

SOLUTIONS MANUAL



LINEAR ALGEBRA
WITH
APPLICATIONS 3E
OTTO BRETSCHER

Chapter 2

2.1

1. Not a linear transformation, since $y_2 = x_2 + 2$ is not linear in our sense.

2. Linear, with matrix
$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

3. Not linear, since $y_2 = x_1x_3$ is nonlinear.

4.
$$A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$$

5. By Fact 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}.$$

6. Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

7. Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \cdots \vec{v}_m] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m]$.

8. Reducing the system $\begin{cases} x_1 + 7x_2 = y_1 \\ 3x_1 + 20x_2 = y_2 \end{cases}$, we obtain $\begin{cases} x_1 = -20y_1 + 7y_2 \\ x_2 = 3y_1 - y_2 \end{cases}$.

9. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2 . Reducing

the system $\begin{cases} 2x_1 + 3x_2 = y_1 \\ 6x_1 + 9x_2 = y_2 \end{cases}$ we obtain $\begin{cases} x_1 + 1.5x_2 = 0.5y_1 \\ 0 = -3y_1 + y_2 \end{cases}$.

No unique solution (x_1, x_2) can be found for a given (y_1, y_2) ; the matrix is noninvertible.

10. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2 . Reducing

the system $\left| \begin{array}{l} x_1 + 2x_2 = y_1 \\ 4x_1 + 9x_2 = y_2 \end{array} \right|$ we find that $\left| \begin{array}{l} x_1 = 9y_1 + 2y_2 \\ x_2 = -4y_1 + y_2 \end{array} \right|$ or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The inverse matrix is $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$.

11. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2 . Reducing

the system $\left| \begin{array}{l} x_1 + 2x_2 = y_1 \\ 3x_1 + 9x_2 = y_2 \end{array} \right|$ we find that $\left| \begin{array}{l} x_1 = 3y_1 - \frac{2}{3}y_2 \\ x_2 = -y_1 + \frac{1}{3}y_2 \end{array} \right|$. The

inverse matrix is $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$.

12. Reducing the system $\left| \begin{array}{l} x_1 + kx_2 = y_1 \\ x_2 = y_2 \end{array} \right|$ we find that $\left| \begin{array}{l} x_1 = y_1 - ky_2 \\ x_2 = y_2 \end{array} \right|$. The

inverse matrix is $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$.

13. a. First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2 .

$$\left| \begin{array}{l} ax_1 + bx_2 = y_1 \\ cx_1 + dx_2 = y_2 \end{array} \right| \div a \rightarrow \left| \begin{array}{l} x_1 + \frac{b}{a}x_2 = \frac{1}{a}y_1 \\ cx_1 + dx_2 = y_2 \end{array} \right| -c(I) \rightarrow$$

$$\left| \begin{array}{l} x_1 + \frac{b}{a}x_2 = \frac{1}{a}y_1 \\ (d - \frac{bc}{a})x_2 = -\frac{c}{a}y_1 + y_2 \end{array} \right| \rightarrow$$

$$\left| \begin{array}{l} x_1 + \frac{b}{a}x_2 = \frac{1}{a}y_1 \\ (\frac{ad-bc}{a})x_2 = -\frac{c}{a}y_1 + y_2 \end{array} \right|$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.

If $a = 0$, then we have to consider the system

$$\left| \begin{array}{l} bx_2 = y_1 \\ cx_1 + dx_2 = y_2 \end{array} \right| \text{swap : } I \leftrightarrow II \left| \begin{array}{l} cx_1 + dx_2 = y_2 \\ bx_2 = y_1 \end{array} \right|$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

- b. First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\left| \begin{array}{rcl} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 + y_2 \end{array} \right| \cdot \frac{a}{D} \rightarrow$$

$$\left| \begin{array}{rcl} x_1 + \frac{b}{a}x_2 & = & \frac{1}{a}y_1 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right| -\frac{b}{a}(II) \rightarrow$$

$$\left| \begin{array}{rcl} x_1 & = & (\frac{1}{a} + \frac{bc}{aD})y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

$$\left| \begin{array}{rcl} x_1 & = & \frac{d}{D}y_1 - \frac{b}{D}y_2 \\ x_2 & = & -\frac{c}{D}y_1 + \frac{a}{D}y_2 \end{array} \right|$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

$$\left| \begin{array}{rcl} cx_1 + dx_2 & = & y_2 \\ bx_2 & = & y_1 \end{array} \right| \div c$$

$$\left| \begin{array}{rcl} x_1 + \frac{d}{c}x_2 & = & \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{array} \right| -\frac{d}{c}(II)$$

$$\left| \begin{array}{rcl} x_1 & = & -\frac{d}{bc}y_1 + \frac{1}{c}y_2 \\ x_2 & = & \frac{1}{b}y_1 \end{array} \right|$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

14. a. By Exercise 13a, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ is invertible if (and only if) $2k - 15 \neq 0$, or $k \neq 7.5$.

b. By Exercise 13b, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.

If all entries of this inverse are integers, then $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is a (nonzero) integer n , so that $2k - 15 = \frac{1}{n}$ or $k = 7.5 + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$ is an integer as well, n must be odd.

We have shown: If all entries of the inverse are integers, then $k = 7.5 + \frac{1}{2n}$, where n is an odd integer. The converse is true as well: If k is chosen in this way, then the entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ will be integers.

15. By Exercise 13a, the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible if (and only if) $a^2 + b^2 \neq 0$, which is the case unless $a = b = 0$. If $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible, then its inverse is $\frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, by Exercise 13b.
16. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then $A\vec{x} = 3\vec{x}$ for all \vec{x} in \mathbb{R}^2 , so that A represents a scaling by a factor of 3. Its inverse is a scaling by a factor of $\frac{1}{3}$: $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$.
17. If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A\vec{x} = -\vec{x}$ for all \vec{x} in \mathbb{R}^2 , so that A represents a reflection about the origin.
- This transformation is its own inverse: $A^{-1} = A$.
18. Compare with Exercise 16: This matrix represents a scaling by the factor of $\frac{1}{2}$; the inverse is a scaling by 2.
19. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so that A represents the orthogonal projection onto the \vec{e}_1 axis. (See Figure 2.1.) This transformation is not invertible, since the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many solutions \vec{x} .

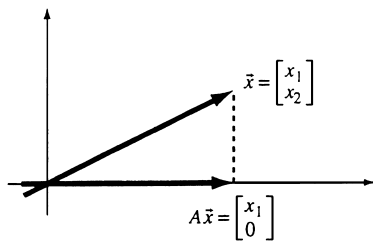


Figure 2.1: for Problem 2.1.19.

20. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$, so that A represents the reflection about the line $x_2 = x_1$. This transformation is its own inverse: $A^{-1} = A$.

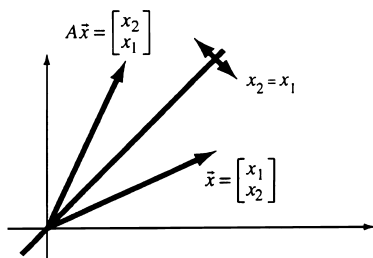


Figure 2.2: for Problem 2.1.20.

21. Compare with Example 5.

If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$. Note that the vectors \vec{x} and $A\vec{x}$ are perpendicular and have the same length. If \vec{x} is in the first quadrant, then $A\vec{x}$ is in the fourth. Therefore, A represents the rotation through an angle of 90° in the clockwise direction.

(See Figure 2.3.) The inverse $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation through 90° in the counterclockwise direction.

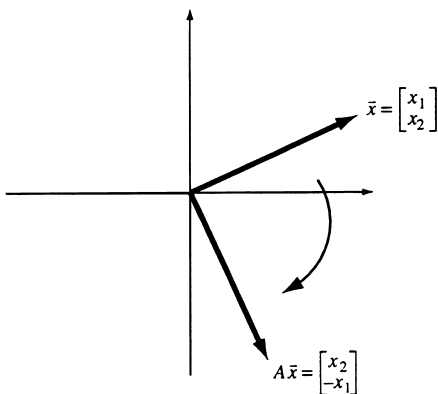


Figure 2.3: for Problem 2.1.21.

22. If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$, so that A represents the reflection about the

\vec{e}_1 axis. This transformation is its own inverse: $A^{-1} = A$.

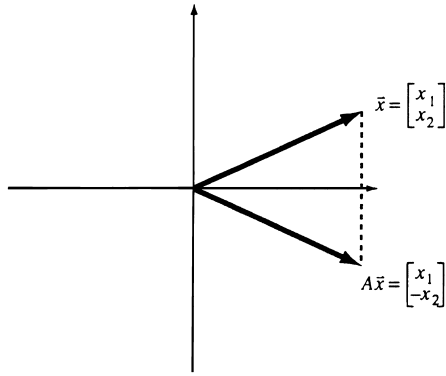


Figure 2.4: for Problem 2.1.22.

23. Compare with Exercise 21.

Note that $A = 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that A represents a rotation through an angle of 90° in the clockwise direction, followed by a scaling by the factor of 2.

The inverse $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ represents a rotation through an angle of 90° in the counterclockwise direction, followed by a scaling by the factor of $\frac{1}{2}$.

24. Compare with Example 5.

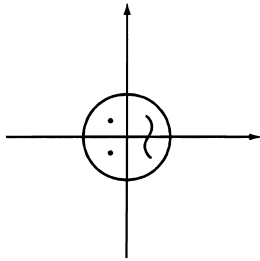


Figure 2.5: for Problem 2.1.24.

25. The matrix represents a scaling by the factor of 2. (See Figure 2.6.)

26. This matrix represents a reflection about the line $x_2 = x_1$.

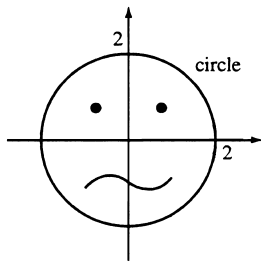


Figure 2.6: for Problem 2.1.25.

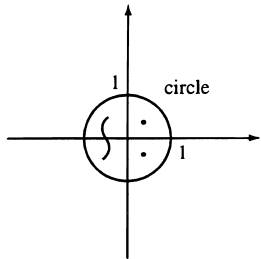


Figure 2.7: for Problem 2.1.26.

27. This matrix represents a reflection about the \vec{e}_1 axis. (See Figure 2.8.)

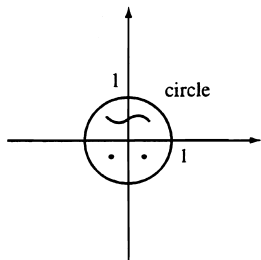


Figure 2.8: for Problem 2.1.27.

28. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged.

29. This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.10.)

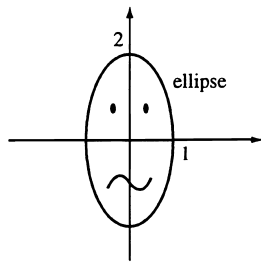


Figure 2.9: for Problem 2.1.28.

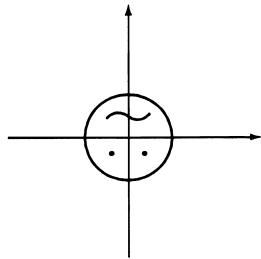


Figure 2.10: for Problem 2.1.29.

30. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the \vec{e}_2 axis.

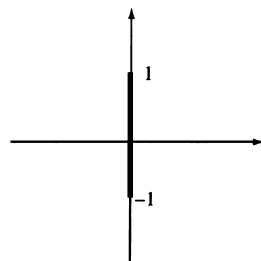


Figure 2.11: for Problem 2.1.30.

31. The image must be reflected about the \vec{e}_2 axis, that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be transformed into $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$: This can be accomplished by means of the linear transformation $T(\vec{x}) =$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}.$$

32. Using Fact 2.1.2, we find $A = \begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}$. This matrix has 3's on the diagonal and 0's everywhere else.

33. By Fact 2.1.2, $A = \left[T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$. (See Figure 2.12.)

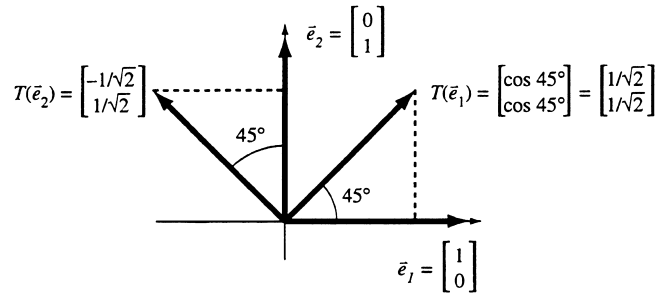


Figure 2.12: for Problem 2.1.33.

$$\text{Therefore, } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

34. As in Exercise 33, we find $T(\vec{e}_1)$ and $T(\vec{e}_2)$; then by Fact 2.1.2, $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$.

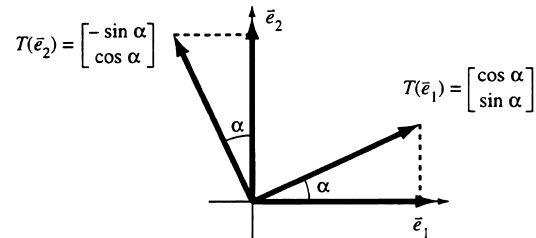


Figure 2.13: for Problem 2.1.34.

$$\text{Therefore, } A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

35. We want to find a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A \begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$ and $A \begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

This amounts to solving the system
$$\left| \begin{array}{rcl} 5a + 42b & = & 89 \\ 6a + 41b & = & 88 \\ & 5c + 42d & = 52 \\ & 6c + 41d & = 53 \end{array} \right|.$$

(Here we really have two systems with two unknowns each.)

The unique solution is $a = 1$, $b = 2$, $c = 2$, and $d = 1$, so that $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

36. First we draw \vec{w} in terms of \vec{v}_1 and \vec{v}_2 so that $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some c_1 and c_2 . Then, we scale the \vec{v}_2 -component by 3, so our new vector equals $c_1\vec{v}_1 + 3c_2\vec{v}_2$.
37. Since $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, we have $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v})) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$, by Fact 2.1.3

Since k is between 0 and 1, the tip of this vector $T(\vec{x})$ is on the line segment connecting the tips of $T(\vec{v})$ and $T(\vec{w})$. (See Figure 2.14.)

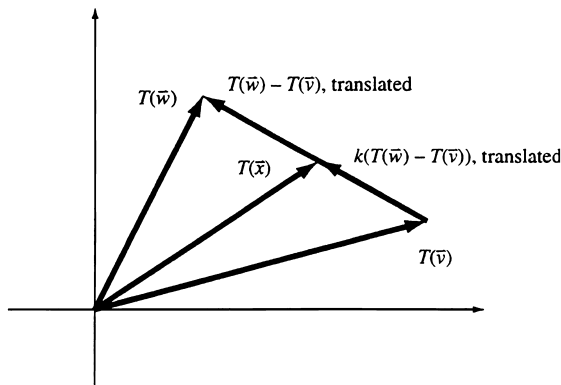


Figure 2.14: for Problem 2.1.37.

38. $T \begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2 = 2\vec{v}_1 + (-\vec{v}_2)$

39. By Fact 2.1.2, we have $T \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_m) \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1T(\vec{e}_1) + \dots + x_mT(\vec{e}_m).$

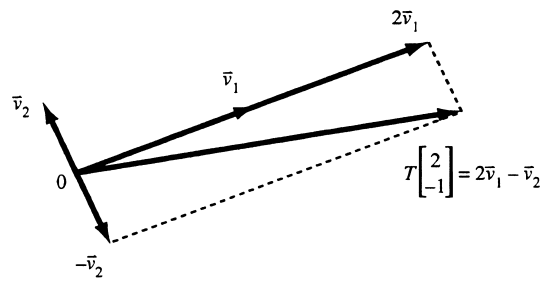


Figure 2.15: for Problem 2.1.38.

