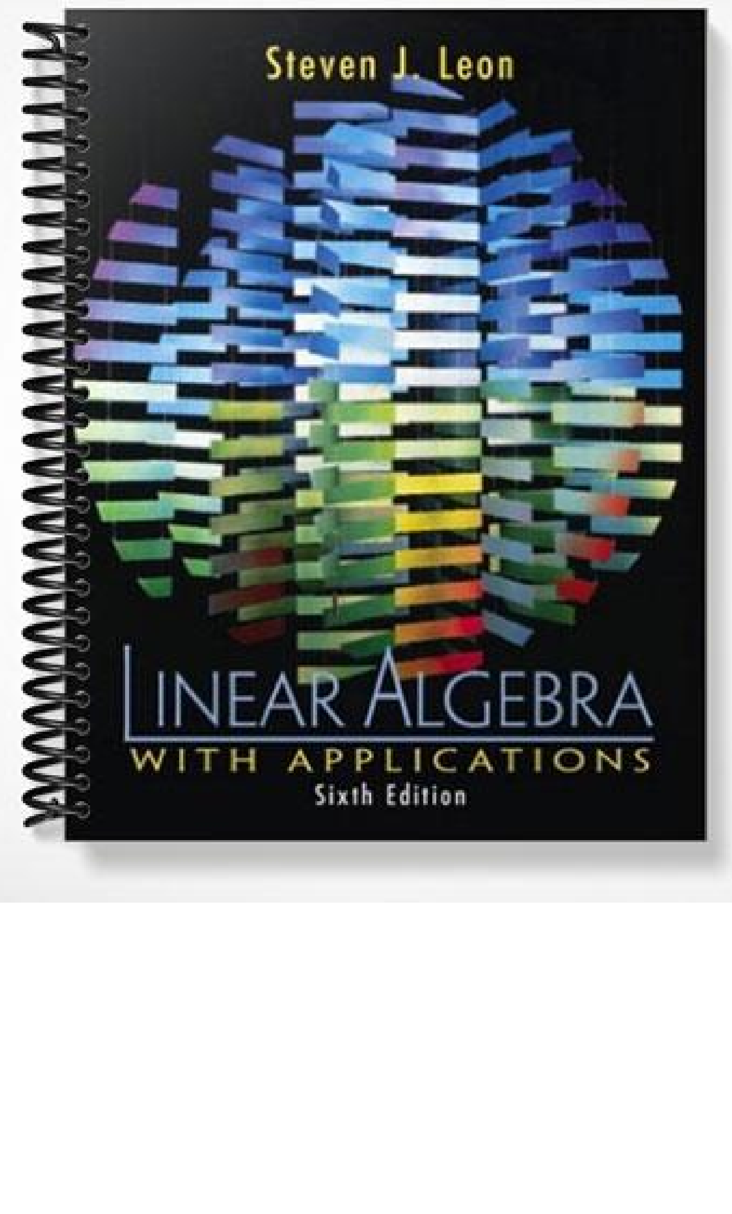
SOLUTIONS MANUAL



Instructor's Solutions Manual

Steven J. Leon

LINEAR ALGEBRA
WITH APPLICATIONS
Sixth Edition

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PREFACE

This solutions manual is designed to accompany the sixth edition of *Linear Algebra with Applications* by Steven J. Leon. The answers in this manual supplement those given in the answer key in the text. In addition this manual contains the complete solutions to all of the nonroutine exercises in the text.

At the end of each chapter of the textbook there is a Chapter Test and a section of computer exercises to be solved using MATLAB. The chapter test questions are to be answered as either *true* or *false*. Although the true-false answers are given in the Answer Section of the textbook, students are required to explain or prove their answers. This manual includes explanations, proofs, and counterexamples for all chapter test questions.

