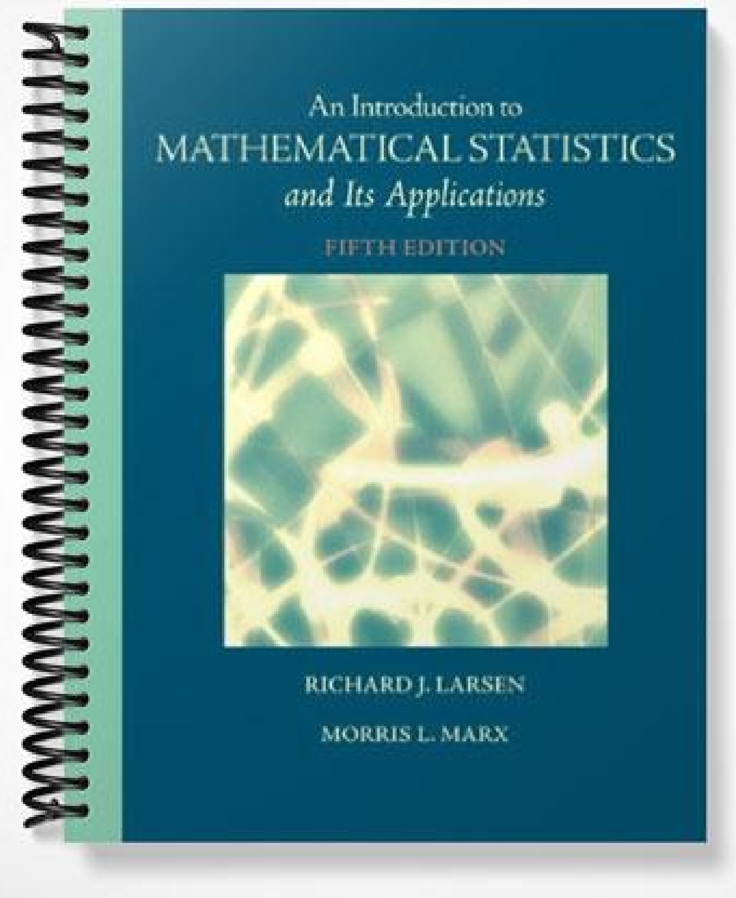
SOLUTIONS MANUAL



Chapter 2: Probability

Section 2.2: Sample Spaces and the Algebra of Sets

2.2.1 $S = \{(s,s,s), (s,s,f), (s,f,s), (f,s,s), (s,f,f), (f,s,f), (f,f,s), (f,f,f)\}$
$A = \{ (s,f,s), (f,s,s)\}$; $B = \{(f,f,f)\}$

2.2.2 Let (*x, y, z*) denote a red *x*, a blue *y*, and a green *z*. Then
$A = \{(2,2,1), (2,1,2), (1,2,2), (1,1,3), (1,3,1), (3,1,1)\}$

2.2.3 (1,3,4), (1,3,5), (1,3,6), (2,3,4), (2,3,5), (2,3,6)

2.2.4 There are 16 ways to get an ace and a 7, 16 ways to get a 2 and a 6, 16 ways to get a 3 and a 5, and 6 ways to get two 4's, giving 54 total.

2.2.5 The outcome sought is (4, 4). It is "harder" to obtain than the set {(5, 3), (3, 5), (6, 2), (2, 6)} of other outcomes making a total of 8.

2.2.6 The set *N* of five card hands in hearts that are not flushes are called *straight flushes*. These are five cards whose denominations are consecutive. Each one is characterized by the lowest value in the hand. The choices for the lowest value are A, 2, 3, …, 10. (Notice that an ace can be high or low). Thus, *N* has 10 elements.

2.2.7 P = {right triangles with sides (5, a, b): $a^2 + b^2 = 25$}

2.2.8 *A* = {*SSBBBB*, *SBSBBB*, *SBBSBB*, *SBBBSB*, *BSSBBB*, *BSBSBB*, *BSBBSB*, *BBSSBB*, *BBSBSB*, *BBBSSB*}

2.2.9 **(a)** *S* = {(0, 0, 0, 0) (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1,), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1,)}

(b) *A* = {(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0,)}

(c) $1 + k$

2.2.10 **(a)** *S* = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4)}

(b) {2, 3, 4, 5, 6, 8}

2.2.11 Let p_1 and p_2 denote the two perpetrators and i_1, i_2, and i_3, the three in the lineup who are innocent.
Then $S = \{(p_1,i_1), (p_1,i_2), (p_1,i_3), (p_2,i_1), (p_2,i_2), (p_2,i_3), (p_1,p_2), (i_1,i_2), (i_1,i_3), (i_2,i_3)\}$.
The event *A* contains every outcome in *S* except (p_1, p_2).

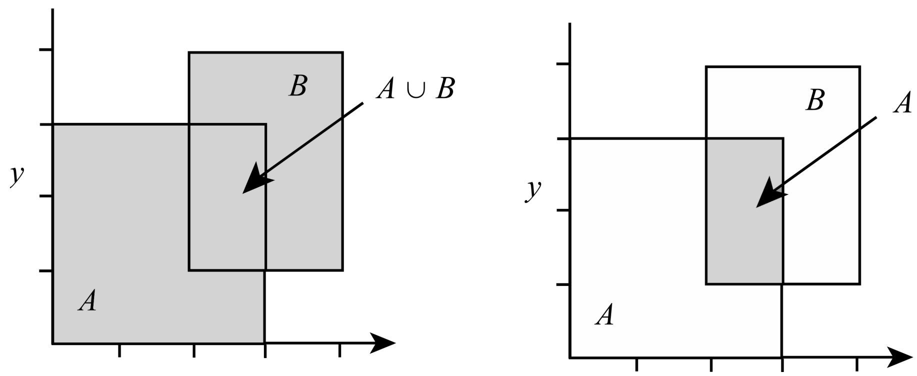
2.2.12 The quadratic equation will have complex roots—that is, the event *A* will occur—if $b^2 - 4ac < 0$.

2.2.13 In order for the shooter to win with a point of 9, one of the following (countably infinite) sequences of sums must be rolled: (9,9), (9, no 7 or no 9,9), (9, no 7 or no 9, no 7 or no 9,9), …

2.2.14 Let (x, y) denote the strategy of putting x white chips and y black chips in the first urn (which results in $10 - x$ white chips and $10 - y$ black chips being in the second urn). Then $S = \{(x, y): x = 0,1,...,10,\ \ y = 0,1,...,10,\ \text{and}\ 1 \le x + y \le 19\}$. Intuitively, the optimal strategies are (1, 0) and (9, 10).

2.2.15 Let A_k be the set of chips put in the urn at $1/2^k$ minute until midnight. For example, $A_1 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. Then the set of chips in the urn at midnight is $\bigcup_{k=1}^{\infty}(A_k - \{k+1\}) = \varnothing$.

2.2.16

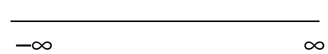


2.2.17 If $x^2 + 2x \le 8$, then $(x + 4)(x - 2) \le 0$ and $A = \{x: -4 \le x \le 2\}$. Similarly, if $x^2 + x \le 6$, then $(x + 3)(x - 2) \le 0$ and $B = \{x: -3 \le x \le 2)$. Therefore, $A \cap B = \{x: -3 \le x \le 2\}$ and $A \cup B = \{x: -4 \le x \le 2\}$.

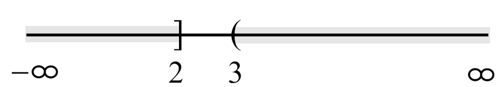
2.2.18 $A \cap B \cap C = \{x: x = 2, 3, 4\}$

2.2.19 The system fails if either the first pair fails or the second pair fails (or both pairs fail). For either pair to fail, though, both of its components must fail. Therefore, $A = (A_{11} \cap A_{21}) \cup (A_{12} \cap A_{22})$.

2.2.20 **(a)** 

(b) 

(c) empty set

(d) 

2.2.21 40

2.2.22 **(a)** $\{E1, E2\}$ **(b)** $\{S1, S2, T1, T2\}$ **(c)** $\{A, I\}$

2.2.23 **(a)** If s is a member of $A \cup (B \cap C)$ then s belongs to A or to $B \cap C$. If it is a member of A or of $B \cap C$, then it belongs to $A \cup B$ and to $A \cup C$. Thus, it is a member of $(A \cup B) \cap (A \cup C)$. Conversely, choose s in $(A \cup B) \cap (A \cup C)$. If it belongs to A, then it belongs to $A \cup (B \cap C)$. If it does not belong to A, then it must be a member of $B \cap C$. In that case it also is a member of $A \cup (B \cap C)$.

(b) If s is a member of $A \cap (B \cup C)$ then s belongs to A and to $B \cup C$. If it is a member of B, then it belongs to $A \cap B$ and, hence, $(A \cap B) \cup (A \cap C)$. Similarly, if it belongs to C, it is a member of $(A \cap B) \cup (A \cap C)$. Conversely, choose s in $(A \cap B) \cup (A \cap C)$. Then it belongs to A. If it is a member of $A \cap B$ then it belongs to $A \cap (B \cup C)$. Similarly, if it belongs to $A \cap C$, then it must be a member of $A \cap (B \cup C)$.

2.2.24 Let $B = A_1 \cup A_2 \cup \ldots \cup A_k$. Then $A_1^C \cap A_2^C \cap \ldots \cap A_k^C = (A_1 \cup A_2 \cup \ldots \cup A_k)^C = B^C$. Then the expression is simply $B \cup B^C = S$.

2.2.25 **(a)** Let s be a member of $A \cup (B \cup C)$. Then s belongs to either A or $B \cup C$ (or both). If s belongs to A, it necessarily belongs to $(A \cup B) \cup C$. If s belongs to $B \cup C$, it belongs to B or C or both, so it must belong to $(A \cup B) \cup C$. Now, suppose s belongs to $(A \cup B) \cup C$. Then it belongs to either $A \cup B$ or C or both. If it belongs to C, it must belong to $A \cup (B \cup C)$. If it belongs to $A \cup B$, it must belong to either A or B or both, so it must belong to $A \cup (B \cup C)$.

(b) Suppose s belongs to $A \cap (B \cap C)$, so it is a member of A and also $B \cap C$. Then it is a member of A and of B and C. That makes it a member of $(A \cap B) \cap C$. Conversely, if s is a member of $(A \cap B) \cap C$, a similar argument shows it belongs to $A \cap (B \cap C)$.

