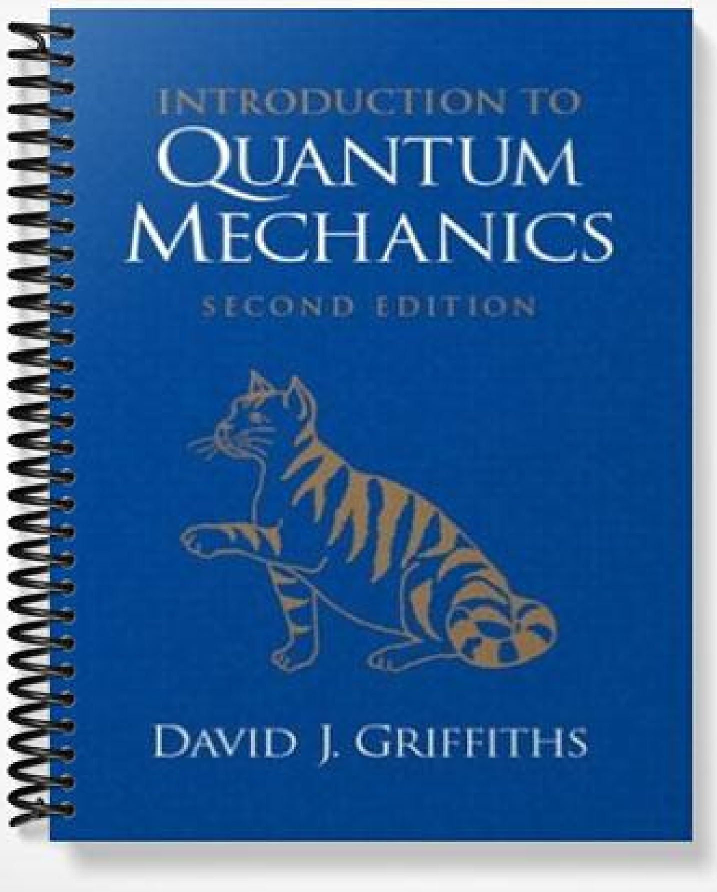
SOLUTIONS MANUAL



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Preface

These are my own solutions to the problems in *Introduction to Quantum Mechanics, 2nd ed.* I have made every effort to insure that they are clear and correct, but errors are bound to occur, and for this I apologize in advance. I would like to thank the many people who pointed out mistakes in the solution manual for the first edition, and encourage anyone who finds defects in this one to alert me (griffith@reed.edu). I'll maintain a list of errata on my web page (http://academic.reed.edu/physics/faculty/griffiths.html), and incorporate corrections in the manual itself from time to time. I also thank my students at Reed and at Smith for many useful suggestions, and above all Neelaksh Sadhoo, who did most of the typesetting.

At the end of the manual there is a grid that correlates the problem numbers in the second edition with those in the first edition.

David Griffiths

Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j\rangle^2 = 21^2 = \boxed{441.}$$

$$\langle j^2\rangle = \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[(14^2) + (15^2) + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2)\right]$$
$$= \frac{1}{14}(196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14} = \boxed{459.571.}$$

(b)

j	$\Delta j = j - \langle j\rangle$
14	$14 - 21 = -7$
15	$15 - 21 = -6$
16	$16 - 21 = -5$
22	$22 - 21 = 1$
24	$24 - 21 = 3$
25	$25 - 21 = 4$

$$\sigma^2 = \frac{1}{N}\sum(\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2 + (-6)^2 + (-5)^2\cdot 3 + (1)^2\cdot 2 + (3)^2\cdot 2 + (4)^2\cdot 5\right]$$
$$= \frac{1}{14}(49 + 36 + 75 + 2 + 18 + 80) = \frac{260}{14} = \boxed{18.571.}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571. \quad \text{[Agrees with (b).]}$$

Problem 1.2

(a)

$$\langle x^2\rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\left(\frac{2}{5}x^{5/2}\right)\Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3}\right)^2 = \frac{4}{45}h^2 \;\Rightarrow\; \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}}dx = 1 - \frac{1}{2\sqrt{h}}(2\sqrt{x})\Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}}\left(\sqrt{x_+} - \sqrt{x_-}\right).$$

$$x_+ \equiv \langle x\rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x\rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx. \qquad \text{Let } u \equiv x-a,\ du = dx,\ u: -\infty \to \infty.$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\frac{\pi}{\lambda}} \quad \Rightarrow \quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