In the MATLAB exercises most of the computations are straightforward. Consequently they have not been included in this solutions manual. On the other hand, the text also includes questions related to the computations. The purpose of the questions is to emphasize the significance of the computations. The solutions manual does provide the answers to most of these questions. There are some questions for which it is not possible to provide a single answer. For example answers to questions involving significant digits depend on the floating point arithmetic of the particular computer that is used. Similarly if an exercise involves randomly generated matrices, the answer may depend on the particular random matrices that were generated.

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CHAPTER 1

SECTION 1

2. (d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

5. (a) $3x_1 + 2x_2 = 8$
$x_1 + 5x_2 = 7$

(b) $5x_1 - 2x_2 + x_3 = 3$
$2x_1 + 3x_2 - 4x_3 = 0$

(c) $2x_1 + x_2 + 4x_3 = -1$
$4x_1 - 2x_2 + 3x_3 = 4$
$5x_1 + 2x_2 + 6x_2 = -1$

(d) $4x_1 - 3x_2 + x_3 + 2x_4 = 4$
$3x_1 + x_2 - 5x_3 + 6x_4 = 5$
$x_1 + x_2 + 2x_3 + 4x_4 = 8$
$5x_1 + x_2 + 3x_3 - 2x_4 = 7$

9. Given the system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

one can eliminate the variable x_2 by subtracting the first row from the second. One then obtains the equivalent system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (m_1 - m_2)x_1 &= b_2 - b_1 \end{aligned}$$

(a) If $m_1 \neq m_2$, then one can solve the second equation for x_1

$$x_1 = \frac{b_2 - b_1}{m_1 - m_2}$$

One can then plug this value of x_1 into the first equation and solve for x_2. Thus, if $m_1 \neq m_2$, there will be a unique ordered pair (x_1, x_2) that satisfies the two equations.

(b) If $m_1 = m_2$, then the x_1 term drops out in the second equation

$$0 = b_2 - b_1$$

This is possible if and only if $b_1 = b_2$.

(c) If $m_1 \neq m_2$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_1 = m_2$ and $b_1 = b_2$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_1 = m_2$ and $b_1 \neq b_2$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.

10. The system must be consistent since $(0, 0)$ is a solution.

11. A linear equation in 3 unknowns represents a plane in three space. The solution set to a 3×3 linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point then the solution set will contain only that point.

SECTION 2

7. A homogeneous linear equation in 3 unknowns corresponds to a plane that passes through the origin in 3-space. Two such equations would correspond to two planes through the origin. If one equation is a multiple of the other, then both represent the same plane through the origin and every point on that plane will be a solution to the system. If one equation is not a multiple of the other, then we have two distinct planes that intersect in a line through the origin. Every point on the line of intersection will be a solution to the linear system. So in either case the system must have infinitely many solutions.

In the case of a nonhomogeneous 2×3 linear system, the equations correspond to planes that do not both pass through the origin. If one equation is a multiple of the other, then both represent the same plane and there are infinitely many solutions. If the equations represent planes that are parallel, then they do not intersect and hence the system will not have any solutions. If the equations represent distinct planes that are not parallel, then they must intersect in a line and hence there will be infinitely many solutions.

So the only possibilities for a nonhomogeneous 2×3 linear system are 0 or infinitely many solutions.

9. (a) Since the system is homogeneous it must be consistent.

15. If (c_1, c_2) is a solution, then

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 &= 0 \\ a_{21}c_1 + a_{22}c_2 &= 0 \end{aligned}$$

Multiplying both equations through by α, one obtains

$$\begin{aligned} a_{11}(\alpha c_1) + a_{12}(\alpha c_2) &= \alpha \cdot 0 = 0 \\ a_{21}(\alpha c_1) + a_{22}(\alpha c_2) &= \alpha \cdot 0 = 0 \end{aligned}$$

Thus $(\alpha c_1, \alpha c_2)$ is also a solution.

