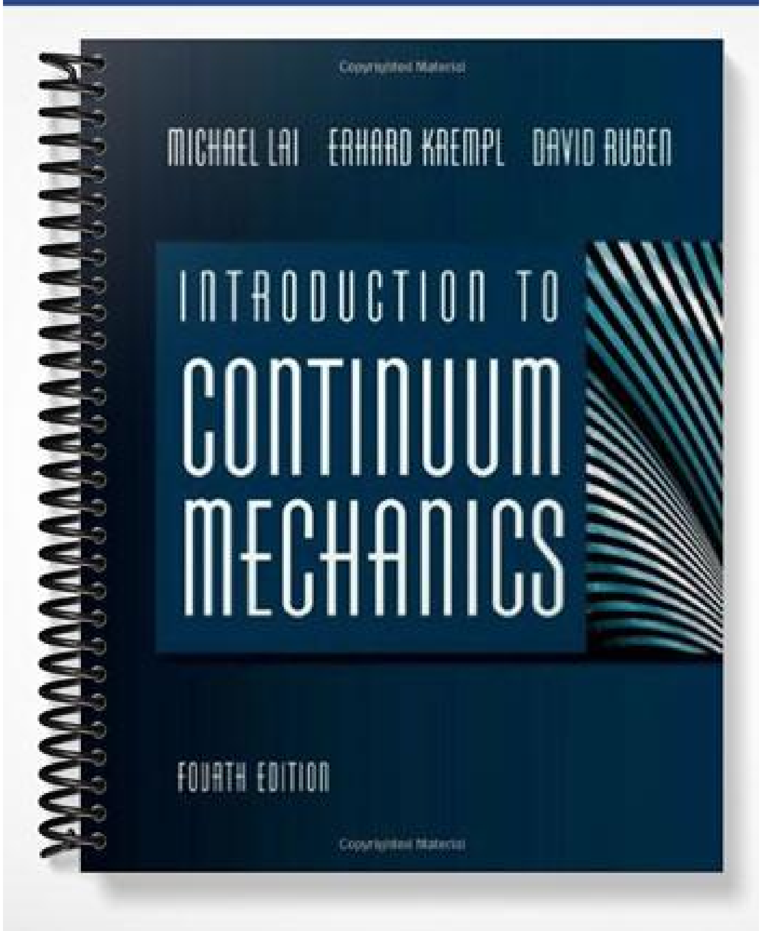
SOLUTIONS MANUAL



CHAPTER 2, PART A

2.1 Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \text{ and } [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$ (e) $a_m a_m$, (f) $S_{mn}a_m a_n$, (g) $S_{nm}a_m a_n$

Ans. (a) $S_{ii} = S_{11} + S_{22} + S_{33} = 1+1+3 = 5$.

(b) $S_{ij}S_{ij} = S_{11}^2 + S_{12}^2 + S_{13}^2 + S_{21}^2 + S_{22}^2 + S_{23}^2 + S_{31}^2 + S_{32}^2 + S_{33}^2 =$

$1+0+4+0+1+4+9+0+9 = 28$.

(c) $S_{ji}S_{ji} = S_{ij}S_{ij} = 28$.

(d) $S_{jk}S_{kj} = S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3}$

$= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{31}S_{13} + S_{32}S_{23} + S_{33}S_{33}$

$= (1)(1) + (0)(0) + (2)(3) + (0)(0) + (1)(1) + (2)(0) + (3)(2) + (0)(2) + (3)(3) = 23$.

(e) $a_m a_m = a_1^2 + a_2^2 + a_3^2 = 1+4+9 = 14$.

(f) $S_{mn}a_m a_n = S_{1n}a_1 a_n + S_{2n}a_2 a_n + S_{3n}a_3 a_n =$

$S_{11}a_1a_1 + S_{12}a_1a_2 + S_{13}a_1a_3 + S_{21}a_2a_1 + S_{22}a_2a_2 + S_{23}a_2a_3 + S_{31}a_3a_1 + S_{32}a_3a_2 + S_{33}a_3a_3$

$= (1)(1)(1) + (0)(1)(2) + (2)(1)(3) + (0)(2)(1) + (1)(2)(2) + (2)(2)(3) + (3)(3)(1)$

$+ (0)(3)(2) + (3)(3)(3) = 1+0+6+0+4+12+9+0+27 = 59$.

(g) $S_{nm}a_m a_n = S_{mn}a_m a_n = 59$.

2.2 Determine which of these equations have an identical meaning with $a_i = Q_{ij}a'_j$.

(a) $a_p = Q_{pm}a'_m$, (b) $a_p = Q_{qp}a'_q$, (c) $a_m = a'_n Q_{mn}$.

Ans. (a) and (c)

2.3 Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Demonstrate the equivalence of the subscripted equations and corresponding matrix equations in the following two problems.

(a) $b_i = B_{ij}a_j$ and $[b] = [B][a]$, (b) $s = B_{ij}a_i a_j$ and $s = [a]^T[B][a]$

Ans. (a)

$b_i = B_{ij}a_j \to b_1 = B_{1j}a_j = B_{11}a_1 + B_{12}a_2 + B_{13}a_3 = (2)(1) + (3)(0) + (0)(2) = 2$

$b_2 = B_{2j}a_j = B_{21}a_1 + B_{22}a_2 + B_{23}a_3 = 2, \quad b_3 = B_{3j}a_j = B_{31}a_1 + B_{32}a_2 + B_{33}a_3 = 2$.

$$[b]=[B][a]=\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}.$$ Thus, $b_i=B_{ij}a_j$ gives the same results as $[b]=[B][a]$

(b)

$s=B_{ij}a_ia_j=B_{11}a_1a_1+B_{12}a_1a_2+B_{13}a_1a_3++B_{21}a_2a_1+B_{22}a_2a_2+B_{23}a_2a_3$

$+B_{31}a_3a_1+B_{32}a_3a_2+B_{33}a_3a_3=(2)(1)(1)+(3)(1)(0)+(0)(1)(2)+(0)(0)(1)$

$+(5)(0)(0)+(1)(0)(2)+(0)(2)(1)+(2)(2)(0)+(1)(2)(2)=2+4=6.$

and $s=[a]^{\mathrm{T}}[B][a]=[1\quad 0\quad 2]\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=[1\quad 0\quad 2]\begin{bmatrix}2\\2\\2\end{bmatrix}=2+4=6.$

2.4 Write in indicial notation the matrix equation (a) $[A]=[B][C]$, (b) $[D]=[B]^{\mathrm{T}}[C]$ and (c) $[E]=[B]^{\mathrm{T}}[C][F]$.

Ans. (a) $[A]=[B][C]\rightarrow A_{ij}=B_{im}C_{mj}$, (b) $[D]=[B]^{\mathrm{T}}[C]\rightarrow A_{ij}=B_{mi}C_{mj}$.

(c) $[E]=[B]^{\mathrm{T}}[C][F]\rightarrow E_{ij}=B_{mi}C_{mk}F_{kj}$.

2.5 Write in indicial notation the equation (a) $s=A_1^2+A_2^2+A_3^2$ and (b) $\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}=0$.

Ans. (a) $s=A_1^2+A_2^2+A_3^2=A_iA_i$. (b) $\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}=0\rightarrow\frac{\partial^2\phi}{\partial x_i\partial x_i}=0$.

2.6 Given that $S_{ij}=a_ia_j$ and $S'_{ij}=a'_ia'_j$, where $a'_i=Q_{mi}a_m$ and $a'_j=Q_{nj}a_n$, and $Q_{ik}Q_{jk}=\delta_{ij}$. Show that $S'_{ii}=S_{ii}$.

Ans. $S'_{ij}=Q_{mi}a_mQ_{nj}a_n=Q_{mi}Q_{nj}a_ma_n\rightarrow S'_{ii}=Q_{mi}Q_{ni}a_ma_n=\delta_{mn}a_ma_n=a_ma_m=S_{mm}=S_{ii}$.

2.7 Write $a_i=\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}$ in long form.

Ans.

$i=1\rightarrow a_1=\frac{\partial v_1}{\partial t}+v_j\frac{\partial v_1}{\partial x_j}=\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}$.

$i=2\rightarrow a_2=\frac{\partial v_2}{\partial t}+v_j\frac{\partial v_2}{\partial x_j}=\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}$.

$i=3\rightarrow a_3=\frac{\partial v_3}{\partial t}+v_j\frac{\partial v_3}{\partial x_j}=\frac{\partial v_3}{\partial t}+v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}$.

