrightarrow 2,$
$2 \leftrightarrow 3, ..., w \leftrightarrow w+1, ...$ shows $W \sim N$.

(b) The correspondence $1 \leftrightarrow 2, 3 \leftrightarrow 4, \ldots,$
$n \leftrightarrow n+1, \ldots$ shows $D \sim E$.

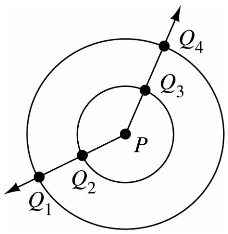
(c) The correspondence
$1 \leftrightarrow 10, 2 \leftrightarrow 100 = 10^2, \ldots, n \leftrightarrow 10^n, \ldots$
shows that the sets are equivalent.

7. **(a)** Finite. The number of grains of sand is large, but finite.

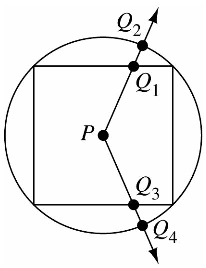
(b) Infinite

(c) Infinite

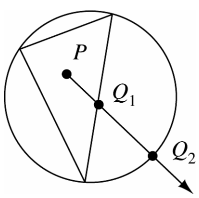
8. **(a)** Answers will vary. For example, $Q_1 \leftrightarrow Q_2$, $Q_3 \leftrightarrow Q_4$, and so on. (Note that P need not be the center—it can be any fixed point inside the small circle.)



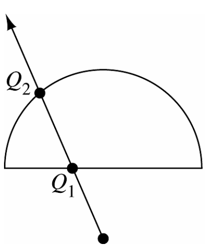
(b) Answers will vary. For example, $Q_1 \leftrightarrow Q_2$, $Q_3 \leftrightarrow Q_4$, and so on.



(c) Answers will vary. For example, $Q_1 \leftrightarrow Q_2$, etc.



(d) Answers will vary. For example, $Q_1 \leftrightarrow Q_2$, etc.



9. **(a)** True. A set B cannot have fewer elements than its subset A.

(b) False. Possible counterexample: $A = \{1\}$, $B = \{2, 3\}$. Then $n(A) = 1$ and $n(B) = 2$, so $n(A) < n(B)$, but $A \not\subseteq B$.

(c) True. The union $A \cup B$ does not include any more elements than just A, so the elements of B must already be elements of A.

(d) True. If an element of A was not also an element of B, then this element would be missing in $A \cap B$ but included in A.

10. **(a)** $n(A \cap B) \le n(A)$. The set $A \cap B$ contains only the elements of A that are also elements of B. That is, $A \cap B \subseteq A$. Thus, $A \cap B$ cannot have more elements than A.

(b) $n(A) \le n(A \cup B)$. The set $A \cup B$ contains all of the elements of the set A *and* any additional elements of B that are not already included. Then $A \subseteq A \cup B$. Therefore, $A \cup B$ must have at least as many elements as A.

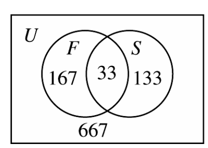
(c) Since $n(A \cap B) = n(A \cup B)$ and $A \cap B \subseteq A \subseteq A \cup B$, we can conclude that $A \cap B = A \cup B$. Thus $A \cup B$ has no additional elements besides those in $A \cap B$, and so neither A nor B has any additional elements. Since $A = A \cap B$ and $B = A \cap B$, we conclude that $A = B$. (*Caution*: This reasoning would not be valid if infinite sets were allowed.)

11. Three of the four regions in the 2-loop diagram account for 500 of the households. Thus there are 200 households in the overlapped region representing the number of households having both a TV and a computer.

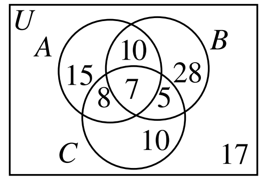
12. **(a)** $1000 \div 6 = 166.666\ldots$, so the largest element of S is $166 \cdot 6 = 996$. Therefore, $n(S) = 166$.

(b) The elements in $F \cap S$ are the multiples of 30. Since $1000 \div 30 = 33.33\ldots$, the largest element of $F \cap S$ is $33 \cdot 30 = 990$. Therefore $n(F \cap S) = 33$.

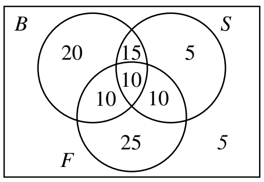
(c)



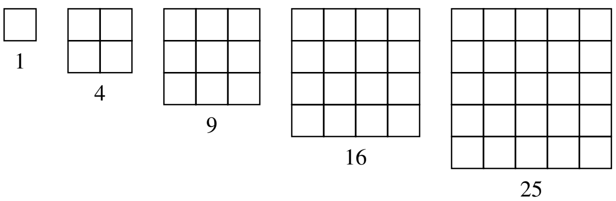
13. Use the fact that the eight regions pictured below are mutually disjoint sets and apply the strategy of working backwards, that is, start with $n(A \cap B \cap C) = 7$. Next use $n(A \cap B)$, $n(B \cap C)$, and $n(A \cap C)$ to find the values 10, 5, and 8, respectively. Then use $n(A)$, $n(B)$, and $n(C)$ to find the values 15, 28, and 10, respectively, and finally $n(U)$ to find the value 17.



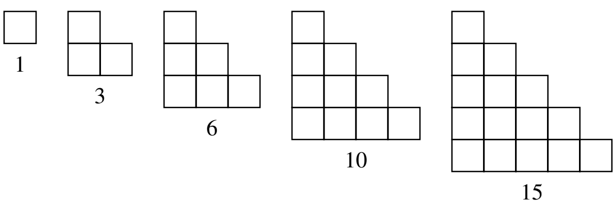
14. The 3-loop Venn diagram can be filled in the manner of the preceding problem. We now see there are 20 + 5 + 25 = 50 percent of the students who like just one sport and 5 percent do not like any of the three sports.



15. **(a)**



(b)



16. **(a)** 0101101

(b) 1100011

(c) 0100110

17. **(a)** 1110100* (last bit corrected)

(b) 0010*111 (fourth bit corrected)

(c) 1101000 is correct

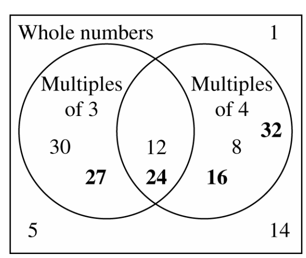
(d) 11111*11 (fifth bit corrected)

18. Answers will vary. Both “number” and “color” are abstract concepts that can be taught by giving concrete examples.

19. Answers will vary. One possible answer, using a cup with 5 marbles. Ask “how many marbles are in the cup?” to get the response “five.” Now remove a marble, and again ask how many marbles are in the cup, eliciting the response “four.” Continue to remove one marble at a time, until no marbles remain in the cup. Explain that the number of marbles in the empty cup is “zero.”

20. Answers will vary, but should include the concept that if one strip is shorter than a second strip, and the second strip is shorter than the third, then the first must be shorter than the third.

21. **(a)** The numbers that have been inserted are bold. Explanations to Zack will vary.



(b) Answers will vary

22. (a) Six ways: yrg, rgy, gyr, ygr, gry, ryg

(b) Six. The numbers 1, 2, 3 can be viewed as the bottom, middle, top positions for balls in the can. Therefore, each of the six ways to place the balls in the can are a distinct one-to-one correspondence.

(c) There are 12 ways to place the first egg, and 11 empty spaces in which to place the second egg. Therefore, the first two eggs can be put into the carton in 12×11 ways. This leaves 10 empty choices for the third egg, so there are $12 \times 11 \times 10$ ways to place the first three eggs. Continuing in this way shows there are $12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479{,}001{,}600$ ways to place the dozen eggs.

(d) If {1, 2, 3, …, 12} and {a, b, c, d, e, f, g, h, i, j, k, l} represent the set of eggs and the set of spaces in the carton from part (c), the number of one-to-one correspondences is the same as the number of ways to put the eggs into the carton, namely $12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479{,}001{,}600$.