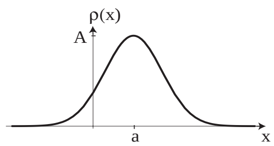
(b)

$$\begin{aligned}\langle x\rangle &= A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du \\ &= A\left[\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right] = A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a.}\end{aligned}$$

$$\begin{aligned}\langle x^2\rangle &= A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}dx \\ &= A\left\{\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\} \\ &= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\lambda}.}\end{aligned}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

(c)



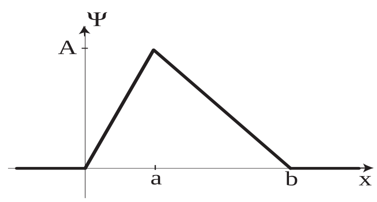
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2\left\{\frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b\right\}$$

$$= |A|^2\left[\frac{a}{3}+\frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A=\sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x=a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.}\begin{cases} P=1 & \text{if } b=a, \checkmark \\ P=1/2 & \text{if } b=2a. \checkmark\end{cases}$$

(e)

$$\langle x\rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Bigg|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

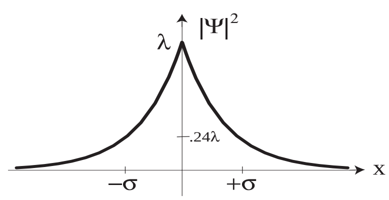
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3}\right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Bigg|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2) dx \neq (x|\Psi|^2)\Big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$

$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2amx^2/\hbar} dx = 2|A|^2 \frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\begin{aligned}\langle p^2\rangle &= \int \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi dx = -\hbar^2 \int \Psi^* \frac{\partial^2\Psi}{\partial x^2} dx \\ &= -\hbar^2 \int \Psi^* \left[-\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi\right] dx = 2am\hbar \left\{\int |\Psi|^2 dx - \frac{2am}{\hbar}\int x^2|\Psi|^2 dx\right\} \\ &= 2am\hbar\left(1 - \frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1 - \frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}\end{aligned}$$

(d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Longrightarrow \boxed{\sigma_x = \sqrt{\frac{\hbar}{4am}}}; \quad \sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Longrightarrow \boxed{\sigma_p = \sqrt{am\hbar}.}$$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \frac{\hbar}{2}$. This *is* (just barely) consistent with the uncertainty principle.

Problem 1.10

From Math Tables: $\pi = 3.141592653589793238462643\cdots$

(a)

$P(0) = 0$	$P(1) = 2/25$	$P(2) = 3/25$	$P(3) = 5/25$	$P(4) = 3/25$
$P(5) = 3/25$	$P(6) = 3/25$	$P(7) = 1/25$	$P(8) = 2/25$	$P(9) = 3/25$

In general, $P(j) = \frac{N(j)}{N}$.

(b) *Most probable*: $\boxed{3.}$ *Median*: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4.}$

Average: $\langle j\rangle = \frac{1}{25}[0\cdot 0 + 1\cdot 2 + 2\cdot 3 + 3\cdot 5 + 4\cdot 3 + 5\cdot 3 + 6\cdot 3 + 7\cdot 1 + 8\cdot 2 + 9\cdot 3]$

$= \frac{1}{25}[0 + 2 + 6 + 15 + 12 + 15 + 18 + 7 + 16 + 27] = \frac{118}{25} = \boxed{4.72.}$

(c) $\langle j^2\rangle = \frac{1}{25}[0 + 1^2\cdot 2 + 2^2\cdot 3 + 3^2\cdot 5 + 4^2\cdot 3 + 5^2\cdot 3 + 6^2\cdot 3 + 7^2\cdot 1 + 8^2\cdot 2 + 9^2\cdot 3]$

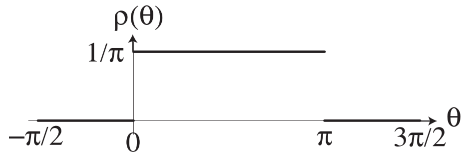
$= \frac{1}{25}[0 + 2 + 12 + 45 + 48 + 75 + 108 + 49 + 128 + 243] = \frac{710}{25} = \boxed{28.4.}$

$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216;\quad \sigma = \sqrt{6.1216} = \boxed{2.474.}$

Problem 1.11

(a) Constant for $0 \le \theta \le \pi$, otherwise zero. In view of Eq. 1.16, the constant is $1/\pi$.

$$\boxed{\rho(\theta) = \begin{cases} 1/\pi, & \text{if } 0 \le \theta \le \pi, \\ 0, & \text{otherwise.} \end{cases}}$$