2.8 Given that $T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}$, show that

(a) $T_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda\left(E_{kk}\right)^2$ and (b) $T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2)$

Ans. (a)

$$T_{ij}E_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}\delta_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}E_{ii} = 2\mu E_{ij}E_{ij} + \lambda(E_{kk})^2$$

(b)

$$T_{ij}T_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})(2\mu E_{ij} + \lambda E_{kk}\delta_{ij}) = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ij}E_{kk}\delta_{ij} + 2\mu\lambda E_{kk}\delta_{ij}E_{ij}$$
$$+\lambda^2\left(E_{kk}\right)^2\delta_{ij}\delta_{ij} = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ii}E_{kk} + 2\mu\lambda E_{kk}E_{ii} + \lambda^2\left(E_{kk}\right)^2\delta_{ii}$$
$$= 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2).$$

2.9 Given that $a_i = T_{ij}b_j$, and $a_i' = T_{ij}'b_j'$, where $a_i = Q_{im}a_m'$ and $T_{ij} = Q_{im}Q_{jn}T_{mn}'$.

(a) Show that $Q_{im}T_{mn}'b_n' = Q_{im}Q_{jn}T_{mn}'b_j$ and (b) if $Q_{ik}Q_{im} = \delta_{km}$, then $T_{kn}'(b_n' - Q_{jn}b_j) = 0$.

Ans. (a) Since $a_i = Q_{im}a_m'$ and $T_{ij} = Q_{im}Q_{jn}T_{mn}'$, therefore, $a_i = T_{ij}b_j \rightarrow$.

$Q_{im}a_m' = Q_{im}Q_{jn}T_{mn}'b_j$ (1), Now, $a_i' = T_{ij}'b_j' \rightarrow a_m' = T_{mj}'b_j' = T_{mn}'b_n'$, therefore, Eq. (1) becomes $Q_{im}T_{mn}'b_n' = Q_{im}Q_{jn}T_{mn}'b_j$. (2)

(b) To remove Q_{im} from Eq. (2), we make use of $Q_{ik}Q_{im} = \delta_{km}$ by multiplying the above equation, Eq.(2) with Q_{ik}. That is,

$$Q_{ik}Q_{im}T_{mn}'b_n' = Q_{ik}Q_{im}Q_{jn}T_{mn}'b_j \rightarrow \delta_{km}T_{mn}'b_n' = \delta_{km}Q_{jn}T_{mn}'b_j \rightarrow T_{kn}'b_n' = Q_{jn}T_{kn}'b_j$$
$$\rightarrow T_{kn}'(b_n' - Q_{jn}b_j) = 0.$$

2.10 Given $\left[a_i\right] = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\left[b_i\right] = \begin{bmatrix}0\\2\\3\end{bmatrix}$ Evaluate $[d_i]$, if $d_k = \varepsilon_{ijk}a_ib_j$ and show that this result is the same as $d_k = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_k$.

Ans. $d_k = \varepsilon_{ijk}a_ib_j \rightarrow$

$$d_1 = \varepsilon_{ij1}a_ib_j = \varepsilon_{231}a_2b_3 + \varepsilon_{321}a_3b_2 = a_2b_3 - a_3b_2 = (2)(3) - (0)(2) = 6$$
$$d_2 = \varepsilon_{ij2}a_ib_j = \varepsilon_{312}a_3b_1 + \varepsilon_{132}a_1b_3 = a_3b_1 - a_1b_3 = (0)(0) - (1)(3) = -3$$
$$d_3 = \varepsilon_{ij3}a_ib_j = \varepsilon_{123}a_1b_2 + \varepsilon_{213}a_2b_1 = a_1b_2 - a_2b_1 = (1)(2) - (2)(0) = 2$$

Next, $\left(\mathbf{a}\times\mathbf{b}\right) = \left(\mathbf{e}_1 + 2\mathbf{e}_2\right)\times\left(2\mathbf{e}_2 + 3\mathbf{e}_3\right) = 6\mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

$d_1 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_1 = 6,\quad d_2 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_2 = -3,\quad d_3 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_3 = 2$.

2.11 (a) If $\varepsilon_{ijk}T_{ij} = 0$, show that $T_{ij} = T_{ji}$, and (b) show that $\delta_{ij}\varepsilon_{ijk} = 0$

Ans. (a) *for* $k=1,\ \varepsilon_{ij1}T_{ij}=0\to\varepsilon_{231}T_{23}+\varepsilon_{321}T_{32}=0\to T_{23}-T_{32}\to T_{23}=T_{32}$.

for $k=2,\ \varepsilon_{ij2}T_{ij}=0\to\varepsilon_{312}T_{31}+\varepsilon_{132}T_{13}=0\to T_{31}-T_{13}\to T_{31}=T_{13}$.

for $k=3,\ \varepsilon_{ij3}T_{ij}=0\to\varepsilon_{123}T_{12}+\varepsilon_{213}T_{21}=0\to T_{12}-T_{21}\to T_{12}=T_{21}$.

(b) $\delta_{ij}\varepsilon_{ijk}=\delta_{11}\varepsilon_{11k}+\delta_{22}\varepsilon_{22k}+\delta_{33}\varepsilon_{33k}=(1)(0)+(1)(0)+(1)(0)=0$.

2.12 Verify the following equation: $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$.

(Hint): there are 6 cases to be considered (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$.

Ans. There are 4 free indices in the equation. Therefore, there are the following 6 cases to consider: (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$. We consider each case below where we use LS for left side, RS for right side and repeated indices with parenthesis are not sum:

(1) For $i=j$, LS=$\varepsilon_{(i)(i)m}\varepsilon_{klm}=0$, $RS=\delta_{(i)k}\delta_{(i)l}-\delta_{(i)l}\delta_{(i)k}=0$.

(2) For $i=k$, LS=$\varepsilon_{(i)j1}\varepsilon_{(i)l1}+\varepsilon_{(i)j2}\varepsilon_{(i)l2}+\varepsilon_{(i)j3}\varepsilon_{(i)l3}$, $RS=\delta_{(i)(i)}\delta_{jl}-\delta_{(i)l}\delta_{j(i)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq l\\ 0 & \text{if } j=l=i\\ 1 & \text{if } j=l\neq i\end{cases}.$$

(3) For $i=l$, LS=$\varepsilon_{(i)jm}\varepsilon_{k(i)m}$, $RS=\delta_{(i)k}\delta_{j(i)}-\delta_{(i)(i)}\delta_{jk}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq k\\ 0 & \text{if } j=k=i\\ -1 & \text{if } j=k\neq i\end{cases}$$

(4) For $j=k$, LS=$\varepsilon_{i(j)m}\varepsilon_{(j)lm}$, $RS=\delta_{i(j)}\delta_{(j)l}-\delta_{il}\delta_{(j)(j)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq l\\ 0 & \text{if } i=l=j\\ -1 & \text{if } i=l\neq j\end{cases}$$

(5) For $j=l$, LS=$\varepsilon_{i(j)m}\varepsilon_{k(j)m}$, $RS=\delta_{ik}\delta_{(j)(j)}-\delta_{i(j)}\delta_{(j)k}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq k\\ 0 & \text{if } i=k=j\\ 1 & \text{if } i=k\neq j\end{cases}$$

(6) For $k=l$, LS=$\varepsilon_{ijm}\varepsilon_{(k)(k)m}$=0, $RS=\delta_{i(k)}\delta_{j(k)}-\delta_{i(k)}\delta_{j(k)}=0$

2.13 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ as a short cut to obtain the following results: (a) $\varepsilon_{ilm}\varepsilon_{jlm}=2\delta_{ij}$ and (b) $\varepsilon_{ijk}\varepsilon_{ijk}=6$.

Ans. (a) $\varepsilon_{ilm}\varepsilon_{jlm}=\delta_{ij}\delta_{ll}-\delta_{il}\delta_{lj}=3\delta_{ij}-\delta_{ij}=2\delta_{ij}$.

(b) $\varepsilon_{ijk}\varepsilon_{ijk}=\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ji}=(3)(3)-\delta_{ii}=9-3=6$.

2.14 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ to show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Ans. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = a_m\mathbf{e}_m \times(\varepsilon_{ijk}b_jc_k\mathbf{e}_i) = \varepsilon_{ijk}a_mb_jc_k(\mathbf{e}_m\times\mathbf{e}_i)$

$= \varepsilon_{ijk}a_mb_jc_k(\varepsilon_{nmi}\mathbf{e}_n) = \varepsilon_{ijk}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n = \varepsilon_{jki}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n$

$= (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})a_mb_jc_k\mathbf{e}_n = \delta_{jn}\delta_{km}a_mb_jc_k\mathbf{e}_n - \delta_{jm}\delta_{kn}a_mb_jc_k\mathbf{e}_n$

$= a_kb_nc_k\mathbf{e}_n - a_jb_jc_n\mathbf{e}_n = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

2.15 (a) Show that if $T_{ij} = -T_{ji}$, $T_{ij}a_ia_j = 0$ and (b) if $T_{ij} = -T_{ji}$, and $S_{ij} = S_{ji}$, then $T_{ij}S_{ij} = 0$

Ans. Since $T_{ij}a_ia_j = T_{ji}a_ja_i$ (switching the original dummy index i to j and the original index j to i), therefore $T_{ij}a_ia_j = T_{ji}a_ja_i = -T_{ij}a_ja_i = -T_{ij}a_ia_j \to 2T_{ij}a_ia_j = 0 \to T_{ij}a_ia_j = 0$.