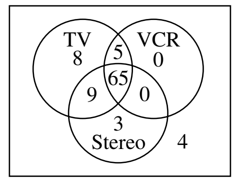
23. (a)

Row 0	1				
Row 1	1	1			
Row 2	1	2	1		
Row 3	1	3	3	1	
Row 4	1	4	6	4	1

(b) The table is the same as Pascal's triangle each entry in the table is the sum of the numbers in the row just above that are directly above and one column to the left. Using this pattern, we get
Row 5: 1 5 10 10 5 1 and
Row 6: 1 6 15 20 15 6 1.

24. Since we are given that $k < l$ and $l < m$, we can choose sets K, L, and M satisfying $K \subset L \subset M$ and $n(K) = k$, $n(L) = l$, and $n(M) = m$. By the transitive property of set inclusion (see Section 2.1, or just look at a Venn diagram) we know that $K \subset M$ and so $k < m$.

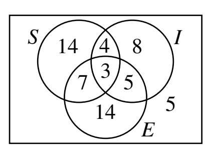
25. Evelyn's assumptions give the following diagram.



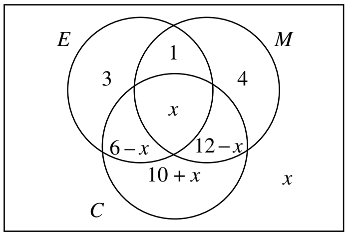
The zero values are obtained from Evelyn's assumption that anyone with a VCR also has a TV. Evelyn should question the survey because only 94 of the 100 households are represented. (Evelyn's assumption may be wrong—for example, there may be people who use a monitor instead of a TV to view videotapes—but these are very unlikely to account for 6% of a given sample.)

26. Consider the 35 students taking Arabic: 20 take only Arabic, 7 take both Arabic and Bulgarian (and possibly Chinese), telling us that 8 students are taking Arabic and Chinese and not Bulgarian. Similarly, there are 5 students taking Bulgarian and Chinese but not Arabic. The rest of the values in the Venn diagram are now easy to fill in. In particular, 3 students take all three languages, and 26 are not taking any of the three languages.

27. Construct a Venn diagram like the one shown, where S, I, and E are the sets of voters willing to raise sales, income, and excise taxes, respectively. Use a guess and check strategy in which you guess how many voters are willing to raise all three taxes and complete the diagram accordingly. The only way to account for all 60 voters is to start with 3 voters approving all three taxes.



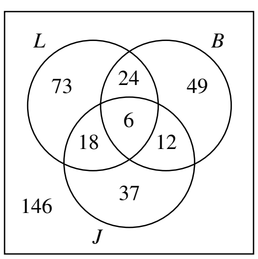
28. **(a)** The diagram shown below gives $36 + x = 40$, so $x = 4$. (This result can also be obtained using guess and check.) 4 students have been to all three countries.



(b) 10 + 4 = 14 students have been only to Canada.

29. **(a)** All three friends will meet at the mall every $3 \times 4 \times 5 = 60$th day, so there will be 6 days when all three get together. ($365 \div 60 = 6$ with a remainder of 5.)

(b) Since Letitia and Brianne are both at the mall 30 days, there are 24 of these days they will not be joined by Jake. Letitia and Jake will both be at the mall every $3 \times 5 = 15$th day, and $365 \div 15$ is 24 with a remainder of 5. This means Letitia and Jake are at the mall 18 days by themselves and the other 6 days are also joined by Brianne. Similar reasoning fills in the remaining regions of the Venn diagram shown.



(c) Jake will be at the mall by himself 37 days.

(d) None of the three is at the mall on 146 days of the year.

30. **(a)** The words 01 and 10 are incorrect, but cannot be corrected. Both can be changed to either 00 or 11 by changing a single bit.

(b) The words 001, 010, 100, 011, 101, and 110 are incorrect and each can be corrected by changing a single bit in just one way. The one bit different from the other matching bits is the bit in error.

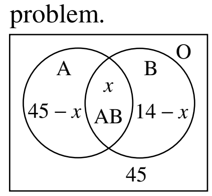
31. **(a)** Before the extension, the *A*, *B*, *C*, and *D* loops have five, four, five, and two 1s, respectively. Placing 1s in regions w and y and 0s in regions x and z makes all the loops contain an even number of 1s.
Loop A—w:1, a:1, b:0, d:1, e:1, h:1, i;1, k:0
Loop B—z:0, a:1, b:0, c:0, d:1, e:1, f:0, g:1
Loop C—y:1, d:1, e:1, f:0, g:1, h:1, i:1, j:0
Loop D—x:0, b:0, c:0, e:1, f:0, i:1, j:0, k:0

(b) The *A*, *B*, and *C* loops each have an odd number of 1s, and the *D* loop has an even number of 1s. The region in the Venn diagram inside loops *A*, *B*, and *C* but outside *D* is region d, the fourth bit of the word. Therefore, the bit in region d is incorrect and the corrected word is 001000111100100.

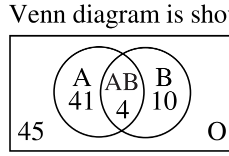
(c) The *A* and *C* loops each have five 1s, and the *B* and *D* loops each have six 1s. The region inside *A* and *C* and outside of *B* and *D* is region h, the eighth bit of the word. The corrected word is therefore 110011111101101.

32. Answers will vary.

33. Use the Venn diagram below to solve the problem.



$104 - x = 100$
$x = 4$
Four people are of type AB. The completed Venn diagram is shown below.



34. Answers will vary. Consult foreign language dictionaries for assistance.

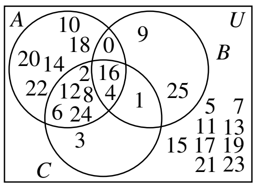
35. C, because *X* is mid-way between 0 and 600, and *W* is closer to *X* than *Y* is.

36. A. Seth read a total of 5 + 9 = 14 books and Anna read 16 books.

37. B. The problem condition is Max's age (M) is greater than Sam's age (S).

38. B. 7 = 4 + 3. The students taking French and algebra are the four students taking all three courses and the three students taking only French and algebra.

39. $U = \{0, 1, 2, 3, ..., 25\}$,
$A = \{0, 2, 4, ..., 24\}$,
$B = \{0, 1, 4, 9, 16, 25\}$,
$C = \{1, 2, 3, 4, 6, 8, 12, 16, 24\}$



40. **(a)** $\overline{A} = \{b, c, f, g\}$

(b) $A \cap B = \{e\}$

(c) $A \cup \overline{B} = \{a, b, d, e\}$

(d) $A \cap \overline{B} = \{a, d\}$

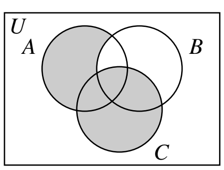
(e) $\overline{A} \cap \overline{B} = \{b\}$

(f) $\overline{A \cup B} = \{b\}$

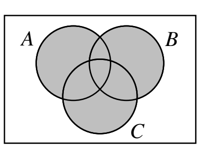
(g) $\overline{A \cap B} = \{a, b, c, d, f, g\}$

(h) $(A \cup B) \cap \overline{(A \cap B)} = \{a, c, d, f, g\}$

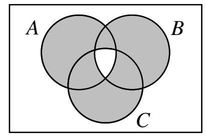
41. **(a)** $(A \cap \overline{B}) \cup C$



(b) $(A \cup B \cup C)$



(c) $(A \cup B \cup C) \cap \left(\overline{A \cap B \cap C}\right)$



COOPERATIVE INVESTIGATION
(page 110)
Counting Cars and Trains

1. 8 trains: P, (LG)W, W(LG), RR, RWW, WRW, WWR, WWWW

2. 16 trains: Y, PW, WP, (LG)R, R(LG), (LG)WW, W(LG)W, WW(LG), RRW, RWR, WRR, RWWW, WRWW, WWRW, WWWR, WWWWW

3. The number of trains of lengths 3, 4, and 5 are respectively 4, 8, and 16, revealing a doubling pattern. Continuation of the pattern shows there will be 32, 64, and 128 trains of respective lengths 6, 7, and 8.