(b)

$$\langle \theta \rangle = \int \theta \rho(\theta)\, d\theta = \frac{1}{\pi}\int_0^\pi \theta d\theta = \frac{1}{\pi}\left(\frac{\theta^2}{2}\right)\Big|_0^\pi = \boxed{\frac{\pi}{2}} \quad \text{[of course].}$$

$$\langle \theta^2 \rangle = \frac{1}{\pi}\int_0^\pi \theta^2\, d\theta = \frac{1}{\pi}\left(\frac{\theta^3}{3}\right)\Big|_0^\pi = \boxed{\frac{\pi^2}{3}.}$$

$$\sigma^2 = \langle \theta^2 \rangle - \langle \theta \rangle^2 = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12}; \quad \boxed{\sigma = \frac{\pi}{2\sqrt{3}}.}$$

(c)

$$\langle \sin\theta \rangle = \frac{1}{\pi}\int_0^\pi \sin\theta\, d\theta = \frac{1}{\pi}\left(-\cos\theta\right)\big|_0^\pi = \frac{1}{\pi}(1-(-1)) = \boxed{\frac{2}{\pi}.}$$

$$\langle \cos\theta \rangle = \frac{1}{\pi}\int_0^\pi \cos\theta\, d\theta = \frac{1}{\pi}\left(\sin\theta\right)\big|_0^\pi = \boxed{0.}$$

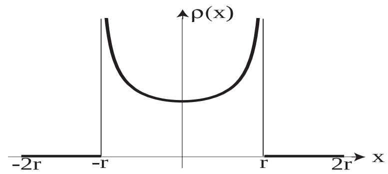
$$\langle \cos^2\theta \rangle = \frac{1}{\pi}\int_0^\pi \cos^2\theta\, d\theta = \frac{1}{\pi}\int_0^\pi (1/2) d\theta = \boxed{\frac{1}{2}.}$$

[Because $\sin^2\theta + \cos^2\theta = 1$, and the integrals of $\sin^2$ and $\cos^2$ are equal (over suitable intervals), one can replace them by 1/2 in such cases.]

Problem 1.12

(a) $x = r\cos\theta \Rightarrow dx = -r\sin\theta\, d\theta$. The probability that the needle lies in range $d\theta$ is $\rho(\theta)d\theta = \frac{1}{\pi}d\theta$, so the probability that it's in the range dx is

$$\rho(x)dx = \frac{1}{\pi}\frac{dx}{r\sin\theta} = \frac{1}{\pi}\frac{dx}{r\sqrt{1-(x/r)^2}} = \frac{dx}{\pi\sqrt{r^2-x^2}}.$$



$$\therefore \boxed{\rho(x) = \begin{cases} \frac{1}{\pi\sqrt{r^2-x^2}}, & \text{if } -r < x < r, \\ 0, & \text{otherwise.} \end{cases}}$$ [*Note:* We want the *magnitude* of dx here.]

Total: $\int_{-r}^{r} \frac{1}{\pi\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\int_0^r \frac{1}{\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\sin^{-1}\left.\frac{x}{r}\right|_0^r = \frac{2}{\pi}\sin^{-1}(1) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$✓

(b)

$$\langle x\rangle = \frac{1}{\pi}\int_{-r}^{r} x\frac{1}{\sqrt{r^2-x^2}}dx = \boxed{0} \quad \text{[odd integrand, even interval].}$$

$$\langle x^2\rangle = \frac{2}{\pi}\int_0^r \frac{x^2}{\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\left[-\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]\Bigg|_0^r = \frac{2}{\pi}\frac{r^2}{2}\sin^{-1}(1) = \boxed{\frac{r^2}{2}.}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = r^2/2 \Longrightarrow \boxed{\sigma = r/\sqrt{2}.}$$

To get $\langle x\rangle$ and $\langle x^2\rangle$ from Problem 1.11(c), use $x = r\cos\theta$, so $\langle x\rangle = r\langle\cos\theta\rangle = 0$, $\langle x^2\rangle = r^2\langle\cos^2\theta\rangle = r^2/2$.

Problem 1.13

Suppose the eye end lands a distance y up from a line ($0 \le y < l$), and let x be the projection along that same direction ($-l \le x < l$). The needle crosses the line above if $y + x \ge l$ (i.e. $x \ge l - y$), and it crosses the line below if $y + x < 0$ (i.e. $x < -y$). So for a given value of y, the probability of crossing (using Problem 1.12) is

$$P(y) = \int_{-l}^{-y}\rho(x)dx + \int_{l-y}^{l}\rho(x)dx = \frac{1}{\pi}\left\{\int_{-l}^{-y}\frac{1}{\sqrt{l^2-x^2}}dx + \int_{l-y}^{l}\frac{1}{\sqrt{l^2-x^2}}dx\right\}$$