40. These linear transformations are of the form $[y] = [a][x]$, or $y = ax$. The graph of such a function is a line through the origin.
41. These linear transformations are of the form $[y] = [a \ b] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, or $y = ax_1 + bx_2$. The graph of such a function is a plane through the origin.
42. a. See Figure 2.16.

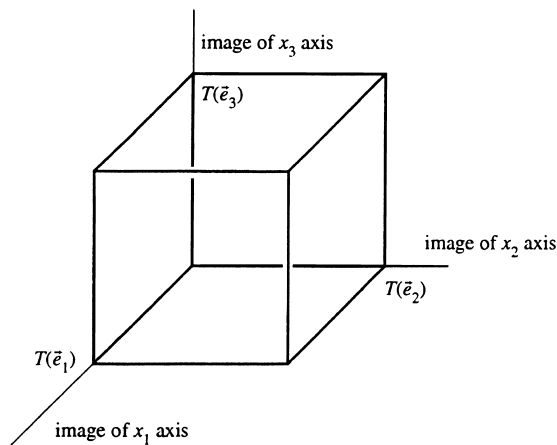


Figure 2.16: for Problem 2.1.42.

- b. The image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

c. Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\begin{cases} -\frac{1}{2}x_1 + x_2 = 0 \\ -\frac{1}{2}x_1 + x_3 = 0 \end{cases}$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$, where t is an arbitrary real number.

For example, for $t = \frac{1}{2}$, we find the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ considered in part b. These points are on the line through the origin and the observer's eye.

43. a. $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2 \ 3 \ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix $[2 \ 3 \ 4]$.

b. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1 \ v_2 \ v_3]$, as in part (a).

c. Let $[a \ b \ c]$ be the matrix of T . Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a \ b \ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

44. $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

45. Yes, $\vec{z} = L(T(\vec{x}))$ is also linear, which we will verify using Fact 2.1.3. Part a holds, since $L(T(\vec{v} + \vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w}))$, and part b also works, because $L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v}))$.

$$46. T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B \left(A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = B \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} pa + qc \\ ra + sc \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B \left(A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = B \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$$

$$\text{So, } T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \left(T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + x_2 \left(T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} pb + qd \\ rb + sd \end{bmatrix}$$

47. Write \vec{w} as a linear combination of \vec{v}_1 and \vec{v}_2 : $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. (See Figure 2.17.)

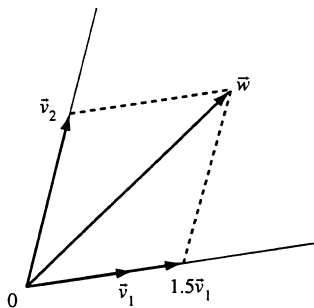


Figure 2.17: for Problem 2.1.47.

Measurements show that we have *roughly* $\vec{w} = 1.5\vec{v}_1 + \vec{v}_2$.

Therefore, by linearity, $T(\vec{w}) = T(1.5\vec{v}_1 + \vec{v}_2) = 1.5T(\vec{v}_1) + T(\vec{v}_2)$. (See Figure 2.18.)

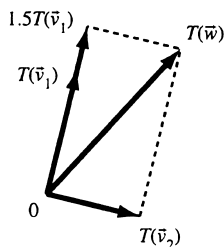


Figure 2.18: for Problem 2.1.47.

48. Let \vec{x} be some vector in \mathbb{R}^2 . Since \vec{v}_1 and \vec{v}_2 are not parallel, we can write \vec{x} in terms of components of \vec{v}_1 and \vec{v}_2 . So, let c_1 and c_2 be scalars such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then, by Fact 2.1.3, $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$. So $T(\vec{x}) = L(\vec{x})$ for all \vec{x} in \mathbb{R}^2 .

49. a. Let x_1 be the number of 2 Franc coins, and x_2 be the number of 5 Franc coins. Then
- $$\begin{cases} 2x_1 + 5x_2 = 144 \\ x_1 + x_2 = 51 \end{cases}$$

From this we easily find our solution vector to be $\begin{bmatrix} 37 \\ 14 \end{bmatrix}$.

b.
$$\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix} = \begin{bmatrix} 2x_1 & +5x_2 \\ x_1 & +x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So, $A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$.

- c. By Exercise 13, matrix A is invertible (since $ad - bc = -3 \neq 0$), and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}$.

Then $-\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 144 \\ 51 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 144 & -5(51) \\ -144 & +2(51) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -111 \\ -42 \end{bmatrix} = \begin{bmatrix} 37 \\ 14 \end{bmatrix}$, which was the vector we found in part a.

50. a. Let $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{mass of the platinum alloy} \\ \text{mass of the silver alloy} \end{bmatrix}$. Using the definition density = mass/volume, or volume = mass/density, we can set up the system:

$$\begin{cases} p + s = 5,000 \\ \frac{p}{20} + \frac{s}{10} = 370 \end{cases}, \text{ with the solution } p = 2,600 \text{ and } s = 2,400. \text{ We see that the platinum alloy makes up only 52 percent of the crown; this gold smith is a crook!}$$

- b. We seek the matrix A such that $A \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} p + s \\ \frac{p}{20} + \frac{s}{10} \end{bmatrix}$. Thus
- $$A = \begin{bmatrix} 1 & 1 \\ \frac{1}{20} & \frac{1}{10} \end{bmatrix}.$$

- c. Yes. By Exercise 13, $A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$. Applied to the case considered in part a, we find that $\begin{bmatrix} p \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix} \begin{bmatrix} 5,000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2,600 \\ 2,400 \end{bmatrix}$, confirming our answer in part a.

51. a.
$$\begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F - 32) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}F - \frac{160}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix}.$$

$$\text{So } A = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}.$$

b. Using Exercise 13, we find $\frac{5}{9}(1) - (-\frac{160}{9})0 = \frac{5}{9} \neq 0$, so A is invertible.

$$A^{-1} = \frac{9}{5} \begin{bmatrix} 1 & \frac{160}{9} \\ 0 & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}. \text{ So, } F = \frac{9}{5}C + 32.$$

52. a. $A\vec{x} = \begin{bmatrix} 420 \\ 2100 \end{bmatrix}$, which is the total value of our money in terms of C\$ and R.

b. From Exercise 13, we test the value $ad - bc$ and find it to be zero. Thus A is not invertible. To determine when A is consistent, we begin to compute $\text{rref} \begin{bmatrix} A:\vec{b} \end{bmatrix}$:

$$\begin{bmatrix} 1 & \frac{1}{5} & \vdots & b_1 \\ 5 & 1 & \vdots & b_2 \end{bmatrix} \xrightarrow{-5I} \begin{bmatrix} 1 & \frac{1}{5} & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 - 5b_1 \end{bmatrix}.$$

Thus, the system is consistent only when $b_2 = 5b_1$. This makes sense, since b_2 is the total value of our money in terms of Rand, while b_1 is the value in terms of Canadian dollars. Consider the example in part a.

If the system $A\vec{x} = \vec{b}$ is consistent, then there will be infinitely many solutions \vec{x} , representing various compositions of our portfolio in terms of Rand and Canadian dollars, all representing the same total value.

53. First we notice that all entries along the diagonal must be 1, since those represent converting one currency to itself. Also, since $a_{34} = 200$, $\mathcal{L}1 = \mathcal{Y}200$, so $\mathcal{Y}1 = \mathcal{L}\frac{1}{200}$. So $a_{43} = \frac{1}{200}$. Using this same approach, we can find a_{21} and a_{41} as well.

$$\text{So far, } A = \begin{bmatrix} 1 & 0.8 & * & 1.5 \\ 1.25 & 1 & * & * \\ * & * & 1 & 200 \\ \frac{2}{3} & * & \frac{1}{200} & 1 \end{bmatrix}.$$

Now, using a_{43} and a_{14} , $\mathcal{Y}1 = \mathcal{L}\frac{1}{200}$ and $\mathcal{L}1 = 1.5$ Euros. So, $\mathcal{Y}1 = \frac{1}{200}(1.5)\text{Euros} = \frac{3}{400}$ Euros, meaning that $a_{13} = \frac{3}{400}$.

We use this same approach to see that $a_{24} = a_{21}a_{14} = \frac{5}{4}(\frac{3}{2}) = \frac{15}{8}$, and $a_{23} = a_{21}a_{13} = \frac{5}{4}(\frac{3}{400}) = \frac{3}{320}$.

Then, using our method from above to find a_{43} , we can find a_{31} , a_{42} and a_{32} .

$$\text{Thus, } A = \begin{bmatrix} 1 & \frac{4}{5} & \frac{3}{400} & \frac{3}{2} \\ \frac{5}{4} & 1 & \frac{3}{320} & \frac{15}{8} \\ \frac{400}{3} & \frac{320}{3} & 1 & 200 \\ \frac{2}{3} & \frac{8}{15} & \frac{1}{200} & 1 \end{bmatrix}.$$

54. a. 1: this represents converting a currency to itself.

b. a_{ij} is the reciprocal of a_{ji} , meaning that $a_{ij}a_{ji} = 1$. This represents converting on currency to another, then converting it back.

c. Note that a_{ik} is the conversion factor from currency k to currency i , meaning that 1 unit of currency $k = a_{ik}$ units of currency i .

Likewise, 1 unit of currency $j = a_{kj}$ units of currency k .

It follows that 1 unit of currency $j = a_{kj}a_{ik}$ units of currency $i = a_{ij}$ units of currency i ,

so that $a_{ik}a_{kj} = a_{ij}$.

d. The rank of A is only 1, because every row is simply a scalar multiple of the top row. More precisely, since $a_{ij} = a_{i1}a_{1j}$, by part c, the i^{th} row is a_{i1} times the top row. When we compute the rref, every row but the top will be removed in the first step. Thus, $\text{rref}(A)$ is a matrix with the top row of A and zeroes for all other entries.

2.2

1. The standard L is transformed into a distorted L whose foot is the vector $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Meanwhile, the back becomes the vector $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

2. By Fact 2.2.3, this matrix is $\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

3. If \vec{x} is in the unit square in \mathbb{R}^2 , then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ with $0 \leq x_1, x_2 \leq 1$, so that

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2).$$

The image of the unit square is a parallelogram in \mathbb{R}^3 ; two of its sides are $T(\vec{e}_1)$ and $T(\vec{e}_2)$, and the origin is one of its vertices. (See Figure 2.19.)

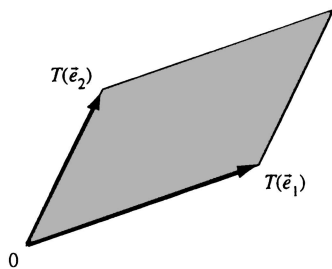


Figure 2.19: for Problem 2.2.3.

4. By Fact 2.2.4, this is a rotation combined with a scaling. The transformation rotates 45 degrees counterclockwise, and has a scaling factor of $\sqrt{2}$.
5. Note that $\cos(\theta) = -0.8$, so that $\theta = \arccos(-0.8) \approx 2.498$.

6. By Fact 2.2.1, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u}$, where \vec{u} is a unit vector on L . To get \vec{u} , we

normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$$\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ so that } \text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}.$$

7. According to the discussion on page 61, $\text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where \vec{u} is a

unit vector on L . To get \vec{u} , we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$$\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ so that } \text{ref}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\left(\frac{5}{3}\right)\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{1}{9} \\ \frac{11}{9} \end{bmatrix}.$$

8. From Definition 2.2.2, we can see that this is a reflection about the line $x_1 = -x_2$.

9. By Fact 2.2.5, this is a vertical shear.

10. By Fact 2.2.1, $\text{proj}_L \vec{x} = (\vec{u} \cdot \vec{x})\vec{u}$, where \vec{u} is a unit vector on L . We can choose $\vec{u} = \frac{1}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$.

$$\begin{aligned} \text{Then } \text{proj}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \left(\begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} = (0.8x_1 + 0.6x_2) \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.64x_1 + 0.48x_2 \\ 0.48x_1 + 0.36x_2 \end{bmatrix} = \\ &= \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

$$\text{The matrix is } A = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}.$$

11. In Exercise 10 we found the matrix $A = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$ of the projection onto the line L . By Fact 2.2.2,

$\text{ref}_L \vec{x} = 2(\text{proj}_L \vec{x}) - \vec{x} = 2A\vec{x} - \vec{x} = (2A - I_2)\vec{x}$, so that the matrix of the reflection is

$$2A - I_2 = \begin{bmatrix} 0.28 & 0.96 \\ 0.96 & -0.28 \end{bmatrix}.$$

12. From Definition 2.2.1, we can figure out the terms of this matrix from a unit vector. Converting \vec{v} to a unit vector is simple, as we will just divide by its length: $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} =$

$$\frac{\vec{v}}{\sqrt{v_1^2 + v_2^2}} = \begin{bmatrix} \frac{v_1}{\sqrt{v_1^2 + v_2^2}} \\ \frac{v_2}{\sqrt{v_1^2 + v_2^2}} \end{bmatrix}.$$

$$\text{So, } u_1 = \frac{v_1}{\sqrt{v_1^2 + v_2^2}}, u_2 = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}. \text{ Then } A = \begin{bmatrix} v_1^2/(v_1^2 + v_2^2) & v_1v_2/(v_1^2 + v_2^2) \\ v_1v_2/(v_1^2 + v_2^2) & v_2^2/(v_1^2 + v_2^2) \end{bmatrix} = \frac{1}{v_1^2 + v_2^2} \begin{bmatrix} v_1^2 & v_1v_2 \\ v_1v_2 & v_2^2 \end{bmatrix}.$$

13. By Fact 2.2.2,

$$\begin{aligned} \text{ref}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 2 \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2(u_1x_1 + u_2x_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}. \end{aligned}$$

The matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$. Note that the sum of the diagonal entries is $a + d = 2(u_1^2 + u_2^2) - 2 = 0$, since \vec{u} is a unit vector. It follows that $d = -a$.

Since $c = b$, A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Also, $a^2 + b^2 = (2u_1^2 - 1)^2 + 4u_1^2u_2^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1 - u_1^2) = 1$, as claimed.

14. a. Proceeding as on Page 58 of the text, we find that A is the matrix whose ij th entry is $u_i u_j$:

$$A = \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 & u_2 u_3 \\ u_n u_1 & u_n u_2 & u_3^2 \end{bmatrix}$$

- b. The sum of the diagonal entries is $u_1^2 + u_2^2 + u_3^2 = 1$, since \vec{u} is a unit vector.

15. According to the discussion on Page 61, $\text{ref}_L(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$

$$\begin{aligned} &= 2(x_1 u_1 + x_2 u_2 + x_3 u_3) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 u_1^2 & +2x_2 u_2 u_1 & +2x_3 u_3 u_1 & -x_1 \\ 2x_1 u_1 u_2 & +2x_2 u_2^2 & +2x_3 u_3 u_2 & -x_2 \\ 2x_1 u_1 u_3 & +2x_2 u_2 u_3 & +2x_3 u_3^2 & -x_3 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 & +2u_2 u_1 x_2 & +2u_1 u_3 x_3 \\ 2u_1 u_2 x_1 & +(2u_2^2 - 1)x_2 & +2u_2 u_3 x_3 \\ 2u_1 u_3 x_1 & +2u_2 u_3 x_2 & +(2u_3^2 - 1)x_3 \end{bmatrix}. \\ \text{So } A &= \begin{bmatrix} (2u_1^2 - 1) & 2u_2 u_1 & 2u_1 u_3 \\ 2u_1 u_2 & (2u_2^2 - 1) & 2u_2 u_3 \\ 2u_1 u_3 & 2u_2 u_3 & (2u_3^2 - 1) \end{bmatrix}. \end{aligned}$$

16. a. See Figure 2.20.