2.2.26 **(a)** $A^C \cap B^C \cap C^C$
(b) $A \cap B \cap C$
(c) $A \cap B^C \cap C^C$
(d) $(A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)$
(e) $(A \cap B \cap C^C) \cup (A \cap B^C \cap C) \cup (A^C \cap B \cap C)$

2.2.27 A is a subset of B.

2.2.28 **(a)** $\{0\} \cup \{x\colon 5 \le x \le 10\}$ **(b)** $\{x\colon 3 \le x < 5\}$ **(c)** $\{x\colon 0 < x \le 7\}$
(d) $\{x\colon 0 < x < 3\}$ **(e)** $\{0\} \cup \{x: 3 \le x \le 10\}$ **(f)** $\{0\} \cup \{x: 7 < x \le 10\}$

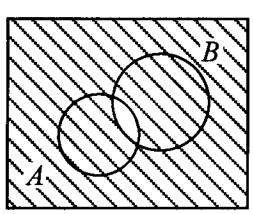
2.2.29 **(a)** B and C **(b)** B is a subset of A.

2.2.30 **(a)** $A_1 \cap A_2 \cap A_3$
(b) $A_1 \cup A_2 \cup A_3$

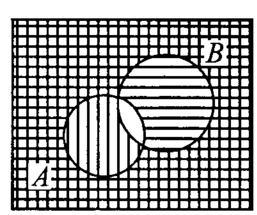
The second protocol would be better if speed of approval matters. For very important issues, the first protocol is superior.

2.2.31 Let A and B denote the students who saw the movie the first time and the second time, respectively. Then $N(\mathbf{a}) = 850$, $N(\mathbf{b}) = 690$, and $N[(A \cup B)^C] = 4700$ (implying that $N(A \cup B) = 1300$). Therefore, $N(A \cap B)$ = number who saw movie twice $= 850 + 690 - 1300 = 240$.

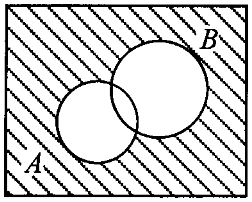
2.2.32 **(a)**



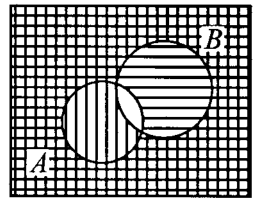
$(A \cap B)^C = A^C \cup B^C$



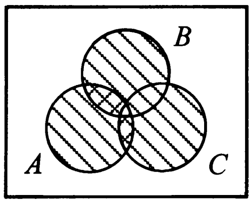
(b)



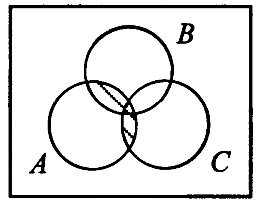
$(A \cup B)^C = A^C \cap B^C$



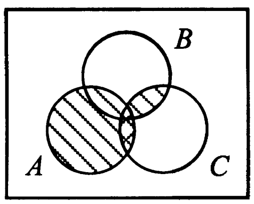
2.2.33 **(a)**



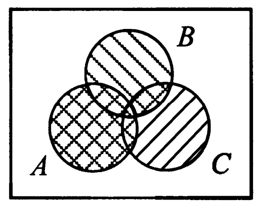
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



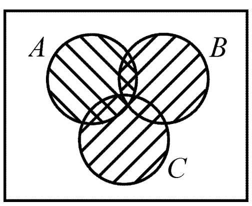
(b)



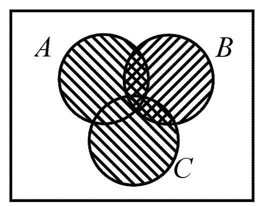
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



2.2.34 **(a)**

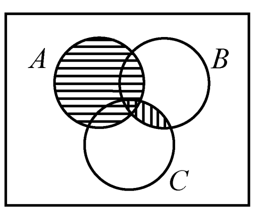


$A \cup (B \cup C)$

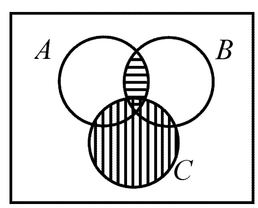


$(A \cup B) \cup C$

(b)



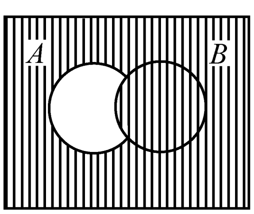
$A \cap (B \cap C)$



$(A \cap B) \cap C$

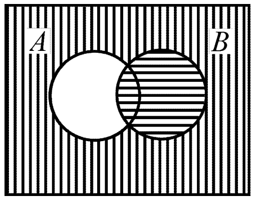
2.2.35 A and B are subsets of $A \cup B$.

2.2.36 **(a)**



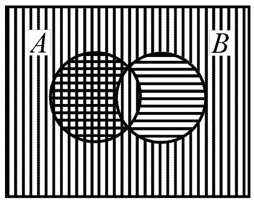
$(A \cap B^C)^C = A^C \cup B$

(b)



$B \cup (A \cup B)^C = A^C \cup B$

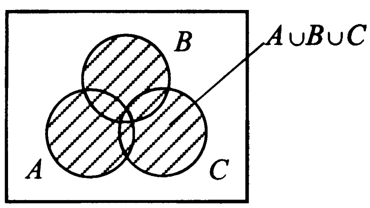
(c)



$A \cap (A \cap B)^C = A \cap B^C$

2.2.37 Let A be the set of those with MCAT scores ≥ 27 and B be the set of those with GPAs ≥ 3.5. We are given that N(a) = 1000, N(b) = 400, and $N(A \cap B) = 300$. Then
$N(A^C \cap B^C) = N[(A \cup B)^C] = 1200 - N(A \cup B) = 1200 - [(N(\text{a}) + N(\text{b}) - N(A \cap B)]$
$= 1200 - [(1000 + 400 - 300] = 100$. The requested proportion is 100/1200.

2.2.38



$N(A \cup B \cup C) = N(\text{a}) + N(\text{b}) + N(\text{c}) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$

2.2.39 Let A be the set of those saying "yes" to the first question and B be the set of those saying "yes" to the second question. We are given that N(a) = 600, N(b) = 400, and $N(A^C \cap B) = 300$. Then $N(A \cap B) = N(\text{b}) - N(A^C \cap B) = 400 - 300 = 100$. $N(A \cap B^C) = N(\text{a}) - N(A \cap B) = 600 - 100$ $= 500$.

2.2.40 $N[(A \cap B)^C] = 120 - N(A \cup B) = 120 - [N(A^C \cap B) + N(A \cap B^C) + N(A \cap B)]$
$= 120 - [50 + 15 + 2] = 53$

Section 2.3: The Probability Function

2.3.1 Let L and V denote the sets of programs with offensive language and too much violence, respectively. Then $P(L) = 0.42$, $P(V) = 0.27$, and $P(L \cap V) = 0.10$. Therefore, P(program complies) $= P((L \cup V)^C) = 1 - [P(L) + P(V) - P(L \cap V)] = 0.41$.

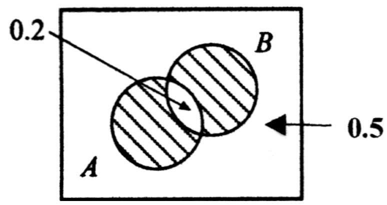
2.3.2 $P(A \text{ or } B \text{ but not both}) = P(A \cup B) - P(A \cap B) = P(\text{a}) + P(\text{b}) - P(A \cap B) - P(A \cap B)$
$= 0.4 + 0.5 - 0.1 - 0.1 = 0.7$

2.3.3 **(a)** $1 - P(A \cap B)$ **(b)** $P(\text{b}) - P(A \cap B)$

2.3.4 $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = 0.3$; $P(\text{a}) - P(A \cap B) = 0.1$. Therefore, $P(\text{b}) = 0.2$.

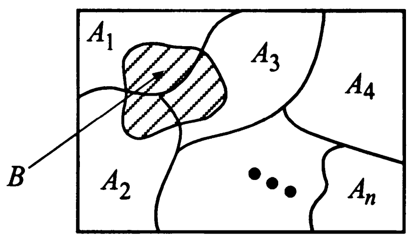
2.3.5 No. $P(A_1 \cup A_2 \cup A_3) = P(\text{at least one "6" appears}) = 1 - P(\text{no 6's appear}) = 1 - \left(\frac{5}{6}\right)^3 \neq \frac{1}{2}$.
The A_i's are not mutually exclusive, so $P(A_1 \cup A_2 \cup A_3) \neq P(A_1) + P(A_2) + P(A_3)$.

2.3.6



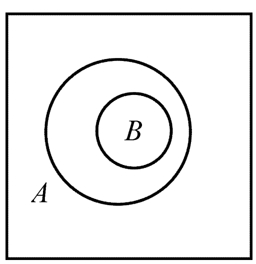
$P(A$ or B but not both$) = 0.5 - 0.2 = 0.3$

2.3.7

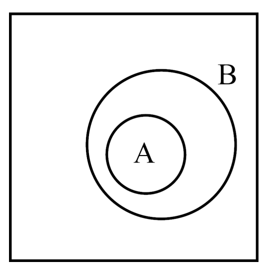


By inspection, $B = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n)$.

2.3.8 **(a)**



(b)



2.3.9 $P(\text{odd man out}) = 1 - P(\text{no odd man out}) = 1 - P(HHH \text{ or } TTT) = 1 - \frac{2}{8} = \frac{3}{4}$

2.3.10 $A = \{2, 4, 6, \ldots, 24\}$; $B = \{3, 6, 9, \ldots, 24)$; $A \cap B = \{6, 12, 18, 24\}$.
Therefore, $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = \frac{12}{24} + \frac{8}{24} - \frac{4}{24} = \frac{16}{24}$.

2.3.11 Let A: State wins Saturday and B: State wins next Saturday. Then $P(\text{a}) = 0.10$, $P(\text{b}) = 0.30$, and $P(\text{lose both}) = 0.65 = 1 - P(A \cup B)$, which implies that $P(A \cup B) = 0.35$. Therefore, $P(A \cap B) = 0.10 + 0.30 - 0.35 = 0.05$, so $P(\text{State wins exactly once}) = P(A \cup B) - P(A \cap B) = 0.35 - 0.05 = 0.30$.