(b) $T_{ij}S_{ij} = T_{ji}S_{ji}$ (switching the original dummy index i to j and the original index j to i), therefore, $T_{ij}S_{ij} = T_{ji}S_{ji} = -T_{ij}S_{ji} = -T_{ij}S_{ij} \to 2T_{ij}S_{ij} = 0 \to T_{ij}S_{ij} = 0$.

2.16 Let $T_{ij} = \left(S_{ij} + S_{ji}\right)/2$ and $R_{ij} = \left(S_{ij} - S_{ji}\right)/2$, show that $T_{ij} = T_{ji}$, $R_{ij} = -R_{ji}$, and $S_{ij} = T_{ij} + \mathrm{R}_{ij}$.

Ans. $T_{ij} = \left(S_{ij} + S_{ji}\right)/2 \to\ T_{ji} = \left(S_{ji} + S_{ij}\right)/2 = T_{ij}$.

$R_{ij} = \left(S_{ij} - S_{ji}\right)/2 \to R_{ji} = \left(S_{ji} - S_{ij}\right)/2 = -\left(S_{ij} - S_{ji}\right)/2 = -R_{ij}$.

$T_{ij}+R_{ij}=\left(S_{ij} + S_{ji}\right)/2+\left(S_{ij} - S_{ji}\right)/2 = S_{ij}$.

2.17 Let $f(x_1,x_2,x_3)$ be a function of $x_1, x_2, \text{and } x_3$ and $v_i(x_1,x_2,x_3)$ be three functions of $x_1, x_2, \text{and } x_3$. Express the total differential df and dv_i in indicial notation.

Ans. $df = \dfrac{\partial f}{\partial x_1}dx_1 + \dfrac{\partial f}{\partial x_2}dx_2 + \dfrac{\partial f}{\partial x_3}dx_3 = \dfrac{\partial f}{\partial x_i}dx_i$.

$dv_i = \dfrac{\partial v_i}{\partial x_1}dx_1 + \dfrac{\partial v_i}{\partial x_2}dx_2 + \dfrac{\partial v_i}{\partial x_3}dx_3 = \dfrac{\partial v_i}{\partial x_m}dx_m$.

2.18 Let $\left|A_{ij}\right|$ denote that determinant of the matrix $\left[A_{ij}\right]$. Show that $\left|A_{ij}\right| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

Ans. $\varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \varepsilon_{1jk}A_{11}A_{j2}A_{k3} + \varepsilon_{2jk}A_{21}A_{j2}A_{k3} + \varepsilon_{3jk}A_{31}A_{j2}A_{k3}$

$= \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{132}A_{11}A_{32}A_{23} + \varepsilon_{231}A_{21}A_{32}A_{13} + \varepsilon_{213}A_{21}A_{12}A_{33} + \varepsilon_{312}A_{31}A_{12}A_{23} + \varepsilon_{321}A_{31}A_{22}A_{13}$

$= A_{11}A_{22}A_{33} - A_{11}A_{32}A_{23} + A_{21}A_{32}A_{13} - A_{21}A_{12}A_{33} + A_{31}A_{12}A_{23} - A_{31}A_{22}A_{13}$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

CHAPTER 2, PART B

2.19 A transformation $\mathbf{T}$ operate on any vector $\mathbf{a}$ to give $\mathbf{Ta} = \mathbf{a} / |\mathbf{a}|$, where $|\mathbf{a}|$ is the magnitude of $\mathbf{a}$. Show that $\mathbf{T}$ is not a linear transformation.

Ans. Since $\mathbf{Ta} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ for any $\mathbf{a}$, therefore $\mathbf{T}(\mathbf{a}+\mathbf{b}) = \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$. Now $\mathbf{Ta}+\mathbf{Tb} = \dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|}$ therefore $\mathbf{T}(\mathbf{a}+\mathbf{b}) \neq \mathbf{Ta}+\mathbf{Tb}$ and $\mathbf{T}$ is not a linear transformation.

2.20 (a) A tensor $\mathbf{T}$ transforms every vector $\mathbf{a}$ into a vector $\mathbf{Ta} = \mathbf{m} \times \mathbf{a}$ where $\mathbf{m}$ is a specified vector. Show that $\mathbf{T}$ is a linear transformation and (b) If $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$, find the matrix of the tensor $\mathbf{T}$.

Ans. (a) $\mathbf{T}(\alpha\mathbf{a}+\beta\mathbf{b}) = \mathbf{m}\times(\alpha\mathbf{a}+\beta\mathbf{b}) = \mathbf{m}\times\alpha\mathbf{a}+\mathbf{m}\times\beta\mathbf{b} = \alpha\mathbf{m}\times\mathbf{a}+\beta\mathbf{m}\times\mathbf{b} = \alpha\mathbf{Ta}+\beta\mathbf{Tb}.$ Thus, the given $\mathbf{T}$ is a linear transformation.

(b) $\mathbf{Te}_1 = \mathbf{m}\times\mathbf{e}_1 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_1 = -\mathbf{e}_3$, $\quad \mathbf{Te}_2 = \mathbf{m}\times\mathbf{e}_2 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_2 = \mathbf{e}_3$,

$\mathbf{Te}_3 = \mathbf{m}\times\mathbf{e}_3 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_3 = -\mathbf{e}_2+\mathbf{e}_1$. Thus,

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

2.21 A tensor $\mathbf{T}$ transforms the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $\mathbf{Te}_1 = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{Te}_2 = \mathbf{e}_1 - \mathbf{e}_2$. If $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2$ and $\mathbf{b} = 3\mathbf{e}_1 + 2\mathbf{e}_2$, use the linear property of $\mathbf{T}$ to find (a) $\mathbf{Ta}$,(b) $\mathbf{Tb}$, and (c) $\mathbf{T}(\mathbf{a}+\mathbf{b})$.

Ans.

(a)$\mathbf{Ta} = \mathbf{T}(2\mathbf{e}_1+3\mathbf{e}_2) = 2\mathbf{Te}_1+3\mathbf{Te}_2 = 2(\mathbf{e}_1+\mathbf{e}_2)+3(\mathbf{e}_1-\mathbf{e}_2) = 5\mathbf{e}_1-\mathbf{e}_2$.

(b)$\mathbf{Tb} = \mathbf{T}(3\mathbf{e}_1+2\mathbf{e}_2) = 3\mathbf{Te}_1+2\mathbf{Te}_2 = 3(\mathbf{e}_1+\mathbf{e}_2)+2(\mathbf{e}_1-\mathbf{e}_2) = 5\mathbf{e}_1+\mathbf{e}_2$.

(c)$\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta}+\mathbf{Tb} = (5\mathbf{e}_1-\mathbf{e}_2)+(5\mathbf{e}_1+\mathbf{e}_2) = 10\mathbf{e}_1$.

2.22 Obtain the matrix for the tensor $\mathbf{T}$ which transforms the base vectors as follows:
$\mathbf{Te}_1 = 2\mathbf{e}_1+\mathbf{e}_3$, $\mathbf{Te}_2 = \mathbf{e}_2+3\mathbf{e}_3$, $\mathbf{Te}_3 = -\mathbf{e}_1+3\mathbf{e}_2$.

Ans. $$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix}.$$

2.23 Find the matrix of the tensor $\mathbf{T}$ which transforms any vector $\mathbf{a}$ into a vector $\mathbf{b} = \mathbf{m}(\mathbf{a}\cdot\mathbf{n})$ where $\mathbf{m} = (\sqrt{2}/2)(\mathbf{e}_1+\mathbf{e}_2)$ and $\mathbf{n} = (\sqrt{2}/2)(-\mathbf{e}_1+\mathbf{e}_3)$.

Ans. $\mathbf{Te}_1 = \mathbf{m}(\mathbf{e}_1\cdot\mathbf{n}) = n_1\mathbf{m} = (-\sqrt{2}/2)\left[(\sqrt{2}/2)(\mathbf{e}_1+\mathbf{e}_2)\right] = -(\mathbf{e}_1+\mathbf{e}_2)/2$.