$$= \frac{1}{\pi}\left\{\sin^{-1}\left(\frac{x}{l}\right)\Big|_{-l}^{-y} + \sin^{-1}\left(\frac{x}{l}\right)\Big|_{l-y}^{l}\right\} = \frac{1}{\pi}\left[-\sin^{-1}(y/l) + 2\sin^{-1}(1) - \sin^{-1}(1-y/l)\right]$$

$$= 1 - \frac{\sin^{-1}(y/l)}{\pi} - \frac{\sin^{-1}(1-y/l)}{\pi}.$$

Now, all values of y are equally likely, so $\rho(y) = 1/l$, and hence the probability of crossing is

$$P = \frac{1}{\pi l}\int_0^l\left[\pi - \sin^{-1}\left(\frac{y}{l}\right) - \sin^{-1}\left(\frac{l-y}{l}\right)\right]dy = \frac{1}{\pi l}\int_0^l\left[\pi - 2\sin^{-1}(y/l)\right]dy$$

$$= \frac{1}{\pi l}\left[\pi l - 2\left(y\sin^{-1}(y/l) + l\sqrt{1-(y/l)^2}\right)\Big|_0^l\right] = 1 - \frac{2}{\pi l}[l\sin^{-1}(1) - l] = 1 - 1 + \frac{2}{\pi} = \boxed{\frac{2}{\pi}.}$$

Problem 1.14

(a) $P_{ab}(t) = \int_a^b |\Psi(x,t)^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial t}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so J has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

(b) Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi\frac{\partial\Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial\Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

Problem 1.15

(a) Eq. 1.24 now reads $\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^* - V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0 + i\Gamma - V_0 + i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and Eq. 1.27 becomes $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2 dx = -\frac{2\Gamma}{\hbar}P.$ QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Longrightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Longrightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar},} \text{ so } \boxed{\tau = \frac{\hbar}{2\Gamma}.}$$

Problem 1.16

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\Psi_1^*\Psi_2)\,dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2 + \Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\Psi_1^*\right)\Psi_2 + \Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - \frac{i}{\hbar}V\Psi_2\right)\right]dx \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2 - \Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}\right)dx \\
&= -\frac{i\hbar}{2m}\left[\frac{\partial\Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx - \Psi_1^*\frac{\partial\Psi_2}{\partial x}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx\right] = 0. \text{ QED}
\end{aligned}$$

Problem 1.17

(a)

$$1 = |A|^2 \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx = 2|A|^2 \int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx = 2|A|^2 \left[a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5}\right]\Bigg|_0^a$$

$$= 2|A|^2 a^5 \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15} a^5 |A|^2, \text{ so } \boxed{A = \sqrt{\frac{15}{16a^5}}.}$$

(b)

$$\langle x \rangle = \int_{-a}^{a} x |\Psi|^2 \, dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

(c)

$$\langle p \rangle = \frac{\hbar}{i} A^2 \int_{-a}^{a} \left(a^2 - x^2\right) \underbrace{\frac{d}{dx}\left(a^2 - x^2\right)}_{-2x} dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

Since we only know $\langle x \rangle$ at $t = 0$ we cannot calculate $d\langle x \rangle / dt$ directly.

(d)

$$\langle x^2 \rangle = A^2 \int_{-a}^{a} x^2 \left(a^2 - x^2\right)^2 dx = 2A^2 \int_0^a \left(a^4 x^2 - 2a^2 x^4 + x^6\right) dx$$

$$= 2 \frac{15}{16a^5} \left[a^4 \frac{x^3}{3} - 2a^2 \frac{x^5}{5} + \frac{x^7}{7}\right]\Bigg|_0^a = \frac{15}{8a^5} \left(a^7\right) \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)$$

$$= \frac{\cancel{15}a^2}{8} \left(\frac{35 - 42 + 15}{\cancel{3} \cdot \cancel{5} \cdot 7}\right) = \frac{a^2}{8} \cdot \frac{8}{7} = \boxed{\frac{a^2}{7}.}$$

(e)

$$\langle p^2 \rangle = -A^2 \hbar^2 \int_{-a}^{a} \left(a^2 - x^2\right) \underbrace{\frac{d^2}{dx^2}\left(a^2 - x^2\right)}_{-2} dx = 2A^2 \hbar^2 2 \int_0^a \left(a^2 - x^2\right) dx$$

$$= 4 \cdot \frac{15}{16a^5} \hbar^2 \left(a^2 x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{15\hbar^2}{4a^5} \left(a^3 - \frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2} \cdot \frac{2}{3} = \boxed{\frac{5}{2} \frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{7} a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5}{2} \frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}} \frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}} \cdot \sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