2.3.12 Since A_1 and A_2 are mutually exclusive and cover the entire sample space, $p_1 + p_2 = 1$. But $3p_1 - p_2 = \frac{1}{2}$, so $p_2 = \frac{5}{8}$.

2.3.13 Let F: female is hired and T: minority is hired. Then $P(\text{f}) = 0.60$, $P(T) = 0.30$, and $P(F^C \cap T^C) = 0.25 = 1 - P(F \cup T)$. Since $P(F \cup T) = 0.75$, $P(F \cap T)$ $= 0.60 + 0.30 - 0.75 = 0.15$.

2.3.14 The smallest value of $P[(A \cup B \cup C)^C]$ occurs when $P(A \cup B \cup C)$ is as large as possible. This, in turn, occurs when A, B, and C are mutually disjoint. The largest value for $P(A \cup B \cup C)$ is $P(\text{a}) + P(\text{b}) + P(\text{c}) = 0.2 + 0.1 + 0.3 = 0.6$. Thus, the smallest value for $P[(A \cup B \cup C)^C]$ is 0.4.

2.3.15 **(a)** $X^C \cap Y = \{(H, T, T, H), (T, H, H, T)\}$, so $P(X^C \cap Y) = 2/16$
(b) $X \cap Y^C = \{(H, T, T, T), (T, T, T, H), (T, H, H, H), (H, H, H, T)\}$ so $P(X \cap Y^C) = 4/16$

2.3.16 $A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
$A \cap B^C = \{(1, 5), (3, 3), (5, 1)\}$, so $P(A \cap B^C) = 3/36 = 1/12$.

2.3.17 $A \cap B$, $(A \cap B) \cup (A \cap C)$, A, $A \cup B$, S

2.3.18 Let A be the event of getting arrested for the first scam; B, for the second. We are given $P(\mathbf{a}) = 1/10$, $P(\text{b}) = 1/30$, and $P(A \cap B) = 0.0025$. Her chances of not getting arrested are $P[(A \cup B)^C] = 1 - P(A \cup B) = 1 - [P(\text{a}) + P(\text{b}) - P(A \cap B)] = 1 - [1/10 + 1/30 - 0.0025] = 0.869$

Section 2.4: Conditional Probability

2.4.1 $$P(\text{sum} = 10 | \text{sum exceeds } 8) = \frac{P(\text{sum} = 10 \text{ and sum exceeds } 8)}{P(\text{sum exceeds } 8)}$$
$$= \frac{P(\text{sum} = 10)}{P(\text{sum } = 9, 10, 11, \text{ or } 12)} = \frac{3/36}{4/36 + 3/36 + 2/36 + 1/36} = \frac{3}{10}.$$

2.4.2 $P(A|B) + P(B|A) = 0.75 = \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)} = \frac{10P(A \cap B)}{4} + 5P(A \cap B)$, which implies that $P(A \cap B) = 0.1$.

2.4.3 If $P(A|B) = \frac{P(A \cap B)}{P(B)} < P(A)$, then $P(A \cap B) < P(\text{a}) \cdot P(\text{b})$. It follows that
$$P(B|A) = \frac{P(A \cap B)}{P(A)} < \frac{P(A) \cdot P(B)}{P(A)} = P(\text{b}).$$

2.4.4 $$P(E|A \cup B) = \frac{P(E \cap (A \cup B))}{P(A \cup B)} = \frac{P(E)}{P(A \cup B)} = \frac{P(A \cup B) - P(A \cap B)}{P(A \cup B)} = \frac{0.4 - 0.1}{0.4} = \frac{3}{4}.$$

2.4.5 The answer would remain the same. Distinguishing only three family types does not make them equally likely; (girl, boy) families will occur twice as often as either (boy, boy) or (girl, girl) families.

2.4.6 $P(A \cup B) = 0.8$ and $P(A \cup B) - P(A \cap B) = 0.6$, so $P(A \cap B) = 0.2$. Also, $P(A|B) = 0.6 = \dfrac{P(A \cap B)}{P(B)}$, so $P(b) = \dfrac{0.2}{0.6} = \dfrac{1}{3}$ and $P(a) = 0.8 + 0.2 - \dfrac{1}{3} = \dfrac{2}{3}$.

2.4.7 Let R_i be the event that a red chip is selected on the ith draw, $i = 1, 2$.
Then $P(\text{both are red}) = P(R_1 \cap R_2) = P(R_2 \mid R_1)P(R_1) = \dfrac{3}{4} \cdot \dfrac{1}{2} = \dfrac{3}{8}$.

2.4.8 $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) + P(B) - P(A \cup B)}{P(B)} = \dfrac{a + b - P(A \cup B)}{b}$.
But $P(A \cup B) \leq 1$, so $P(A|B) \geq \dfrac{a + b - 1}{b}$.

2.4.9 Let W_i be the event that a white chip is selected on the ith draw, $i = 1,2$. Then
$P(W_2|W_1) = \dfrac{P(W_1 \cap W_2)}{P(W_1)}$. If both chips in the urn are white, $P(W_1) = 1$;
if one is white and one is black, $P(W_1) = \dfrac{1}{2}$. Since each chip distribution is equally likely,
$P(W_1) = 1 \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{3}{4}$. Similarly, $P(W_1 \cap W_2) = 1 \cdot \dfrac{1}{2} + \dfrac{1}{4} \cdot \dfrac{1}{2} = \dfrac{5}{8}$, so $P(W_2|W_1) = \dfrac{5/8}{3/4} = \dfrac{5}{6}$.

2.4.10 $P[(A \cap B)| (A \cup B)^C] = \dfrac{P[(A \cap B) \cap (A \cup B)^C]}{P[(A \cup B)^C]} = \dfrac{P(\varnothing)}{P[(A \cup B)^C]} = 0$

2.4.11 **(a)** $P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - [P(a) + P(b) - P(A \cap B)] = 1 - [0.65 + 0.55 - 0.25] = 0.05$

(b) $P[(A^C \cap B) \cup (A \cap B^C)] = P(A^C \cap B) + P(A \cap B^C) = [P(a) - P(A \cap B)] + [P(b) - P(A \cap B)]$
$= [0.65 - 0.25] + [0.55 - 0.25] = 0.70$

(c) $P(A \cup B) = 0.95$

(d) $P[(A \cap B)^C] = 1 - P(A \cap B) = 1 - 0.25 = 0.75$

(e) $P\{[(A^C \cap B) \cup (A \cap B^C)]| A \cup B\} = \dfrac{P[(A^C \cap B) \cup (A \cap B^C)]}{P(A \cup B)} = 0.70/0.95 = 70/95$

(f) $P(A \cap B)| A \cup B) = P(A \cap B)/P(A \cup B) = 0.25/0.95 = 25/95$

(g) $P(B|A^C) = P(A^C \cap B)/P(A^C)\ ] = [P(b) - P(A \cap B)]/[1 - P(a)] = [0.55 - 0.25]/[1 - 0.65]$
$= 30/35$

2.4.12 $P(\text{No. of heads} \geq 2| \text{ No. of heads} \leq 2)$
$= P(\text{No. of heads} \geq 2 \text{ and No. of heads} \leq 2)/P(\text{No. of heads} \leq 2)$
$= P(\text{No. of heads} = 2)/P(\text{No. of heads} \leq 2) = (3/8)/(7/8) = 3/7$

2.4.13 $P(\text{first die} \geq 4 | \text{sum} = 8) = P(\text{first die} \geq 4 \text{ and sum} = 8)/P(\text{sum} = 8)$
$= P(\{(4, 4), (5, 3), (6, 2)\}/P(\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}) = 3/5$

2.4.14 There are 4 ways to choose three aces (count which one is left out). There are 48 ways to choose the card that is not an ace, so there are $4 \times 48 = 192$ sets of cards where exactly three are aces. That gives 193 sets where there are at least three aces. The conditional probability is (1/270,725)/(193/270,725) = 1/193.

2.4.15 First note that $P(A \cup B) = 1 - P[(A \cup B)^C] = 1 - 0.2 = 0.8$.
Then $P(\text{b}) = P(A \cup B) - P(A \cap B^C) - P(A \cap B) = 0.8 - 0.3 - 0.1 = 0.5$. Finally
$P(A|B) = P(A \cap B)/P(\text{b}) = 0.1/0.5 = 1/5$

2.4.16 $P(A|B) = 0.5$ implies $P(A \cap B) = 0.5P(\text{b})$. $P(B|A) = 0.4$ implies $P(A \cap B) = (0.4)P(\text{a})$.
Thus, $0.5P(\text{b}) = 0.4P(\text{a})$ or $P(\text{b}) = 0.8P(\text{a})$.
Then, $0.9 = P(\text{a}) + P(\text{b}) = P(\text{a}) + 0.8P(\text{a})$ or $P(\text{a}) = 0.9/1.8 = 0.5$.

2.4.17 $P[(A \cap B)^C] = P[(A \cup B)^C] + P(A \cap B^C) + P(A^C \cap B) = 0.2 + 0.1 + 0.3 = 0.6$
$P(A \cup B|(A \cap B)^C) = P[(A \cap B^C) \cup (A^C \cap B)]/P((A \cap B)^C) = [0.1 + 0.3]/0.6 = 2/3$

2.4.18 $P(\text{sum} \geq 8 | \text{at least one die shows 5})$
$= P(\text{sum} \geq 8 \text{ and at least one die shows 5})/P(\text{at least one die shows 5})$
$= P(\{(5, 3), (5, 4), (5, 6), (3, 5), (4, 5), (6, 5), (5, 5)\})/(11/36) = 7/11$

2.4.19 *P*(Outandout wins|Australian Doll and Dusty Stake don't win)
= *P*(Outandout wins and Australian Doll and Dusty Stake don't win)/*P*(Australian Doll and Dusty Stake don't win) = 0.20/0.55 = 20/55

2.4.20 Suppose the guard will randomly choose to name Bob or Charley if they are the two to go free. Then the probability the guard will name Bob, for example, is
P(Andy, Bob) + (1/2)*P*(Bob, Charley) = 1/3 + (1/2)(1/3) = 1/2.
The probability Andy will go free given the guard names Bob is *P*(Andy, Bob)/*P*(Guard names Bob) = (1/3)/(1/2) = 2/3. A similar argument holds for the guard naming Charley. Andy's concern is not justified.