- b. By Fact 2.1.2, the matrix of T is $[T(\vec{e}_1) \quad T(\vec{e}_2)]$.

$T(\vec{e}_2)$ is the unit vector in the fourth quadrant perpendicular to $T(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$,

so that

$$T(\vec{e}_2) = \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}. \text{ The matrix of } T \text{ is therefore } \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Alternatively, we can use the result of Exercise 13, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ to find the matrix

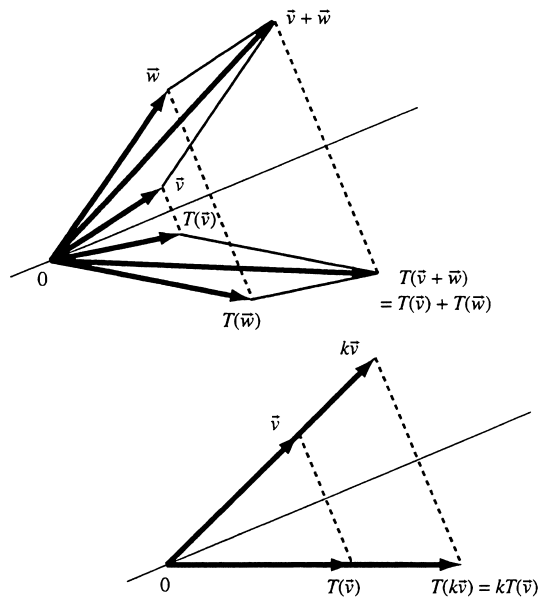


Figure 2.20: for Problem 2.2.16a.

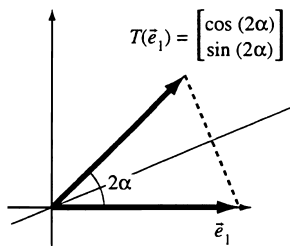


Figure 2.21: for Problem 2.2.16b.

$$\begin{bmatrix} 2 \cos^2 \theta - 1 & 2 \cos \theta \sin \theta \\ 2 \cos \theta \sin \theta & 2 \sin^2 \theta - 1 \end{bmatrix}.$$

You can use trigonometric identities to show that the two results agree.

17. We want, $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ bv_1 - av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$

Now, $(a - 1)v_1 + bv_2 = 0$ and $bv_1 - (a + 1)v_2 = 0$, which is a system with solutions of the

form $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$, where t is an arbitrary constant.

Let's choose $t = 1$, making $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$.

Similarly, we want $A\vec{w} = -\vec{w}$. We perform a computation as above to reveal $\vec{w} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ as a possible choice. A quick check of $\vec{v} \cdot \vec{w} = 0$ reveals that they are indeed perpendicular.

Now, any vector \vec{x} in \mathbb{R}^2 can be written in terms of components with respect to $L = \text{span}(\vec{v})$ as $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$. Then, $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{\parallel} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$, by Definition 2.2.2.

(The vectors \vec{v} and \vec{w} constructed above are both zero in the special case that $a = 1$ and $b = 0$. In that case, we can let $\vec{v} = \vec{e}_1$ and $\vec{w} = \vec{e}_2$ instead.)

18. From Exercise 17, we know that the reflection is about the line parallel to $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix} = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, every point on this line can be described as $\begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, $y = k = \frac{1}{2}x$, and $y = \frac{1}{2}x$ is the line we are looking for.

19. $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{0}$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

20. $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = -\vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

21. $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (See Figure 2.22.)

22. Sketch the $\vec{e}_1 - \vec{e}_3$ plane, as viewed from the positive \vec{e}_2 axis.

Since $T(\vec{e}_2) = \vec{e}_2$, the matrix is $\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$.

23. $T(\vec{e}_1) = \vec{e}_3$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_1$, so that the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. (See Figure 2.24.)

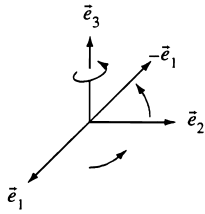


Figure 2.22: for Problem 2.2.21.

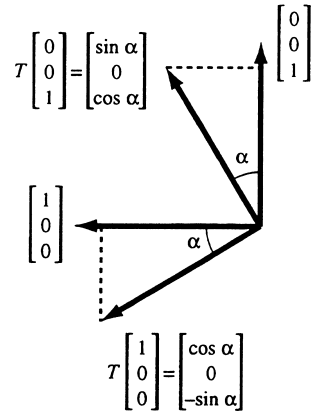


Figure 2.23: for Problem 2.2.22.

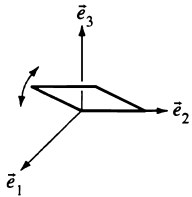


Figure 2.24: for Problem 2.2.23.

24. a. $A = [\vec{v} \ \vec{w}]$, so $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}$. Since A preserves length, both \vec{v} and \vec{w} must be unit vectors. Furthermore, since A preserves angles and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are clearly perpendicular, \vec{v} and \vec{w} must also be perpendicular.

- b. Since \vec{w} is a unit vector perpendicular to \vec{v} , it can be obtained by rotating \vec{v} through 90 degrees, either in the counterclockwise or in the clockwise direction. Using the corresponding rotation matrices, we see that $\vec{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\vec{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$.
- c. Following part b, A is either of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, representing a rotation, or $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, representing a reflection.
25. The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, and its inverse $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ represents such a shear as well, but “the other way.”
26. a. $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$. So $k = 4$ and $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.
- b. This is the orthogonal projection onto the horizontal axis, with matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
- c. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5b \\ 5a \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. So $a = \frac{4}{5}$, $b = -\frac{3}{5}$, and $C = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a rotation matrix.
- d. Since the x_1 term is being modified, this must be a horizontal shear. Then $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + 3k \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. So $k = 2$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
- e. $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7a + b \\ 7b - a \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$. So $a = -\frac{4}{5}$, $b = \frac{3}{5}$, and $E = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a reflection matrix.
27. Matrix B clearly represents a scaling.
 Matrix C represents a projection, by Definition 2.2.1, with $u_1 = 0.6$ and $u_2 = 0.8$.
 Matrix E represents a shear, by Fact 2.2.5.
 Matrix A represents a reflection, by Definition 2.2.2.

Matrix D represents a rotation, by Definition 2.2.3.

28. a. D is a scaling, being of the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.
- b. E is the shear, since it is the only matrix which has the proper form (Fact 2.2.5).
- c. C is the rotation, since it fits Fact 2.2.3.
- d. A is the projection, following the form given in Definition 2.2.1.
- e. F is the reflection, using Definition 2.2.2.

29. To check that L is linear, we verify the two parts of Fact 2.1.3.

- a. Use the hint and apply L on both sides of the equation $\vec{x} + \vec{y} = T(L(\vec{x}) + L(\vec{y}))$:

$$L(\vec{x} + \vec{y}) = L(T(L(\vec{x}) + L(\vec{y}))) = L(\vec{x}) + L(\vec{y}), \text{ as claimed.}$$

- b. $L(k\vec{x}) = L(kT(L(\vec{x}))) = L(T(kL(\vec{x}))) = kL(\vec{x})$, as claimed.

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & & \end{array}$$

$$\vec{x} = T(L(\vec{x})) \quad T \text{ is linear.}$$

30. Write $A = [\vec{v}_1 \ \vec{v}_2]$; then $A\vec{x} = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$. We must choose \vec{v}_1 and \vec{v}_2

in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2$ is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all x_1 and x_2 .

This is the case if (and only if) both \vec{v}_1 and \vec{v}_2 are scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For example, choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

31. Write $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$; then $A\vec{x} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$.

We must choose \vec{v}_1, \vec{v}_2 , and \vec{v}_3 in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ is perpendicular to

$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for all x_1, x_2 , and x_3 . This is the case if (and only if) all the vectors \vec{v}_1, \vec{v}_2 , and

\vec{v}_3 are perpendicular to \vec{w} , that is, if $\vec{v}_1 \cdot \vec{w} = \vec{v}_2 \cdot \vec{w} = \vec{v}_3 \cdot \vec{w} = 0$.

For example, we can choose $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \vec{v}_3 = \vec{0}$, so that $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

32. a. See Figure 2.25.

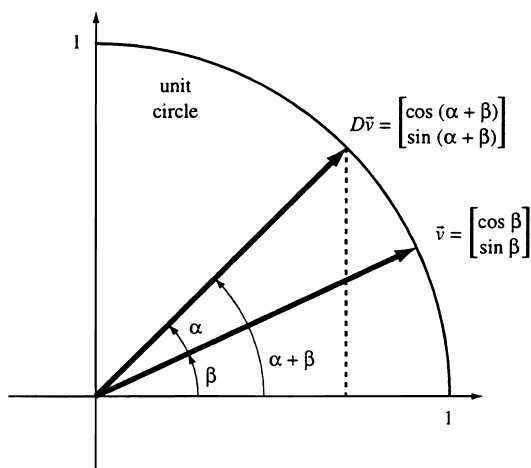


Figure 2.25: for Problem 2.2.32a.

b. Compute $D\vec{v} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{bmatrix}$.

Comparing this result with our finding in part (a), we get the addition theorems

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

33. Geometrically, we can find the representation $\vec{v} = \vec{v}_1 + \vec{v}_2$ by means of a parallelogram, as shown in Figure 2.26.

To show the existence and uniqueness of this representation algebraically, choose a nonzero vector \vec{w}_1 in L_1 and a nonzero \vec{w}_2 in L_2 . Then the system $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$ or $[\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ has only the solution $x_1 = x_2 = 0$ (if $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$ then $x_1\vec{w}_1 = -x_2\vec{w}_2$ is both in L_1 and in L_2 , so that it must be the zero vector).

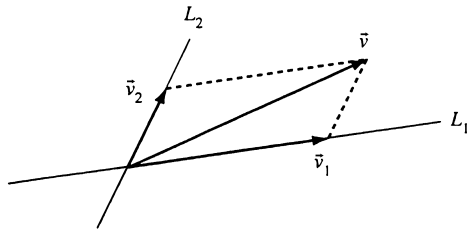


Figure 2.26: for Problem 2.2.33.

Therefore, the system $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{v}$ or $[\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v}$ has a unique solution x_1, x_2 for all \vec{v} in \mathbb{R}^2 (by Fact 1.3.4). Now set $\vec{v}_1 = x_1\vec{w}_1$ and $\vec{v}_2 = x_2\vec{w}_2$ to obtain the desired representation $\vec{v} = \vec{v}_1 + \vec{v}_2$. (Compare with Exercise 1.3.57.)

To show that the transformation $T(\vec{v}) = \vec{v}_1$ is linear, we will verify the two parts of Fact 2.1.3.

Let $\vec{v} = \vec{v}_1 + \vec{v}_2$, $\vec{w} = \vec{w}_1 + \vec{w}_2$, so that $\vec{v} + \vec{w} = (\vec{v}_1 + \vec{w}_1) + (\vec{v}_2 + \vec{w}_2)$ and $k\vec{v} = k\vec{v}_1 + k\vec{v}_2$.

$$\begin{array}{ccccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{in } L_1 & \text{in } L_2 & \text{in } L_1 & \text{in } L_2 & \text{in } L_1 & \text{in } L_2 & \text{in } L_1 & \text{in } L_2 \end{array}$$

a. $T(\vec{v} + \vec{w}) = \vec{v}_1 + \vec{w}_1 = T(\vec{v}) + T(\vec{w})$, and

b. $T(k\vec{v}) = k\vec{v}_1 = kT(\vec{v})$, as claimed.

34. Keep in mind that the columns of the matrix of a linear transformation T from \mathbb{R}^3 to \mathbb{R}^3 are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

If T is the orthogonal projection onto a line L , then $T(\vec{x})$ will be on L for all \vec{x} in \mathbb{R}^3 ; in particular, the three columns of the matrix of T will be on L , and therefore pairwise parallel. This is the case only for matrix B : B represents an orthogonal projection onto a line.

A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E : E represents the reflection about a line.

35. If the vectors \vec{v}_1 and \vec{v}_2 are defined as shown in Figure 2.27, then the parallelogram P consists of all vectors of the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \leq c_1, c_2 \leq 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

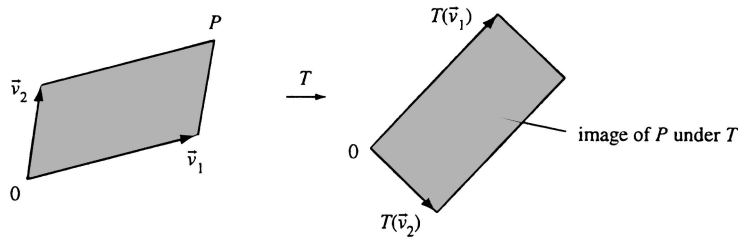


Figure 2.27: for Problem 2.2.35.

These vectors form the parallelogram shown in Figure 2.27 on the right.

36. If the vectors \vec{v}_0 , \vec{v}_1 , and \vec{v}_2 are defined as shown in Figure 2.28, then the parallelogram P consists of all vectors \vec{v} of the form $\vec{v} = \vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \leq c_1, c_2 \leq 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2) = T(\vec{v}_0) + c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

These vectors form the parallelogram shown in Figure 2.28 on the right.

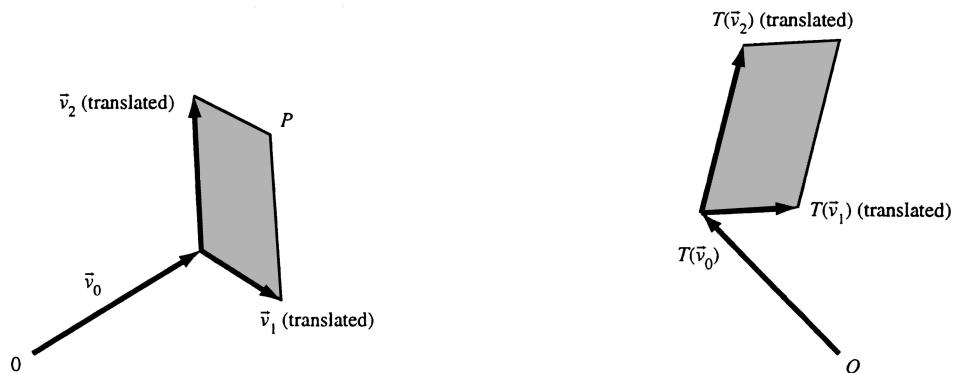


Figure 2.28: for Problem 2.2.36.

37. a. By Definition 2.2.1, a projection has a matrix of the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector.

So the trace is $u_1^2 + u_2^2 = 1$.