4. 2^{n-1} of length n can be formed. It must be shown that there are twice as many trains of length n as there are of length $n - 1$. There are two ways to lengthen a train of length $n - 1$ by one unit: (1) add a white caboose, giving us 2^{n-2} trains of length n, all of which have white cabooses; and (2) lengthen the caboose by one unit, giving us another 2^{n-2} trains of length n, all of which have nonwhite cabooses. Altogether, the two methods give us $2^{n-2} + 2^{n-2} = 2(2^{n-2}) = 2^{n-1}$ different trains of length n. This counts all of the trains of length n, since any train of length n corresponds to a unique train of length $n - 1$ by either removing a white caboose or shortening a nonwhite caboose by one unit. Since the number of trains of length one is 2^0, there are $n - 1$ doublings to give us 2^{n-1} trains of length n. Alternatively, think of starting with a train of n white cars. Those could be glued together in all possible ways to form all possible trains of length n. To do this, at each of the $n - 1$ junctions between two white cars one must make one of two choices—to glue or not to glue. Hence, there are 2^{n-1} possibilities in all, as above.

5. **(i)** 5 trains: WWWW, WWR, WRW, RWW, RR

(ii) 8 trains: WWWWW, WWWR, WWRW, WRWW, RWWW, WRR, RWR, RRW

(iii) The number of RW-trains of length eight is $F_9 = 34$; 10 of these trains have five cars (WWRRR and its rearrangements).

(iv) The number of RW-trains of length 1, 2, 3, 4, 5, ... is 1, 2, 3, 5, 8, ..., the Fibonacci numbers beginning with F_2. Recall that $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Note that adding a white caboose to the trains of length $n + 1$ and adding a red caboose to trains of length n yields all of the RW-trains of length $n + 2$. Thus, the number of trains of length n is F_{n+1}.

6. **(i)** 2 trains: P, RR

(ii) 3 trains: Y, (LG)R, R(LG)

(iii) 13 trains: Br, (DG)R, R(DG), Y(LG), (LG)Y, PP, PRR, RPR, RRP, (LG)(LG)R, (LG)R(LG), R(LG)(LG), RRRR. None of these trains contain five cars.

(iv) The number of $\overline{\text{W}}$-trains of length 1, 2, 3, 4, 5, ... is 0, 1, 1, 2, 3, ..., again the Fibonacci numbers where we agree to set $F_0 = 0$. Note that adding cabooses of length $n - 1$, $n - 2$, $n - 3$, ..., 2 to the $\overline{\text{W}}$-trains of length 2, 3, 4, .., $n - 1$ gives all the $\overline{\text{W}}$-trains of length $n + 1$, except for the one additional train that consists of a single rod. This corresponds to the formula $1 + F_1 + F_2 + \cdots + F_{n-2} = F_n$. Thus the number of $\overline{\text{W}}$-trains of length n is F_{n-1}.

COOPERATIVE INVESTIGATION (page 115)
Diffy

1. It appears that the process will always stop.

2. Answers will vary. One possibility is given in each case.

(i) 1, 1, 1, 1

(ii) 1, 2, 1, 2

(iii) 3, 2, 0, 1

3. It looks as though four or five steps should suffice, but we should probably try some more extreme examples.

4. Answers will vary. Four such numbers are not too easy to find. One possibility is 6, 9, 17, 32.

5.

17	32	58	107
15	26	49	90
11	23	41	75
12	18	34	64
6	16	30	52
10	14	22	46
4	8	24	36
4	16	12	32
12	4	20	28
8	16	8	16
8	8	8	8
0	0	0	0

Problem Set 2.3 (page 116)

1. **(a)** **(i)** $A \cup B = \{$apple, berry, peach, lemon, lime$\}$
so $n(A \cup B) = 5$.

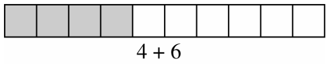
(ii) $A \cup C = \{$apple, berry, peach, lemon, prune$\}$
so $n(A \cup C) = 5$.

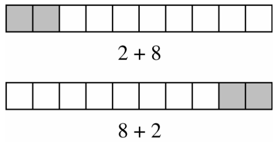
(iii) $B \cup C = \{$lemon, lime, berry, prune$\}$
so $n(B \cup C) = 4$ since lemon is a member of both sets.

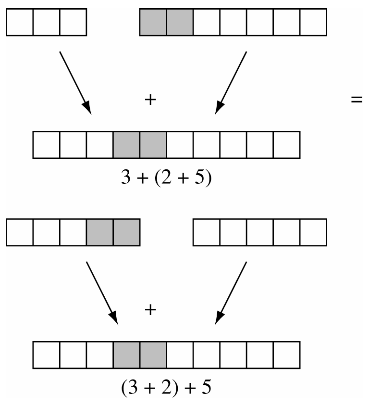
(b) (ii) and (iii) because the two sets in each case are not disjoint, that is, they have at least one member in common.

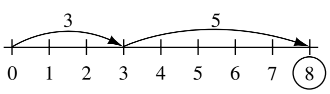
2. **(a)** B may contain 4, 5, 6, 7, or 8 elements. If B were to contain more than 8 elements, then $n(A \cup B)$ would be greater than 8 which contradicts the fact that $n(A \cup B) = 8$. Similarly, if B were to contain less than 4 elements, $n(A \cup B)$ would be less than 8.

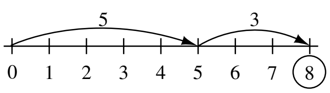
(b) If $(A \cap B) = \varnothing$, then $n(A) + n(B) = n(A \cup B)$, so $n(B) = 4$.

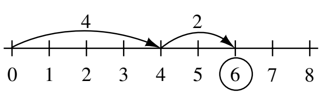
3. **(a)** 

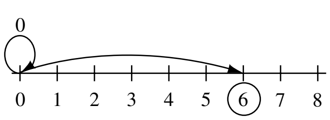
(b) 

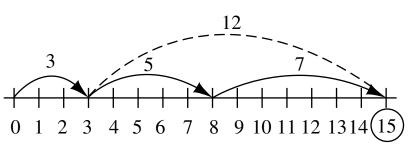
(c) 

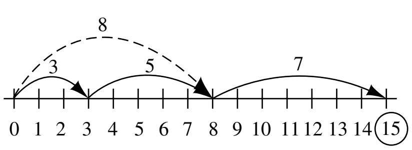
4. **(a)** 

(b) 

(c) 

(d) 

(e) 

(f) 

5. Answers will vary. One possibility is given. Art has 30 marbles and Barbara has 28 marbles. How many marbles do they have all together?

6. Answers will vary. One possibility is given. Town B is 18 miles due east of Town A. Town C is 25 miles due east of Town B. How far is Town C from Town A?

7. **(a)** Closed (since the sum of two positive multiples of 5 is a larger multiple of 5)

(b) Not closed; for example, 1 + 1000 = 1001

(c) Closed (since 0 + 0 = 0)

(d) Not closed; for example, 1 + 6 = 7

(e) Closed (since the sum of two numbers that are ≥ 19 is a larger number, which must be ≥ 19)

(f) Closed (since the sum of two multiples of 3 is a multiple of 3: $3a + 3b = 3(a + b)$)

8. **(a)** Commutative property of addition

(b) Closure property

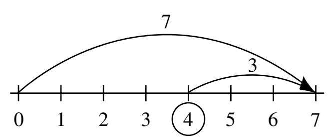
(c) Additive-identity property of zero

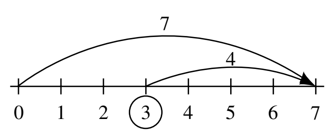
(d) Associative and commutative properties

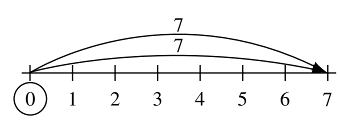
(e) Associative and commutative properties

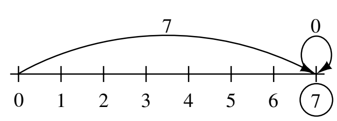
9. **(a)**
$$\begin{aligned}&(1+20)+(2+19)+(3+18)+\cdots+(10+11)\\ &\quad = 21+21+21+\cdots+21 \text{ (for 10 terms)}\\ &\quad = (10)(21) = 210\end{aligned}$$

(b) Associative and commutative properties

10. **(a)** 

(b) 

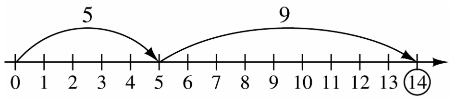
(c) 

(d) 