2.4.21 $P(BBRWW) = P(\text{b})P(B|B)P(R|BB)P(W|BBR)P(W|BBRW) = \frac{4}{15} \cdot \frac{3}{14} \cdot \frac{5}{13} \cdot \frac{6}{12} \cdot \frac{5}{11}$
$=.0050$. $P(2, 6, 4, 9, 13) = \frac{1}{15} \cdot \frac{1}{14} \cdot \frac{1}{13} \cdot \frac{1}{12} \cdot \frac{1}{11} = \frac{1}{360{,}360}$.

2.4.22 Let K_i be the event that the ith key tried opens the door, $i = 1, 2, \ldots, n$. Then *P*(door opens first time with 3rd key) = $P(K_1^C \cap K_2^C \cap K_3) = P(K_1^C) \cdot P(K_2^C \mid K_1^C) \cdot P(K_3 \mid K_1^C \cap K_2^C) =$
$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n}$.

2.4.23 (1/52)(1/51)(1/50)(1/49) = 1/6,497,400

2.4.24 (1/2)(1/2)(1/2)(2/3)(3/4) = 1/16

2.4.25 Let A_i be the event "Bearing came from supplier i", i = 1, 2, 3. Let B be the event "Bearing in toy manufacturer's inventory is defective."
Then $P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$ and $P(B|A_1) = 0.02$, $P(B|A_2) = 0.03$, $P(B|A_3) = 0.04$
Combining these probabilities according to Theorem 2.4.1 gives
$P(\text{b}) = (0.02)(0.5) + (0.03)(0.3) + (0.04)(0.2) = 0.027$
meaning that the manufacturer can expect 2.7% of her ball-bearing stock to be defective.

2.4.26 Let B be the event that the face (or sum of faces) equals 6. Let A_1 be the event that a Head comes up and A_2, the event that a Tail comes up. Then $P(\text{b}) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$
$= \frac{1}{6}\cdot\frac{1}{2}+\frac{5}{36}\cdot\frac{1}{2} = 0.15$.

2.4.27 Let B be the event that the countries go to war. Let A be the event that terrorism increases. Then $P(\text{b}) = P(B|A)P(\text{a}) + P(B|A^C)P(A^C) = (0.65)(0.30) + (0.05)(0.70) = 0.23$.

2.4.28 Let B be the event that a donation is received; let A_1, A_2, and A_3 denote the events that the call is placed to Belle Meade, Oak Hill, and Antioch, respectively.
Then $P(\text{b}) = \sum_{i=1}^{3} P(B|A_i)P(A_i) = (0.60)\cdot\frac{1000}{4000}+(0.55)\cdot\frac{1000}{4000}+(0.35)\cdot\frac{2000}{4000} = 0.46$.

2.4.29 Let B denote the event that the person interviewed answers truthfully, and let A be the event that the person interviewed is a man.
Then $P(\text{b}) = P(B|A)P(\text{a}) + P(B|A^C)P(A^C) = (0.78)(0.47) + (0.63)(0.53) = 0.70$.

2.4.30 Let B be the event that a red chip is ultimately drawn from Urn I. Let A_{RW}, for example, denote the event that a red is transferred from Urn I and a white is transferred from Urn II. Then
$P(\text{b}) = P(B|A_{RR})P(A_{RR}) + P(B|A_{RW})P(A_{RW}) + P(B|A_{WR})P(A_{WR}) + P(B|A_{WW})P(A_{WW})$
$= \frac{3}{4}\left(\frac{3}{4}\cdot\frac{2}{4}\right)+\frac{2}{4}\left(\frac{3}{4}\cdot\frac{2}{4}\right)+1\left(\frac{1}{4}\cdot\frac{2}{4}\right)+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{2}{4}\right)=\frac{11}{16}$.

2.4.31 Let B denote the event that someone will test positive, and let A denote the event that someone is infected. Then $P(\text{b}) = P(B|A)P(\text{a}) + P(B|A^C)P(A^C) = (0.999)(0.0001) + (0.0001)(0.9999)$
$= 0.00019989$.

2.4.32 The optimal allocation has 1 white chip in one urn and the other 19 chips (9 white and 10 black) in the other urn. Then $P(\text{white is drawn}) = 1 \cdot \frac{1}{2}+\frac{9}{19}\cdot\frac{1}{2} = 0.74$.

2.4.33 If B is the event that Backwater wins and A is the event that their first-string quarterback plays, then $P(\text{b}) = P(B|A)P(\text{a}) + P(B|A^C)P(A^C) = (0.75)(0.70) + (0.40)(0.30) = 0.645$.

2.4.34 Since the identities of the six chips drawn are not known, their selection does not affect any probability associated with the seventh chip. Therefore,
$P(\text{seventh chip drawn is red}) = \text{P(first chip drawn is red)} = \frac{40}{100}$.

2.4.35 No. Let B denote the event that the person calling the toss is correct. Let A_H be the event that the coin comes up Heads and let A_T be the event that the coin comes up Tails.
Then $P(b) = P(B|A_H)P(A_H) + P(B|A_T)P(A_T) = (0.7)\left(\frac{1}{2}\right) + (0.3)\left(\frac{1}{2}\right) = \frac{1}{2}$.

2.4.36 Let B be the event of a guilty verdict; let A be the event that the defense can discredit the police. Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = 0.15(0.70) + 0.80(0.30) = 0.345$

2.4.37 Let A_1 be the event of a 3.5-4.0 GPA; A_2, of a 3.0-3.5 GPA; and A_3, of a GPA less than 3.0. If B is the event of getting into medical school, then
$P(b) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = (0.8)(0.25) + (0.5)(0.35) + (0.1)(0.40)$
$= 0.415$

2.4.38 Let B be the event of early release; let A be the event that the prisoner is related to someone on the governor's staff.
Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.90)(0.40) + (0.01)(0.60) = 0.366$

2.4.39 Let A_1 be the event of being a Humanities major; A_2, of being a Natural Science major; A_3, of being a History major; and A_4, of being a Social Science major. If B is the event of a male student, then $P(b) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$
$= (0.40)(0.4) + (0.85)(0.1) + (0.55)(0.3) + (0.25)(0.2) = 0.46$

2.4.40 Let B denote the event that the chip drawn from Urn II is red; let A_R and A_W denote the events that the chips transferred are red and white, respectively.
Then $P(A_W \mid B) = \dfrac{P(B \mid A_W)P(A_W)}{P(B \mid A_R)P(A_R) + P(B \mid A_W)P(A_W)} = \dfrac{(2/4)(2/3)}{(3/4)(1/3)+(2/4)(2/3)} = \dfrac{4}{7}$

2.4.41 Let A_i be the event that Urn i is chosen, i = I, II, III. Then, $P(A_i) = 1/3$, i = I, II, III. Suppose B is the event a red chip is drawn. Note that $P(B|A_1) = 3/8$, $P(B|A_2) = 1/2$ and $P(B|A_3) = 5/8$.

$$P(A_3 \mid B) = \frac{P(B \mid A_3)P(A_3)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3)}$$

$$= \frac{(5/8)(1/3)}{(3/8)(1/3) + (1/2)(1/3) + (5/8)(1/3)} = 5/12.$$

2.4.42 If B is the event that the warning light flashes and A is the event that the oil pressure is low, then

$$P(A|B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^C)P(A^C)} = \frac{(0.99)(0.10)}{(0.99)(0.10) + (0.02)(0.90)} = 0.85$$

2.4.43 Let B be the event that the basement leaks, and let A_T, A_W, and A_H denote the events that the house was built by Tara, Westview, and Hearthstone, respectively. Then $P(B|A_T) = 0.60$, $P(B|A_W) = 0.50$, and $P(B|A_H) = 0.40$. Also, $P(A_T) = 2/11$, $P(A_W) = 3/11$, and $P(A_H) = 6/11$. Applying Bayes' rule to each of the builders shows that $P(A_T|B) = 0.24$, $P(A_W|B) = 0.29$, and $P(A_H|B) = 0.47$, implying that Hearthstone is the most likely contractor.

2.4.44 Let B denote the event that Francesca passed, and let A_X and A_Y denote the events that she was enrolled in Professor X's section and Professor Y's section, respectively. Since $P(B|A_X) = 0.85$, $P(B|A_Y) = 0.60$, $P(A_X) = 0.4$, and $P(A_Y) = 0.6$,

$$P(A_X|B) = \frac{(0.85)(0.4)}{(0.85)(0.4)+(0.60)(0.6)} = 0.486$$

2.4.45 Let B denote the event that a check bounces, and let A be the event that a customer wears sunglasses. Then $P(B|A) = 0.50$, $P(B|A^C) = 1 - 0.98 = 0.02$, and P(**a**) = 0.10, so

$$P(A|B) = \frac{(0.50)(0.10)}{(0.50)(0.10)+(0.02)(0.90)} = 0.74$$

2.4.46 Let B be the event that Basil dies, and define A_1, A_2, and A_3 to be the events that he ordered cherries flambe, chocolate mousse, or no dessert, respectively. Then $P(B|A_1) = 0.60$, $P(B|A_2) = 0.90$, $P(B|A_3) = 0$, $P(A_1) = 0.50$, $P(A_2) = 0.40$, and $P(A_3) = 0.10$. Comparing $P(A_1|B)$ and $P(A_2|B)$ suggests that Margo should be considered the prime suspect:

$$P(A_1|B) = \frac{(0.60)(0.50)}{(0.60)(0.50)+(0.90)(0.40)+(0)(0.10)} = 0.45$$

$$P(A_2|B) = \frac{(0.90)(0.40)}{(0.60)(0.50)+(0.90)(0.40)+(0)(0.10)} = 0.55$$

2.4.47 Define B to be the event that Josh answers a randomly selected question correctly, and let A_1 and A_2 denote the events that he was 1) unprepared for the question and 2) prepared for the question, respectively. Then $P(B|A_1) = 0.20$, $P(B|A_2) = 1$, $P(A_2) = p$, $P(A_1) = 1 - p$, and

$$P(A_2|B) = 0.92 = \frac{P(B\mid A_2)P(A_2)}{P(B\mid A_1)P(A_1)+P(B\mid A_2)P(A_2)} = \frac{1\cdot p}{(0.20)(1-p)+(1\cdot p)}$$

which implies that $p = 0.70$ (meaning that Josh was prepared for (0.70)(20) = 14 of the questions).