$\mathbf{Te}_2 = \mathbf{m}(\mathbf{e}_2 \cdot \mathbf{n}) = n_2\mathbf{m} = 0\mathbf{m} = \mathbf{0}$.

$\mathbf{Te}_3 = \mathbf{m}(\mathbf{e}_3 \cdot \mathbf{n}) = n_3\mathbf{m} = \left(\sqrt{2}/2\right)\left[\left(\sqrt{2}/2\right)(\mathbf{e}_1 + \mathbf{e}_2)\right] = (\mathbf{e}_1 + \mathbf{e}_2)/2$.

Thus, $[\mathbf{T}] = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$.

2.24 (a) A tensor $\mathbf{T}$ transforms every vector into its mirror image with respect to the plane whose normal is $\mathbf{e}_2$. Find the matrix of $\mathbf{T}$. (b) Do part (a) if the plane has a normal in the $\mathbf{e}_3$ direction.

Ans. (a) $\mathbf{Te}_1 = \mathbf{e}_1$, $\mathbf{Te}_2 = -\mathbf{e}_2$, $\mathbf{Te}_3 = \mathbf{e}_3$, thus, $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{Te}_1 = \mathbf{e}_1$, $\mathbf{Te}_2 = \mathbf{e}_2$, $\mathbf{Te}_3 = -\mathbf{e}_3$, thus, $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

2.25 (a) Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_1-axis. Find the matrix of $\mathbf{R}$. (b) do part (a) if the rotation is about the x_2-axis. The coordinates are right-handed.

Ans.(a) $\mathbf{Re}_1 = \mathbf{e}_1$, $\mathbf{Re}_2 = 0\mathbf{e}_1 + \cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3$, $\mathbf{Re}_3 = 0\mathbf{e}_1 - \sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3$. Thus,

$$[\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

(b) $\mathbf{Re}_1 = -\sin\theta\mathbf{e}_3 + \cos\theta\mathbf{e}_1$, $\mathbf{Re}_2 = \mathbf{e}_2$, $\mathbf{Re}_3 = \cos\theta\mathbf{e}_3 + \sin\theta\mathbf{e}_1$. Thus,

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2.26 Consider a plane of reflection which passes through the origin. Let $\mathbf{n}$ be a unit normal vector to the plane and let $\mathbf{r}$ be the position vector for a point in space. (a) Show that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{T}$ is the transformation that corresponds to the reflection. (b) Let $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$, find the matrix of $\mathbf{T}$. (c) Use this linear transformation to find the mirror image of the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$.

Ans. (a) Let the vector $\mathbf{r}$ be decomposed into two vectors $\mathbf{r}_n$ and $\mathbf{r}_t$, where $\mathbf{r}_n$ is in the direction of $\mathbf{n}$ and $\mathbf{r}_t$ is in a direction perpendicular to $\mathbf{n}$. That is, $\mathbf{r}_n$ is normal to the plane of reflection and $\mathbf{r}_t$ is on the plane of reflection and $\mathbf{r} = \mathbf{r}_t + \mathbf{r}_n$. In the reflection given by $\mathbf{T}$, we have,

$\mathbf{Tr}_n = -\mathbf{r}_n$ and $\mathbf{Tr}_t = \mathbf{r}_t$, so that $\mathbf{Tr} = \mathbf{Tr}_t + \mathbf{Tr}_n = \mathbf{r}_t - \mathbf{r}_n = (\mathbf{r} - \mathbf{r}_n) - \mathbf{r}_n = \mathbf{r} - 2\mathbf{r}_n = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$.

(b) $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3} \rightarrow \mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = \mathbf{e}_3\cdot\mathbf{n} = 1/\sqrt{3}$.

$\mathbf{Te}_1=\mathbf{e}_1-2(\mathbf{e}_1\cdot\mathbf{n})\mathbf{n}=\mathbf{e}_1-2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}\right]=\left(\mathbf{e}_1-2\mathbf{e}_2-2\mathbf{e}_3\right)/3$.

$\mathbf{Te}_2=\mathbf{e}_2-2(\mathbf{e}_2\cdot\mathbf{n})\mathbf{n}=\mathbf{e}_2-2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}\right]=\left(-2\mathbf{e}_1+\mathbf{e}_2-2\mathbf{e}_3\right)/3$.

$\mathbf{Te}_3=\mathbf{e}_3-2(\mathbf{e}_3\cdot\mathbf{n})\mathbf{n}=\mathbf{e}_3-2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}\right]=\left(-2\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3\right)/3$.

$$[\mathbf{T}]=\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{bmatrix}.$$

(c) $$[\mathbf{T}][\mathbf{a}]=\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}-3\\-2\\-1\end{bmatrix}\rightarrow\mathbf{Ta}=-\left(3\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\right).$$

2.27 Knowing that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr}=\mathbf{r}-2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$ (see the previous problem), where $\mathbf{T}$ is the transformation that corresponds to the reflection and $\mathbf{n}$ is the normal to the mirror, show that in dyadic notation, the reflection tensor is given by $\mathbf{T}=\mathbf{I}-2\mathbf{nn}$ and find the matrix of $\mathbf{T}$ if the normal of the mirror is given by $\mathbf{n}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$,

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Ans. From the definition of dyadic product, we have ,

$\mathbf{Tr}=\mathbf{r}-2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}=\mathbf{r}-2(\mathbf{nn})\mathbf{r}=(\mathbf{Ir}-2(\mathbf{nn})\mathbf{r})=(\mathbf{I}-2\mathbf{nn})\mathbf{r}\rightarrow\mathbf{T}=\mathbf{I}-2\mathbf{nn}$.

For $\mathbf{n}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}\rightarrow[2\mathbf{nn}]=\frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix}=\frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$.

$$\rightarrow[\mathbf{T}]=[\mathbf{I}]-[2\mathbf{nn}]=\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{bmatrix}.$$

2.28 A rotation tensor $\mathbf{R}$ is defined by the relation $\mathbf{Re}_1=\mathbf{e}_2$, $\mathbf{Re}_2=\mathbf{e}_3$, $\mathbf{Re}_3=\mathbf{e}_1$ (a) Find the matrix of $\mathbf{R}$ and verify that $\mathbf{R}^{\mathrm{T}}\mathbf{R}=\mathbf{I}$ and $\det\mathbf{R}=1$ and (b) find a unit vector in the direction of the axis of rotation that could have been used to effect this particular rotation.

Ans. (a) $[\mathbf{R}]=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\rightarrow[\mathbf{R}]^{\mathrm{T}}[\mathbf{R}]=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\det[\mathbf{R}]=\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix}=1$.

(b) Let the axis of rotation be $\mathbf{n}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3$, then

$$\mathbf{Rn}=\mathbf{n}\rightarrow[\mathbf{R}-\mathbf{I}][\mathbf{n}]=[\mathbf{0}]\rightarrow\begin{bmatrix}-1&0&1\\1&-1&0\\0&1&-1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\rightarrow-\alpha_1+\alpha_3=0,\quad\alpha_1-\alpha_2=0,\quad\alpha_2-\alpha_3=0\ .$$

Thus, $\alpha_1=\alpha_2=\alpha_3$, so that a unit vector in the direction of the axis of rotation is

$\mathbf{n}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$.

2.29 A rigid body undergoes a right hand rotation of angle θ about an axis which is in the direction of the unit vector $\mathbf{m}$. Let the origin of the coordinates be on the axis of rotation and $\mathbf{r}$ be the position vector for a typical point in the body. (a) show that the rotated vector of $\mathbf{r}$ is given by: $\mathbf{Rr}=(1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta(\mathbf{m}\times\mathbf{r})$, where $\mathbf{R}$ is the rotation tensor. (b) Let $\mathbf{m}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, find the matrix for $\mathbf{R}$.