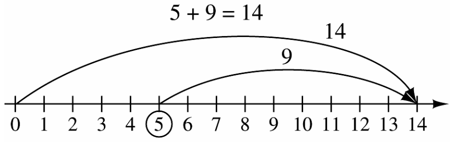
11. **(a)** $5 + 7 = 12$ $\quad 12 - 7 = 5$
$7 + 5 = 12$ $\quad 12 - 5 = 7$

(b) $4 + 8 = 12$ $\quad 12 - 8 = 4$
$8 + 4 = 12$ $\quad 12 - 4 = 8$

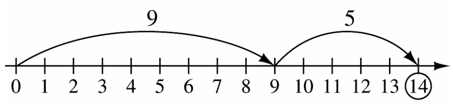
12. (a)



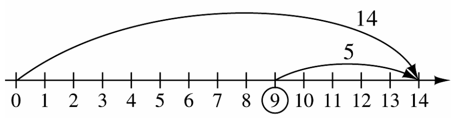
5 + 9 = 14



14 ± 9 = 5

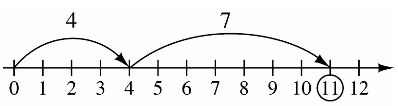


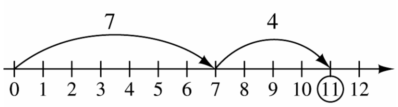
9 + 5 = 14

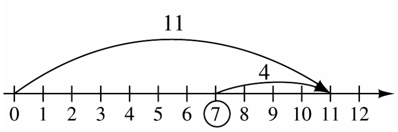


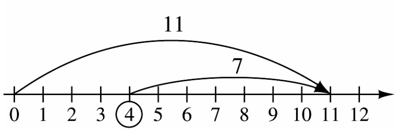
14 ± 5 = 9

(b)









13. (a) comparison

(b) measurement (number-line)

(c) missing addend

(d) take-away

14. Answers will vary. For example,

(a) *Take-away*: Maritza bought a booklet of 20 tickets for the amusement park rides. She used 6 tickets for the roller coaster. How many tickets does she have left?

(b) *Missing Addend*: Nancy's school has 23 proof-of-purchase coupons for graphing calculators. Her school will be entitled to a free overhead projection calculator with 40 proof-of-purchase coupons. How many more coupons do they need to get the free calculator?

(c) *Comparison:* Oak Ridge Elementary School has an enrollment of 482 students and Crest Hill School has an enrollment of 393 students. How many more students does Oak Ridge have than Crest Hill?

(d) *Measurement*: A fireman climbed up 11 rungs on a ladder, but the smoke was too thick and he came down 3 rungs. How many rungs up the ladder is the fireman?

15. Jeff has read through page 240. Therefore, the number of pages is 257 – 240 = 17 pages or 257 – 241 + 1 = 17 pages.

16. Use the guess and check method. Some answers will vary.

(a) (8 – 5) – (2 – 1) = 2

(b) 8 – (5 – 2) – 1 = 4

(c) ((8 – 5) – 2) – 1 = 0

(d) 8 + (5 – 2) + 1 = 12

(e) (8 + 5) – (2 + 1) = 10

17. (a) First fill in the squares by noting that 3 – 1 = 2, 4 – 2 = 2, and 7 – 2 = 5. Then complete the circles.

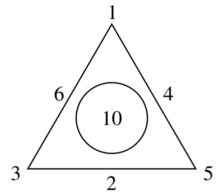
5	2	7
1	2	3
6	4	7

(b) Use the guess and check method.

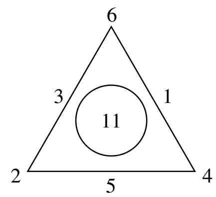
4	2	6
5	3	8
9	5	7

18. Write all possible combinations of 3 different numbers whose sum is the indicated number in the triangle. Then, place numbers which occur in more than one sum at the vertices as shown on the next page.

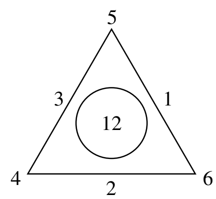
(a) 1 + 3 + 6
2 + 3 + 5
4 + 5 + 1



(b) 1 + 4 + 6
2 + 3 + 6
5 + 4 + 2



(c) 1 + 5 + 6
2 + 4 + 6
3 + 4 + 5



19. Blake has one more marble than before, and Andrea has one less than before, so now Blake has two more marbles than Andrea. Use a small number of marbles, say five each, to demonstrate what happened. After giving Blake one marble, he has 6 and Andrea has 4, and thus clearly Blake has two more marbles than Andrea.

20. Answers will vary.

21. Answers will vary.

22. **(a)** 0 is the cardinal number of the empty set, $n\{\varnothing\} = 0$. If the set A has a elements in it (i.e., $n(A) = a$), then $n(A) + n(\varnothing) = n(A \cup \varnothing)$. But, $A \cup \varnothing = A$, since there are no elements in $\varnothing$ to add to the set A. Thus, $n(A \cup \varnothing) = n(A) = a$.

(b) Answers will vary, but the major idea is to give an example of starting with a group of children in the room, playing the role of A, and adding all others in the room who are over 80 years old. As there is no addition to A, the number of elements in it doesn't change, and so $a + 0 = a$.

23. Answers will vary but should include that given two of the minuend, subtrahend, and answer, the third can be found. Furthermore, students often struggle when problems are presented in "non-conventional" ways. (This is worth the struggle as it is the beginning of algebra.) Many elementary school students have difficulty when there is a blank in the front of the number sentence.

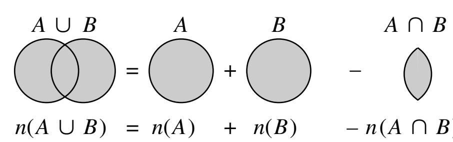
24. They must both love some of the same mice. In fact, there are five mice that Angel and Jane both love since $8 = 7 + 6 - n$ (where there are n mice they both love).

25. **(a)** This student is "subtracting up", which is a common mistake.

(b) Answers will vary. For example: After reminding her about place value, ask her to do a two-digit subtraction such as $73 - 35$ and reflect on the similarities between how to subtract two-digit numbers and how to subtract three-digit numbers. That could be followed with $736 - 327 = ?$ as a next level problem that is easier than the one in the problem.

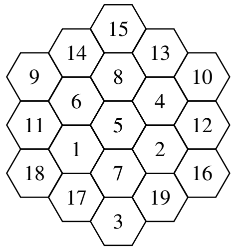
26. Phillip does two things incorrectly. First, he sees that he can't subtract the 9 from the 6, so he goes to the next column to do an exchange. He changes the 8 to a 7, but then, instead of having 16 ones, he adds 1 to 6 to give 7. He has misunderstood that the 1 he got from the exchange is actually 10, not 1 unit. Phillip then proceeds to subtract up anyway (he mistakenly writes $7 - 9$ as 2).

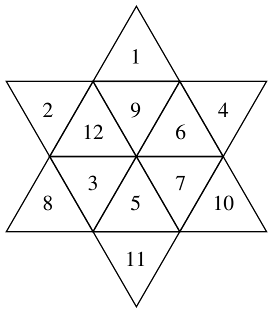
27.



Note that elements of $A \cap B$ are counted twice when calculating $n(A) + n(B)$. We compensate by subtracting $n(A \cap B)$, giving $n(A \cup B)$.

28. Let
$T = \{n \in N \mid n \text{ is a multiple of 12 and } n \le 200\}$
and
$F = \{n \in N \mid n \text{ is a multiple of 5 and } n \le 200\}$.
The number we need to find is
$n(T \cup F) = n(T) + n(F) - n(T \cap F)$
$= 16 + 40 - 3 = 53$.

29. 

30. 

31. {0}, since 0 – 0 = 0.

32. **(a)** {2, 3, 4, 5, 6, 7, 8, 9, …} since 2 + 2 = 4, 2 + 3 = 5, 2 + 4 = 6, etc.

(b) {0, 1}

(c) No, because the set of all whole numbers, {0, 1, 2, …}, fits the given description.

(d) Whole numbers that must be in C are all even numbers ≥ 2. Zero and odd numbers may or may not be in C.