2.4.48 Let B denote the event that the program diagnoses the child as abused, and let A be the event that the child <u>is</u> abused. Then P(a) = 1/90, $P(B|A) = 0.90$, and $P(B|A^C) = 0.03$, so

$$P(A|B) = \frac{(0.90)(1/90)}{(0.90)(1/90)+(0.03)(89/90)} = 0.25$$

If P(a) = 1/1000, $P(A|B) = 0.029$; if P(a) = 1/50, $P(A|B) = 0.38$.

2.4.49 Let A_1 be the event of being a Humanities major; A_2, of being a History and Culture major; and A_3, of being a Science major. If B is the event of being a woman, then

$$P(A_2|B) = \frac{(0.45)(0.5)}{(0.75)(0.3)+(0.45)(0.5)+(0.30)(0.2)} = 225/510$$

2.4.50 Let B be the event that a 1 is received. Let A be the event that a 1 was sent. Then

$$P(A^C|B) = \frac{(0.10)(0.3)}{(0.95)(0.7)+(0.10)(0.3)} = 30/695$$

2.4.51 Let B be the event that Zach's girlfriend responds promptly. Let A be the event that Zach sent an e-mail, so A^C is the event of leaving a message. Then

$$P(A|B) = \frac{(0.8)(2/3)}{(0.8)(2/3)+(0.9)(1/3)} = 16/25$$

2.4.52 Let A be the event that the shipment came from Warehouse A with events B and C defined similarly. Let D be the event of a complaint.

$$P(C|D) = \frac{P(D\,|\,C)P(C)}{P(D\,|\,A)P(A)+P(D\,|\,B)P(B)+P(D\,|\,C)P(C)}$$

$$= \frac{(0.02)(0.5)}{(0.03)(0.3)+(0.05)(0.2)+(0.02)(0.5)} = 10/29$$

2.4.53 Let A_i be the event that Drawer i is chosen, i, = 1, 2, 3. If B is the event a silver coin is selected, then $P(A_3|B) = \frac{(0.5)(1/3)}{(0)(1/3)+(1)(1/3)+(0.5)(1/3)} = 1/3$

Section 2.5: Independence

2.5.1 **(a)** No, because $P(A \cap B) > 0$.
(b) No, because $P(A \cap B) = 0.2 \neq P(\text{a}) \cdot P(\text{b}) = (0.6)(0.5) = 0.3$
(c) $P(A^C \cup B^C) = P((A \cap B)^C) = 1 - P(A \cap B) = 1 - 0.2 = 0.8$.

2.5.2 Let C and M be the events that Spike passes chemistry and mathematics, respectively. Since $P(C \cap M) = 0.12 \neq P(\text{c}) \cdot P(M) = (0.35)(0.40) = 0.14$, C and M are not independent.
P(Spike fails both) = 1 − P(Spike passes at least one) = $1 - P(C \cup M)$
$= 1 - [P(\text{c}) + P(M) - P(C \cap M)] = 0.37$.

2.5.3 P(one face is twice the other face) = P((1, 2), (2, 1), (2, 4), (4, 2), (3, 6), (6, 3)) = $\frac{6}{36}$.

2.5.4 Let R_i, B_i, and W_i be the events that red, black, and white chips are drawn from urn i, i = 1, 2. Then P(both chips drawn are same color) = $P((R_1 \cap R_2) \cup (B_1 \cap B_2) \cup (W_1 \cap W_2))$
$= P(R_1) \cdot P(R_2) + P(B_1) \cdot P(B_2) + P(W_1) \cdot P(W_2)$ [because the intersections are mutually exclusive and the individual draws are independent]. But $P(R_1) \cdot P(R_2) + P(B_1) \cdot P(B_2) + P(W_1) \cdot P(W_2)$

$$= \left(\frac{3}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{4}{9}\right)+\left(\frac{5}{10}\right)\left(\frac{3}{9}\right) = 0.32.$$

2.5.5 P(Dana wins at least 1 game out of 2) = 0.3, which implies that P(Dana loses 2 games out of 2) = 0.7. Therefore, P(Dana wins at least 1 game out of 4) = 1 − P(Dana loses all 4 games)
= 1 − P(Dana loses first 2 games <u>and</u> Dana loses second 2 games) = 1 − (0.7)(0.7) = 0.51.

2.5.6 Six equally-likely orderings are possible for any set of three distinct random numbers: $x_1 < x_2 < x_3$, $x_1 < x_3 < x_2$, $x_2 < x_1 < x_3$, $x_2 < x_3 < x_1$, $x_3 < x_1 < x_2$, and $x_3 < x_2 < x_1$. By inspection, $P(\text{a}) = \frac{2}{6}$, and $P(\text{b}) = \frac{1}{6}$, so $P(A \cap B) = P(\text{a}) \cdot P(\text{b}) = \frac{1}{18}$.

2.5.7 **(a)** 1. $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = 1/4 + 1/8 + 0 = 3/8$
2. $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(\text{a})P(\text{b}) = 1/4 + 1/8 - (1/4)(1/8) = 11/32$

(b) 1. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{P(B)} = 0$

2. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)} = P(A) = 1/4$

2.5.8 **(a)** $P(A \cup B \cup C) = P(\text{a}) + P(\text{b}) + P(\text{c}) - P(\text{a})P(\text{b}) - P(\text{a})P(\text{c}) - P(\text{b})P(\text{c}) + P(\text{a})P(\text{b})P(\text{c})$
(b) $P(A \cup B \cup C) = 1 - P[(A \cup B \cup C)^C] = 1 - P(A^C \cap B^C \cap C^C) = 1 - P(A^C)P(B^C)P(C^C)$

2.5.9 Let A_i be the event of i heads in the first two tosses, $i = 0, 1, 2$. Let B_i be the event of i heads in the last two tosses, $i = 0, 1, 2$. The A's and B's are independent. The event of interest is $(A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (A_2 \cap B_2)$ and $P[(A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (A_2 \cap B_2)]$
$= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) = (1/4)(1/4) + (1/2)(1/2) + (1/4)(1/4) = 6/16$

2.5.10 A and B are disjoint, so they cannot be independent.

2.5.11 Equation 2.5.3: $P(A \cap B \cap C) = P(\{1, 3)\}) = 1/36 = (2/6)(3/6)(6/36) = P(\text{a})P(\text{b})P(\text{c})$
Equation 2.5.4: $P(B \cap C) = P(\{1, 3), (5,6)\}) = 2/36 \neq (3/6)(6/36) = P(\text{b})P(\text{c})$

2.5.12 Equation 2.5 3: $P(A \cap B \cap C) = P(\{2, 4, 10, 12)\}) = 4/36 \neq (1/2)(1/2)(1/2) = P(\text{a})P(\text{b})P(\text{c})$
Equation 2.5.4: $P(A \cap B) = P(\{2, 4, 10, 12, 24, 26, 32, 34, 36)\}) = 9/36 = 1/4 = (1/2)(1/2)$
$= P(\text{a})P(\text{b})$
$P(A \cap C) = P(\{1, 2, 3, 4, 5, 10, 11, 12, 13)\}) = 9/36 = 1/4 = (1/2)(1/2) = P(\text{a})P(\text{c})$
$P(B \cap C) = P(\{2, 4, 6, 8, 10, 12, 14, 16, 18)\}) = 9/36 = 1/4 = (1/2)(1/2) = P(\text{a})P(\text{c})$

2.5.13 11 [= 6 verifications of the form $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ + 4 verifications of the form $P(A_i \cap A_j \cap A_k) = P(A_i) \cdot P(A_j) \cdot P(A_k)$ + 1 verification that $P(A_1 \cap A_2 \cap A_3 \cap A_4)$
$= P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4)$].

2.5.14 $P(\text{a}) = \frac{3}{6}$, $P(\text{b}) = \frac{2}{6}$, $P(\text{c}) = \frac{6}{36}$, $P(A \cap B) = \frac{6}{36}$, $P(A \cap C) = \frac{3}{36}$, $P(B \cap C) = \frac{2}{36}$, and $P(A \cap B \cap C) = \frac{1}{36}$. It follows that A, B, and C are mutually independent because $P(A \cap B \cap C) = \frac{1}{36} = P(\text{a}) \cdot P(\text{b}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{6}{36}$, $P(A \cap B) = \frac{6}{36} = P(\text{a}) \cdot P(\text{b}) = \frac{3}{6} \cdot \frac{2}{6}$, $P(A \cap C) = \frac{3}{36} = P(\text{a}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{6}{36}$, and $P(B \cap C) = \frac{2}{36} = P(\text{b}) \cdot P(\text{c}) = \frac{2}{6} \cdot \frac{6}{36}$.

2.5.15 $P(A \cap B \cap C) = 0$ (since the sum of two odd numbers is necessarily even) $\neq P(\text{a}) \cdot P(\text{b}) \cdot P(\text{c}) > 0$, so A, B, and C are not mutually independent. However, $P(A \cap B) = \frac{9}{36} = P(\text{a}) \cdot P(\text{b}) = \frac{3}{6} \cdot \frac{3}{6}$, $P(A \cap C) = \frac{9}{36} = P(\text{a}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{18}{36}$, and $P(B \cap C) = \frac{9}{36} = P(\text{b}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{18}{36}$, so A, B, and C are pairwise independent.

2.5.16 Let R_i and G_i be the events that the ith light is red and green, respectively, $i = 1, 2, 3, 4$. Then $P(R_1) = P(R_2) = \frac{1}{3}$ and $P(R_3) = P(R_4) = \frac{1}{2}$. Because of the considerable distance between the intersections, what happens from light to light can be considered independent events. P(driver stops at least 3 times) = P(driver stops exactly 3 times) + P(driver stops all 4 times)
$= P((R_1 \cap R_2 \cap R_3 \cap G_4) \cup (R_1 \cap R_2 \cap G_3 \cap R_4) \cup (R_1 \cap G_2 \cap R_3 \cap R_4) \cup (G_1 \cap R_2 \cap R_3 \cap R_4) \cup (R_1 \cap R_2 \cap R_3 \cap R_4)) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
$+ \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{7}{36}$.