SECTION 3

1. (e) $\begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6 \end{pmatrix}$

5. (a) $5A = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$

$$2A + 3A = \begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} + \begin{pmatrix} 9 & 12 \\ 3 & 3 \\ 6 & 21 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$$

(b) $6A = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$

$$3(2A) = 3\begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$$

(c) $A^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}$

$$(A^T)^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{pmatrix} = A$$

6. (a) $A + B = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} = B + A$

(b) $3(A + B) = \begin{pmatrix} 15 & 12 & 18 \\ 0 & 15 & 3 \end{pmatrix}$

$$3A+3B=\begin{pmatrix}12&3&18\\6&9&15\end{pmatrix}+\begin{pmatrix}3&9&0\\-6&6&-12\end{pmatrix}$$
$$=\begin{pmatrix}15&12&18\\0&15&3\end{pmatrix}$$

(c) $(A+B)^T=\begin{pmatrix}5&4&6\\0&5&1\end{pmatrix}^T=\begin{pmatrix}5&0\\4&5\\6&1\end{pmatrix}$

$$A^T+B^T=\begin{pmatrix}4&2\\1&3\\6&5\end{pmatrix}+\begin{pmatrix}1&-2\\3&2\\0&-4\end{pmatrix}=\begin{pmatrix}5&0\\4&5\\6&1\end{pmatrix}$$

7. (a) $3(AB)=3\begin{pmatrix}5&14\\15&42\\0&16\end{pmatrix}=\begin{pmatrix}15&42\\45&126\\0&48\end{pmatrix}$

$$(3A)B=\begin{pmatrix}6&3\\18&9\\-6&12\end{pmatrix}\begin{pmatrix}2&4\\1&6\end{pmatrix}=\begin{pmatrix}15&42\\45&126\\0&48\end{pmatrix}$$

$$A(3B)=\begin{pmatrix}2&1\\6&3\\-2&4\end{pmatrix}\begin{pmatrix}6&12\\3&18\end{pmatrix}=\begin{pmatrix}15&42\\45&126\\0&48\end{pmatrix}$$

(b) $(AB)^T=\begin{pmatrix}5&14\\15&42\\0&16\end{pmatrix}^T=\begin{pmatrix}5&15&0\\14&42&16\end{pmatrix}$

$$B^TA^T=\begin{pmatrix}2&1\\4&6\end{pmatrix}\begin{pmatrix}2&6&-2\\1&3&4\end{pmatrix}=\begin{pmatrix}5&15&0\\14&42&16\end{pmatrix}$$

8. (a) $(A+B)+C=\begin{pmatrix}0&5\\1&7\end{pmatrix}+\begin{pmatrix}3&1\\2&1\end{pmatrix}=\begin{pmatrix}3&6\\3&8\end{pmatrix}$

$$A+(B+C)=\begin{pmatrix}2&4\\1&3\end{pmatrix}+\begin{pmatrix}1&2\\2&5\end{pmatrix}=\begin{pmatrix}3&6\\3&8\end{pmatrix}$$

(b) $(AB)C=\begin{pmatrix}-4&18\\-2&13\end{pmatrix}\begin{pmatrix}3&1\\2&1\end{pmatrix}=\begin{pmatrix}24&14\\20&11\end{pmatrix}$

$$A(BC)=\begin{pmatrix}2&4\\1&3\end{pmatrix}\begin{pmatrix}-4&-1\\8&4\end{pmatrix}=\begin{pmatrix}24&14\\20&11\end{pmatrix}$$

(c) $A(B+C)=\begin{pmatrix}2&4\\1&3\end{pmatrix}\begin{pmatrix}1&2\\2&5\end{pmatrix}=\begin{pmatrix}10&24\\7&17\end{pmatrix}$

$$AB+AC=\begin{pmatrix}-4&18\\-2&13\end{pmatrix}+\begin{pmatrix}14&6\\9&4\end{pmatrix}=\begin{pmatrix}10&24\\7&17\end{pmatrix}$$