33. **(a)** Use the formula $t_n = \frac{n(n+1)}{2}$.

n	1	2	3	4	5	6
t_n	1	3	6	10	15	21
n	7	8	9	10	11	12
t_n	28	36	45	55	66	78
n	13	14	15			
t_n	91	105	120			

(b) 11 = 10 + 1
12 = 6 + 6
13 = 10 + 3
14 = 10 + 3 + 1
15 = 15
16 = 15 + 1
17 = 15 + 1 + 1
18 = 15 + 3
19 = 10 + 6 + 3
20 = 10 + 10
21 = 21
22 = 21 + 1
23 = 10 + 10 + 3
24 = 21 + 3
25 = 15 + 10

(c) For example, choose 73 and 74.
73 = 45 + 28, and 74 = 45 + 28 + 1. Three is the maximum number of triangular numbers needed.

(d) Any whole number may be written as the sum of at most three triangular numbers.

34. It can be shown that Sameer is correct. The easiest way to find a sum is to use the largest Fibonacci number possible as the next summand, a procedure sometimes called a “greedy algorithm.” For example, to express 100 as a Fibonacci sum, first write down 89 since it is the largest Fibonacci number less than 100. This leaves 11, so next write down 8. The 3 that still remains to be accounted for is a Fibonacci number, so 100 = 89 + 8 + 3.

35. First find the top row and the left column. For example, the fourth entry in the left column must be $6 - 2 = 4$, and then the third entry in the top row is $5 - 4 = 1$, and so on. The completed table is shown.

+	5	4	1	6	9	2	0	8	7	3
3	8	7	4	9	12	5	3	11	10	6
9	14	13	10	15	18	11	9	17	16	12
6	11	10	7	12	15	8	6	14	13	9
4	9	8	5	10	13	6	4	12	11	7
0	5	4	1	6	9	2	0	8	7	3
7	12	11	8	13	16	9	7	15	14	10
5	10	9	6	11	14	7	5	13	12	8
2	7	6	3	8	11	4	2	10	9	5
1	6	5	2	7	10	3	1	9	8	4
8	13	12	9	14	17	10	8	16	15	11

36. Answers will vary.

37. **(a)** $0 = 5 - (1 + 4)$, $1 = 5 - 4$, $2 = (1 + 5) - 4$, $3 = 4 - 1$, $4 = 5 - 1$, $5 = 1 + 4$, $6 = 1 + 5$

(b) The single roll suffices: $0 = 5 - (2 + 3)$, $1 = 3 - 2$, $2 = 5 - 3$, $3 = 5 - 2$, $4 = (5 + 2) - 3$, $5 = 2 + 3$, $6 = (5 + 3) - 2$.

(c) Only the even numbers 0, 2, 4 and 6 can be formed, so at least one more roll is required.

38. Answers will vary. Two examples: A train is called colorful if no two consecutive cars are the same color. Answer questions 1 through 4 for a colorful train. Suppose that Larisa won't ride in any train with a red car. Answer questions 1 through 4 if Larisa is to go on the train. There are many more responses possible.

39. Answers will vary.

40. **(a)** Corey is correct. The sum of two even numbers can be thought of as the sum of two rows having m columns with two rows having n columns. The result is two rows having $m + n$ columns. Since the result can always be represented using two rows each having the same number of columns, the result is always an even number.

(b) Maya is mistaken. This can be shown by a counterexample. For example, the sum of the odd numbers 3 and 5 is the even number 8.

41. Statement B does not correspond to the number sentence $15 - 8 = \square$. The other problems illustrate take-away (A), comparison (C), and missing addend (D).

42. C. Jordan started with his allowance, x, and subtracted \$4 from it because he spent that amount. He is left with $x - 4$ which is the \$16 he has left. Thus $x - 4 = 16$.

43. B. Corazon added the two numbers. They are depicted as -4 and then moving from -4 to 2, which is 6. Therefore, the answer is $-4 + 6 = 2$.

44. **(a)** $n(A \cap B) = 0 + 2 = 2$

(b) $n(B \cup C) = 0 + 3 + 2 + 4 + 1 + 6 = 16$

(c) $n(\overline{A} \cup C) = 1 + 2 + 3 + 4 + 6 + 7 = 23$

(d) $n(B \cap C) = 2 + 4 = 6$

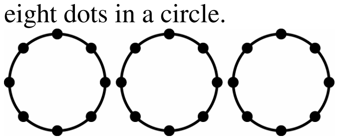
(e) $n(A \cap \overline{C}) = 5 + 0 = 5$

(f) $n(\overline{A \cup B} \cup C) = 7 + 6 + 1 + 2 + 4 = 20$

45. $100 - 10 - 2 \cdot 20 = 50$ students take only geometry and $300 - 3 \cdot 50 - 3 \cdot 20 - 10 = 80$ students are not enrolled in any of these three courses.

SCHOOL BOOK PAGE (page 129)
Modeling Division in Grade Four

1. A fourth grader might arrange each group of eight dots in a circle.



2. Make any arrangement of 24 red counters. Remove six counters and place them in a pile. Then remove six more counters, and place them in another pile. Continue in this manner until there are no counters left in the original arrangement. Discuss with students that the counters are now in four piles of six counters each, so
$24 \div 6 = 4$

3. Arrange the counters in piles of two. Then have students count by 2's while pointing at each pile. When students reach 24, have them count the number of piles of two. Have them discuss why $2 \times 12 = 24$ means that
$24 \div 2 = 12$.

4. Give students use 36 counters, separated into piles of four. Then have them count the number of piles of four.

Problem Set 2.4 (page 136)

1. **(a)** $3 \times 5 = 15$, set model (repeated addition)

   (b) $6 \times 3 = 18$, number-line model

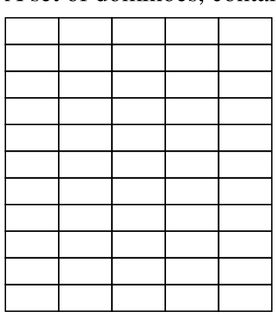
   (c) $5 \times 3 = 15$, set model

   (d) $3 \times 6 = 18$, number-line model

   (e) $8 \times 4 = 32$, rectangular area model

   (f) $3 \times 2 = 6$, multiplication tree model

2. **(a)** A set of dominoes, containing 11 stacks of 5 dominoes is modeled by a rectangular array.



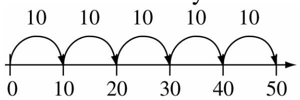
$11 \times 5 = 55$
The set contains 55 dominoes.

   (b) The outfits that can be made from 3 skirts and 6 blouses is modeled by a Cartesian product.

	b_1	b_2	b_3	b_4	b_5	b_6
s_1	(s_1, b_1)	(s_1, b_2)	(s_1, b_3)	(s_1, b_4)	(s_1, b_5)	(s_1, b_6)
s_2	(s_2, b_1)	(s_2, b_2)	(s_2, b_3)	(s_2, b_4)	(s_2, b_5)	(s_2, b_6)
s_3	(s_3, b_1)	(s_3, b_2)	(s_3, b_3)	(s_3, b_4)	(s_3, b_5)	(s_3, b_6)

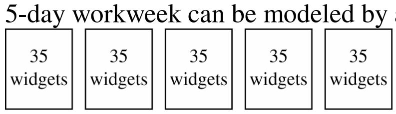
$3 \times 6 = 18$
She has 18 outfits she can wear.

(c) A hike of 10 miles each day for 5 days can be modeled by a number line.



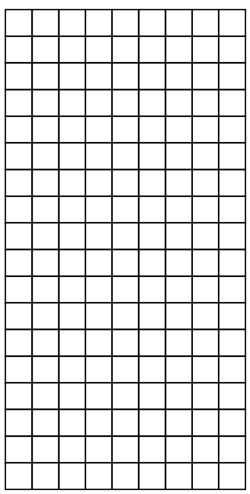
$5 \cdot 10 = 50$
He hiked 50 miles.

(d) The production of 35 widgits a day in a 5-day workweek can be modeled by a set.



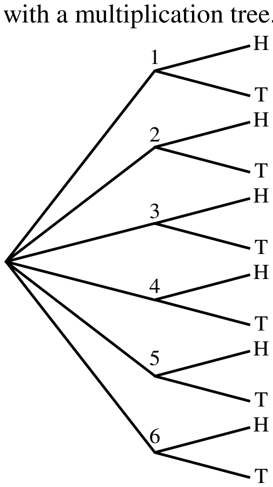
$5 \cdot 35 = 175$
The company makes 175 widgits in a 5-day workweek.