Ans. (a) Let the vector $\mathbf{r}$ be decomposed into two vectors $\mathbf{r}_m$ and $\mathbf{r}_p$, where $\mathbf{r}_m$ is in the direction of $\mathbf{m}$ and $\mathbf{r}_p$ is in a direction perpendicular to $\mathbf{m}$, that is, $\mathbf{r}=\mathbf{r}_p+\mathbf{r}_m$. Let $\mathbf{p}\equiv\mathbf{r}_p/|\mathbf{r}_p|$ be the unit vector in the direction of $\mathbf{r}_p$, and let $\mathbf{q}\equiv\mathbf{m}\times\mathbf{p}$. Then, $(\mathbf{m},\mathbf{p},\mathbf{q})$ forms an orthonormal set of vectors which rotates an angle of θ about the unit vector $\mathbf{m}$. Thus,

$\mathbf{Rr}_m=\mathbf{r}_m$ and $\mathbf{Rr}_p=|\mathbf{r}_p|(\cos\theta\mathbf{p}+\sin\theta\mathbf{q})$. From $\mathbf{r}=\mathbf{r}_p+\mathbf{r}_m$, we have,

$$\mathbf{Rr}=\mathbf{Rr}_p+\mathbf{Rr}_m=|\mathbf{r}_p|(\cos\theta\mathbf{p}+\sin\theta\mathbf{q})+\mathbf{r}_m=\left\{\cos\theta|\mathbf{r}_p|\mathbf{p}+\sin\theta|\mathbf{r}_p|(\mathbf{m}\times\mathbf{p})\right\}+\mathbf{r}_m$$

$$=\left\{\cos\theta\mathbf{r}_p+\sin\theta(\mathbf{m}\times\mathbf{r}_p)\right\}+\mathbf{r}_m=\left\{\cos\theta(\mathbf{r}-\mathbf{r}_m)+\sin\theta(\mathbf{m}\times(\mathbf{r}-\mathbf{r}_m))\right\}+\mathbf{r}_m$$

$$=\mathbf{r}\cos\theta+\mathbf{r}_m(1-\cos\theta)+\sin\theta\mathbf{m}\times(\mathbf{r}-\mathbf{r}_m)=\mathbf{r}\cos\theta+\mathbf{r}_m(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$$

We note that $\mathbf{r}_m=(\mathbf{r}\cdot\mathbf{m})\mathbf{m}$, so that $\mathbf{Rr}=\mathbf{r}\cos\theta+(\mathbf{r}\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$.

(b) Use the result of (a), that is, $\mathbf{Rr}=\mathbf{r}\cos\theta+(\mathbf{r}\cdot\mathbf{m})(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$, we have,

$\mathbf{Re}_1=\mathbf{e}_1\cos\theta+(\mathbf{e}_1\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_1$,

$\mathbf{Re}_2=\mathbf{e}_2\cos\theta+(\mathbf{e}_2\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_2$,

$\mathbf{Re}_3=\mathbf{e}_3\cos\theta+(\mathbf{e}_3\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_3$.

Now, $\mathbf{m}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, therefore, $\mathbf{m}\cdot\mathbf{e}_1=\mathbf{m}\cdot\mathbf{e}_2=\mathbf{m}\cdot\mathbf{e}_3=1/\sqrt{3}$

$\mathbf{m}\times\mathbf{e}_1=(1/\sqrt{3})(-\mathbf{e}_3+\mathbf{e}_2)$, $\mathbf{m}\times\mathbf{e}_2=(1/\sqrt{3})(\mathbf{e}_3-\mathbf{e}_1)$, $\mathbf{m}\times\mathbf{e}_3=(1/\sqrt{3})(-\mathbf{e}_2+\mathbf{e}_1)$. Thus,

$$\mathbf{Re}_1=\mathbf{e}_1\cos\theta+(\mathbf{e}_1\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_1$$

$$=\mathbf{e}_1\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(-\mathbf{e}_3+\mathbf{e}_2)$$

$$=(1/3)\{1+2\cos\theta\}\mathbf{e}_1+\mathbf{e}_2\left\{(1/3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})\right\}+\mathbf{e}_3\left\{(1/3)(1-\cos\theta)-\sin\theta(1/\sqrt{3})\right\}$$

$$\mathbf{Re}_2=\mathbf{e}_2\cos\theta+(\mathbf{e}_2\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_2$$

$$=\mathbf{e}_2\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(\mathbf{e}_3-\mathbf{e}_1)$$

$$=\left\{(1/3)(1-\cos\theta)-(1/\sqrt{3})\sin\theta\right\}\mathbf{e}_1+(1/3)(1+2\cos\theta)\mathbf{e}_2+\left\{(1/3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})\right\}\mathbf{e}_3$$

$$\mathbf{Re}_3=\mathbf{e}_3\cos\theta+(\mathbf{e}_3\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_3$$

$$=\mathbf{e}_3\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(-\mathbf{e}_2+\mathbf{e}_1)$$

$$=\left\{(1/3)(1-\cos\theta)+(1/\sqrt{3})\sin\theta\right\}\mathbf{e}_1+\left\{(1/3)(1-\cos\theta)-\sin\theta(1/\sqrt{3})\right\}\mathbf{e}_2+(1/3)(1+2\cos\theta)\mathbf{e}_3$$

Thus,

$$[\mathbf{T}]=\frac{1}{3}\begin{bmatrix} 1+2\cos\theta & (1-\cos\theta)-\sqrt{3}\sin\theta & (1-\cos\theta)+\sqrt{3}\sin\theta \\ (1-\cos\theta)+\sqrt{3}\sin\theta & (1+2\cos\theta) & (1-\cos\theta)-\sqrt{3}\sin\theta \\ (1-\cos\theta)-\sqrt{3}\sin\theta & (1-\cos\theta)+\sqrt{3}\sin\theta & (1+2\cos\theta) \end{bmatrix}.$$

2.30 For the rotation about an arbitrary axis $\mathbf{m}$ by an angle θ, (a) show that the rotation tensor is given by $\mathbf{R} = (1-\cos\theta)(\mathbf{mm}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}$, where $\mathbf{mm}$ denotes that dyadic product of $\mathbf{m}$ and $\mathbf{E}$ is the antisymmetric tensor whose dual vector (or axial vector) is $\mathbf{m}$, (b) find the $\mathbf{R}^{\mathrm{A}}$, the antisymmetric part of $\mathbf{R}$ and (c) show that the dual vector for $\mathbf{R}^{\mathrm{A}}$ is given by $(\sin\theta)\mathbf{m}$. Hint, $\mathbf{Rr} = (1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta(\mathbf{m}\times\mathbf{r})$ (see previous problem).

Ans. (a) We have, from the previous problem, $\mathbf{Rr} = (1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta(\mathbf{m}\times\mathbf{r})$. Now, by the definition of dyadic product, we have $(\mathbf{m}\cdot\mathbf{r})\mathbf{m} = (\mathbf{mm})\mathbf{r}$, and by the definition of dual vector we have, $\mathbf{m}\times\mathbf{r} = \mathbf{Er}$, thus $\mathbf{Rr} = (1-\cos\theta)(\mathbf{mm})\mathbf{r} + \cos\theta\mathbf{r} + \sin\theta\mathbf{Er}$ $= \{(1-\cos\theta)(\mathbf{mm}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}\}\mathbf{r}$, from which, $\mathbf{R} = (1-\cos\theta)(\mathbf{mm}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}$.

(b) $\mathbf{R}^{\mathrm{A}} = (\mathbf{R}-\mathbf{R}^{\mathrm{T}})/2 \rightarrow$

$2\mathbf{R}^{\mathrm{A}} = \{(1-\cos\theta)(\mathbf{mm}) + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}\} - \{(1-\cos\theta)(\mathbf{mm})^{\mathrm{T}} + \cos\theta\mathbf{I} + \sin\theta\mathbf{E}^{\mathrm{T}}\}$. Now

$[\mathbf{mm}] = [m_i m_j] = [m_j m_i] = [\mathbf{mm}]^{\mathrm{T}}$, and the tensor $\mathbf{E}$, being antisymmetric, $\mathbf{E} = -\mathbf{E}^{\mathrm{T}}$, therefore,

$2\mathbf{R}^{\mathrm{A}} = 2\sin\theta\mathbf{E}$, that is, $\mathbf{R}^{\mathrm{A}} = \sin\theta\mathbf{E}$.

(c) dual vector of $\mathbf{R}^{\mathrm{A}} = (\sin\theta)(\text{dual vector of }\mathbf{E}) = \sin\theta\mathbf{m}$.

2.31 (a) Given a mirror whose normal is in the direction of $\mathbf{e}_2$. Find the matrix of the tensor $\mathbf{S}$ which first transforms every vector into its mirror image and then transforms them by a 45° right-hand rotation about the $\mathbf{e}_1$-axis. (b) Find the matrix of the tensor $\mathbf{T}$ which first transforms every vector by a 45° right-hand rotation about the $\mathbf{e}_1$-axis, and then transforms them by a reflection with respect to the mirror (whose normal is $\mathbf{e}_2$). (c) Consider the vector $\mathbf{a} = (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$, find the transformed vector by using the transformation $\mathbf{S}$.

(d) For the same vector $\mathbf{a} = (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$, find the transformed vector by using the transformation $\mathbf{T}$.