- b. By Definition 2.2.2, reflection matrices look like $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so the trace is $a - a = 0$.
- c. According to Fact 2.2.3, a rotation matrix has the form $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, so the trace is $\cos \theta + \cos \theta = 2 \cos \theta$ for some θ . Thus, the trace is in the interval $[-2, 2]$.
- d. By Fact 2.2.5, the matrix of a shear appears as either $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, depending on whether it represents a vertical or horizontal shear. In both cases, however, the trace is $1 + 1 = 2$.
38. a. $A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$, so $\det(A) = u_1^2 u_2^2 - u_1 u_2 u_1 u_2 = 0$.
- b. $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so $\det(A) = -a^2 - b^2 = -(a^2 + b^2) = -1$.
- c. $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so $\det(A) = a^2 - (-b^2) = a^2 + b^2 = 1$.
- d. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, both of which have determinant equal to $1^2 - 0 = 1$.
39. a. Note that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. The matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ represents an orthogonal projection (Definition 2.2.1), with $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. So, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ represents a projection combined with a scaling by a factor of 2.
- b. This looks similar to a shear, with the one zero off the diagonal. Since the two diagonal entries are identical, we can write $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$, showing that this matrix represents a vertical shear combined with a scaling by a factor of 3.
- c. We are asked to write $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = k \begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$, with our scaling factor k yet to be determined. This matrix, $\begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$ has the form of a reflection matrix $\left(\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \right)$. This form further requires that $1 = a^2 + b^2 = \left(\frac{3}{k}\right)^2 + \left(\frac{4}{k}\right)^2$, or $k = 5$. Thus, the matrix represents a reflection combined with a scaling by a factor of 5.

40. $\vec{x} = \text{proj}_P \vec{x} + \text{proj}_Q \vec{x}$, as illustrated in Figure 2.29.

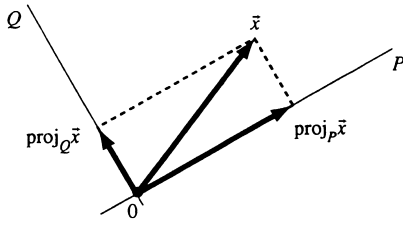


Figure 2.29: for Problem 2.2.40.

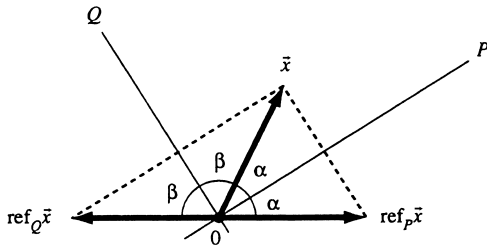


Figure 2.30: for Problem 2.2.41.

41. $\text{ref}_Q \vec{x} = -\text{ref}_P \vec{x}$ since $\text{ref}_Q \vec{x}$, $\text{ref}_P \vec{x}$, and \vec{x} all have the same length, and $\text{ref}_Q \vec{x}$ and $\text{ref}_P \vec{x}$ enclose an angle of $2\alpha + 2\beta = 2(\alpha + \beta) = \pi$. (See Figure 2.30.)
42. $T(\vec{x}) = T(T(\vec{x}))$ since $T(\vec{x})$ is on L hence the projection of $T(\vec{x})$ onto L is $T(\vec{x})$ itself.
43. Since $\vec{y} = A\vec{x}$ is obtained from \vec{x} by a rotation through θ in the counterclockwise direction, \vec{x} is obtained from \vec{y} by a rotation through θ in the *clockwise* direction, that is, a rotation through $-\theta$. (See Figure 2.31.)

Therefore, the matrix of the inverse transformation is $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. You can use the formula in Exercise 2.1.13b to check this result.

44. By Exercise 1.1.13b, $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

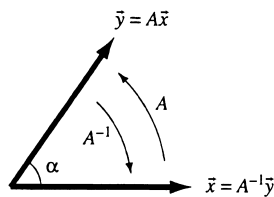


Figure 2.31: for Problem 2.2.43.

If A represents a rotation through θ followed by a scaling by r , then A^{-1} represents a rotation through $-\theta$ followed by a scaling by $\frac{1}{r}$.

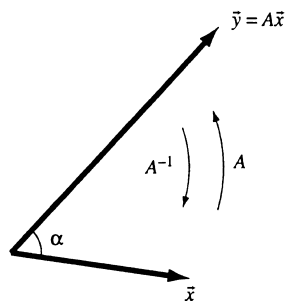


Figure 2.32: for Problem 2.2.44.

45. By Exercise 2.1.13, $A^{-1} = \frac{1}{-a^2-b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \frac{1}{-(a^2+b^2)} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = -1 \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

So $A^{-1} = A$, which makes sense. Reflecting a vector twice about the same line will return it to its original state.

46. We want to write $A = k \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$, where the matrix $B = \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$ represents a reflection. It is required that $(\frac{a}{k})^2 + (\frac{b}{k})^2 = 1$, meaning that $a^2 + b^2 = k^2$, or, $k = \sqrt{a^2 + b^2}$.

Now $A^{-1} = \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \frac{1}{k^2} A = \frac{1}{k} B$, for the reflection matrix B and the scaling factor k introduced above. In summary: If A represents a reflection combined with a scaling by k , then A^{-1} represents the same reflection combined with a scaling by $\frac{1}{k}$.

47. Write $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$.

$$\begin{aligned} \text{a. } f(t) &= \left(T \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \right) \cdot \left(T \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right) = \begin{bmatrix} a \cos t + b \sin t \\ c \cos t + d \sin t \end{bmatrix} \cdot \begin{bmatrix} -a \sin t + b \cos t \\ -c \sin t + d \cos t \end{bmatrix} \\ &= (a \cos t + b \sin t)(-a \sin t + b \cos t) + (c \cos t + d \sin t)(-c \sin t + d \cos t) \end{aligned}$$

This function $f(t)$ is continuous, since $\cos(t)$, $\sin(t)$, and constant functions are continuous, and sums and products of continuous functions are continuous.

$$\text{b. } f\left(\frac{\pi}{2}\right) = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} -1 \\ 0 \end{bmatrix} = - \left(T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \text{ since } T \text{ is linear.}$$

$$f(0) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ The claim follows.}$$

- c. By part (b), the numbers $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have different signs (one is positive and the other negative), or they are both zero. Since $f(t)$ is continuous, by part (a), we can apply the intermediate value theorem. (See Figure 2.33.)

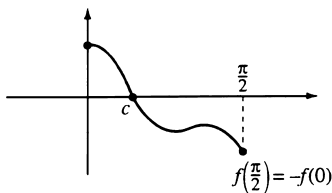


Figure 2.33: for Problem 2.2.47c.

- d. Note that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ are perpendicular unit vectors, for any t . If we set

$$\vec{v}_1 = \begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -\sin(c) \\ \cos(c) \end{bmatrix}, \text{ with the number } c \text{ we found in part (c), then}$$

$$f(c) = T(\vec{v}_1) \cdot T(\vec{v}_2) = 0, \text{ so that } T(\vec{v}_1) \text{ and } T(\vec{v}_2) \text{ are perpendicular, as claimed. Note that } T(\vec{v}_1) \text{ or } T(\vec{v}_2) \text{ may be zero.}$$

48. We find

$$\begin{aligned} f(t) &= \left(\begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \right) \\ &= \begin{bmatrix} 4 \sin(t) \\ 5 \cos(t) - 3 \sin(t) \end{bmatrix} \cdot \begin{bmatrix} 4 \cos(t) \\ -5 \sin(t) - 3 \cos(t) \end{bmatrix} \end{aligned}$$

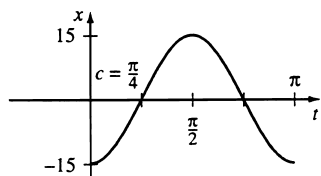


Figure 2.34: for Problem 2.2.48.

$$= 15(\sin^2 t - \cos^2 t) = 15(2 \sin^2 t - 1). \text{ See Figure 2.34.}$$

The only zero of $f(t)$ between 0 and $\frac{\pi}{2}$ is at $c = \frac{\pi}{4}$.

Therefore, $\vec{v}_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -\sin(\frac{\pi}{4}) \\ \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ work. Note

$$\text{that } T(\vec{v}_1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

and $T(\vec{v}_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ are indeed perpendicular. See Figure 2.35.

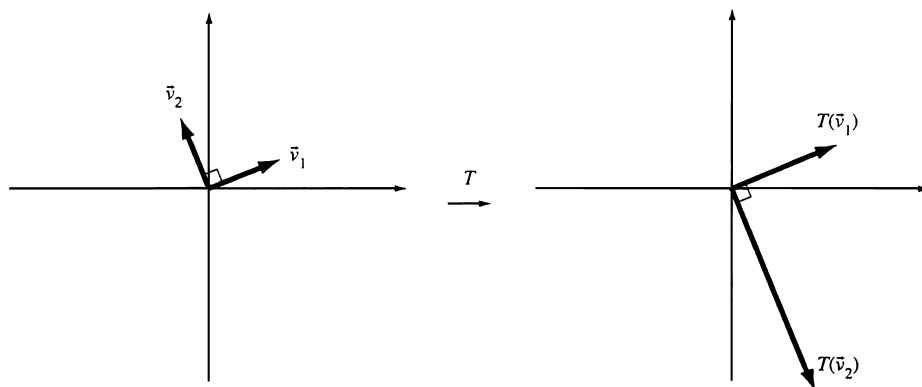


Figure 2.35: for Problem 2.2.48.

49. If $\vec{x} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ then $T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} 5 \cos(t) \\ 2 \sin(t) \end{bmatrix} = \cos(t) \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$

These vectors form an ellipse; consider the characterization of an ellipse given in the footnote on page 70, with $\vec{w}_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. (See Figure 2.36.)

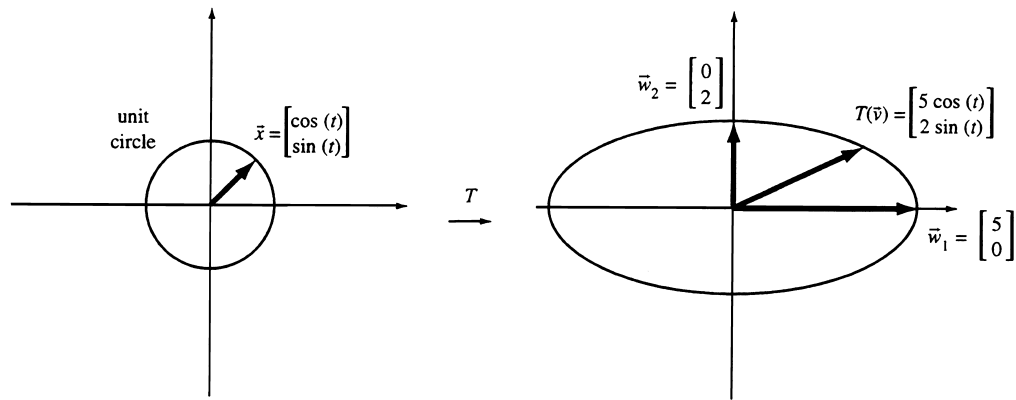


Figure 2.36: for Problem 2.2.49.

50. Use the hint: Since the vectors on the unit circle are of the form $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$, the image of the unit circle consists of the vectors of the form $T(\vec{v}) = T(\cos(t)\vec{v}_1 + \sin(t)\vec{v}_2) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$.

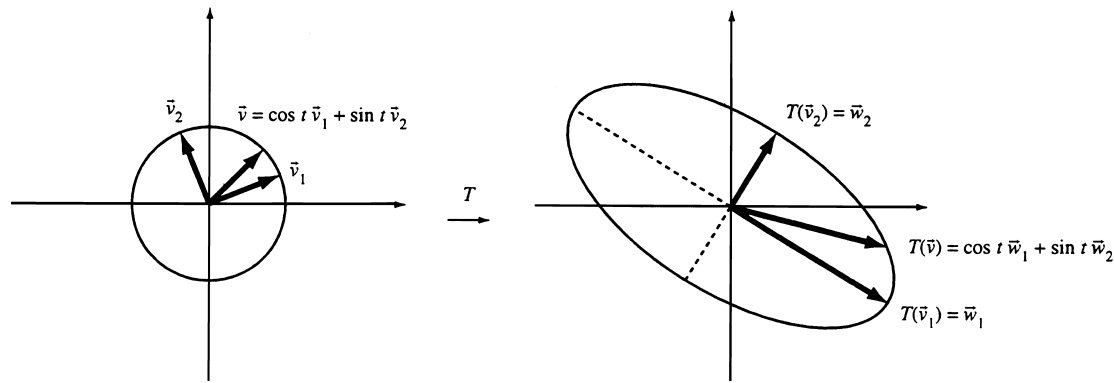


Figure 2.37: for Problem 2.2.50.

These vectors form an ellipse: Consider the characterization of an ellipse given in the footnote, with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$. The key point is that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular.

51. Consider the linear transformation T with matrix $A = [\vec{w}_1 \quad \vec{w}_2]$, that is,

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\vec{w}_1 \quad \vec{w}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{w}_1 + x_2 \vec{w}_2.$$

The curve C is the image of the unit circle under the transformation T : if $\vec{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$

is on the unit circle, then $T(\vec{v}) = \cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$ is on the curve C . Therefore, C is an ellipse, by Exercise 50. (See Figure 2.38.)

52. By definition, the vectors \vec{v} on an ellipse E are of the form $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$, for some perpendicular vectors \vec{v}_1 and \vec{v}_2 . Then the vectors on the image C of E are of the form $T(\vec{v}) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$. These vectors form an ellipse, by Exercise 51 (with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$).

2.3

1. $\text{rref} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 5 & 8 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 8 & -3 \\ 0 & 1 & -5 & 2 \end{bmatrix}$, so that $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$.

2. $\text{rref} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.

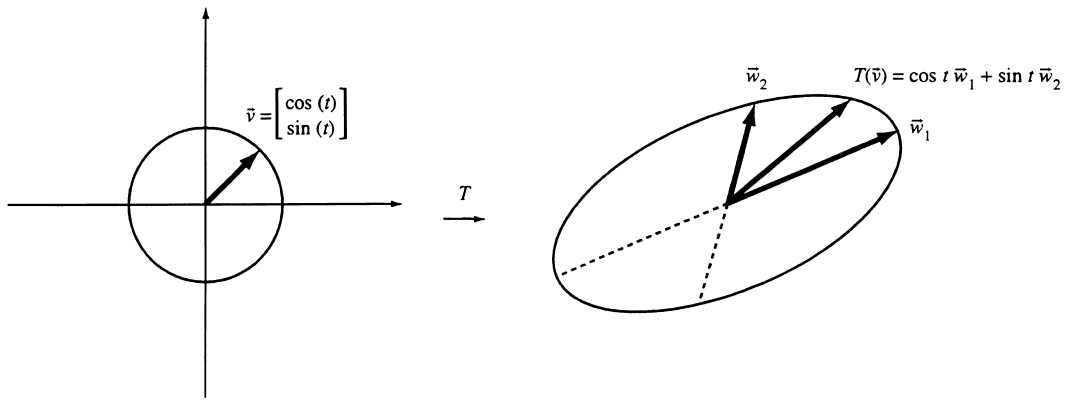


Figure 2.38: for Problem 2.2.51.

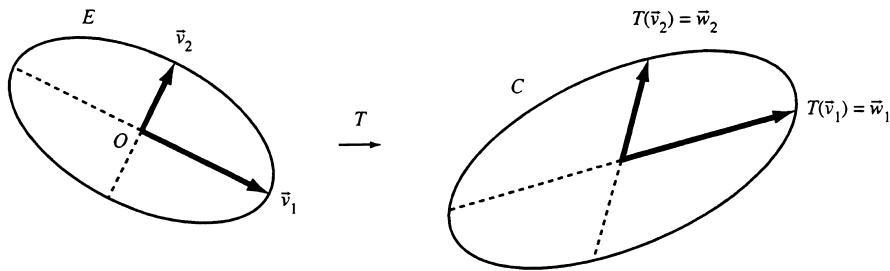


Figure 2.39: for Problem 2.2.52.