(e) The sunroom can be modeled by a rectangular area.



$9 \times 18 = 162$
She needs 162 1-square-foot tiles.

(f) The outcomes that result from rolling a die and flipping a coin can be modeled with a multiplication tree.



$6 \times 2 = 12$
12 outcomes are possible.

3. **(a)** Answers will vary.

(b) **(i)** $4 \cdot 9 = 9 + 9 + 9 + 9 = 36$

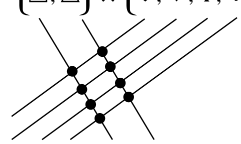
(ii) 7×536
$= 536 + 536 + 536 + 536 + 536$
$+ 536 + 536 = 3752$

(iii) $6 \times 47{,}819$
$= 47{,}819 + 47{,}819 + 47{,}819$
$+ 47{,}819 + 47{,}819$
$+ 47{,}819 = 286{,}914$

(iv) Using the commutative property of multiplication,
$56{,}108 \times 6 = 6 \times 56{,}108$
$= 56{,}108 + 56{,}108 + 56{,}108$
$+ 56{,}108 + 56{,}108$
$+ 56{,}108 = 336{,}648$

4. **(a)** Each of the *a* lines from set *A* intersects each of the *b* lines from set *B*. Since the lines for *A* are parallel and the lines for *B* are parallel, the intersection points will all be distinct.

(b) $\{\square, \triangle\} \times \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$



5. **(a)** Not closed. For example, $2 \times 2 = 4$, which is not in the set.

(b) Closed. $0 \times 0 = 0, 0 \times 1 = 0, 1 \times 1 = 1,$ $1 \times 0 = 0$. All products are in the set.

(c) Not closed. For example, $2 \times 4 = 8$, which is not in the set.

(d) Closed. The product of any two even whole numbers is always another even whole number.

(e) Closed. The product of any two odd whole numbers is always another odd whole number.

(f) Not closed. For example, $2 \times 2^3 = 2^4$, which is not in the set.

(g) Closed. $2^a \times 2^b = 2^{a+b}$ for any whole numbers a and b.

(h) Closed. $7^a \times 7^b = 7^{a+b}$ for any whole numbers a and b.

6. (a) Closed. The product of any two whole numbers in the set $W - \{5\}$ is also in the set, and 5 is not a product of any numbers in the set (5 is prime).

(b) Not closed, since $2 \times 3 = 6$.

(c) Closed. The product of any two whole numbers in the set $W - \{2, 3\}$ is also in the set, and 2 and 3 are not products of numbers in the set (2 and 3 are prime).

7. (a) Commutative property of multiplication

(b) Distributive property of multiplication over addition

(c) Multiplication-by-zero property

(d) Distributive property of multiplication over addition

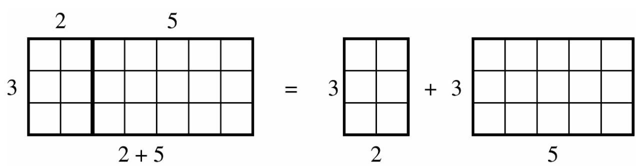
(e) Associative property of multiplication

(f) Multiplicative identity property of one

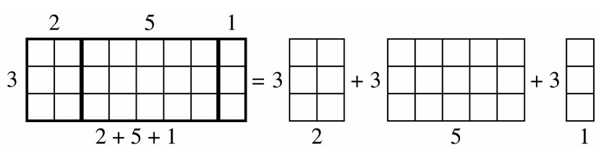
8. (a) Commutative property: $5 \times 3 = 3 \times 5$

(b) Commutative property: $6 \times 2 = 2 \times 6$

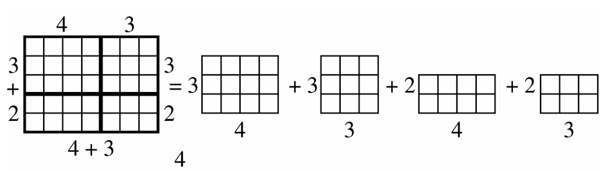
9. (a)



(b)



(c)



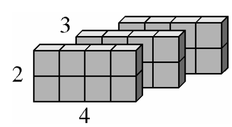
10. The rectangle is $a + b$ by $c + d$, so its area is $(a + b) \times (c + d)$. The rectangle labeled F is a by c so its area is ac. Similarly, the areas of the rectangles O, I, and L are given by the respective products ad, bc, and bd. Summing the areas of the four rectangles labeled F, O, I, and L gives the area of the large rectangle, so $(a + b) \times (c + d) = ac + ad + bc + bd$.

11.

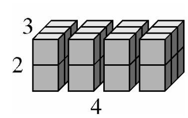
A	$a \cdot d$	$a \cdot e$	$a \cdot f$
b	$b \cdot d$	$b \cdot e$	$b \cdot f$
c	$c \cdot d$	$c \cdot e$	$c \cdot f$
	d	e	f

$$(a + b + c) \cdot (d + e + f)$$
$$= a \cdot d + a \cdot e + a \cdot f + b \cdot d + b \cdot e + b \cdot f + c \cdot d + c \cdot e + c \cdot f$$

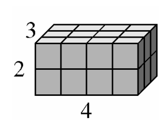
12.



$3 \cdot (2 \cdot 4)$



$(3 \cdot 2) \cdot 4$



$3 \cdot 2 \cdot 4$

13. **(a)** Distributive property of multiplication over addition:
$7 \cdot 19 + 3 \cdot 19 = (7 + 3) \cdot 19$
$= 10 \cdot 19 = 190$

(b) Distributive property of multiplication over addition:
$24 \cdot 17 + 24 \cdot 3 = 24 \cdot (17 + 3) = 24 \cdot 20$
$= 480$

(c) Distributive property, associative property, and/or multiplication property of zero:
$36 \cdot 15 - 12 \cdot 45 = (12 \cdot 3) \cdot 15 - 12 \cdot (3 \cdot 15) = 12 \cdot (3 \cdot 15) - 12 \cdot (3 \cdot 15)$
$= (12 - 12) \cdot (3 \cdot 15) = 0 \cdot (3 \cdot 15) = 0$

14. **(a)** $18 \div 6 = 3$

(b) $18 \div 3 = 6$

15. **(a)** $4 \times 8 = 32$, $8 \times 4 = 32$, $32 \div 8 = 4$, $32 \div 4 = 8$

(b) $6 \times 5 = 30$, $5 \times 6 = 30$, $30 \div 5 = 6$, $30 \div 6 = 5$

16. **(a)** Repeated subtraction

(b) Partition

(c) Missing $1/n$ factor or repeated subtraction

17. **(a)** $19 - 5 = 14$, which is greater than 5, so we must subtract 5 again. $14 - 5 = 9$, which is still greater than 5. Subtracting 5 again, $9 - 5 = 4$, which is less than 4, so we are done. Because we subtracted 5 three times, the quotient is 3. The remainder is 4. This is represented as
$a = 19 \overset{-5}{\Rightarrow} 19 - 5 = 14 \overset{-5}{\Rightarrow} 14 - 5 = 9 \overset{-5}{\Rightarrow}$
$9 - 5 = 4$ (done)

(b) $18 - 9 = 9$. Since $9 \le 9$, subtract 9 again to get $9 - 9 = 0$. The remainder is 0. We subtracted twice, so the quotient is 2.

(c) $25 - 8 = 17$, which is greater than 8, so we must subtract 8 again. $17 - 8 = 9$, which is still greater than 8. Subtracting 8 again gives $9 - 8 = 1$, which is less than 8, so we are done. Because we subtracted 8 three times, the quotient is 3. The remainder is 1.

(d) $14 - 7 = 7$. Since $7 \le 7$, subtract 7 again to get $7 - 7 = 0$. The remainder is 0. We subtracted twice, so the quotient is 2.

(e) 7 is already less than 14, so there is no subtraction. The remainder is 7 and, since there were no subtractions, the quotient is 0.

(f) Answered in each part separately.

18. Answers will vary, but should include the fact that, if we are dividing a by b, then if $a < b$, the process is finished. If $a = b$ or $a > b$, the either $a - b = 0$ or $a - b > 0$. The remainder cannot be negative in either case because the process stops before that can happen.