Ans. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ correspond to the reflection and the rotation respectively. We have

$$\mathbf{T}_1\mathbf{e}_1 = \mathbf{e}_1,\ \mathbf{T}_1\mathbf{e}_2 = -\mathbf{e}_2,\ \mathbf{T}_1\mathbf{e}_3 = \mathbf{e}_3 \rightarrow [\mathbf{T}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{T}_2\mathbf{e}_1 = \mathbf{e}_1,\ \mathbf{T}_2\mathbf{e}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3),\ \mathbf{T}_2\mathbf{e}_3 = \frac{1}{\sqrt{2}}(-\mathbf{e}_2 + \mathbf{e}_3) \rightarrow [\mathbf{T}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(a) $[\mathbf{S}]=[\mathbf{T}_2][\mathbf{T}_1]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$.

(b) $[\mathbf{T}]=[\mathbf{T}_1][\mathbf{T}_2]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$.

(c) $[\mathbf{b}]=[\mathbf{S}][\mathbf{a}]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}=\begin{bmatrix}1\\ -5/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$.

(d) $[\mathbf{c}]=[\mathbf{T}][\mathbf{a}]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}=\begin{bmatrix}1\\ 1/\sqrt{2}\\ 5/\sqrt{2}\end{bmatrix}$.

2.32 Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_3-axis (a) find the matrix of $\mathbf{R}^2$. (b) Show that $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis (c) Find the matrix of $\mathbf{R}^n$ for any integer n.

Ans. (a) $[\mathbf{R}]=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

$\rightarrow\left[\mathbf{R}^2\right]=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta & 0\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

(b)

$\left[\mathbf{R}^2\right]=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta & 0\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos 2\theta & -\sin 2\theta & 0\\ \sin 2\theta & \cos 2\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

Thus, $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis

(c) $\left[\mathbf{R}^n\right]=\begin{bmatrix}\cos n\theta & -\sin n\theta & 0\\ \sin n\theta & \cos n\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

2.33 Rigid body rotations that are small can be described by an orthogonal transformation $\mathbf{R}=\mathbf{I}+\varepsilon\mathbf{R}^*$ where $\varepsilon\rightarrow 0$ as the rotation angle approaches zero. Consider two successive small rotations $\mathbf{R}_1$ and $\mathbf{R}_2$, show that the final result does not depend on the order of rotations.

Ans. $\mathbf{R}_2\mathbf{R}_1 = \left(\mathbf{I}+\varepsilon\mathbf{R}_2^*\right)\left(\mathbf{I}+\varepsilon\mathbf{R}_1^*\right) = \mathbf{I}+\varepsilon\mathbf{R}_2^*+\varepsilon\mathbf{R}_1^*+\varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^* = \mathbf{I}+\varepsilon\left(\mathbf{R}_2^*+\mathbf{R}_1^*\right)+\varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^*$.

As $\varepsilon \to 0$, $\mathbf{R}_2\mathbf{R}_1 \approx \mathbf{I}+\varepsilon\left(\mathbf{R}_2^*+\mathbf{R}_1^*\right) = \mathbf{R}_1\mathbf{R}_2$.

2.34 Let $\mathbf{T}$ and $\mathbf{S}$ be any two tensors. Show that (a) $\mathbf{T}^{\mathrm{T}}$ is a tensor, (b) $\mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}} = (\mathbf{T}+\mathbf{S})^{\mathrm{T}}$ and (c) $(\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

Ans. Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be three arbitrary vectors and α β be any two scalars, then

(a) $\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}(\alpha\mathbf{b}+\beta\mathbf{c}) = (\alpha\mathbf{b}+\beta\mathbf{c})\cdot\mathbf{Ta} = \alpha\mathbf{b}\cdot\mathbf{Ta}+\beta\mathbf{c}\cdot\mathbf{Ta} = \alpha\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{c}$

$= \mathbf{a}\cdot\left(\alpha\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{T}^{\mathrm{T}}\mathbf{c}\right) \to \mathbf{T}^{\mathrm{T}}(\alpha\mathbf{b}+\beta\mathbf{c}) = \left(\alpha\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{T}^{\mathrm{T}}\mathbf{c}\right)$. Thus, $\mathbf{T}^{\mathrm{T}}$ is a linear transformation, i.e., tensor.

(b) $\mathbf{a}\cdot(\mathbf{T}+\mathbf{S})^{\mathrm{T}}\mathbf{b} = \mathbf{b}\cdot(\mathbf{T}+\mathbf{S})\mathbf{a} = \mathbf{b}\cdot\mathbf{Ta}+\mathbf{b}\cdot\mathbf{Sa} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b}+\mathbf{a}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{b}$

$= \mathbf{a}\cdot(\mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}})\mathbf{b} \to (\mathbf{T}+\mathbf{S})^{\mathrm{T}} = \mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}$.

(c) $\mathbf{a}\cdot(\mathbf{TS})^{\mathrm{T}}\mathbf{b} = \mathbf{b}\cdot(\mathbf{TS})\mathbf{a} = \mathbf{b}\cdot\mathbf{T}(\mathbf{Sa}) = (\mathbf{Sa})\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b} = \mathbf{a}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{b} \to (\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

2.35 For arbitrary tensors $\mathbf{T}$ and $\mathbf{S}$, without relying on the component form, prove that (a) $(\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$ and (b) $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$

Ans. (a) $\mathbf{TT}^{-1} = \mathbf{I} \to (\mathbf{TT}^{-1})^{\mathrm{T}} = \mathbf{I} \to (\mathbf{T}^{-1})^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = \mathbf{I} \to (\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$.

(b) $(\mathbf{TS})(\mathbf{S}^{-1}\mathbf{T}^{-1}) = \mathbf{T}(\mathbf{SS}^{-1})\mathbf{T}^{-1} = \mathbf{TT}^{-1} = \mathbf{I}$, thus, $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$.

2.36 Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ be two Rectangular Cartesian base vectors. (a) Show that if $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$, then $\mathbf{e}_i = Q_{im}\mathbf{e}_m'$ and (b) verify $Q_{mi}Q_{mj} = \delta_{ij} = Q_{im}Q_{jm}$.

Ans. (a) $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m \to \mathbf{e}_i'\cdot\mathbf{e}_j = Q_{mi}\mathbf{e}_m\cdot\mathbf{e}_j = Q_{mi}\delta_{mj} = Q_{ji} \to \mathbf{e}_j = Q_{jm}\mathbf{e}_m' \to \mathbf{e}_i = Q_{im}\mathbf{e}_m'$.

(b) We have, $\mathbf{e}_i'\cdot\mathbf{e}_j' = \delta_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j$, thus,

$\delta_{ij} = \mathbf{e}_i'\cdot\mathbf{e}_j' = Q_{mi}\mathbf{e}_m\cdot Q_{nj}\mathbf{e}_n = Q_{mi}Q_{nj}\mathbf{e}_m\cdot\mathbf{e}_n = Q_{mi}Q_{nj}\delta_{mn} = Q_{mi}Q_{mj}$. And

$\delta_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j = Q_{im}\mathbf{e}_m\cdot Q_{jn}\mathbf{e}_n = Q_{im}Q_{jn}\mathbf{e}_m\cdot\mathbf{e}_n = Q_{im}Q_{jn}\delta_{\mathrm{mn}} = Q_{im}Q_{jm}$.

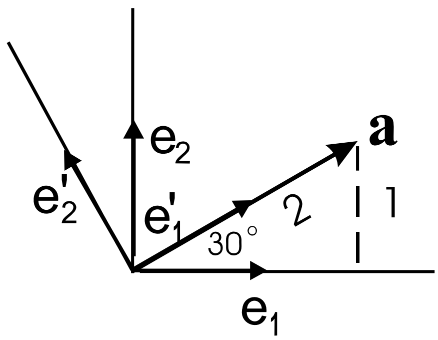
2.37 The basis $\{\mathbf{e}_i'\}$ is obtained by a 30^{o} counterclockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis. (a) Find the transformation matrix $[\mathbf{Q}]$ relating the two sets of basis, (b) by using the vector transformation law, find the components of $\mathbf{a} = \sqrt{3}\mathbf{e}_1+\mathbf{e}_2$ in the primed basis, i.e., find a_i' and (c) do part (b) geometrically.

Ans. (a) $\mathbf{e}_1' = \cos 30^{\mathrm{o}}\mathbf{e}_1+\sin 30^{\mathrm{o}}\mathbf{e}_2$, $\mathbf{e}_2' = -\sin 30^{\mathrm{o}}\mathbf{e}_1+\cos 30^{\mathrm{o}}\mathbf{e}_2$, $\mathbf{e}_3' = \mathbf{e}_3$. Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^{\mathrm{o}} & -\sin 30^{\mathrm{o}} & 0 \\ \sin 30^{\mathrm{o}} & \cos 30^{\mathrm{o}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $[\mathbf{a}]_{\mathbf{e}_i'} = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = 2\mathbf{e}_1'$

(c) Clearly $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$ is a vector in the same direction as $\mathbf{e}_1'$ and has a length of 2. See figure below



2.38 Do the previous problem with the $\{\mathbf{e}_i'\}$ basis obtained by a 30° clockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis.