2.5.17 Let M, L, and G be the events that a student passes the mathematics, language, and general knowledge tests, respectively. Then $P(M) = \frac{6175}{9500}$, $P(L) = \frac{7600}{9500}$, and $P(\text{g}) = \frac{8075}{9500}$.
P(student fails to qualify) = P(student fails at least one exam)
= 1 − P(student passes all three exams) $= 1 - P(M \cap L \cap G) = 1 - P(M) \cdot P(L) \cdot P(\text{g}) = 0.56$.

2.5.18 Let A_i denote the event that switch A_i closes, $i = 1, 2, 3, 4$. Since the A_i's are independent events, P(circuit is completed) $= P((A_1 \cap A_2) \cup (A_3 \cap A_4)) = P(A_1 \cap A_2) + P(A_3 \cap A_4) - P((A_1 \cap A_2) \cap (A_3 \cap A_4)) = 2p^2 - p^4$.

2.5.19 Let p be the probability of having a winning game card.
Then 0.32 = P(winning at least once in 5 tries) = 1 − P(not winning in 5 tries) $= 1 - (1 - p)^5$, so $p = 0.074$

2.5.20 Let A_H, A_T, B_H, B_T, C_H, and C_T denote the events that players A, B, and C throw heads and tails on individual tosses. Then P(A throws first head) $= P(A_H \cup (A_T \cap B_T \cap C_T \cap A_H) \cup \cdots)$
$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{8}\right)^2 + \cdots = \frac{1}{2}\left(\frac{1}{1 - 1/8}\right) = \frac{4}{7}$. Similarly, P(B throws first head)
$= P((A_T \cap B_H) \cup (A_T \cap B_T \cap C_T \cap A_T \cap B_H) \cup \ldots) = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{8}\right) + \frac{1}{4}\left(\frac{1}{8}\right)^2 + \ldots = \frac{1}{4}\left(\frac{1}{1 - 1/8}\right) = \frac{2}{7}$.
P(C throws first head) $= 1 - \frac{4}{7} - \frac{2}{7} = \frac{1}{7}$.

2.5.21 P(at least one child becomes adult) = 1 − P(no child becomes adult) $= 1 - 0.2^n$.
Then $1 - 2^n \geq 0.75$ implies $n \geq \frac{\ln 0.25}{\ln 0.2}$ or $n \geq 0.86$, so take $n = 1$.

2.5.22 P(at least one viewer can name actor) = 1 − P(no viewer can name actor) = 1 − $(0.85)^{10}$ = 0.80.
P(exactly one viewer can name actor) = 10 (0.15) $(0.85)^9$ = 0.347.

2.5.23 Let B be the event that no heads appear, and let A_i be the event that i coins are tossed, i = 1, 2, …, 6. Then $P(b) = \sum_{i=1}^{6} P(B \mid A_i)P(A_i) = \frac{1}{2}\left(\frac{1}{6}\right)+\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)+\ldots+\left(\frac{1}{2}\right)^6\left(\frac{1}{6}\right)=\frac{63}{384}$.

2.5.24 P(at least one red chip is drawn from at least one urn) = 1 − P(all chips drawn are white)
$= 1-\left(\frac{4}{7}\right)^r\cdot\left(\frac{4}{7}\right)^r\cdots\left(\frac{4}{7}\right)^r = 1-\left(\frac{4}{7}\right)^{rm}$.

2.5.25 P(at least one double six in n throws) = 1 − P(no double sixes in n throws) = $1-\left(\frac{35}{36}\right)^n$. By trial and error, the smallest n for which P(at least one double six in n throws) exceeds 0.50 is 25 [$1-\left(\frac{35}{36}\right)^{24}$ = 0.49; $1-\left(\frac{35}{36}\right)^{25}$ = 0.51].

2.5.26 Let A be the event that a sum of 8 appears before a sum of 7. Let B be the event that a sum of 8 appears on a given roll and let C be the event that the sum appearing on a given roll is neither 7 nor 8. Then $P(b) = \frac{5}{36}$, $P(c) = \frac{25}{36}$, and $P(a) = P(b) + P(c)P(b) + P(c)P(c)P(b)$
$+ \cdots = \frac{5}{36}+\frac{25}{36}\frac{5}{36}+\left(\frac{25}{36}\right)^2\frac{5}{36}+\cdots=\frac{5}{36}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^k=\frac{5}{36}\left(\frac{1}{1\,-\,25/36}\right)=\frac{5}{11}$.

2.5.27 Let W, B, and R denote the events of getting a white, black and red chip, respectively, on a given draw. Then P(white appears before red) = $P(W \cup (B \cap W) \cup (B \cap B \cap W) \cup \cdots)$
$= \frac{w}{w+b+r}+\frac{b}{w+b+r}\cdot\frac{w}{w+b+r}+\left(\frac{b}{w+b+r}\right)^2\cdot\frac{w}{w+b+r}+\cdots$
$= \frac{w}{w+b+r}\cdot\left(\frac{1}{1\,-\,b/(w+b+r)}\right)=\frac{w}{w+r}$.

2.5.28 $P(B|A_1) = 1 - P$(all m I-teams fail) $= 1-(1-r)^m$; similarly, $P(B|A_2) = 1-(1-r)^{n-m}$. From Theorem 2.4.1, $P(b) = [1-(1-r)^m]p + [1-(1-r)^{n-m}](1-p)$. Treating m as a continuous variable and differentiating $P(b)$ gives
$\frac{dP(B)}{dm} = -p(1-r)^m\cdot\ln(1-r) + (1-p)(1-r)^{n-m}\cdot\ln(1-r)$. Setting $\frac{dP(B)}{dm} = 0$ implies that
$m = \frac{n}{2}+\frac{\ln[(1-p)/p]}{2\ln(1-r)}$.

2.5.29 P(at least one four) = 1 − P(no fours) = $1-(0.9)^n$. $1-(0.9)^n \geq 0.7$ implies $n = 12$

2.5.30 Let B be the event that all *n* tosses come up heads. Let A_1 be the event that the coin has two heads, and let A_2 be the event the coin is fair. Then

$$P(A_2 \mid B) = \frac{(1/2)^n(8/9)}{1(1/9)+(1/2)^n(8/9)} = \frac{8(1/2)^n}{1+8(1/2)^n}$$

By inspection, the limit of $P(A_2 \mid B)$ as n goes to infinity is 0.

Section 2.6: Combinatorics

2.6.1 $2 \cdot 3 \cdot 2 \cdot 2 = 24$

2.6.2 $20 \cdot 9 \cdot 6 \cdot 20 = 21{,}600$

2.6.3 $3 \cdot 3 \cdot 5 = 45$. Included will be aeu and cdx.

2.6.4 **(a)** $26^2 \cdot 10^4 = 6{,}760{,}000$
(b) $26^2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3{,}407{,}040$
(c) The total number of plates *with* four zeros is $26 \cdot 26$, so the total number *not* having four zeros must be $26^2 \cdot 10^4 - 26^2 = 6{,}759{,}324$.

2.6.5 There are 9 choices for the first digit (1 through 9), 9 choices for the second digit (0 + whichever eight digits are not appearing in the hundreds place), and 8 choices for the last digit. The number of admissible integers, then, is $9 \cdot 9 \cdot 8 = 648$. For the integer to be odd, the last digit must be either 1, 3, 5, 7, or 9. That leaves 8 choices for the first digit and 8 choices for the second digit, making a total of 320 (= $8 \cdot 8 \cdot 5$) odd integers.

2.6.6 For each topping, the customer has 2 choices: "add" or "do not add." The eight available toppings, then, can produce a total of $2^8 = 256$ different hamburgers.

2.6.7 The bases can be occupied in any of 2^7 ways (each of the seven can be either "empty" or "occupied"). Moreover, the batter can come to the plate facing any of five possible "out" situations (0 through 4). It follows that the number of base-out configurations is $5 \cdot 2^7$, or 640.

2.6.8 With 4 choices for the first digit, 1 for the third digit, 5 for the last digit, and 10 for each of the remaining six digits, the total number of admissible zip codes is 20,000,000(= $4 \cdot 10^6 \cdot 1 \cdot 5$).

2.6.9 $4 \cdot 14 \cdot 6 + 4 \cdot 6 \cdot 5 + 14 \cdot 6 \cdot 5 + 4 \cdot 14 \cdot 5 = 1156$

2.6.10 There are two mutually exclusive sets of ways for the black and white keys to alternate—the black keys can be 1st, 3rd, 5th, and 7th notes in the melody, or the 2nd, 4th 6th, and 8th. Since there are 5 black keys and 7 white keys, there are $5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7$ variations in the first set and $7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5$ in the second set. The total number of alternating melodies is the sum $5^4 7^4 + 7^4 5^4 = 3{,}001{,}250$.

2.6.11 The number of usable garage codes is $2^8 - 1 = 255$, because the "combination" where none of the buttons is pushed is inadmissible (recall Example 2.6.3). Five additional families can be added before the eight-button system becomes inadequate.

2.6.12 4, because $2^1 + 2^2 + 2^3 < 26$ but $2^1 + 2^2 + 2^3 + 2^4 \geq 26$.

2.6.13 In order to exceed 256, the binary sequence of coins must have a head in the ninth position and at least one head somewhere in the first eight tosses. The number of sequences satisfying those conditions is $2^8 - 1$, or 255. (The "1" corresponds to the sequences TTTTTTTTH, whose value would not exceed 256.)

2.6.14 There are 3 choices for the vowel and 4 choices for the consonant, so there are $3 \cdot 4 = 12$ choices, if order doesn't matter. If we are taking ordered arrangements, then there are 24 ways, since each unordered selection can be written vowel first or consonant first.

2.6.15 There are $1 \cdot 3$ ways if the ace of clubs is the first card and $12 \cdot 4$ ways if it is not. The total is then $3 + 12 \cdot 4 = 51$

2.6.16 Monica has $3 \cdot 5 \cdot 2 = 30$ routes from Nashville to Anchorage, so there are $30 \cdot 30 = 900$ choices of round trips.

2.6.17 ${}_6P_3 = 6 \cdot 5 \cdot 4 = 120$

2.6.18 ${}_4P_4 = 4! = 24$; ${}_2P_2 \cdot {}_2P_2 = 4$

2.6.19 $\log_{10}(30!) \doteq \log_{10}\left(\sqrt{2\pi}\right) + \left(30 + \frac{1}{2}\right)\log_{10}(30) - 30\log_{10}e = 32.42246$, which implies that $30! \doteq 10^{32.42246} = 2.645 \times 10^{32}$.