19. **(a)** Since 78 – 13 = = = = = = gives 0, $78 \div 13 = 6$.

(b) Since 832 – 52 = = = = = = = = = = = = = = = = gives 0, $832 \div 52 = 16$.

(c) Since 96 – 14 = = = = = = gives 12, $96 \div 14 = 6$ R 12.
(*Note*: Stop pressing = when the result is less than the divisor, 14.)

(d) Since 548,245 – 45,687 = = = = = = = = = = = = gives 1, $548{,}245 \div 45{,}687 = 12$ R 1.

20. **(a)** $y = (5 \cdot 5) + 4 = 25 + 4 = 29$

(b) $3x + 2 = 20$, so $x = 6$.

21. **(a)** $3^{20} \cdot 3^{15} = 3^{20+15} = 3^{35}$

(b) $4^8 \cdot 7^8 = (4 \cdot 7)^8 = 28^8$

(c) $(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$

(d) $x^7 \cdot x^9 = x^{7+9} = x^{16}$

(e) $y^3 \cdot z^3 = (y \cdot z)^3$ or $(yz)^3$

(f) $(t^3)^4 = t^{3 \cdot 4} = t^{12}$

22. **(a)** $8 = 2 \cdot 2 \cdot 2 = 2^3$

(b) $4 \cdot 8 = (2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^5$

(c) $1024 = 2^{10}$

(d) $8^4 = (2^3)^4 = 2^{3\cdot 4} = 2^{12}$

23. Use the guess and check method.

(a) $m = 4$

(b) $n = 12$

(c) $p = 10$

(d) $q = 20$

24. Answers will vary. The following are sample answers.

(a) Have children hold hands in, say, groups of four. Have one group come to the front of the class, and verify that $1 \times 4 = 4$. Next have a second group come forward, and verify that $2 \times 4 = 8$. And so on.

(b) Have the children form rectangles of various sizes, and then count off row by row to obtain the total number of children in the array.

(c) Choose how many teams to be formed (the divisor) and see how large the teams will be, with an equal number on each team.

(d) For a given number of children on a team, add teams until the total number of children reaches a certain total. Then count how many teams were needed.

25. Answers will vary. These are sample problems:
Peter has a board 14 feet long, and each box he makes requires 3 feet of board. How many completed boxes can be made? Answer = 4.
Tina has a collection of 14 antique dolls. A display box can hold at most three dolls. How many boxes does Tina require to display her entire collection? Answer = 5.
Andrea has 14 one-by-one foot paving stones to place on her 3-foot-wide walkway. How many feet of walk can she pave, and how many stones will she have left over for a future project? Answer = 4 feet of walk paved, with 2 stones remaining.

26. Answers will vary, but should make the point that the division $a \div b$ is approached by forming a train of length a, and then seeing how many cars of length b are needed to form an equally long train.

27. As listed in the theorem, starting with the second one,

- It doesn't matter in which order you multiply two numbers.
- It doesn't matter which way you group the terms when multiplying three numbers.
- Multiplying by 1 never changes the number (and 1 is the only number for which that happens.)
- Zero times any number is zero.

28. The student struggles to see the relationship between multiplication and division exercises. In addition, students typically have more difficulty when the blank or box is at the beginning of the equation.

29. Answers will vary, but you could first ask for her solution. She may say that since the total cost of one nut and one bolt together is \$1.00, and she is buying 18 of them, the total cost is \$18.00. This is a wonderful opportunity for the Mathematical Habit of the Mind because you could then ask her to justify her response:

$$\begin{aligned} 18\cdot 86 + 18\cdot 14 &= 18\cdot(86+14) \\ &= 18\cdot 100 = 1800 \text{ cents or \$18.00} \end{aligned}$$

You could then ask her a question that would lead to her saying it is the distributive property.

30. The student subtracted the number of cupcakes Nelson baked in each pan instead of dividing to find the number of pans needed to bake all of the cupcakes. This is common because students understand they need to do something with the numbers, but they aren't sure what. In this case, the student is not thinking of grouping the cupcakes into pans, which should be a signal that he should divide rather than subtract, which suggests removal.

31. The operation is closed, commutative, and associative. The circle is the identity, since if it is either of the "factors" the outcome of the operation is the other factor.

32. (a) The magic multiplication constant is $4096 = 2^{12}$.

(b) The exponents form a magic addition square with the magic addition constant 12. It is clear how the multiplication square has been formed. For example, the product of the upper row is $2^3 \times 2^8 \times 2^1 = 2^{3+8+1} = 2^{12}$. That is, the product is always 2^{12}.

2^3	2^8	2^1
2^2	2^4	2^6
2^7	2^0	2^5

3	8	1
2	4	6
7	0	5

(c)

$27 = 3^3$	$6561 = 3^8$	$3 = 3^1$
$9 = 3^2$	$81 = 3^4$	$729 = 3^6$
$2187 = 3^7$	$1 = 3^0$	$243 = 3^5$

33. **(a)** Since $2 \times 185 = 370, 500 - 370 = 130,$ and $130 \div 2 = 65,$ the question is "How many tickets must still be sold?'

(b) $67 \div 12 = 5 \text{ R } 7.$ Thus 5 answers "How many full cartons of a dozen eggs can be filled?' and 7 answers "How many eggs are in the partially filled carton?'

34. Note that $318 \div 14 = 22 \text{ R } 10.$

(a) 22 is the answer to "How many full rows of chairs are set up in the auditorium?

(b) 10 answers "How many chairs are in the incomplete row at the rear of the auditorium?"

35. When divided by 5 the remainder is 4, so the number is in the list 4, 9, 14, 19, ..., 94, 99. When divided by 4 the remainder is 3, the list of remainders are 0, 1, 2, 3, 0, 1, 2, 3, ..., 0, 1, 2, 3, so the number we seek is one of 19, 39, 59, 79, or 99. When divided by 3 the remainders in this list of 5 numbers is 1, 0, 2, 1, 0. Thus the number we seek is 59, which we note has a remainder of 1 when divided by 2.

36. $0 = 2 \times (3-3), 1 = 2^{(3-3)}, 2 = 2 + (3-3),$
$3 = 3^{(3-2)}.4 = 3 + (3-2), 5 = 2^3 - 3,$
$6 = 3^2 - 3, 7 = 3 \times 3 - 2, 8 = 3 + 3 + 2,$
$9 = ?, 10 = ?, 11 = 2^3 + 3 \text{ or } (3 \times 3) + 2,$
$12 = 3^2 + 3, 13 = ?, 14 = ?, 15 = 3 \times (3+2),$
$16 = ?, 17 = ?, 18 = 2 \times 3 \times 3$

37. **(a)** Note that $T_{n+1} = T_n + t_{n+1}$ for $n \geq 1.$

$$T_4 = 10 + 10 = 20 = \frac{4 \cdot 5 \cdot 6}{6}$$
$$T_5 = 20 + 15 = 35 = \frac{5 \cdot 6 \cdot 7}{6}$$
$$T_6 = 35 + 21 = 56 = \frac{6 \cdot 7 \cdot 8}{6}$$
$$T_7 = 56 + 28 = 84 = \frac{7 \cdot 8 \cdot 9}{6}$$
$$T_8 = 84 + 36 = 120 = \frac{8 \cdot 9 \cdot 10}{6}$$
$$T_9 = 120 + 45 = 165 = \frac{9 \cdot 10 \cdot 11}{6}$$
$$T_{10} = 165 + 55 = 220 = \frac{10 \cdot 11 \cdot 12}{6}$$
$$T_{11} = 220 + 66 = 286 = \frac{11 \cdot 12 \cdot 13}{6}$$
$$T_{12} = 286 + 78 = 364 = \frac{12 \cdot 13 \cdot 14}{6}$$

The pattern continues to hold.

(b) $T_{100} = \dfrac{100 \cdot 101 \cdot 102}{6} = 171,700$

38. $t_{49} = \dfrac{49 \cdot 50}{2} = 49 \cdot 25 = 7^2 \cdot 5^2 = 35^2$

$t_{288} = \dfrac{288 \cdot 289}{2} = 144 \cdot 289 = 12^2 \cdot 17^2 = 204^2$

$t_{1681} = \dfrac{1681 \cdot 1682}{2} = 1681 \cdot 841$
$= 41^2 \cdot 29^2 = 1189^2$

39. D. $12 = 2 \cdot 3 \cdot 2$

40. B. $3 \times (4 \times 4)$

41. C. $10 = 2 \times 20 \div 4$

42. D. It is an array of 3 by 4. State assessments are nearly always done in April, so it is probably not a Valentine's Day problem.