(d) $(A+B)C=\begin{pmatrix}0&5\\1&7\end{pmatrix}\begin{pmatrix}3&1\\2&1\end{pmatrix}=\begin{pmatrix}10&5\\17&8\end{pmatrix}$

$$AC+BC=\begin{pmatrix}14&6\\9&4\end{pmatrix}+\begin{pmatrix}-4&-1\\8&4\end{pmatrix}=\begin{pmatrix}10&5\\17&8\end{pmatrix}$$

9. Let

$$D = (AB)C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

It follows that

$$\begin{aligned} d_{11} &= (a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21} \\ &= a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} \\ d_{12} &= (a_{11}b_{11} + a_{12}b_{21})c_{12} + (a_{11}b_{12} + a_{12}b_{22})c_{22} \\ &= a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\ d_{21} &= (a_{21}b_{11} + a_{22}b_{21})c_{11} + (a_{21}b_{12} + a_{22}b_{22})c_{21} \\ &= a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} \\ d_{22} &= (a_{21}b_{11} + a_{22}b_{21})c_{12} + (a_{21}b_{12} + a_{22}b_{22})c_{22} \\ &= a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22} \end{aligned}$$

If we set

$$E = A(BC) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

then it follows that

$$\begin{aligned} e_{11} &= a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) \\ &= a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} \\ e_{12} &= a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\ &= a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\ e_{21} &= a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) \\ &= a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} \\ e_{22} &= a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\ &= a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22} \end{aligned}$$

Thus

$$d_{11} = e_{11} \qquad d_{12} = e_{12} \qquad d_{21} = e_{21} \qquad d_{22} = e_{22}$$

and hence

$$(AB)C = D = E = A(BC)$$

12.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^4 = O$. If $n > 4$, then

$$A^n = A^{n-4}A^4 = A^{n-4}O = O$$

15. If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$ then

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left[\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right]$$

$$= \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$$

Therefore

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

16. Since

$$A^{-1}A = AA^{-1} = I$$

it follows from the definition that A^{-1} is nonsingular and A is its inverse.

17. Since

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I \end{aligned}$$

it follows that

$$(A^{-1})^T = (A^T)^{-1}$$

18. For $m = 1$,

$$(A^1)^{-1} = A^{-1} = (A^{-1})^1$$

Assume the result holds in the case $m = k$, that is,

$$(A^k)^{-1} = (A^{-1})^k$$

It follows that

$$(A^{-1})^{k+1}A^{k+1} = A^{-1}(A^{-1})^k A^k A = A^{-1}A = I$$

and

$$A^{k+1}(A^{-1})^{k+1} = AA^k(A^{-1})^k A^{-1} = AA^{-1} = I$$

Therefore

$$(A^{-1})^{k+1} = (A^{k+1})^{-1}$$

and the result follows by mathematical induction.

19. (a) $(A+B)^2 = (A+B)(A+B) = (A+B)A+(A+B)B = A^2+BA+AB+B^2$
In the case of real numbers $ab + ba = 2ab$, however, with matrices $AB + BA$ is generally not equal to $2AB$.

(b)

$$\begin{aligned} (A+B)(A-B) &= (A+B)(A-B) \\ &= (A+B)A - (A+B)B \\ &= A^2 + BA - AB - B^2 \end{aligned}$$

In the case of real numbers $ab - ba = 0$, however, with matrices $AB - BA$ is generally not equal to O.

20. There are many possible choices for A and B. For example, one could choose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

More generally if

$$A = \begin{pmatrix} a & b \\ ca & cb \end{pmatrix} \qquad B = \begin{pmatrix} db & eb \\ -da & -ea \end{pmatrix}$$

then $AB = O$ for any choice of the scalars a, b, c, e.

21. To construct nonzero matrices A, B, C with the desired properties, first find nonzero matrices C and D such that $DC = O$ (see Exercise 20). Next, for any nonzero matrix A, set $B = A + D$. It follows that

$$BC = (A + D)C = AC + DC = AC + O = AC$$

22. A 2×2 symmetric matrix is one of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus

$$A^2 = \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix}$$

If $A^2 = O$, then its diagonal entries must be 0.

$$a^2 + b^2 = 0 \quad \text{and} \quad b^2 + c^2 = 0$$

Thus $a = b = c = 0$ and hence $A = O$.