3. $\text{rref} \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$, so that $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

4. Use Fact 2.3.5; the inverse is $\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$.

5. $\text{rref} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

6. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

7. $\text{rref} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

8. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

9. $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Fact 2.3.3.

10. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

11. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

12. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$.

13. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

14. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

15. Use Fact 2.3.5; the inverse is $\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}$.

16. Solving for x_1 and x_2 in terms of y_1 and y_2 we find that

$$\begin{aligned} x_1 &= -8y_1 + 5y_2 \\ x_2 &= 5y_1 - 3y_2 \end{aligned}$$

17. We make an attempt to solve for x_1 and x_2 in terms of y_1 and y_2 :

$$\left| \begin{array}{rcl} x_1 + 2x_2 & = & y_1 \\ 4x_1 + 8x_2 & = & y_2 \end{array} \right| \xrightarrow{-4(I)} \left| \begin{array}{rcl} x_1 + 2x_2 & = & y_1 \\ 0 & = & -4y_1 + y_2 \end{array} \right|.$$

This system has no solutions (x_1, x_2) for some (y_1, y_2) , and infinitely many solutions for others; the transformation fails to be invertible.

18. Solving for x_1, x_2 , and x_3 in terms of y_1, y_2 , and y_3 we find that

$$\begin{aligned} x_1 &= y_3 \\ x_2 &= y_1 \\ x_3 &= y_2 \end{aligned}$$

19. Solving for x_1, x_2 , and x_3 in terms of y_1, y_2 , and y_3 , we find that

$$\begin{aligned} x_1 &= 3y_1 - \frac{5}{2}y_2 + \frac{1}{2}y_3 \\ x_2 &= -3y_1 + 4y_2 - y_3 \\ x_3 &= y_1 - \frac{3}{2}y_2 + \frac{1}{2}y_3 \end{aligned}$$

20. Solving for x_1, x_2 , and x_3 in terms of y_1, y_2 , and y_3 we find that

$$\begin{aligned} x_1 &= -8y_1 - 15y_2 + 12y_3 \\ x_2 &= 4y_1 + 6y_2 - 5y_3 \\ x_3 &= -y_1 - y_2 + y_3 \end{aligned}$$

21. $f(x) = x^2$ fails to be invertible, since the equation $f(x) = x^2 = 1$ has two solutions, $x = \pm 1$.

22. $f(x) = 2^x$ fails to be invertible, since the equation $f(x) = 2^x = 0$ has no solution x .

23. Note that $f'(x) = 3x^2 + 1$ is always positive; this implies that the function $f(x) = x^3 + x$ is increasing throughout. Therefore, the equation $f(x) = b$ has *at most* one solution x for all b . (See Figure 2.40.)

Now observe that $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$; this implies that the equation $f(x) = b$ has at least one solution x for a given b (for a careful proof, use the intermediate value theorem; compare with Exercise 2.2.47c).

24. We can write $f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$.

The equation $f(x) = 0$ has three solutions, $x = 0, 1, -1$, so that $f(x)$ fails to be invertible.

25. Invertible, with inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_1} \\ y_2 \end{bmatrix}$

26. Invertible, with inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_2 - y_1} \\ y_1 \end{bmatrix}$

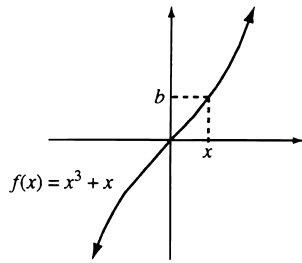


Figure 2.40: for Problem 2.3.23.

27. This transformation fails to be invertible, since the equation $\begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution.

28. We are asked to find the inverse of the matrix $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$.

$$\text{We find that } A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & 80 & 222 \end{bmatrix}.$$

T^{-1} is the transformation from \mathbb{R}^4 to \mathbb{R}^4 with matrix A^{-1} .

29. Use Fact 2.3.3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{bmatrix} \xrightarrow{\substack{-II \\ -3(II)}} \begin{bmatrix} 1 & 0 & 2-k \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix}$$

The matrix is invertible if (and only if) $k^2 - 3k + 2 = (k - 2)(k - 1) \neq 0$, in which case we can further reduce it to I_3 . Therefore, the matrix is invertible if $k \neq 1$ and $k \neq 2$.

30. Use Fact 2.3.3:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{I \leftrightarrow II} \begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{+b(I) + c(II)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix fails to be invertible, regardless of the values of b and c .

31. Use Fact 2.3.3; first assume that $a \neq 0$.

$$\begin{aligned} \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} & \xrightarrow{\substack{\text{swap:} \\ I \leftrightarrow II}} \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-a)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{+b(I)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \xrightarrow{\div a} \\ \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} & \xrightarrow{+c(II)} \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Now consider the case when $a = 0$:

$$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\substack{\text{swap:} \\ I \leftrightarrow III}} \begin{bmatrix} -b & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & b \end{bmatrix} : \text{The second entry on the diagonal of ref}$$

will be 0.

It follows that the matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ fails to be invertible, regardless of the values of a , b , and c .

32. Use Fact 2.3.6.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix such that $ad - bc = 1$ and $A^{-1} = A$, then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ so that } b = 0, c = 0, \text{ and } a = d.$$

The condition $ad - bc = a^2 = 1$ now implies that $a = d = 1$ or $a = d = -1$.

This leaves only two matrices A , namely, I_2 and $-I_2$. Check that these two matrices do indeed satisfy the given requirements.

33. Use Fact 2.3.6.

The requirement $A^{-1} = A$ means that $-\frac{1}{a^2+b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. This is the case if (and only if) $a^2 + b^2 = 1$.

34. a. By Fact 2.3.3, A is invertible if (and only if) a , b , and c are all nonzero. In this case,

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}.$$

- b. In general, a diagonal matrix is invertible if (and only if) all of its diagonal entries are nonzero.
35. a. A is invertible if (and only if) all its diagonal entries, a , d , and f , are nonzero.
- b. As in part (a): if all the diagonal entries are nonzero.
- c. Yes, A^{-1} will be upper triangular as well; as you construct $\text{rref}[A:I_n]$, you will perform only the following row operations:
- divide rows by scalars
 - subtract a multiple of the j th row from the i th row, where $j > i$.
- Applying these operations to I_n , you end up with an upper triangular matrix.
- d. As in part (b): if all diagonal entries are nonzero.

36. If a matrix A can be transformed into B by elementary row operations, then A is invertible if (and only if) B is invertible. The claim now follows from Exercise 35, where we show that a triangular matrix is invertible if (and only if) its diagonal entries are nonzero.

37. Make an attempt to solve the linear equation $\vec{y} = (cA)\vec{x} = c(A\vec{x})$ for \vec{x} :

$$A\vec{x} = \frac{1}{c}\vec{y}, \text{ so that } \vec{x} = A^{-1}\left(\frac{1}{c}\vec{y}\right) = \left(\frac{1}{c}A^{-1}\right)\vec{y}.$$

This shows that cA is indeed invertible, with $(cA)^{-1} = \frac{1}{c}A^{-1}$.

38. Use Fact 2.3.6; $A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} (= A)$.

39. Suppose the ij th entry of M is k , and all other entries are as in the identity matrix. Then we can find $\text{rref}[M:I_n]$ by subtracting k times the j th row from the i th row. Therefore, M is indeed invertible, and M^{-1} differs from the identity matrix only at the ij th entry; that entry is $-k$. (See Figure 2.41.)

40. If you apply an elementary row operation to a matrix with two equal columns, then the resulting matrix will also have two equal columns. Therefore, $\text{rref}(A)$ has two equal columns, so that $\text{rref}(A) \neq I_n$. Now use Fact 2.3.3.

41. a. Invertible: the transformation is its own inverse.

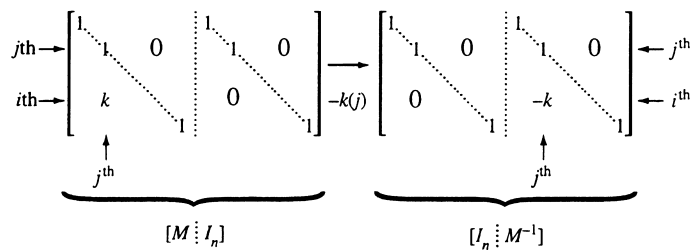


Figure 2.41: for Problem 2.3.39.

- b. Not invertible: the equation $T(\vec{x}) = \vec{b}$ has infinitely many solutions if \vec{b} is on the plane, and none otherwise.
- c. Invertible: The inverse is a scaling by $\frac{1}{5}$ (that is, a contraction by 5). If $\vec{y} = 5\vec{x}$, then $\vec{x} = \frac{1}{5}\vec{y}$.
- d. Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.
42. Permutation matrices are invertible since they row reduce to I_n in an obvious way, just by row swaps. The inverse of a permutation matrix A is also a permutation matrix since $\text{rref}[A: I_n] = [I_n: A^{-1}]$ is obtained from $[A: I_n]$ by a sequence of row swaps.
43. We make an attempt to solve the equation $\vec{y} = A(B\vec{x})$ for \vec{x} :

$$B\vec{x} = A^{-1}\vec{y}, \text{ so that } \vec{x} = B^{-1}(A^{-1}\vec{y}).$$

44. a. $\text{rref}(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so that $\text{rank}(M_4) = 2$.

- b. To simplify the notation, we introduce the row vectors $\vec{v} = [1 \ 1 \ \dots \ 1]$ and $\vec{w} = [0 \ n \ 2n \ \dots \ (n-1)n]$

with n components. Then we can write M_n in terms of its rows as $M_n = \begin{bmatrix} \vec{v} + \vec{w} \\ 2\vec{v} + \vec{w} \\ \dots \\ n\vec{v} + \vec{w} \end{bmatrix} \begin{matrix} -2(I) \\ \dots \\ -n(I) \end{matrix}$.

Applying the Gauss-Jordan algorithm to the first column we get

$$\begin{bmatrix} \vec{v} + \vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \dots \\ -(n-1)\vec{w} \end{bmatrix}.$$

All the rows below the second are scalar multiples of the second; therefore, $\text{rank}(M_n) = 2$.

- c. By part (b), the matrix M_n is invertible only if $n = 1$ or $n = 2$.
45. a. Each of the three row divisions requires three multiplicative operations, and each of the six row subtractions requires three multiplicative operations as well; altogether, we have $3 \cdot 3 + 6 \cdot 3 = 9 \cdot 3 = 3^3 = 27$ operations.
- b. Suppose we have already taken care of the first m columns: $[A:I_n]$ has been reduced the matrix in Figure 2.42.

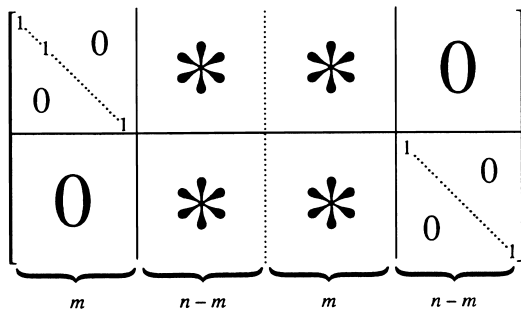


Figure 2.42: for Problem 2.3.45b.

Here, the stars represent arbitrary entries.

Suppose the $(m+1)$ th entry on the diagonal is k . Dividing the $(m+1)$ th row by k requires n operations: $n-m-1$ to the left of the dotted line (not counting the computation $\frac{k}{k} = 1$), and $m+1$ to the right of the dotted line (including $\frac{1}{k}$). Now the matrix has the form shown in Figure 2.43.

Eliminating each of the other $n-1$ components of the $(m+1)$ th column now requires n multiplicative operations ($n-m-1$ to the left of the dotted line, and $m+1$ to the right). Altogether, it requires $n + (n-1)n = n^2$ operations to process the m th column. To process all n columns requires $n \cdot n^2 = n^3$ operations.

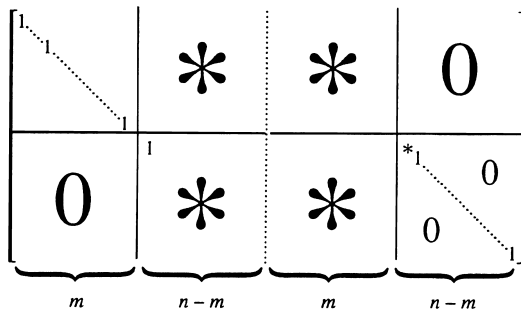


Figure 2.43: for Problem 2.3.45b.

- c. The inversion of a 12×12 matrix requires $12^3 = 4^3 3^3 = 64 \cdot 3^3$ operations, that is, 64 times as much as the inversion of a 3×3 matrix. If the inversion of a 3×3 matrix takes one second, then the inversion of a 12×12 matrix takes 64 seconds.
46. Computing $A^{-1}\vec{b}$ requires $n^3 + n^2$ operations: First, we need n^3 operations to find A^{-1} (see Exercise 45b) and then n^2 operations to compute $A^{-1}\vec{b}$ (n multiplications for each component).

How many operations are required to perform Gauss-Jordan eliminations on $[A:\vec{b}]$? Let us count these operations “column by column.” If m columns of the coefficient matrix are left, then processing the next column requires nm operations (compare with Exercise 45b). To process all the columns requires

$$n \cdot n + n(n-1) + \cdots + n \cdot 2 + n \cdot 1 = n(n + n-1 + \cdots + 2 + 1) = n \frac{n(n+1)}{2} = \frac{n^3 + n^2}{2}$$

operations.

only half of what was required to compute $A^{-1}\vec{b}$.

We mention in passing that one can reduce the number of operations further (by about 50% for large matrices) by performing the steps of the row reduction in a different order.

47. Let $f(x) = x^2$; the equation $f(x) = 0$ has the unique solution $x = 0$.

48. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note that Fact 2.3.4 applies to *square* matrices only.

$$49. \text{ a. } A = \begin{bmatrix} 0.293 & 0 & 0 \\ 0.014 & 0.207 & 0.017 \\ 0.044 & 0.01 & 0.216 \end{bmatrix}, \quad I_3 - A = \begin{bmatrix} 0.707 & 0 & 0 \\ -0.014 & 0.793 & -0.017 \\ -0.044 & -0.01 & 0.784 \end{bmatrix}$$

$$(I_3 - A)^{-1} = \begin{bmatrix} 1.41 & 0 & 0 \\ 0.0267 & 1.26 & 0.0274 \\ 0.0797 & 0.0161 & 1.28 \end{bmatrix}$$

b. We have $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\vec{x} = (I_3 - A)^{-1}\vec{e}_1 = \text{first column of } (I_3 - A)^{-1} \approx \begin{bmatrix} 1.41 \\ 0.0267 \\ 0.0797 \end{bmatrix}$.

c. As illustrated in part (b), the i th column of $(I_3 - A)^{-1}$ gives the output vector required to satisfy a consumer demand of 1 unit on industry i , in the absence of any other consumer demands. In particular, the i th diagonal entry of $(I_3 - A)^{-1}$ gives the output of industry i required to satisfy this demand. Since industry i has to satisfy the consumer demand of 1 as well as the interindustry demand, its total output will be at least 1.

d. Suppose the consumer demand increases from \vec{b} to $\vec{b} + \vec{e}_2$ (that is, the demand on manufacturing increases by one unit). Then the output must change from $(I_3 - A)^{-1}\vec{b}$ to

$$(I_3 - A)^{-1}(\vec{b} + \vec{e}_2) = (I_3 - A)^{-1}\vec{b} + (I_3 - A)^{-1}\vec{e}_2 = (I_3 - A)^{-1}\vec{b} + (\text{second column of } (I_3 - A)^{-1}).$$

The components of the second column of $(I_3 - A)^{-1}$ tells us by how much each industry has to increase its output.

e. The ij th entry of $(I_n - A)^{-1}$ gives the required increase of the output x_i of industry i to satisfy an increase of the consumer demand b_j on industry j by one unit. In the language of multivariable calculus, this quantity is $\frac{\partial x_i}{\partial b_j}$.