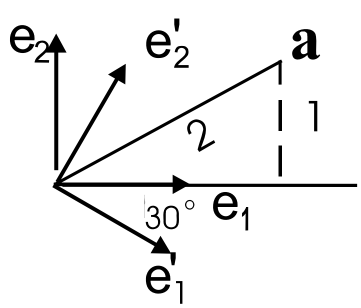
Ans.

(a) $\mathbf{e}_1' = \cos 30^{\circ}\mathbf{e}_1 - \sin 30^{\circ}\mathbf{e}_2$, $\mathbf{e}_2' = \sin 30^{\circ}\mathbf{e}_1 + \cos 30^{\circ}\mathbf{e}_2$, $\mathbf{e}_3' = \mathbf{e}_3$. Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^{\circ} & \sin 30^{\circ} & 0 \\ -\sin 30^{\circ} & \cos 30^{\circ} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $[\mathbf{a}]_{\mathbf{e}_i'} = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = \mathbf{e}_1' + \sqrt{3}\mathbf{e}_2'$

(c) See figure below



2.39 The matrix of a tensor $\mathbf{T}$ with respect to the basis $\{\mathbf{e}_i\}$ is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find T'_{11}, T'_{12} and T'_{31} with respect to a right-handed basis $\{\mathbf{e}'_i\}$ where $\mathbf{e}'_1$ is in the direction of $-\mathbf{e}_2 + 2\mathbf{e}_3$ and $\mathbf{e}'_2$ is in the direction of $\mathbf{e}_1$.

Ans. The basis $\{\mathbf{e}'_i\}$ is given by:

$$\mathbf{e}'_1 = (-\mathbf{e}_2 + 2\mathbf{e}_3)/\sqrt{5},\ \ \mathbf{e}'_2 = \mathbf{e}_1,\ \ \ \mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2 = (2\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{5}\ .$$

$$T'_{11} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 4/5.$$

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -15/\sqrt{5}\ .$$

$$T'_{31} = \mathbf{e}'_3 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 2/5\ .$$

2.40 (a) For the tensor of the previous problem, find $\left[T'_{ij}\right]$, i.e., $[\mathbf{T}]_{\mathbf{e}'_i}$ if $\{\mathbf{e}'_i\}$ is obtained by a 90° right hand rotation about the $\mathbf{e}_3$ axis and (b) obtain T'_{ii} and the determinant $\left|T'_{ij}\right|$ and compare them with T_{ii} and $\left|T_{ij}\right|$.

Ans. (a) $\mathbf{e}'_1 = \mathbf{e}_2,\ \mathbf{e}'_2 = -\mathbf{e}_1,\ \ \mathbf{e}'_3 = \mathbf{e}_3 \rightarrow [\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[T'_{ij}\right] = [\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 0 \\ -5 & 1 & 5 \\ 0 & 5 & 1 \end{bmatrix}$$

(b) $T'_{ii} = T'_{11} + T'_{22} + T'_{33} = 0 + 1 + 1 = 2,\ \ \ \left|T'_{ij}\right| = -25$.

$T_{ii} = T_{11} + T_{22} + T_{33} = 1 + 0 + 1 = 2,\ \ \ \left|T_{ij}\right| = -25$.

2.41 The dot product of two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$ is equal to $a_i b_i$. Show that the dot product is a scalar invariant with respect to orthogonal transformations of coordinates.

Ans. From $a'_i = Q_{mi}a_m$ and $b'_i = Q_{mi}b_m$, we have,

$$a'_i b'_i = Q_{mi}a_m Q_{ni}b_n = Q_{mi}Q_{ni}a_m b_n = \delta_{mn}a_m b_n = a_m b_m = a_i b_i\ .$$

2.42 If T_{ij} are the components of a tensor (a) show that $T_{ij}T_{ij}$ is a scalar invariant with respect to orthogonal transformations of coordinates, (b) evaluate $T_{ij}T_{ij}$ with respect to the basis $\{\mathbf{e}_i\}$ for $[\mathbf{T}]=\begin{bmatrix}1&0&0\\1&2&5\\1&2&3\end{bmatrix}_{\mathbf{e}_i}$, (c) find $[\mathbf{T}]'$, if $\mathbf{e}_i'=\mathbf{Q}\mathbf{e}_i$, where $[\mathbf{Q}]=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}_{\mathbf{e}_i}$ and

(d) verify for the above $[\mathbf{T}]$and$[\mathbf{T}]'$ that $T_{ij}'T_{ij}'=T_{ij}T_{ij}$.

Ans. (a) Since T_{ij} are the components of a tensor, $T_{ij}'=Q_{mi}Q_{nj}T_{mn}$. Thus,

$T_{ij}'T_{ij}'=Q_{mi}Q_{nj}T_{mn}(Q_{pi}Q_{qj}T_{pq})=(Q_{mi}Q_{pi})(Q_{nj}Q_{qj})T_{mn}T_{pq}=\delta_{mp}\delta_{nq}T_{mn}T_{pq}=T_{mn}T_{mn}$

(b) $T_{ij}T_{ij}=T_{11}^2+T_{12}^2+T_{13}^2+T_{21}^2+T_{22}^2+T_{23}^2+T_{31}^2+T_{32}^2+T_{33}^2=1+1+4+25+1+4+9=45$.

(c) $[\mathbf{T}]'=[\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}]=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&2&5\\1&2&3\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\2&5&1\\2&3&1\end{bmatrix}=\begin{bmatrix}2&5&1\\2&3&1\\0&0&1\end{bmatrix}$

(d) $T_{ij}'T_{ij}'=4+25+1+4+9+1+1=45$.

2.43 Let $[\mathbf{T}]$and$[\mathbf{T}]'$ be two matrices of the same tensor $\mathbf{T}$, show that $\det[\mathbf{T}]=\det[\mathbf{T}]'$.

Ans. $[\mathbf{T}]'=[\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}]\rightarrow\det[\mathbf{T}]'=\det[\mathbf{Q}]^{\mathrm{T}}\det[\mathbf{Q}]\det[\mathbf{T}]=(\pm1)(\pm1)\det[\mathbf{T}]=\det[\mathbf{T}]$.

2.44 (a) If the components of a third order tensor are R_{ijk}, show that R_{iik} are components of a vector, (b) if the components of a fourth order tensor are R_{ijkl}, show that R_{iikl} are components of a second order tensor and (c) what are components of $R_{iik\ldots}$, if $R_{ijk\ldots}$ are components of a tensor of n^{th} order?

Ans. (a) Since R_{ijk} are components of a third order tensor, therefore,

$R_{ijk}'=Q_{mi}Q_{nj}Q_{pk}R_{mnp}\rightarrow R_{iik}'=Q_{mi}Q_{ni}Q_{pk}R_{mnp}=\delta_{mn}Q_{pk}R_{mnp}=Q_{pk}R_{nnp}$, therefore, R_{iik} are components of a vector.

(b) Consider a 4^{th} order tensor R_{ijkl}, we have,

$R_{ijkl}'=Q_{mi}Q_{nj}Q_{pk}Q_{ql}R_{mnpq}\rightarrow R_{iikl}'=Q_{mi}Q_{ni}Q_{pk}Q_{ql}R_{mnpq}=\delta_{mn}Q_{pk}Q_{ql}R_{mnpq}=Q_{pk}Q_{ql}R_{nnpq}$, therefore, R_{iikl} are components of a second order tensor.

(c) $R_{iik\ldots}$ are components of a tensor of the $(n-2)^{th}$ order.

2.45 The components of an arbitrary vector $\mathbf{a}$ and an arbitrary second tensor $\mathbf{T}$ are related by a triply subscripted quantity R_{ijk} in the manner $a_i=R_{ijk}T_{jk}$ for any rectangular Cartesian basis $\{\mathbf{e}_i\}$. Prove that R_{ijk} are the components of a third-order tensor.