23. For most pairs of symmetric matrices A and B the product AB will not be symmetric. For example

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 5 & 4 \end{pmatrix}$$

See Exercise 25 for a characterization of the conditions under which the product will be symmetric.

24. (a) A^T is an $n \times m$ matrix. Since A^T has m columns and A has m rows, the multiplication $A^T A$ is possible. The multiplication AA^T is possible since A has n columns and A^T has n rows.

(b) $(A^T A)^T = A^T (A^T)^T = A^T A$
$(AA^T)^T = (A^T)^T A^T = AA^T$

25. Let A and B be symmetric $n \times n$ matrices. If $(AB)^T = AB$ then

$$BA = B^T A^T = (AB)^T = AB$$

Conversely if $BA = AB$ then

$$(AB)^T = B^T A^T = BA = AB$$

27. **(a)**

$$\begin{aligned} B^T &= (A+A^T)^T = A^T + A^{T^T} = A^T + A = B \\ C^T &= (A-A^T)^T = A^T - A^{T^T} = A^T - A = -C \end{aligned}$$

(b) $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

30. The search vector is $\mathbf{x} = (1,0,1,0,1,0)^T$. The search result is given by the vector

$$\mathbf{y} = A^T\mathbf{x} = (1,2,2,1,1,2,1)^T$$

The ith entry of $\mathbf{y}$ is equal to the number of search words in the title of the ith book.

32. (b) The $(1,j)$ entry of A^2 represents the number of walks of length 2 from V_1 to V_j.

33. If $\alpha = a_{21}/a_{11}$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} + b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \alpha a_{12} + b \end{pmatrix}$$

The product will equal A provided

$$\alpha a_{12} + b = a_{22}$$

Thus we must choose

$$b = a_{22} - \alpha a_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

SECTION 4

2. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, type I

(b) The given matrix is not an elementary matrix. Its inverse is given by

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$, type II

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, type III

5. (c) Since

$$C = FB = FEA$$

where F and E are elementary matrices it follows that C is row equivalent to A.

6. (b) $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

The product $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is lower triangular.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

8. (a) $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

11. (b) $XA + B = C$

$X = (C - B)A^{-1}$

$= \begin{pmatrix} 8 & -14 \\ -13 & 19 \end{pmatrix}$

(d) $XA + C = X$

$XA - XI = -C$

$X(A - I) = -C$

$X = -C(A - I)^{-1}$

$= \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$

12. (a) If E is an elementary matrix of type I or type II then E is symmetric. Thus $E^T = E$ is an elementary matrix of the same type. If E is the elementary matrix of type III formed by adding α times the ith row of the identity matrix to the jth row, then E^T is the elementary matrix of type III formed from the identity matrix by adding α times the jth row to the ith row.

(b) In general the product of two elementary matrices will not be an elementary matrix. Generally the product of two elementary matrices will be a matrix formed from the identity matrix by the performance of two row operations. For example, if

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

then E_1 and E_2 are elementary matrices, but

$$E_1E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is not an elementary matrix.

13. If $T = UR$, then

$$t_{ij} = \sum_{k=1}^{n} u_{ik}r_{kj}$$

Since U and R are upper triangular

$$\begin{aligned} u_{i1} &= u_{i2} = \cdots = u_{i,i-1} = 0 \\ r_{j+1,j} &= r_{j+2,j} = \cdots - r_{nj} = 0 \end{aligned}$$

If $i > j$, then

$$\begin{aligned} t_{ij} &= \sum_{k=1}^{j} u_{ik}r_{kj} + \sum_{k=j+1}^{n} u_{ik}r_{kj} \\ &= \sum_{k=1}^{j} 0\,r_{kj} + \sum_{k=j+1}^{n} u_{ik}0 \\ &= 0 \end{aligned}$$

Therefore T is upper triangular.