50. Recall that $1 + k + k^2 + \cdots = \frac{1}{1-k}$.

The top left entry of $I_3 - A$ is $1 - k$, and the top left entry of $(I_3 - A)^{-1}$ will therefore be $\frac{1}{1-k}$, as claimed:

$$\begin{bmatrix} 1-k & 0 & 0 & \vdots & 1 & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\div(1-k)} \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{1-k} & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{bmatrix}$$

→ ... (first row will remain unchanged).

In terms of economics, we can explain this fact as follows: The top left entry of $(I_3 - A)^{-1}$ is the output of industry 1 (Agriculture) required to satisfy a consumer demand of 1 unit on industry 1. Producing this one unit to satisfy the consumer demand will generate an extra demand of $k = 0.293$ units on industry 1. Producing these k units in turn will generate an extra demand of $k \cdot k = k^2$ units, and so forth. We are faced with an infinite series of (ever smaller) demands, $1 + k + k^2 + \dots$.

51. a. Since $\text{rank}(A) < n$, the matrix $E = \text{rref}(A)$ will not have a leading one in the last row, and all entries in the last row of E will be zero.

Let $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Then the last equation of the system $E\vec{x} = \vec{c}$ reads $0 = 1$, so this system

is inconsistent.

Now, we can “rebuild” \vec{b} from \vec{c} by performing the reverse row-operations in the opposite order on $\begin{bmatrix} E & \vec{c} \end{bmatrix}$ until we reach $\begin{bmatrix} A & \vec{b} \end{bmatrix}$. Since $E\vec{x} = \vec{c}$ is inconsistent, $A\vec{x} = \vec{b}$ is inconsistent as well.

- b. Since $\text{rank}(A) \leq \min(n, m)$, and $m < n$, $\text{rank}(A) < n$ also. Thus, by part a, there is a \vec{b} such that $A\vec{x} = \vec{b}$ is inconsistent.

52. Let $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & \vdots & 0 \\ 0 & 2 & 4 & \vdots & 0 \\ 0 & 3 & 6 & \vdots & 1 \\ 1 & 4 & 8 & \vdots & 0 \end{bmatrix}$. We find that $\text{rref}\begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$,

which has an inconsistency in the third row.

53. a. $A - \lambda I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 3 & 5 - \lambda \end{bmatrix}$.

This fails to be invertible when $(3 - \lambda)(5 - \lambda) - 3 = 0$,

or $15 - 8\lambda + \lambda^2 - 3 = 0$,

or $12 - 8\lambda + \lambda^2 = 0$

or $(6 - \lambda)(2 - \lambda) = 0$. So $\lambda = 6$ or $\lambda = 2$.

- b. For $\lambda = 6$, $A - \lambda I_2 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$.

The system $(A - 6I_2)\vec{x} = \vec{0}$ has the solutions $\begin{bmatrix} t \\ 3t \end{bmatrix}$, where t is an arbitrary constant. Pick $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for example.

For $\lambda = 2$, $A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$.

The system $(A - 2I_2)\vec{x} = \vec{0}$ has the solutions $\begin{bmatrix} t \\ -t \end{bmatrix}$, where t is an arbitrary constant.

Pick $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for example.

c. For $\lambda = 6$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = 2$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

54. $A - \lambda I_2 = \begin{bmatrix} 1 - \lambda & 10 \\ -3 & 12 - \lambda \end{bmatrix}$. This fails to be invertible when $\det(A - \lambda I_2) = 0$,

so $0 = (1 - \lambda)(12 - \lambda) + 30 = 12 - 13\lambda + \lambda^2 + 30 = \lambda^2 - 13\lambda + 42 = (\lambda - 6)(\lambda - 7)$. In order for this to be zero, λ must be 6 or 7.

If $\lambda = 6$, then $A - 6I_2 = \begin{bmatrix} -5 & 10 \\ -3 & 6 \end{bmatrix}$. We solve the system $(A - 6I_2)\vec{x} = \vec{0}$ and find that the solutions are of the form $\vec{x} = \begin{bmatrix} 2t \\ t \end{bmatrix}$. For example, when $t = 1$, we find $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

If $\lambda = 7$, then $A - 7I_2 = \begin{bmatrix} -6 & 10 \\ -3 & 5 \end{bmatrix}$. Here we solve the system $(A - 7I_2)\vec{x} = \vec{0}$, this time finding that our solutions are of the form $\vec{x} = \begin{bmatrix} 5t \\ 3t \end{bmatrix}$. For example, for $t = 1$, we find $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

2.4

1. $\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}$

3. Undefined

4. $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

5. $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

7. $\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. $[0 \ 1]$

11. $[10]$

12. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

13. $[h]$

14. $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $BC = [14 \ 8 \ 2]$, $BD = [6]$, $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB =$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix},$$

$$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, EB = [5 \ 10 \ 15], E^2 = [25]$$

15. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; Fact 2.4.9 applies to square matrices only.

16. True; $(I_n - A)(I_n + A) = I_n^2 + A - A - A^2 = I_n - A^2$.
17. Not necessarily true; $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$ if $AB \neq BA$.
18. True; apply Fact 2.4.8 to $B = A$.
19. Not necessarily true; consider the case $A = I_n$ and $B = -I_n$.
20. Not necessarily true; $(A - B)(A + B) = A^2 + AB - BA - B^2 \neq A^2 - B^2$ if $AB \neq BA$.
21. True; $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.
22. Not necessarily true; the equation $ABA^{-1} = B$ is equivalent to $AB = BA$ (multiply by A from the right), which is not true in general.
23. True; $(ABA^{-1})^3 = ABA^{-1}ABA^{-1}ABA^{-1} = AB^3A^{-1}$.
24. True; $(I_n + A)(I_n + A^{-1}) = I_n^2 + A + A^{-1} + AA^{-1} = 2I_n + A + A^{-1}$.
25. True; $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$ (use Fact 2.4.8).

$$26. \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right] + \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \end{array} \right] + \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \end{array} \right] = \left[\begin{array}{c|c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 3 & 5 \\ 7 & 9 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 3 & 5 \\ 7 & 9 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \right] =$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$27. \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \right] + \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 3 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \right] + \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 3 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \right] + \left[\begin{array}{c|c} \begin{bmatrix} 0 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 3 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \right] = \left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 19 \\ 16 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \right] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 19 & 16 \end{bmatrix}$$

28. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is one such matrix.

29. The column vectors of B must be solutions of the system $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solutions are of the form $B = \begin{bmatrix} -3t & -3s \\ t & s \end{bmatrix}$, where t and s are arbitrary constants, with at least one of them being nonzero.

30. Yes; by Fact 2.3.4b, the equation $A\vec{x} = \vec{0}$ has a nonzero solution \vec{v} . Let $B = [\vec{v} \ \vec{v} \ \dots \ \vec{v}]$.

Then $AB = [A\vec{v} \ A\vec{v} \ \dots \ A\vec{v}] = \mathbf{0}$

31. The two column vectors of A must be solutions of the linear systems $B\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $B\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. Each of these systems has *infinitely many solutions*.

The solutions are of the form $\begin{bmatrix} 2+t & -1+s \\ -1-2t & 1-2s \\ t & s \end{bmatrix}$.

32. By Fact 1.3.3, there is a nonzero \vec{x} such that $B\vec{x} = \vec{0}$ and therefore $AB\vec{x} = \vec{0}$. This implies that $AB \neq I_3$, since $I_3\vec{x} = \vec{x} \neq \vec{0}$.
33. By Fact 1.3.3, there is a nonzero \vec{x} such that $B\vec{x} = \vec{0}$ and therefore $AB\vec{x} = \vec{0}$. By Fact 2.3.4b, the 3×3 matrix AB fails to be invertible.
34. We can write $AB(AB)^{-1} = A(B(AB)^{-1}) = I_n$ and $(AB)^{-1}AB = ((AB)^{-1}A)B = I_n$.
By Fact 2.4.9, A and B are invertible.

35. a. Consider a solution \vec{x} of the equation $A\vec{x} = \vec{0}$.

Multiply both sides by B from the left: $BA\vec{x} = B\vec{0} = \vec{0}$, so that $\vec{x} = \vec{0}$ (since $BA = I_m$).

It follows that $\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

- b. $\vec{x} = A\vec{b}$ is a solution, since $B\vec{x} = BA\vec{b} = \vec{b}$ (because $BA = I_m$).
- c. $\text{rank}(A) = m$, by part (a) (all variables are leading).
 $\text{rank}(B) = m$, by part (b) (compare with Exercise 2.3.51a).
- d. $m = \text{rank}(B) \leq (\text{number of columns of } B) = n$

36. The column vectors of X must be solutions of the system $A\vec{x} = \vec{0}$. These solutions are of the form $\begin{bmatrix} -2t \\ t \end{bmatrix}$, where t is arbitrary. Therefore, $X = \begin{bmatrix} -2s & -2t \\ s & t \end{bmatrix}$, where s and t are arbitrary.

37. We want $S^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\text{So } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}.$$

Thus, $c = a$ and $d = -b$. Matrix S must be of the form $\begin{bmatrix} a & b \\ a & -b \end{bmatrix}$ where $-ab - ab \neq 0$, or $-2ab \neq 0$, or $a \neq 0$ and $b \neq 0$.

$$38. \text{ We want } \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} S = S \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Let } S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then we need } \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} a+3c & b+3d \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 2a & b \\ 2c & d \end{bmatrix}.$$

Thus, $d = 0$ and $c = \frac{1}{3}a$. Thus, S must be of the form $\begin{bmatrix} a & b \\ \frac{1}{3}a & 0 \end{bmatrix}$. By inspection, we can see that S will be invertible as long as a and b are not equal to zero.

$$39. \text{ Let } X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \text{ Then we want } X \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X, \text{ or } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

or $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, meaning that $b = c = 0$. Also, we want $X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X$,

or $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, or $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$ so $a = d$. Thus, $X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2$ must be a multiple of the identity matrix. (X will then commute with any 2×2 matrix M , since $XM = aM = MX$.)

$$40. A = (AB)B^{-1} = ((AB)^{-1})^{-1}B^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}.$$

41. a. $D_\alpha D_\beta$ and $D_\beta D_\alpha$ are the same transformation, namely, a rotation through $\alpha + \beta$.

$$\begin{aligned} \text{b. } D_\alpha D_\beta &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \end{aligned}$$

$D_\beta D_\alpha$ yields the same answer.

42. a. See Figure 2.44.

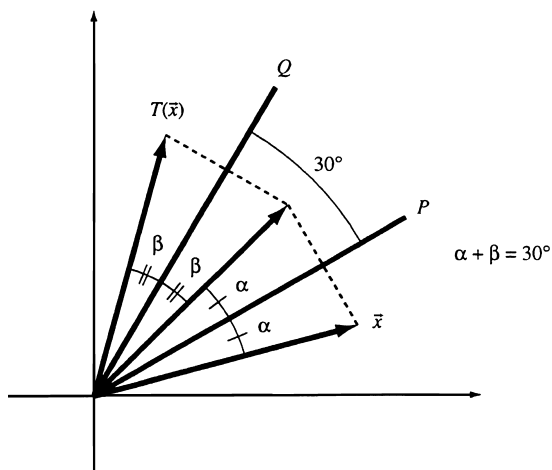


Figure 2.44: for Problem 2.4.42a.

The vectors \vec{x} and $T(\vec{x})$ have the same length (since reflections leave the length unchanged), and they enclose an angle of $2(\alpha + \beta) = 2 \cdot 30^\circ = 60^\circ$

b. Based on the answer in part (a), we conclude that T is a rotation through 60° .

c. The matrix of T is $\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

43. Let A represent the rotation through 120° ; then A^3 represents the rotation through 360° , that is $A^3 = I_2$.

$$A = \begin{bmatrix} \cos(120^\circ) & -\sin(120^\circ) \\ \sin(120^\circ) & \cos(120^\circ) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

44. We want A such that $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, so that $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$.

45. We want A such that $A\vec{v}_i = \vec{w}_i$, for $i = 1, 2, \dots, m$, or $A[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m] = [\vec{w}_1 \ \vec{w}_2 \ \dots \ \vec{w}_m]$, or $AS = B$.

Multiplying by S^{-1} from the right we find the unique solution $A = BS^{-1}$.

46. Use the result of Exercise 45, with $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$;

$$A = BS^{-1} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$$

47. Use the result of Exercise 45, with $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}$;

$$A = BS^{-1} = \frac{1}{5} \begin{bmatrix} 9 & 3 \\ -2 & 16 \end{bmatrix}.$$

48. $P_0 \xrightarrow{T} P_1, P_1 \xrightarrow{T} P_3, P_2 \xrightarrow{T} P_2, P_3 \xrightarrow{T} P_0$

$$P_0 \xrightarrow{L} P_0, P_1 \xrightarrow{L} P_2, P_2 \xrightarrow{L} P_1, P_3 \xrightarrow{L} P_3$$

a. T^{-1} is the rotation about the axis through 0 and P_2 that transforms P_3 into P_1 .

b. $L^{-1} = L$

c. $T^2 = T^{-1}$ (See part (a).)

d. $P_0 \xrightarrow{T \circ L} P_1 \quad P_0 \xrightarrow{L \circ T} P_2$ The transformations $T \circ L$ and $L \circ T$ are not the same.

$$P_1 \longrightarrow P_2 \quad P_1 \longrightarrow P_3$$

$$P_2 \longrightarrow P_3 \quad P_2 \longrightarrow P_1$$

$$P_3 \longrightarrow P_0 \quad P_3 \longrightarrow P_0$$

e.

$$P_0 \xrightarrow{L \circ T \circ L} P_2$$

$$P_1 \longrightarrow P_1$$

$$P_2 \longrightarrow P_3$$

$$P_3 \longrightarrow P_0$$

This is the rotation about the axis through 0 and P_1 that sends P_0 to P_2 .

49. Let A be the matrix of T and C the matrix of L . We want that $AP_0 = P_1, AP_1 = P_3,$

and $AP_2 = P_2$. We can use the result of Exercise 45, with $S = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

$$\text{Then } A = BS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$\text{Using an analogous approach, we find that } C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$50. \text{ a. } EA = \begin{bmatrix} a & b & c \\ d - 3a & e - 3b & f - 3c \\ g & h & k \end{bmatrix}$$

The matrix EA is obtained from A by an elementary row operation: subtract three times the first row from the second.

$$\text{b. } EA = \begin{bmatrix} a & b & c \\ \frac{1}{4}d & \frac{1}{4}e & \frac{1}{4}f \\ g & h & k \end{bmatrix}$$

The matrix EA is obtained from A by dividing the second row of A by 4 (an elementary row operation).

$$\text{c. If we set } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ then } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & k \\ d & e & f \end{bmatrix}, \text{ as desired.}$$

d. An elementary $n \times n$ matrix E has the same form as I_n except that either

- $e_{ij} = k (\neq 0)$ for some $i \neq j$ [as in part (a)], or
- $e_{ii} = k (\neq 0, 1)$ for some i [as in part (b)], or
- $e_{ij} = e_{ji} = 1$, $e_{ii} = e_{jj} = 0$ for some $i \neq j$ [as in part (c)].