Problem 1.18

$$\frac{h}{\sqrt{3mk_BT}} > d \;\Rightarrow\; T < \frac{h^2}{3mk_Bd^2}.$$

(a) Electrons ($m = 9.1 \times 10^{-31}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(9.1\times10^{-31})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{1.3\times10^5 \text{ K.}}$$

Sodium nuclei ($m = 23m_p = 23(1.7\times10^{-27}) = 3.9\times10^{-26}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(3.9\times10^{-26})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{3.0 \text{ K.}}$$

(b) $PV = Nk_BT$; volume occupied by one molecule ($N=1,\ V=d^3$) $\Rightarrow d = (k_BT/P)^{1/3}$.

$$T < \frac{h^2}{2mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \;\Rightarrow\; T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k_B^{5/3}} \;\Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.$$

For helium ($m = 4m_p = 6.8\times10^{-27}$ kg) at 1 atm $= 1.0\times10^5$ N/m^2:

$$T < \frac{1}{(1.4\times10^{-23})}\left(\frac{(6.6\times10^{-34})^2}{3(6.8\times10^{-27})}\right)^{3/5}(1.0\times10^5)^{2/5} = \boxed{2.8 \text{ K.}}$$

For hydrogen ($m = 2m_p = 3.4\times10^{-27}$ kg) with $d = 0.01$ m:

$$T < \frac{(6.6\times10^{-34})^2}{3(3.4\times10^{-27})(1.4\times10^{-23})(10^{-2})^2} = \boxed{3.1\times10^{-14} \text{ K.}}$$

At 3 K it is definitely in the classical regime.

Chapter 2

Time-Independent Schrödinger Equation

Problem 2.1

(a)

$$\Psi(x,t) = \psi(x)e^{-i(E_0+i\Gamma)t/\hbar} = \psi(x)e^{\Gamma t/\hbar}e^{-iE_0t/\hbar} \Longrightarrow |\Psi|^2 = |\psi|^2 e^{2\Gamma t/\hbar}.$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty} |\psi|^2 dx.$$

The second term is independent of t, so if the product is to be 1 for all time, the first term ($e^{2\Gamma t/\hbar}$) must also be constant, and hence $\Gamma = 0$. QED

(b) If ψ satisfies Eq. 2.5, $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + V\psi = E\psi$, then (taking the complex conjugate and noting that V and E are real): $-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{dx^2} + V\psi^* = E\psi^*$, so ψ^* *also* satisfies Eq. 2.5. Now, if ψ_1 and ψ_2 satisfy Eq. 2.5, so too does any linear combination of them ($\psi_3 \equiv c_1\psi_1 + c_2\psi_2$):

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\psi_3}{dx^2} + V\psi_3 &= -\frac{\hbar^2}{2m}\left(c_1\frac{\partial^2\psi_1}{dx^2} + c_2\frac{\partial^2\psi_2}{\partial x^2}\right) + V(c_1\psi_1 + c_2\psi_2) \\
&= c_1\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right] + c_2\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right] \\
&= c_1(E\psi_1) + c_2(E\psi_2) = E(c_1\psi_1 + c_2\psi_2) = E\psi_3.
\end{aligned}$$

Thus, $(\psi + \psi^*)$ and $i(\psi - \psi^*)$ – both of which are *real* – satisfy Eq. 2.5. *Conclusion:* From any complex solution, we can always construct two *real* solutions (of course, if ψ is already real, the second one will be zero). In particular, since $\psi = \frac{1}{2}[(\psi + \psi^*) - i(i(\psi - \psi^*))], \psi$ can be expressed as a linear combination of two real solutions. QED

(c) If $\psi(x)$ satisfies Eq. 2.5, then, changing variables $x \to -x$ and noting that $\partial^2/\partial(-x)^2 = \partial^2/\partial x^2$,

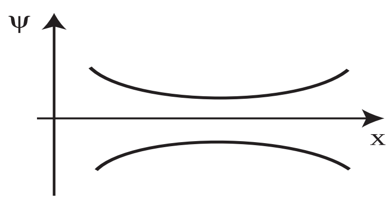
$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x);$$

so if $V(-x) = V(x)$ then $\psi(-x)$ *also* satisfies Eq. 2.5. It follows that $\psi_+(x) \equiv \psi(x) + \psi(-x)$ (which is *even*: $\psi_+(-x) = \psi_+(x)$) and $\psi_-(x) \equiv \psi(x) - \psi(-x)$ (which is *odd*: $\psi_-(-x) = -\psi_-(x)$) both satisfy Eq.