2.6.20 ${}_9P_9 = 9! = 362{,}880$

2.6.21 There are 2 choices for the first digit, 6 choices for the middle digit, and 5 choices for the last digit, so the number of admissible integers that can be formed from the digits 1 through 7 is 60 $(= 2 \cdot 6 \cdot 5)$.

2.6.22 **(a)** ${}_8P_8 = 8! = 40{,}320$

(b) The men can be arranged in, say, the odd-numbered chairs in ${}_4P_4$ ways; for each of those permutations, the women can be seated in the even-numbered chairs in ${}_4P_4$ ways. But the men could also be in the even-numbered chairs. It follows that the total number of alternating seating arrangements is ${}_4P_4 \cdot {}_4P_4 + {}_4P_4 \cdot {}_4P_4 = 1152$.

2.6.23 There are 4 different sets of three semesters in which the electives could be taken. For each of those sets, the electives can be selected and arranged in ${}_{10}P_3$ ways, which means that the number of possible schedules is $4 \cdot {}_{10}P_3$, or 2880.

2.6.24 ${}_6P_6 = 720$; ${}_6P_6 \cdot {}_6P_6 = 518{,}400$; $6!6!2^6$ is the number of ways six male/female cheerleading teams can be positioned along a sideline if each team has the option of putting the male in front or the female in front; $6!6!2^6 2^{12}$ is the number of arrangements subject to the conditions of the previous answer but with the additional option that each cheerleader can face either forwards or backwards.

2.6.25 The number of playing sequences where at least one side is out of order = total number of playing sequences − number of correct playing sequences = ${}_6P_6 - 1 = 719$.

2.6.26 Within each of the n families, members can be lined up in ${}_mP_m = m!$ ways. Since the n families can be permuted in ${}_nP_n = n!$ ways, the total number of admissible ways to arrange the nm people is $n! \cdot (m!)^n$.

2.6.27 There are ${}_2P_2 = 2$ ways for you and a friend to be arranged, ${}_8P_8$ ways for the other eight to be permuted, and six ways for you and a friend to be in consecutive positions in line. By the multiplication rule, the number of admissible arrangements is ${}_2P_2 \cdot {}_8P_8 \cdot 6 = 483{,}840$.

2.6.28 By inspection, ${}_nP_1 = n$. Assume that ${}_nP_k = n(n-1) \cdots (n-k+1)$ is the number of ways to arrange k distinct objects without repetition. Notice that $n - k$ options would be available for a $(k+1)$st object added to the sequences. By the multiplication rule, the number of sequences of length $k + 1$ must be $n(n-1) \cdots (n - \mathrm{k} + 1)(n-k)$. But the latter is the formula for ${}_nP_{k+1}$.

2.6.29 $(13!)^4$

2.6.30 By definition, $(n+1)! = (n+1) \cdot n!$; let $n = 0$.

2.6.31 ${}_9P_2 \cdot {}_4C_1 = 288$

2.6.32 Two people between them: $4 \cdot 2 \cdot 5! = 960$
Three people between them: $3 \cdot 2 \cdot 5! = 720$
Four people between them: $2 \cdot 2 \cdot 5! = 480$
Five people between them: $1 \cdot 2 \cdot 5! = 240$
Total number of ways: 2400

2.6.33 **(a)** $(4!)(5!) = 2880$ **(b)** $6(4!)(5!) = 17,280$
(c) $(4!)(5!) = 2880$ **(d)** $\binom{9}{4}(2)(5!) = 30,240$

2.6.34 TENNESSEE can be permuted in $\frac{9!}{4!2!2!1!} = 3780$ ways;
FLORIDA can be permuted in $7! = 5040$ ways.

2.6.35 If the first digit is a 4, the remaining six digits can be arranged in $\frac{6!}{3!(1!)^3} = 120$ ways; if the first digit is a 5, the remaining six digits can be arranged in $\frac{6!}{2!2!(1!)^2} = 180$ ways. The total number of admissible numbers, then, is $120 + 180 = 300$.

2.6.36 **(a)** $8!/3!3!2! = 560$ **(b)** $8! = 40{,}320$ **(c)** $8!/3!(1!)^5 = 6720$

2.6.37 **(a)** $4! \cdot 3! \cdot 3! = 864$
(b) $3! \cdot 4!3!3! = 5184$ (each of the 3! permutations of the three nationalities can generate 4!3!3! arrangements of the ten people in line)

(c) $10! = 3{,}628{,}800$
(d) $10!/4!3!3! = 4200$

2.6.38 Altogether, the letters in S L U M G U L L I O N can be permuted in $\dfrac{11!}{3!2!(1!)^6}$ ways. The seven consonants can be arranged in $7!/3!(1!)^4$ ways, of which 4! have the property that the three *L*'s come first. By the reasoning used in Example 2.6.13, it follows that the number of admissible arrangements is $4!/(7!/3!) \cdot \dfrac{11!}{3!2!}$, or 95,040.

2.6.39 Imagine a field of 4 entrants (*A*, *B*, *C*, *D*) assigned to positions 1 through 4, where positions 1 and 2 correspond to the opponents for game 1 and positions 3 and 4 correspond to the opponents for game 2. Although the four players can be assigned to the four positions in 4! ways, not all of those permutations yield different tournaments. For example, $\dfrac{B}{1}\dfrac{C}{2}\dfrac{A}{3}\dfrac{D}{4}$ and $\dfrac{A}{1}\dfrac{D}{2}\dfrac{B}{3}\dfrac{C}{4}$ produce the same set of games, as do $\dfrac{B}{1}\dfrac{C}{2}\dfrac{A}{3}\dfrac{D}{4}$ and $\dfrac{C}{1}\dfrac{B}{2}\dfrac{A}{3}\dfrac{D}{4}$. In general, n games can be arranged in $n!$ ways, and the two players in each game can be permuted in 2! ways. Given a field of $2n$ entrants, then, the number of distinct pairings is $(2n)!/n!(2!)^n$, or $1 \cdot 3 \cdot 5 \cdots (2n - 1)$.

2.6.40 Since x^{12} can be the result of the factors $x^6 \cdot x^6 \cdot 1 \cdots 1$ or $x^3 \cdot x^3 \cdot x^3 \cdot x^3 \cdot 1 \cdots 1$ or $x^6 \cdot x^3 \cdot x^3 \cdot 1 \cdots 1$, the analysis described in Example 2.6.16 implies that the coefficient of x^{12} is $\dfrac{18!}{2!16!} + \dfrac{18!}{4!14!} + \dfrac{18!}{1!2!15!} = 5661$.

2.6.41 The letters in E L E E M O S Y N A R Y minus the pair S Y can be permuted in 10!/3! ways. Since S Y can be positioned in front of, within, or behind those ten letters in 11 ways, the number of admissible arrangements is $11 \cdot 10!/3! = 6{,}652{,}800$.

2.6.42 Each admissible spelling of ABRACADABRA can be viewed as a path consisting of 10 steps, five to the right (R) and five to the left (L). Thus, each spelling corresponds to a permutation of the five R's and five L's. There are $\dfrac{10!}{5!5!} = 252$ such permutations.

2.6.43 Six, because the first four pitches must include two balls and two strikes, which can occur in $4!/2!2! = 6$ ways.

2.6.44 $9!/2!3!1!3! = 5040$ (recall Example 2.6.16)

2.6.45 Think of the six points being numbered 1 through 6. Any permutation of three *A*'s and three *B*'s—for example, $\dfrac{A}{1}\dfrac{A}{2}\dfrac{B}{3}\dfrac{B}{4}\dfrac{A}{5}\dfrac{B}{6}$—corresponds to the three vertices chosen for triangle *A* and the three for triangle *B*. It follows that $6!/3!3! = 20$ different sets of two triangles can be drawn.

2.6.46 Consider $k!$ objects categorized into $(k - 1)!$ groups, each group being of size k. By Theorem 2.6.2, the number of ways to arrange the $k!$ objects is $(k!)!/(k!)^{(k-1)!}$, but the latter must be an integer.

2.6.47 There are $\frac{14!}{2!2!1!2!2!3!1!1!}$ total permutations of the letters. There are $\frac{5!}{2!2!1!} = 30$ arrangements of the vowels, only one of which leaves the vowels in their original position. Thus, there are $\frac{1}{30} \cdot \frac{14!}{2!2!1!2!1!3!1!1!} = 30{,}270{,}240$ arrangements of the word leaving the vowels in their original position.

2.6.48 $\frac{15!}{4!3!1!3!1!1!1!1!} = 1, 513, 512, 000$

2.6.49 The three courses with A grades can be: emf, emp, emh, efp, efh, eph, mfp, mfh, mph, fph, or 10 possibilities. From the point of view of Theorem 2.6.2, the grade assignments correspond to the set of permutations of three A's and two B's, which equals $\frac{5!}{3!2!} = 10$.

2.6.50 Since every (unordered) set of two letters describes a different line, the number of possible lines is $\binom{5}{2} = 10$.

2.6.51 To achieve the two-to-one ratio, six pledges need to be chosen from the set of 10 and three from the set of 15, so the number of admissible classes is $\binom{10}{6} \cdot \binom{15}{3} = 95{,}550$.

2.6.52 Of the eight crew members, five need to be on a given side of the boat. Clearly, the remaining three can be assigned to the sides in 3 ways. Moreover, the rowers on each side can be permuted in 4! ways. By the multiplication rule, then, the number of ways to arrange the crew is 1728 (= 3 · 4! · 4!).

2.6.53 **(a)** $\binom{9}{4} = 126$ **(b)** $\binom{5}{2}\binom{4}{2} = 60$ **(c)** $\binom{9}{4} - \binom{5}{4} - \binom{4}{4} = 120$

2.6.54 $\binom{7}{5} = 21$; order does not matter.

2.6.55 Consider a simpler problem: Two teams of two each are to be chosen from a set of four players—*A*, *B*, *C*, and *D*. Although a single team can be chosen in $\binom{4}{2}$ ways, the number of *pairs* of teams is only $\binom{4}{2}\Big/2$, because [(*A B*), (*C D*)] and [(*C D*), (*A B*)] would correspond to the same matchup. Applying that reasoning here means that the ten players can split up in $\binom{10}{5}\Big/2 = 126$ ways.