51. Let E be an elementary $n \times n$ matrix (obtained from I_n by a certain elementary row operation), and let F be the elementary matrix obtained from I_n by the reversed row operation. Our work in Exercise 50 [parts (a) through (c)] shows that $EF = I_n$, so that E is indeed invertible, and $E^{-1} = F$ is an elementary matrix as well.

52. a. The matrix $\text{rref}(A)$ is obtained from A by performing a sequence of p elementary row operations. By Exercise 50 [parts (a) through (c)] each of these operations can be represented by the left multiplication with an elementary matrix, so that $\text{rref}(A) = E_1 E_2 \dots E_p A$.

$$\text{b. } A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \text{ swap rows 1 and 2, represented by } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \div 2, \text{ represented by } \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} -3(II), \text{ represented by } \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Therefore, } \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = E_1 E_2 E_3 A.$$

53. a. Let $S = E_1 E_2 \dots E_p$ in Exercise 52a.

By Exercise 51, the elementary matrices E_i are invertible: now use Fact 2.4.8 repeatedly to see that S is invertible.

$$\text{b. } A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \div 2, \text{ represented by } \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} -4(I), \text{ represented by } \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\text{Therefore, } \text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = E_1 E_2 A = SA, \text{ where}$$

$$S = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix}.$$

(There are other correct answers.)

54. a. By Exercise 52a, $I_n = \text{rref}(A) = E_1 E_2 \dots E_p A$, for some elementary matrices E_1, \dots, E_p . By Exercise 51, the E_i are invertible and their inverses are elementary as well. Therefore,

$A = (E_1 E_2 \dots E_p)^{-1} = E_p^{-1} \dots E_2^{-1} E_1^{-1}$ expresses A as a product of elementary matrices.

b. We can use out work in Exercise 52b:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \left(\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

55. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ represents a vertical shear,
- $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ represents a “scaling in \vec{e}_1 direction” (leaving the \vec{e}_2 component unchanged),
- $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ represents a “scaling in \vec{e}_2 direction” (leaving the \vec{e}_1 component unchanged), and
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents the reflection about the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

56. Performing a sequence of p elementary row operations on a matrix A amounts to multiplying A with $E_1 E_2 \dots E_p$ from the left, where the E_i are elementary matrices. If $I_n = E_1 E_2 \dots E_p A$, then $E_1 E_2 \dots E_p = A^{-1}$, so that

a. $E_1 E_2 \dots E_p A B = B$, and

b. $E_1 E_2 \dots E_p I_n = A^{-1}$.

57. Let A and B be two lower triangular $n \times n$ matrices. We need to show that the ij th entry of AB is 0 whenever $i < j$.

This entry is the dot product of the i th row of A and the j th column of B ,

$$[a_{i1} \ a_{i2} \ \dots \ a_{ii} \ 0 \ \dots \ 0] \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{jj} \\ \vdots \\ b_{nj} \end{bmatrix}, \text{ which is indeed 0 if } i < j.$$

$$58. \text{ a. } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} \begin{matrix} -2I, \\ -2I \end{matrix} \text{ represented by } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix} +II \text{ represented by } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ so that}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}$$

↑
 U

↑
 E_3

↑
 E_2

↑
 E_1

↑
 A

$$\text{b. } A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

↑
 M_1

↑
 M_2

↑
 M_3

↑
 U

$$\text{c. Let } L = M_1 M_2 M_3 \text{ in part (b); we compute } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

$$\text{Then } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

↑
 A

↑
 L

↑
 U

d. We can use the matrix L we found in part (c), but U needs to be modified. Let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(Take the diagonal entries of the matrix U in part (c)).

$$\text{Then } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{array}{cccc} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & A & & L & & D & & U \end{array}$$

59. a. Write the system $L\vec{y} = \vec{b}$ in components:

$$\left\{ \begin{array}{l} y_1 = -3 \\ -3y_1 + y_2 = 14 \\ y_1 + 2y_2 + y_3 = 9 \\ -y_1 + 8y_2 - 5y_3 + y_4 = 33 \end{array} \right., \text{ so that } y_1 = -3, y_2 = 14 + 3y_1 = 5,$$

$$y_3 = 9 - y_1 - 2y_2 = 2, \text{ and } y_4 = 33 + y_1 - 8y_2 + 5y_3 = 0:$$

$$\vec{y} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}.$$

b. Proceeding as in part (a) we find that $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

60. We try to find matrices $L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ and $U = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be + cf \end{bmatrix}.$$

Note that the equations $ad = 0$, $ae = 1$, and $bd = 1$ cannot be solved simultaneously: If $ad = 0$ then a or d is 0 so that ae or bd is zero.

Therefore, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have an LU factorization.

61. a. Write $L = \begin{bmatrix} L^{(m)} & 0 \\ L_3 & L_4 \end{bmatrix}$ and $U = \begin{bmatrix} U^{(m)} & U_2 \\ 0 & U_4 \end{bmatrix}$. Then $A = LU = \begin{bmatrix} L^{(m)}U^{(m)} & L^{(m)}U_2 \\ L_3U^{(m)} & L_3U_2 + L_4U_4 \end{bmatrix}$,
so that $A^{(m)} = L^{(m)}U^{(m)}$, as claimed.

- b. By Exercise 34, the matrices L and U are both invertible. By Exercise 2.3.35, the diagonal entries of L and U are all nonzero. For any m , the matrices $L^{(m)}$ and $U^{(m)}$ are triangular, with nonzero diagonal entries, so that they are invertible. By Fact 2.4.8, the matrix $A^{(m)} = L^{(m)}U^{(m)}$ is invertible as well.

c. Using the hint, we write $A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{w} & k \end{bmatrix} = \begin{bmatrix} L' & 0 \\ \vec{x} & t \end{bmatrix} \begin{bmatrix} U' & \vec{y} \\ 0 & s \end{bmatrix}$.

We are looking for a column vector \vec{y} , a row vector \vec{x} , and scalars t and s satisfying these equations. The following equations need to be satisfied: $\vec{v} = L'\vec{y}$, $\vec{w} = \vec{x}U'$, and $k = \vec{x}\vec{y} + ts$.

We find that $\vec{y} = (L')^{-1}\vec{v}$, $\vec{x} = \vec{w}(U')^{-1}$, and $ts = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$.

We can choose, for example, $s = 1$ and $t = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$, proving that A does indeed have an LU factorization.

Alternatively, one can show that if all principal submatrices are invertible then no row swaps are required in the Gauss-Jordan Algorithm. In this case, we can find an LU -factorization as outlined in Exercise 58.

62. a. If $A = LU$ is an LU factorization, then the diagonal entries of L and U are nonzero (compare with Exercise 61). Let D_1 and D_2 be the diagonal matrices whose diagonal entries are the same as those of L and U , respectively.

Then $A = (LD_1^{-1})(D_1D_2)(D_2^{-1}U)$ is the desired factorization

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{new } L & D & \text{new } U \end{array}$$

(verify that LD_1^{-1} and $D_2^{-1}U$ are of the required form).

- b. If $A = L_1D_1U_1 = L_2D_2U_2$ and A is invertible, then $L_1, D_1, U_1, L_2, D_2, U_2$ are all invertible, so that we can multiply the above equation by $D_2^{-1}L_2^{-1}$ from the left and by U_1^{-1} from the right:

$$D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}.$$

Since products and inverses of upper triangular matrices are upper triangular (and likewise for lower triangular matrices), the matrix $D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}$ is both upper and lower triangular, that is, it is diagonal. Since the diagonal entries of U_2 and U_1 are all 1, so are the diagonal entries of $U_2U_1^{-1}$, that is $U_2U_1^{-1} = I_n$, and thus $U_2 = U_1$.

Now $L_1D_1 = L_2D_2$, so that $L_2^{-1}L_1 = D_2D_1^{-1}$ is diagonal. As above, we have in fact $L_2^{-1}L_1 = I_n$ and therefore $L_2 = L_1$.

63. We will prove that $A(C + D) = AC + AD$, repeatedly using Fact 1.3.9a: $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.

Write $B = [\vec{v}_1 \ \dots \ \vec{v}_m]$ and $C = [\vec{w}_1 \ \dots \ \vec{w}_m]$. Then

$$A(C + D) = A[\vec{v}_1 + \vec{w}_1 \ \dots \ \vec{v}_m + \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \dots \ A\vec{v}_m + A\vec{w}_m],$$

$$AC + AD = A[\vec{v}_1 \ \dots \ \vec{v}_m] + A[\vec{w}_1 \ \dots \ \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \dots \ A\vec{v}_m + A\vec{w}_m].$$

The results agree.

64. The ij th entries of the three matrices are

$$\sum_{h=1}^p (ka_{ih})b_{hj}, \quad \sum_{h=1}^p a_{ih}(kb_{hj}), \quad \text{and} \quad k \left(\sum_{h=1}^p a_{ih}b_{hj} \right)$$

.

The three results agree.

65. Suppose A_{11} is a $p \times p$ matrix and A_{22} is a $q \times q$ matrix. For B to be the inverse of A we must have $AB = I_{p+q}$. Let us partition B the same way as A :

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad \text{where } B_{11} \text{ is } p \times p \text{ and } B_{22} \text{ is } q \times q.$$

$$\text{Then } AB = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \text{ means that}$$

$$A_{11}B_{11} = I_p, \quad A_{22}B_{22} = I_q, \quad A_{11}B_{12} = 0, \quad A_{22}B_{21} = 0.$$

This implies that A_{11} and A_{22} are invertible, and $B_{11} = A_{11}^{-1}$, $B_{22} = A_{22}^{-1}$.

This in turn implies that $B_{12} = 0$ and $B_{21} = 0$.

We summarize: A is invertible if (and only if) both A_{11} and A_{22} are invertible; in this case

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

66. This exercise is very similar to Example 4 in the text. We outline the solution:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \text{ means that}$$

$$A_{11}B_{11} = I_p, \quad A_{11}B_{12} = 0, \quad A_{21}B_{11} + A_{22}B_{21} = 0, \quad A_{21}B_{12} + A_{22}B_{22} = I_q.$$

This implies that A_{11} is invertible, and $B_{11} = A_{11}^{-1}$. Multiplying the second equation with A_{11}^{-1} , we conclude that $B_{12} = 0$. Then the last equation simplifies to $A_{22}B_{22} = I_q$, so that $B_{22} = A_{22}^{-1}$.

Finally, $B_{21} = -A_{22}^{-1}A_{21}B_{11} = -A_{22}^{-1}A_{21}A_{11}^{-1}$.

We summarize: A is invertible if (and only if) both A_{11} and A_{22} are invertible. In this case,

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}.$$

67. Write A in terms of its rows: $A = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \dots \\ \vec{w}_n \end{bmatrix}$ (suppose A is $n \times m$).

We can think of this as a partition into n

$$1 \times m \text{ matrices. Now } AB = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \dots \\ \vec{w}_n \end{bmatrix} B = \begin{bmatrix} \vec{w}_1 B \\ \vec{w}_2 B \\ \dots \\ \vec{w}_n B \end{bmatrix} \text{ (a product of partitioned matrices).}$$

We see that the i th row of AB is the product of the i th row of A and the matrix B .

68. By Exercise 65 or by Example 4 in the text, we find that $S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix}$. Then

$$S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} k & \vec{v} \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} k & \vec{v}R \\ 0 & BR \end{bmatrix} = \begin{bmatrix} k & \vec{v}R \\ 0 & R^{-1}BR \end{bmatrix}.$$

69. Suppose A_{11} is a $p \times p$ matrix. Since A_{11} is invertible, $\text{rref}(A) = \begin{bmatrix} I_p & A_{12} & * \\ 0 & 0 & \text{rref}(A_{23}) \end{bmatrix}$, so that

$$\text{rank}(A) = p + \text{rank}(A_{23}) = \text{rank}(A_{11}) + \text{rank}(A_{23}).$$

70. Try to find a matrix $B = \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix}$ (where X is $n \times n$) such that

$$AB = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix} \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix} = \begin{bmatrix} X + \vec{v}\vec{y} & \vec{x} + t\vec{v} \\ \vec{w}X + \vec{y} & \vec{w}\vec{x} + t \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

We want $X + \vec{v}\vec{y} = I_n$, $\vec{x} + t\vec{v} = \vec{0}$, $\vec{w}X + \vec{y} = \vec{0}$, and $\vec{w}\vec{x} + t = 1$.

Substituting $\vec{x} = -t\vec{v}$ into the last equation we find $-t\vec{w}\vec{v} + t = 1$ or $t(1 - \vec{w}\vec{v}) = 1$.

This equation can be solved only if $\vec{w}\vec{v} \neq 1$, in which case $t = \frac{1}{1-\vec{w}\vec{v}}$. Now substituting $X = I_n - \vec{v}\vec{y}$ into the third equation, we find $\vec{w} - \vec{w}\vec{v}\vec{y} + \vec{y} = \vec{0}$ or $\vec{y} = -\frac{1}{1-\vec{w}\vec{v}}\vec{w} = -t\vec{w}$.

We summarize: A is invertible if (and only if) $\vec{w}\vec{v} \neq 1$. In this case, $A^{-1} = \begin{bmatrix} I_n + t\vec{v}\vec{w} & -t\vec{v} \\ -t\vec{w} & t \end{bmatrix}$, where $t = \frac{1}{1-\vec{w}\vec{v}}$. The same result can be found (perhaps more easily) by working with $\text{rref}[A; I_{n+1}]$, rather than partitioned matrices.

71. Multiplying both sides with A^{-1} we find that $A = I_n$: The identity matrix is the only invertible matrix with this property.

72. Suppose the entries of A are all a , where $a \neq 0$. Then the entries of A^2 are all na^2 . The

equation $na^2 = a$ is satisfied if $a = \frac{1}{n}$. Thus the solution is $A = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ & & \ddots & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$.

73. We must find all S such that $SA = AS$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So $\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$, meaning that $b = 2b$ and $c = 2c$, so b and c must be zero.

We see that all diagonal matrices (those of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$) commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

74. As in Exercise 73, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$, revealing that $c = 0$ (since $a+2c = a$) and $a = d$ (since $b+2d = 2a+b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

75. Again, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} 2b & -2a \\ 2d & -2c \end{bmatrix} = \begin{bmatrix} -2c & -2d \\ 2a & 2b \end{bmatrix}$, meaning that $c = -b$ and $d = a$.

We see that all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ commute with $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

76. Following the form of Exercise 73, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} 2a - 3b & 3a + 2b \\ 2c - 3d & 3c + 2d \end{bmatrix} = \begin{bmatrix} 2a + 3c & 2b + 3d \\ -3a + 2c & -3b + 2d \end{bmatrix}$, revealing that $a = d$ (since $3a + 2b = 2b + 3d$) and $-b = c$ (since $2a + 3c = 2a - 3b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

77. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} a + 2b & 2a - b \\ c + 2d & 2c - d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 2a - c & 2b - d \end{bmatrix}$. So $a + 2b = a + 2c$, or $c = b$, and $2a - b = b + 2d$, revealing $d = a - b$. (The other two equations are redundant.)

All matrices of the form $\begin{bmatrix} a & b \\ b & a - b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

78. As in Exercise 73, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} a + b & a + b \\ c + d & c + d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ a + c & b + d \end{bmatrix}$, revealing that $a = d$ (since $a + b = b + d$) and $b = c$ (since $a + c = a + b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

79. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $\begin{bmatrix} a + 2b & 3a + 6b \\ c + 2d & 3c + 6d \end{bmatrix} = \begin{bmatrix} a + 3c & b + 3d \\ 2a + 6c & 2b + 6d \end{bmatrix}$. So $a + 2b = a + 3c$, or $c = \frac{2}{3}b$, and $3a + 6b = b + 3d$, revealing $d = a + \frac{5}{3}b$. The other two equations are redundant.