2.5. But $\psi(x) = \frac{1}{2}(\psi_+(x) + \psi_-(x))$, so any solution can be expressed as a linear combination of even and odd solutions. QED

Problem 2.2

Given $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$, if $E < V_{\min}$, then ψ'' and ψ always have the same sign: If ψ is positive(negative), then ψ'' is also positive(negative). This means that ψ always curves away from the axis (see Figure). However, it has got to go to zero as $x \to -\infty$ (else it would not be normalizable). At some point it's got to *depart* from zero (if it *doesn't*, it's going to be identically zero *everywhere*), in (say) the positive direction. At this point its slope is positive, and *increasing*, so ψ gets bigger and bigger as x increases. It can't ever "turn over" and head back toward the axis, because that would requuire a negative second derivative—it always has to bend away from the axis. By the same token, if it starts out heading negative, it just runs more and more negative. In neither case is there any way for it to come back to zero, as it must (at $x \to \infty$) in order to be normalizable. QED



Problem 2.3

Equation 2.20 says $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$; Eq. 2.23 says $\psi(0) = \psi(a) = 0$. If $E = 0$, $d^2\psi/dx^2 = 0$, so $\psi(x) = A + Bx$; $\psi(0) = A = 0 \Rightarrow \psi = Bx$; $\psi(a) = Ba = 0 \Rightarrow B = 0$, so $\psi = 0$. If $E < 0$, $d^2\psi/dx^2 = \kappa^2\psi$, with $\kappa \equiv \sqrt{-2mE}/\hbar$ real, so $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$. This time $\psi(0) = A + B = 0 \Rightarrow B = -A$, so $\psi = A(e^{\kappa x} - e^{-\kappa x})$, while $\psi(a) = A\left(e^{\kappa a} - e^{i\kappa a}\right) = 0 \Rightarrow$ either $A = 0$, so $\psi = 0$, or else $e^{\kappa a} = e^{-\kappa a}$, so $e^{2\kappa a} = 1$, so $2\kappa a = \ln(1) = 0$, so $\kappa = 0$, and again $\psi = 0$. In all cases, then, the boundary conditions force $\psi = 0$, which is unacceptable (non-normalizable).

Problem 2.4

$$\langle x \rangle = \int x|\psi|^2 dx = \frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) dx. \qquad \text{Let } y \equiv \frac{n\pi}{a}x, \text{ so } dx = \frac{a}{n\pi}dy; \quad y: 0 \to n\pi.$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y \sin^2 y \, dy = \frac{2a}{n^2\pi^2}\left[\frac{y^2}{4} - \frac{y \sin 2y}{4} - \frac{\cos 2y}{8}\right]\Bigg|_0^{n\pi}$$

$$= \frac{2a}{n^2\pi^2}\left[\frac{n^2\pi^2}{4} - \frac{\cos 2n\pi}{8} + \frac{1}{8}\right] = \boxed{\frac{a}{2}.} \quad \text{(Independent of } n.\text{)}$$

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3\int_0^{n\pi} y^2\sin^2 y\,dy$$
$$= \frac{2a^2}{(n\pi)^3}\left[\frac{y^3}{6} - \left(\frac{y^3}{4}-\frac{1}{8}\right)\sin 2y - \frac{y\cos 2y}{4}\right]_0^{n\pi}$$
$$= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{n\pi\cos(2n\pi)}{4}\right] = \boxed{a^2\left[\frac{1}{3}-\frac{1}{2(n\pi)^2}\right]}.$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad (\textit{Note}: \text{Eq. 1.33 is much faster than Eq. 1.35.})$$

$$\langle p^2\rangle = \int \psi_n^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_n\,dx = -\hbar^2\int\psi_n^*\left(\frac{d^2\psi_n}{dx^2}\right)dx$$
$$= (-\hbar^2)\left(-\frac{2mE_n}{\hbar^2}\right)\int\psi_n^*\psi_n\,dx = 2mE_n = \boxed{\left(\frac{n\pi\hbar}{a}\right)^2}.$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3}-\frac{1}{2(n\pi)^2}-\frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3}-\frac{2}{(n\pi)^2}\right); \quad \boxed{\sigma_x = \frac{a}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}.}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2; \quad \boxed{\sigma_p = \frac{n\pi\hbar}{a}.} \quad \therefore \sigma_x\sigma_p = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3}-2}.}$$

The product $\sigma_x\sigma_p$ is $\boxed{\text{smallest for } n=1;}$ in that case, $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} = (1.136)\hbar/2 > \hbar/2.$ ✓