2.6.56 Number the spaces between the twenty pages from 1 to 19. Choosing any two of these spaces partitions the reading assignment into three non-zero, numbers, x_1, x_2, and x_3, corresponding to the numbers of pages read on Monday, Tuesday, and Wednesday, respectively. Therefore, the number of ways to complete the reading assignment is $\binom{19}{2} = 171$.

2.6.57 The four *I*'s need to occupy any of the $\binom{8}{4}$ sets of four spaces between and around the other seven letters. Since the latter can be permuted in $\frac{7!}{2!4!1!}$ ways, the total number of admissible arrangements is $\binom{8}{4}\cdot\frac{7!}{2!4!1!} = 7350$.

2.6.58 Let $x = y = 1$ in the expansion $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. The total number of hamburgers referred to in Question 2.6.6 ($= 2^8$) must also be equal to the number of ways to choose k condiments, $k =$ 0, 1, 2, …, 8—that is, $\binom{8}{0}+\binom{8}{1}+\ldots+\binom{8}{8}$.

2.6.59 Consider the problem of selecting an unordered sample of n objects from a set of $2n$ objects, where the $2n$ have been divided into two groups, each of size n. Clearly, we could choose n from the first group and 0 from the second group, or $n - 1$ from the first group and 1 from the second group, and so on. Altogether, $\binom{2n}{n}$ must equal $\binom{n}{n}\binom{n}{0}+\binom{n}{n-1}\binom{n}{1}+\ldots+\binom{n}{0}\binom{n}{n}$. But $\binom{n}{j}=\binom{n}{n-j}$, $j = 0, 1, \ldots, n$ so $\binom{2n}{n}=\sum_{j=0}^{n}\binom{n}{j}^2$.

2.6.60 Let $x = y = 1$ in the expansion $(x - y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k(-y)^{n-k}$. Then $x - y = 0$ and the sum reduces to $0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}$, or equivalently, $\binom{n}{1}+\binom{n}{3}+\ldots = \binom{n}{0}+\binom{n}{2}+\ldots$.

2.6.61 The ratio of two successive terms in the sequence is $\binom{n}{j+1}\Big/\binom{n}{j}=\frac{n-j}{j+1}$. For small j, $n - j > j + 1$, implying that the terms are increasing. For $j > \frac{n-1}{2}$, though, the ratio is less than 1, meaning the terms are decreasing.

2.6.62 Four months of daily performance create a need for roughly 120 different sets of jokes. If n denotes the number of different jokes that Mitch has to learn, the question is asking for the smallest n for which $\binom{n}{4}\geq 120$. By trial and error, $n = 9$.

2.6.63 Using Newton's binomial expansion, the equation $(1 + t)^d \cdot (1 + t)^e = (1 + t)^{d+e}$ can be written

$$\left(\sum_{j=0}^{d}\binom{d}{j}t^j\right)\cdot\left(\sum_{j=0}^{e}\binom{e}{j}t^j\right)=\sum_{j=0}^{d+e}\binom{d+e}{j}t^j$$

Since the exponent k can arise as $t^0 \cdot t^k$, $t^1 \cdot t^{k-1}$, … , or $t^k \cdot t^0$, it follows that $\binom{d}{0}\binom{e}{k}+\binom{d}{1}\binom{e}{k-1}+\ldots+\binom{d}{k}\binom{e}{0} = \binom{d+e}{k}$. That is, $\binom{d+e}{k} = \sum_{j=0}^{k}\binom{d}{j}\binom{e}{k-j}$.

Section 2.7: Combinatorial Probability

2.7.1 $\binom{7}{2}\binom{3}{2}\Big/\binom{10}{4}$

2.7.2 $P(\text{sum} = 5) = \dfrac{\text{Number of pairs that sum to 5}}{\text{Total number of pairs}} = 2\Big/\binom{6}{2} = \dfrac{2}{15}$.

2.7.3 P(numbers differ by more than 2) = 1 − P(numbers differ by one) − P(numbers differ by 2)

$= 1 - 19\Big/\binom{20}{2} - 18\Big/\binom{20}{2} = \dfrac{153}{190} = 0.81.$

2.7.4 $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = \binom{4}{4}\binom{48}{9}\Big/\binom{52}{13} + \binom{4}{4}\binom{48}{9}\Big/\binom{52}{13} - \binom{4}{4}\binom{4}{4}\binom{44}{5}\Big/\binom{52}{13}$

2.7.5 Let A_1 be the event that an urn with $3W$ and $3R$ is sampled; let A_2 be the event that the urn with $5W$ and $1R$ is sampled. Let B be the event that the three chips drawn are white. By Bayes' rule,

$$P(A_2 \mid B) = \frac{P(B \mid A_2)P(A_2)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2)}$$

$$= \frac{\left[\binom{5}{3}\binom{1}{0}\Big/\binom{6}{3}\right]\cdot(1/10)}{\left[\binom{3}{3}\binom{3}{0}\Big/\binom{6}{3}\right]\cdot(9/10)+\left[\binom{5}{3}\binom{1}{0}\Big/\binom{6}{3}\right]\cdot(1/10)} = \frac{10}{19}$$

2.7.6 $\binom{2}{1}^{50}\Big/\binom{100}{50}$

2.7.7 $6/6^n = 1/6^{n-1}$

2.7.8 There are 6 faces that could be the "three-of-a-kind" and 5 faces that could be the "two-of-a-kind." Moreover, the five dice bearing those two numbers could occur in any of $5!/2!3! = \binom{5}{2}$ orders. It follows that P("full house") $= 6 \cdot 5 \cdot \binom{5}{2} \Big/ 6^5 = 50/6^4$

2.7.9 By Theorem, 2.6.2, the $2n$ grains of sand can be arranged in $(2n)!/n!n!$ ways. Two of those arrangements have the property that the colors will completely separate. Therefore, the probability of the latter is $2(n!)^2/(2n)!$

2.7.10 P(monkey spells CALCULUS) $= 1/[8!/(2!)^3(1!)^2] = 1/5040$;
P(monkey spells ALGEBRA) $= 1/[7!/2!(1!)^5] = 2/5040$.

2.7.11 P(different floors) $= 7!/7^7$; P(same floor) $= 7/7^7 = 1/7^6$. The assumption being made is that all possible departure patterns are equally likely, which is probably not true, since residents living on lower floors would be less inclined to wait for the elevator than would those living on the top floors.

2.7.12 The total number of distinguishable permutations of the phrase is $\dfrac{23!}{2!2!4!2!1!3!2!4!2!2!1!1!}$. The number of permutations where all of the S's are adjacent is counted by treating the S's as a single letter that appears once. The denominator above will have one of the 4! replaced by 1!. The number of such permutations, then, is $\dfrac{23!}{2!2!4!2!1!3!2!1!2!2!1!1!}$. The probability that the S's are adjacent is then the ratio of these two terms or $4!23!/26! = 1/650$. The requested probability is then the complement, 649/650.

2.7.13 The 10 short pieces and 10 long pieces can be lined up in a row in $20!/(10)!(10)!$ ways. Consider each of the 10 pairs of consecutive pieces as defining the reconstructed sticks. Each of those pairs could combine a short piece (S) and a long piece (L) in two ways: SL or LS. Therefore, the number of permutations that would produce 10 sticks, each having a short and a long component is 2^{10}, so the desired probability is $2^{10} \Big/ \binom{20}{10}$.

2.7.14 $6!/6^6$

2.7.15 Any of $\binom{k}{2}$ people could share any of 365 possible birthdays. The remaining $k-2$ people can generate $364 \cdot 363 \cdots (365-k+2)$ sequences of distinct birthdays. Therefore, P(exactly one match) $= \binom{k}{2} \cdot 365 \cdot 364 \cdots (365-k+2)/365^k$.

2.7.16 The expression $\binom{12}{1}\binom{11}{1}\binom{10}{1}$ <u>orders</u> the denominations of the three single cards—in effect, each set of three denominations would be counted 3! times. The denominator (= $\binom{52}{5}$) in that particular probability calculation, though, does not consider the cards to be ordered. To be consistent, the denominations for the three single cards must be treated as a combination, meaning the number of choices is $\binom{12}{3}$.

2.7.17 To get a flush, Dana needs to draw any three of the remaining eleven diamonds. Since only forty-seven cards are effectively left in the deck (others may already have been dealt, but their identities are unknown), P(Dana draws to flush) = $\binom{11}{3}\Big/\binom{47}{3}$.

2.7.18 P(draws to full house or four-of-a-kind) = P(draws to full house) + P(draws to four-of-a-kind) $= \frac{3}{47} + \frac{1}{47} = \frac{4}{47}$.

2.7.19 There are two pairs of cards that would give Tim a straight flush (5 of clubs and 7 of clubs or 7 of clubs and 10 of clubs). Therefore, P(Tim draws to straight flush) = $2\Big/\binom{47}{2}$. A flush, by definition, consists of five cards in the same suit whose denominations are not all consecutive. It follows that P(Tim draws to flush) = $\left[\binom{10}{2} - 2\right]\Big/\binom{47}{2}$, where the "2" refers to the straight flushes cited earlier.

2.7.20 A sum of 48 requires four 10's and an 8 or three 10's and two 9's; a sum of 49 requires four 10's and a 9; no sums higher than 49 are possible. Therefore,
$P(\text{sum} \geq 48) = \left[\binom{4}{4}\binom{4}{1} + \binom{4}{3}\binom{4}{2} + \binom{4}{4}\binom{4}{1}\right]\Big/\binom{52}{5} = 32\Big/\binom{52}{5}$.

2.7.21 $\binom{5}{3}\binom{4}{2}^3\binom{3}{1}\binom{4}{2}\binom{2}{1}\binom{4}{1}\Big/\binom{52}{9}$

2.7.22 $\binom{32}{13}\Big/\binom{52}{13}$

2.7.23 $\left[\binom{2}{1}\binom{2}{1}\right]^4\binom{32}{4}\Big/\binom{48}{12}$

2.7.24 Any permutation of $\frac{n+r}{2}$ steps forward and $\frac{n-r}{2}$ steps backward will result in a net gain of r steps forward. Since the total number of (equally-likely) paths is 2^n,

$$P(\text{conventioneer ends up } r \text{ steps forward}) = \frac{n!\Big/\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)!}{2^n}.$$