Thus all matrices of the form $\begin{bmatrix} a & b \\ \frac{2}{3}b & a + \frac{5}{3}b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

80. Following the form of Exercise 73, we let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Now we want
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

So,
$$\begin{bmatrix} 2a & 3b & 4c \\ 2d & 3e & 4f \\ 2g & 3h & 4i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{bmatrix},$$
 which forces b, c, d, f, g and h to be zero. a, e and i , however, can be chosen freely.

Thus B is any matrix of the form
$$\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}.$$

81. Now we want
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

or,
$$\begin{bmatrix} 2a & 3b & 2c \\ 2d & 3e & 2f \\ 2g & 3h & 2i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 2g & 2h & 2i \end{bmatrix}.$$
 So, $3b = 2b$, $2d = 3d$, $3f = 2f$ and $3h = 2h$, meaning that b, d, f and h must all be zero.

Thus all matrices of the form
$$\begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix}$$
 commute with
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

82. Following the form of Exercise 73, we let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$

Then we want
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

So,
$$\begin{bmatrix} 2a & 2b & 3c \\ 2d & 2e & 3f \\ 2g & 2h & 3i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{bmatrix}.$$
 Thus c, f, g and h must be zero, leaving B to be

any matrix of the form
$$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{bmatrix}.$$

83. The ij th entry of AB is

$$\sum_{k=1}^n a_{ik}b_{kj}.$$

Then

$$\sum_{k=1}^n a_{ik}b_{kj} \leq \sum_{k=1}^n sb_{kj} = s \left(\sum_{k=1}^n b_{kj} \right) \leq sr.$$

$\begin{matrix} \uparrow & & \uparrow \\ \text{since } a_{ik} \leq s & \text{this is } \leq r, \text{ as it is the } & \\ & j^{\text{th}} \text{ column sum of } B. & \end{matrix}$

84. a. We proceed by induction on m . Since the column sums of A are $\leq r$, the entries of $A^1 = A$ are also $\leq r^1 = r$, so that the claim holds for $m = 1$. Suppose the claim holds for some fixed m . Now write $A^{m+1} = A^m A$; since the entries of A^m are $\leq r^m$ and the column sums of A are $\leq r$, we can conclude that the entries of A^{m+1} are $\leq r^m r = r^{m+1}$, by Exercise 83.

b. For a fixed i and j , let b_m be the ij th entry of A^m . In part (a) we have seen that $0 \leq b_m \leq r^m$.

Note that $\lim_{m \rightarrow \infty} r^m = 0$ (since $r < 1$), so that $\lim_{m \rightarrow \infty} b_m = 0$ as well (this follows from what some calculus texts call the ‘‘squeeze theorem’’).

c. For a fixed i and j , let c_m be the ij th entry of the matrix $I_n + A + A^2 + \cdots + A^m$. By part (a),

$$c_m \leq 1 + r + r^2 + \cdots + r^m < \frac{1}{1-r}.$$

Since the c_m form an increasing bounded sequence, $\lim_{m \rightarrow \infty} c_m$ exists (this is a fundamental fact of calculus).

d. $(I_n - A)(I_n + A + A^2 + \cdots + A^m) = I_n + A + A^2 + \cdots + A^m - A - A^2 - \cdots - A^m - A^{m+1}$
 $= I_n - A^{m+1}$

Now let m go to infinity; use parts (b) and (c). $(I_n - A)(I_n + A + A^2 + \cdots + A^m + \cdots) = I_n$, so that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots.$$

85. a. The components of the j th column of the technology matrix A give the demands industry J_j makes on the other industries, per unit output of J_j . The fact that the j th column sum is less than 1 means that industry J_j *adds value* to the products it produces.

b. A productive economy can satisfy any consumer demand \vec{b} , since the equation

$(I_n - A)\vec{x} = \vec{b}$ can be solved for the output vector \vec{x} : $\vec{x} = (I_n - A)^{-1}\vec{b}$ (compare with Exercise 2.3.49).

- c. The output \vec{x} required to satisfy a consumer demand \vec{b} is

$$\vec{x} = (I_n - A)^{-1}\vec{b} = (I_n + A + A^2 + \cdots + A^m + \cdots)\vec{b} = \vec{b} + A\vec{b} + A^2\vec{b} + \cdots + A^m\vec{b} + \cdots.$$

To interpret the terms in this series, keep in mind that whatever output \vec{v} the industries produce generates an interindustry demand of $A\vec{v}$.

The industries first need to satisfy the consumer demand, \vec{b} . Producing the output \vec{b} will generate an interindustry demand, $A\vec{b}$. Producing $A\vec{b}$ in turn generates an extra interindustry demand, $A(A\vec{b}) = A^2\vec{b}$, and so forth.

For a simple example, see Exercise 2.3.50; also read the discussion of “chains of interindustry demands” in the footnote to Exercise 2.3.49.

86. a. We write our three equations below:

$$\begin{aligned} I &= \frac{1}{3}R + \frac{1}{3}G + \frac{1}{3}B \\ L &= R - G \\ S &= -\frac{1}{2}R - \frac{1}{2}G + B \end{aligned} \quad , \text{ so that the matrix is } P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

b. $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ is transformed into $\begin{bmatrix} R \\ G \\ 0 \end{bmatrix}$, with matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c. This matrix is $PA = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$ (we apply first A , then P .)

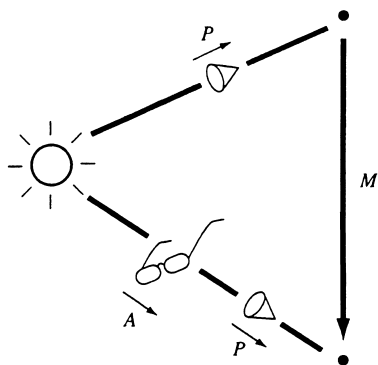


Figure 2.45: for Problem 2.4.86d.

d. See Figure 2.45. A “diagram chase” shows that $M = PAP^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{9} \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}$.

87. a. $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Matrix A^{-1} transforms a wife’s clan into her husband’s clan, and B^{-1} transforms a child’s clan into the mother’s clan.

b. B^2 transforms a women’s clan into the clan of a child of her daughter.

c. AB transforms a woman’s clan into the clan of her daughter-in-law (her son’s wife), while BA transforms a man’s clan into the clan of his children. The two transformations are different. (See Figure 2.46.)

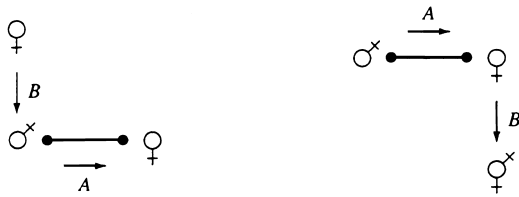


Figure 2.46: for Problem 2.4.87c.

d. The matrices for the four given diagrams (in the same order) are $BB^{-1} = I_3$,

$$BAB^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B(BA)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad BA(BA)^{-1} = I_3.$$

e. Yes; since $BAB^{-1} = A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, in the second case in part (d) the cousin belongs to Bueya’s husband’s clan.

88. a. We need 8 multiplications: 2 to compute each of the four entries of the product.

b. We need n multiplications to compute each of the mp entries of the product, mnp multiplications altogether.

89. $g(f(x)) = x$, for all x , so that $g \circ f$ is the identity, but $f(g(x)) = \begin{cases} x & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$.

90. a. The formula $\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 - Rk & L + R - kLR \\ -k & 1 - kL \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$ is given, which implies that

$$y = (1 - Rk)x + (L + R - kLR)m.$$

In order for y to be independent of x it is required that $1 - Rk = 0$, or $k = \frac{1}{R} = 40$ (diopters).

$\frac{1}{k}$ then equals R , which is the distance between the plane of the lens and the plane on which parallel incoming rays focus at a point; thus the term “focal length” for $\frac{1}{k}$.

b. Now we want y to be independent of the slope m (it must depend on x alone). In view of the formula above, this is the case if $L + R - kLR = 0$, or $k = \frac{L + R}{LR} = \frac{1}{R} + \frac{1}{L} = 40 + \frac{10}{3} \approx 43.3$ (diopters).

c. Here the transformation is

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 - k_1 D & D \\ k_1 k_2 D - k_1 - k_2 & 1 - k_2 D \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

We want the slope n of the outgoing rays to depend on the slope m of the incoming rays alone, and not on x ; this forces $k_1 k_2 D - k_1 - k_2 = 0$, or, $D = \frac{k_1 + k_2}{k_1 k_2} = \frac{1}{k_1} + \frac{1}{k_2}$, the sum of the focal lengths of the two lenses.

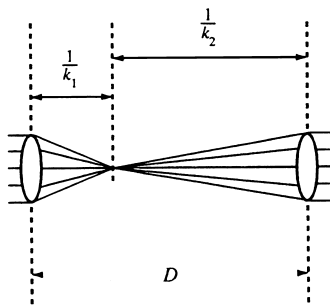


Figure 2.47: for Problem 2.4.90c.

True or False

1. T; The matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
2. F; The columns of a rotation matrix are unit vectors; see Fact 2.2.3.
3. T, by Fact 2.3.3.
4. T; Let $A = B$ in Fact 2.4.8.
5. F, by Fact 2.4.3.
6. T, by Fact 2.4.9.
7. F; Matrix AB will be 3×5 , by Definition 2.4.1b.
8. F; Note that $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A linear transformation transforms $\vec{0}$ into $\vec{0}$.
9. T, by Fact 2.2.4.
10. T, by Fact 2.4.5.
11. F, by Fact 2.3.6. Note that the determinant is 0.
12. T, by Fact 2.3.3.
13. T; The shear matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ works.
14. T; Simplify to see that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ -12x \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
15. T; The equation $\det(A) = k^2 - 6k + 10 = 0$ has no real solution.
16. T; The matrix fails to be invertible for $k = 5$ and $k = -1$, since the determinant is 0 for these values.
17. F; Note that $\det(A) = (k - 2)^2 + 9$ is always positive, so that A is invertible for all values of k .
18. T; Note that the columns are unit vectors, since $(-0.6)^2 + (\pm 0.8)^2 = 1$. The matrix has the form presented in Fact 2.2.3.
19. F; Consider $A = I_2$ (or any other invertible 2×2 matrix).
20. T; Note that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$ is the unique solution.
21. F; For any 2×2 matrix A , the two columns of $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ will be identical.

22. T; One solution is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
23. F; A reflection matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Here, $a^2 + b^2 = 1 + 1 = 2$.
24. T; Just multiply it out.
25. T; The product is $\det(A)I_2$.
26. T; Writing an upper triangular matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and solving the equation $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we find that $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where b is any nonzero constant.
27. T; Note that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation through $\pi/2$. Thus $n = 4$ (or any multiple of 4) works.
28. F; If a matrix A is invertible, then so is A^{-1} . But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.
29. F; If matrix A has two identical rows, then so does AB , for any matrix B . Thus AB cannot be I_n , so that A fails to be invertible.
30. T, by Fact 2.4.9. Note that $A^{-1} = A$ in this case.
31. F; Consider the matrix A that represents a rotation through the angle $2\pi/17$.
32. F; Consider the reflection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
33. T; We have $(5A)^{-1} = \frac{1}{5}A^{-1}$.
34. T; The equation $A\vec{e}_i = B\vec{e}_i$ means that the i th columns of A and B are identical. This observation applies to all the columns.
35. T; Note that $A^2B = AAB = ABA = BAA = BA^2$.
36. T; Multiply both sides of the equation $A^2 = A$ with A^{-1} .
37. F; Consider $A = I_2$ and $B = -I_2$.
38. T; Since $A\vec{x}$ is on the line onto which we project, the vector $A\vec{x}$ remains unchanged when we project again: $A(A\vec{x}) = A\vec{x}$, or $A^2\vec{x} = A\vec{x}$, for all \vec{x} . Thus $A^2 = A$.
39. F; Consider matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, for example.

40. T; Apply Fact 2.4.9 to the equation $(A^2)^{-1}AA = I_n$, with $B = (A^2)^{-1}A$.
41. T; If you reflect twice in a row (about the same line), you will get the original vector back: $A(A\vec{x}) = \vec{x}$, or, $A^2\vec{x} = \vec{x} = I_2\vec{x}$. Thus $A^2 = I_2$ and $A^{-1} = A$.
42. F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.
43. T; Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.
44. F; By Fact 1.3.3, there is a nonzero vector \vec{x} such that $B\vec{x} = \vec{0}$, so that $AB\vec{x} = \vec{0}$ as well. But $I_3\vec{x} = \vec{x} \neq \vec{0}$, so that $AB \neq I_3$.
45. T; We can rewrite the given equation as $A^2 + 3A = -4I_3$ and $-\frac{1}{4}(A + 3I_3)A = I_3$. By Fact 2.4.9, matrix A is invertible, with $A^{-1} = -\frac{1}{4}(A + 3I_3)$.
46. T; Note that $(I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$, so that $(I_n + A)^{-1} = I_n - A$.
47. F; A and C can be two matrices which fail to commute, and B could be I_n , which commutes with anything.
48. F; Consider $T(\vec{x}) = 2\vec{x}$, $\vec{v} = \vec{e}_1$, and $\vec{w} = \vec{e}_2$.
49. F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix fails to be invertible.
50. F; Let $A = B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.
51. F; We will show that $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$ fails to be diagonal, for an arbitrary invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$. Since c and d cannot both be zero (as S must be invertible), at least one of the off-diagonal entries ($-c^2$ and d^2) is nonzero, proving the claim.
52. T; Consider an \vec{x} such that $A^2\vec{x} = \vec{b}$, and let $\vec{x}_0 = A\vec{x}$. Then $A\vec{x}_0 = A(A\vec{x}) = A^2\vec{x} = \vec{b}$, as required.
53. T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $A^{-1} = -A$, or $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. This holds if $ad - bc = 1$ and $d = -a$. These equations have many solutions: for example, $a = d = 0, b = 1, c = -1$. More generally, we can choose an arbitrary a and an arbitrary nonzero b . Then, $d = -a$ and $c = -\frac{1+a^2}{b}$.

54. F; Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We make an attempt to solve the equation $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now the equation $b(a+d) = 0$ implies that $b = 0$ or $d = -a$.

If $b = 0$, then the equation $d^2 + bc = -1$ cannot be solved.

If $d = -a$, then the two diagonal entries of A^2 , $a^2 + bc$ and $d^2 + bc$, will be equal, so that the equations $a^2 + bc = 1$ and $d^2 + bc = -1$ cannot be solved simultaneously.

In summary, the equation $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be solved.

55. T; Recall from Definition 2.2.1 that a projection matrix has the form $\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector. Thus, $a^2 + b^2 + c^2 + d^2 = u_1^4 + (u_1 u_2)^2 + (u_1 u_2)^2 + u_2^4 = u_1^4 + 2(u_1 u_2)^2 + u_2^4 = (u_1^2 + u_2^2)^2 = 1^2 = 1$.
56. T; We observe that the systems $AB\vec{x} = 0$ and $B\vec{x} = 0$ have the same solutions (multiply with A^{-1} and A , respectively, to obtain one system from the other). Then, by True or False Exercise 45 in Chapter 1, $\text{rref}(AB) = \text{rref}(B)$.