Problem 2.5

(a)

$$|\Psi|^2 = \Psi^2\Psi = |A|^2(\psi_1^*+\psi_2^*)(\psi_1+\psi_2) = |A|^2[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2].$$

$$1 = \int|\Psi|^2dx = |A|^2\int[|\psi_1|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + |\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right] \quad \left(\text{but } \frac{E_n}{\hbar} = n^2\omega\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right]}.$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t}+e^{3i\omega t}\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]$$
$$= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]}.$$

(c)

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx$$
$$= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a} - \frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]\Bigg|_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right) + \frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right) - \frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\big(\cos(\pi) - \cos(0)\big) - \frac{a^2}{9\pi^2}\big(\cos(3\pi) - \cos(0)\big)\right] = -\frac{a^2}{\pi^2}\left(1 - \frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\therefore \langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]}.$$

Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603(a/2);}$ *angular frequency:* $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}}.$

(d)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t).}$$

(e) You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H\rangle = \boxed{\frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

Problem 2.6

From Problem 2.5, we see that

$$\Psi(x,t) = \boxed{\tfrac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\tfrac{\pi}{a}x\right) + \sin\left(\tfrac{2\pi}{a}x\right)e^{-3i\omega t}e^{i\phi}\right];}$$

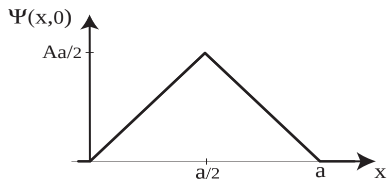
$$|\Psi(x,t)|^2 = \boxed{\tfrac{1}{a}\left[\sin^2\left(\tfrac{\pi}{a}x\right) + \sin^2\left(\tfrac{2\pi}{a}x\right) + 2\sin\left(\tfrac{\pi}{a}x\right)\sin\left(\tfrac{2\pi}{a}x\right)\cos(3\omega t - \phi)\right];}$$

and hence $\boxed{\langle x\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t - \phi)\right].}$ This amounts physically to starting the clock at a different time (i.e., shifting the $t = 0$ point).

If $\phi = \dfrac{\pi}{2}$, so $\Psi(x,0) = A[\psi_1(x) + i\psi_2(x)]$, then $\cos(3\omega t - \phi) = \sin(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}$.

If $\phi = \pi$, so $\Psi(x,0) = A[\psi_1(x) - \psi_2(x)]$, then $\cos(3\omega t - \phi) = -\cos(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}\left(1 + \dfrac{32}{9\pi^2}\right)$.

Problem 2.7



(a)

$$1 = A^2\int_0^{a/2} x^2 dx + A^2\int_{a/2}^{a}(a-x)^2 dx = A^2\left[\left.\frac{x^3}{3}\right|_0^{a/2} - \left.\frac{(a-x)^3}{3}\right|_{a/2}^{a}\right]$$

$$= \frac{A^2}{3}\left(\frac{a^3}{8} + \frac{a^3}{8}\right) = \frac{A^2a^3}{12} \Rightarrow \boxed{A = \frac{2\sqrt{3}}{\sqrt{a^3}}.}$$

(b)

$$c_n = \sqrt{\frac{2}{a}}\frac{2\sqrt{3}}{a\sqrt{a}}\left[\int_0^{a/2} x\sin\left(\frac{n\pi}{a}x\right)dx + \int_{a/2}^{a}(a-x)\sin\left(\frac{n\pi}{a}x\right)dx\right]$$

$$= \frac{2\sqrt{6}}{a^2}\left\{\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{a}x\right) - \frac{xa}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Big|_0^{a/2}\right.$$

$$\left. + a\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Big|_{a/2}^{a} - \left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{a}x\right) - \left(\frac{ax}{n\pi}\right)\cos\left(\frac{n\pi}{a}x\right)\right]\Big|_{a/2}^{a}\right\}$$

$$= \frac{2\sqrt{6}}{a^2}\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{a^2}{n\pi}\cos n\pi} + \cancel{\frac{a^2}{n\pi}\cos\left(\frac{n\pi}{2}\right)}\right.$$

$$\left. + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{a^2}{n\pi}\cos n\pi} - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)}\right]$$

$$= \frac{2\sqrt{6}}{\cancel{a^2}}2\frac{\cancel{a^2}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4\sqrt{6}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even,} \\ (-1)^{(n-1)/2}\frac{4\sqrt{6}}{(n\pi)^2}, & n \text{ odd.}\end{cases}$$

So $\boxed{\Psi(x,t) = \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}}\sum_{n=1,3,5,\dots}(-1)^{(n-1)/2}\frac{1}{n^2}\sin\left(\frac{n\pi}{a}x\right)e^{-E_nt/\hbar},}$ where $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$.