43. $2 + 7 = 9, 7 + 2 = 9, 9 - 7 = 2, 9 - 2 = 7$

44. Since $n(A) + n(B) = n(A \cup B)$, we may conclude that $n(A \cap B) = 0$.

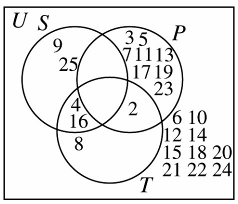
45. $2 \times 3 = 6, 3 \times 2 = 6$

Chapter Review Exercises (page 144)

1. **(a)** $S = \{4, 9, 16, 25\}$
$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$
$T = \{2, 4, 8, 16\}$

(b) $\overline{P} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$
$S \cap T = \{4, 16\}$
$S \cup T = \{2, 4, 8, 9, 16, 25\}$
$S \cap \overline{T} = \{9, 25\}$

2.



3. **(a)** $A \subseteq A \cup B$

(b) If $A \subseteq B$ and $A \neq B$, then $A \subset B$.

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \cup \varnothing = A$

4. $n(S) = 3, n(T) = 6,$
$n(S \cup T) = n(\{\text{s, e, t, h, o, r, y}\}) = 7,$
$n(S \cap T) = n(\{\text{e, t}\}) = 2,$
$n(S \cap \overline{T}) = n(\{s\}) = 1,$
$n(T \cap \overline{S}) = n(\{\text{h, o, r, y}\}) = 4$

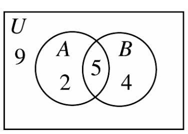
5.

1	4	9	16	25	36	49	64	81	100
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
a	b	c	d	e	f	g	h	i	j

6. There is a one-to-one correspondence between the set of cubes and a proper subset. For example,

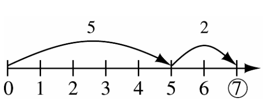
1	8	27	64	125	…	K^3
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		
1	2	3	4	5	…	K

7.



8. **(a)** Suppose $A = \{\text{a, b, c, d, e}\}$ and $B = \{■, ★\}$. Then $n(A) = 5$, $n(B) = 2$, $A \cap B = \varnothing$ and $n(A \cup B) = n(\{\text{a, b, c, d, e, �, �}\}) = 7$.

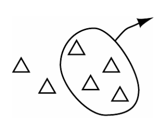
(b)



9. **(a)** Commutative property of addition:
$7 + 3 = 3 + 7$

(b) Additive-identity property of zero:
$7 + 0 = 7$

10. **(a)**



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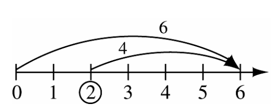
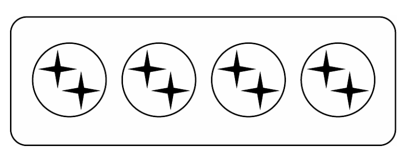


11. **(a)**



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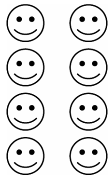

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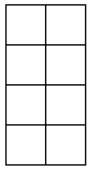

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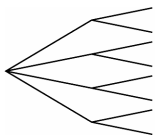

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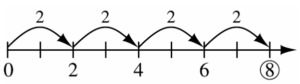


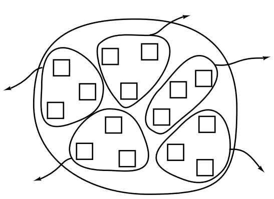
12. (a) $A \times B = \{(p, x), (p, y), (q, x), (q, y), (r, x), (r, y), (s, x), (s, y)\}$

(b) Since $n(A) = 4$, $n(B) = 2$, and $n(A \times B) = 8$, the Cartesian product models $4 \times 2 = 8$.

13. Answers will vary. Since $36 = 3 \times 3 \times 4$, one possibility is $6" \times 6" \times 8"$.

14. Since 92 ÷ 12 = 7 R 8, there are eight rows (7 full rows, and a partial row of 8 soldiers in the back).

15. (a)



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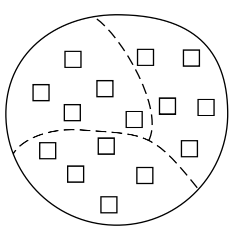
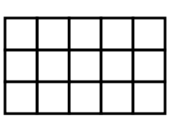


(c)



Chapter Test (page 145)

1. (a) $4 \times 2 = 8$

(b) $12 \div 3 = 4$

(c) $5 \cdot (9 + 2) = 5 \cdot 9 + 5 \cdot 2$

(d) $10 - 4 = 6$

2. 15th—ordinal
1040—nominal
$253—cardinal

3. (a) Yes, because $2^a \cdot 2^b = 2^{a+b}$.

(b) No. For example, $2 \in S$ and $4 \in S$ but $4 + 2 \notin S$.

4. Answers can vary. For example, let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Then $A \subset B$ so $n(A) < n(B)$. That is, $5 < 8$.

5. (a) W is closed under &, since $a + b + ab$ is an element of W for any two whole numbers a and b.

(b) Using the properties of whole number arithmetic, $a \,\&\, b = a + b + ab$ $= b + a + ba = b \,\&\, a$ for all whole numbers a and b, so & is commutative.

(c) Using the properties of whole number arithmetic,

$$\begin{aligned} a \,\&\, (b \,\&\, c) &= a + (b \,\&\, c) + a(b \,\&\, c) \\ &= a + b + c + bc + a(b + c + bc) \\ &= a + b + c + ab + bc + ac + abc \end{aligned}$$

Similarly,

$$\begin{aligned} (a \,\&\, b) \,\&\, c &= (a + b + ab) \,\&\, c \\ &= (a + b + ab) + c + (a + b + ab)c \\ &= a + b + c + ab + bc + ac + abc \end{aligned}$$

Therefore, for all a, b, and c, $a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$ so & is associative.

(d) $0 \,\&\, a = 0 + a + 0(a) = a$ and $a \,\&\, 0 = a + 0 + (a)0 = a$ for all whole numbers a, showing that 0 is an identity for &.

6. (a) Number line

(b) Comparison

(c) Missing addend

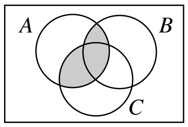
7. (a) Associative property of addition

(b) Distributive property of multiplication over addition

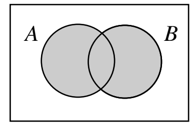
(c) Additive-identity property of zero

(d) Associative property of multiplication

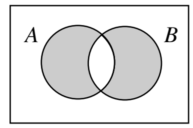
8. **(a)** $A \cap (B \cup C)$



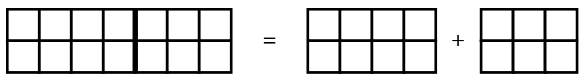
(b) $A \cup B$



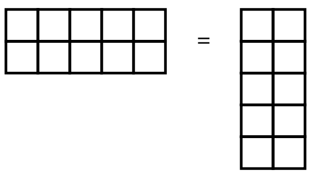
(c) $(A \cup \overline{B}) \cup (B \cap \overline{A})$



9. **(a)**



(b)



10. The possible values are the natural number factors of 21: 1, 3, 7, 21.

11. **(a)** $n(A \cup B) = n(\{\text{w, h, o, l, e, n, u, m, b, r}\}) = 10$

(b) $n(B \cap \overline{C}) = n(\{\text{n, u, m, b}\}) = 4$

(c) $n(A \cap C) = n(\{\text{o, e}\}) = 2$

(d) $n(A \times C) = n(A) \times n(C) = 5 \times 4 = 20$

12. Since $n(A \cup B) = 21 = 12 + 14 - 5$, there must be 5 elements that are in both sets. Thus, $n(A \cap B) = 5$, and $n(\overline{A \cap B}) = 26 - 5 = 21$.

13. Since $A \cap B = A$, we know that $A \subseteq B$. Therefore $A \cap \overline{B} = \varnothing$.

14. **(a)** Since 5 gallons = 20 quarts = 640 ounces, the number of bottles is 640 ÷ 10 = 64.

(b) Grouping