If $i = j$, then

$$\begin{aligned} t_{jj} = t_{ij} &= \sum_{k=1}^{i-1} u_{ik}r_{kj} + u_{jj}r_{jj} + \sum_{k=j+1}^{n} u_{ik}r_{kj} \\ &= \sum_{k=1}^{i-1} 0\,r_{kj} + u_{jj}r_{jj} + \sum_{k=j+1}^{n} u_{ik}0 \\ &= u_{jj}r_{jj} \end{aligned}$$

Therefore

$$t_{jj} = u_{jj}r_{jj} \qquad j = 1, \ldots, n$$

14. If we set $\mathbf{x} = (2, 1-4)^T$, then

$$A\mathbf{x} = 2\mathbf{a}_1 + 1\mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

Thus $\mathbf{x}$ is a nonzero solution to the system $A\mathbf{x} = \mathbf{0}$. But if a homogeneous system has a nonzero solution, then it must have infinitely many solutions. In particular, if c is any scalar, then $c\mathbf{x}$ is also a solution to the system since

$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$$

Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ it follows that the matrix A must be singular. (See Theorem 1.4.3)

15. If B is singular, then it follows from Theorem 1.4.3 that there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. If $C = AB$, then

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

Thus, by Theorem 1.4.3, C must also be singular.

16. (a) If U is upper triangular with nonzero diagonal entries, then using a row operation II, U can be transformed into an upper triangular matrix with 1's on the diagonal. Row operation III can then be used to eliminate all of the entries above the diagonal. Thus U is row equivalent to I and hence is nonsingular.

(b) The same row operations that were used to reduce U to the identity matrix will transform I into U^{-1}. Row operation II applied to I will just change the values of the diagonal entries. When the row operation III steps referred to in part (a) are applied to a diagonal matrix, the entries above the diagonal are filled in. The resulting matrix, U^{-1}, will be upper triangular.

17. Since A is nonsingular it is row equivalent to I. Hence there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_1 A = I$$

It follows that

$$A^{-1} = E_k \cdots E_1$$

and

$$E_k \cdots E_1 B = A^{-1}B = C$$

The same row operations that reduce A to I, will transform B to C. Therefore the reduced row echelon form of $(A \mid B)$ will be $(I \mid C)$.

18. (a) If the diagonal entries of D_1 are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the diagonal entries of D_2 are $\beta_1, \beta_2, \ldots, \beta_n$, then D_1D_2 will be a diagonal matrix with diagonal entries $\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n$ and D_2D_1 will be a diagonal matrix with diagonal entries $\beta_1\alpha_1, \beta_2\alpha_2, \ldots, \beta_n\alpha_n$. Since the two have the same diagonal entries it follows that $D_1D_2 = D_2D_1$.

(b)

$$\begin{aligned} AB &= A(a_0I + a_1A + \cdots + a_kA^k) \\ &= a_0A + a_1A^2 + \cdots + a_kA^{k+1} \\ &= (a_0I + a_1A + \cdots + a_kA^k)A \\ &= BA \end{aligned}$$

19. If A is symmetric and nonsingular, then

$$(A^{-1})^T = (A^{-1})^T(AA^{-1}) = ((A^{-1})^TA^T)A^{-1} = A^{-1}$$

20. If A is row equivalent to B then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_kE_{k-1} \cdots E_1B$$

Each of the E_i's is invertible and E_i^{-1} is also an elementary matrix (Theorem 1.4.2). Thus

$$B = E_1^{-1}E_2^{-1} \cdots E_k^{-1}A$$

and hence B is row equivalent to A.

21. (a) If A is row equivalent to B, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_kE_{k-1} \cdots E_1B$$

Since B is row equivalent to C, there exist elementary matrices $H_1, H_2, \ldots, H_j$ such that

$$B = H_jH_{j-1} \cdots H_1C$$