(c)

$$P_1 = |c_1|^2 = \frac{16 \cdot 6}{\pi^4} = \boxed{0.9855.}$$

(d)

$$\langle H \rangle = \sum |c_n|^2 E_n = \frac{96}{\pi^4} \frac{\pi^2 \hbar^2}{2ma^2} \Bigg(\underbrace{\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}_{\pi^2/8} \Bigg) = \frac{48\hbar^2}{\pi^2 ma^2} \frac{\pi^2}{8} = \boxed{\frac{6\hbar^2}{ma^2}.}$$

Problem 2.8

(a)

$$\boxed{\Psi(x,0) = \begin{cases} A, & 0 < x < a/2; \\ 0, & \text{otherwise.} \end{cases}} \quad 1 = A^2 \int_0^{a/2} dx = A^2 (a/2) \Rightarrow \boxed{A = \sqrt{\frac{2}{a}}.}$$

(b) From Eq. 2.37,

$$c_1 = A\sqrt{\frac{2}{a}} \int_0^{a/2} \sin\left(\frac{\pi}{a}x\right) dx = \frac{2}{a}\left[-\frac{a}{\pi}\cos\left(\frac{\pi}{a}x\right)\right]\Bigg|_0^{a/2} = -\frac{2}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = \frac{2}{\pi}.$$

$$P_1 = |c_1|^2 = \boxed{(2/\pi)^2 = 0.4053.}$$

Problem 2.9

$$\hat{H}\Psi(x,0) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[Ax(a-x)\right] = -A\frac{\hbar^2}{2m}\frac{\partial}{\partial x}(a-2x) = A\frac{\hbar^2}{m}.$$

$$\begin{aligned} \int \Psi(x,0)^* \hat{H}\Psi(x,0)\, dx &= A^2 \frac{\hbar^2}{m} \int_0^a x(a-x)\, dx = A^2 \frac{\hbar^2}{m}\left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^a \\ &= A^2 \frac{\hbar^2}{m}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{30}{a^5}\frac{\hbar^2}{m}\frac{a^3}{6} = \boxed{\frac{5\hbar^2}{ma^2}} \end{aligned}$$

(same as Example 2.3).

Problem 2.10

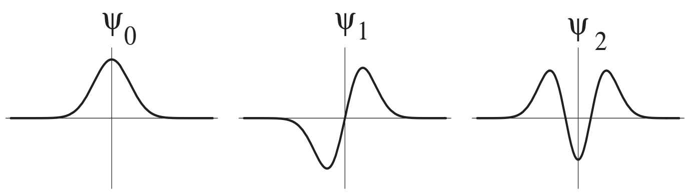
(a) Using Eqs. 2.47 and 2.59,

$$a_+\psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$
$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x + m\omega x\right]e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}.$$
$$(a_+)^2\psi_0 = \frac{1}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega\left(-\hbar\frac{d}{dx} + m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2}$$
$$= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(1 - x\frac{m\omega}{2\hbar}2x\right) + m\omega x^2\right]e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}.$$

Therefore, from Eq. 2.67,

$$\psi_2 = \frac{1}{\sqrt{2}}(a_+)^2\psi_0 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}.}$$

(b)



(c) Since ψ_0 and ψ_2 are even, whereas ψ_1 is odd, $\int \psi_0^*\psi_1 dx$ and $\int \psi_2^*\psi_1 dx$ vanish automatically. The only one we need to check is $\int \psi_2^*\psi_0\, dx$:

$$\int \psi_2^*\psi_0\, dx = \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{\hbar}x^2}dx$$
$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx - \frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx\right)$$
$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right) = 0. \checkmark$$

Problem 2.11

(a) Note that ψ_0 is even, and ψ_1 is odd. In either case $|\psi|^2$ is even, so $\langle x\rangle = \int x|\psi|^2 dx = \boxed{0.}$ Therefore $\langle p\rangle = md\langle x\rangle/dt = \boxed{0.}$ (These results hold for *any* stationary state of the harmonic oscillator.)

From Eqs. 2.59 and 2.62, $\psi_0 = \alpha e^{-\xi^2/2}$, $\psi_1 = \sqrt{2}\alpha\xi e^{-\xi^2/2}$. So

<u>$n = 0$</u>**:**

$$\langle x^2\rangle = \alpha^2\int_{-\infty}^{\infty} x^2 e^{-\xi^2/2}dx = \alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi = \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\frac{\sqrt{\pi}}{2} = \boxed{\frac{\hbar}{2m\omega}.}$$