Ans. Since $a_i = R_{ijk}T_{jk}$ is true for any basis, therefore, $a'_i = R'_{ijk}T'_{jk}$; Since $\mathbf{a}$ is a vector, therefore, $a'_i = Q_{mi}a_m$ and since $\mathbf{T}$ is a second order tensor, therefore, $T'_{ij} = Q_{mi}Q_{nj}T_{mn}$. Thus, $a'_i = Q_{mi}a_m \to R'_{ijk}T'_{jk} = Q_{mi}(R_{mjk}T_{jk})$. Multiply the last equation with Q_{si} and noting that $Q_{si}Q_{mi} = \delta_{sm}$, we have,

$Q_{si}R'_{ijk}T'_{jk} = Q_{si}Q_{mi}(R_{mjk}T_{jk}) \to Q_{si}R'_{ijk}T'_{jk} = \delta_{sm}R_{mjk}T_{jk} \to Q_{si}R'_{ijk}T'_{jk} = R_{sjk}T_{jk}$

$\to Q_{si}R'_{ijk}Q_{mj}Q_{nk}T_{mn} = R_{sjk}T_{jk} \to Q_{si}R'_{ijk}Q_{mj}Q_{nk}T_{mn} = R_{smn}T_{mn}$. Thus,

$\left(R_{smn} - Q_{si}Q_{mj}Q_{nk}R'_{ijk}\right)T_{mn} = 0$. Since this last equation is to be true for all T_{mn}, therefore, $R_{smn} = Q_{si}Q_{mj}Q_{nk}R'_{ijk}$, which is the transformation law for components of a third order tensor.

2.46 For any vector $\mathbf{a}$ and any tensor $\mathbf{T}$, show that (a) $\mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = 0$ and (b) $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{S}\mathbf{a}$, where $\mathbf{T}^{A}$ and $\mathbf{T}^{S}$ are antisymmetric and symmetric part of $\mathbf{T}$ respectively.

Ans. (a) $\mathbf{T}^{A}$ is antisymmetric, therefore, $(\mathbf{T}^{A})^{T} = -\mathbf{T}^{A}$, thus,

$\mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = \mathbf{a}\cdot(\mathbf{T}^{A})^{T}\mathbf{a} = -\mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} \to 2\mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = 0 \to \mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = 0$.

(b) Since $\mathbf{T} = \mathbf{T}^{S} + \mathbf{T}^{A}$, therefore, $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot(\mathbf{T}^{S} + \mathbf{T}^{A})\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{S}\mathbf{a} + \mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{S}\mathbf{a}$.

2.47 Any tensor can be decomposed into a symmetric part and an antisymmetric part, that is $\mathbf{T} = \mathbf{T}^{S} + \mathbf{T}^{A}$. Prove that the decomposition is unique. (Hint, assume that it is not true and show contradiction).

Ans. Suppose that the decomposition is not unique, then ,we have,

$\mathbf{T} = \mathbf{T}^{S} + \mathbf{T}^{A} = \mathbf{S}^{S} + \mathbf{S}^{A} \to (\mathbf{T}^{S} - \mathbf{S}^{S}) + (\mathbf{T}^{A} - \mathbf{S}^{A}) = \mathbf{0}$. Let $\mathbf{a}$ be any arbitrary vector, we have,

$\mathbf{a}\cdot(\mathbf{T}^{S} - \mathbf{S}^{S})\mathbf{a} + \mathbf{a}\cdot(\mathbf{T}^{A} - \mathbf{S}^{A})\mathbf{a} = 0 \to \mathbf{a}\cdot\mathbf{T}^{S}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{S}\mathbf{a} + \mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{A}\mathbf{a} = 0$.

But $\mathbf{a}\cdot\mathbf{T}^{A}\mathbf{a} = \mathbf{a}\cdot\mathbf{S}^{A}\mathbf{a} = 0$ (see the previous problem). Therefore,

$\mathbf{a}\cdot\mathbf{T}^{S}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{S}\mathbf{a} = 0 \to \mathbf{a}\cdot(\mathbf{T}^{S} - \mathbf{S}^{S})\mathbf{a} = 0 \to \mathbf{T}^{S} - \mathbf{S}^{S} = 0 \to \mathbf{T}^{S} = \mathbf{S}^{S}$. It also follows from $(\mathbf{T}^{S} - \mathbf{S}^{S}) + (\mathbf{T}^{A} - \mathbf{S}^{A}) = \mathbf{0}$ that $\mathbf{T}^{A} = \mathbf{S}^{A}$. Thus , the decomposition is unique.

2.48 Given that a tensor $\mathbf{T}$ has the matrix $[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, (a) find the symmetric part and the anti-symmetric part of $\mathbf{T}$ and (b) find the dual vector (or axial vector) of the anti-symmetric part of $\mathbf{T}$.

Ans. (a) $\left[\mathbf{T}^{S}\right] = \frac{1}{2}\left\{[\mathbf{T}] + [\mathbf{T}]^{T}\right\} = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$.

$$\left[\mathbf{T}^A\right] = \frac{1}{2}\left\{[\mathbf{T}] - [\mathbf{T}]^T\right\} = \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 18 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

(b) $\mathbf{t}^A = -(T_{23}^A \mathbf{e}_1 + T_{31}^A \mathbf{e}_2 + T_{12}^A \mathbf{e}_3) = -(-1\mathbf{e}_1 + 2\mathbf{e}_2 - 1\mathbf{e}_3) = \mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$.

2.49 Prove that the only possible real eigenvalues of an orthogonal tensor $\mathbf{Q}$ are $\lambda = \pm 1$. Explain the direction of the eigenvectors corresponding to them for a proper orthogonal(rotation) tensor and for an improper orthogonal (reflection) tensor.

Ans. Since $\mathbf{Q}$ is orthogonal, therefore, for any vector $\mathbf{n}$, we have, $\mathbf{Qn} \cdot \mathbf{Qn} = \mathbf{n} \cdot \mathbf{n}$. Let $\mathbf{n}$ be an eigenvector, then $\mathbf{Qn} = \lambda \mathbf{n}$, so that $\mathbf{Qn} \cdot \mathbf{Qn} = \mathbf{n} \cdot \mathbf{n} \rightarrow$

$\lambda^2 (\mathbf{n} \cdot \mathbf{n}) = (\mathbf{n} \cdot \mathbf{n}) \rightarrow (\lambda^2 - 1)(\mathbf{n} \cdot \mathbf{n}) = 0 \rightarrow \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$.

The eigenvalue $\lambda = 1$ ($\mathbf{Qn} = \mathbf{n}$) corresponds to an eigenvector parallel to the axis of rotation for a proper orthogonal tensor (rotation tensor); Or, it corresponds to an eigenvector parallel to the plane of reflection for an improper orthogonal tensor (reflection tensor). The eigenvalue $\lambda = -1$, ($\mathbf{Qn} = -\mathbf{n}$) corresponds to an eigenvector perpendicular to the axis of rotation for an 180^o rotation; or, it corresponds to an eigenvector perpendicular to the plane of reflection.

2.50 Given the improper orthogonal tensor $[\mathbf{Q}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. (a) Verify that $\det[\mathbf{Q}] = -1$.

(b) Verify that the eigenvalues are $\lambda = 1$ and -1 (c) Find the normal to the plane of reflection (i.e., eigenvectors corresponding to $\lambda = -1$) and (d) find the eigenvectors corresponding $\lambda = 1$ (vectors parallel to the plane of reflection).

Ans. (a) $\det[\mathbf{Q}] = (1/3)^3 (1 - 8 - 8 - 4 - 4 - 4) = (-27)/27 = -1$.

(b) $I_1 = 3/3 = 1, \quad I_2 = (1/3)^2 \{(1-4) + (1-4) + (1-4)\} = -1, \quad I_3 = -1 \rightarrow$

$\lambda^3 - \lambda^2 - \lambda + 1 = 0 \rightarrow (\lambda - 1)(\lambda^2 - 1) = 0 \rightarrow \lambda = 1, 1, -1$

(c) For $\lambda = -1$,

$\left(\frac{1}{3} + 1\right)\alpha_1 - \frac{2}{3}\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 + \left(\frac{1}{3} + 1\right)\alpha_2 - \frac{2}{3}\alpha_3 = 0, \quad -\frac{2}{3}\alpha_1 - \frac{2}{3}\alpha_2 + \left(\frac{1}{3} + 1\right)\alpha_3 = 0$. That is, $2\alpha_1 - \alpha_2 - \alpha_3 = 0, \quad -\alpha_1 + 2\alpha_2 - \alpha_3 = 0, \quad -\alpha_1 - \alpha_2 + 2\alpha_3 = 0$, thus, $\alpha_1 = \alpha_2 = \alpha_3$, therefore, $\mathbf{n} = \pm(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$, this is the normal to the plane of reflection.

(d) For $\lambda = 1$,

$$\left(\frac{1}{3} - 1\right)\alpha_1 - \frac{2}{3}\alpha_2 - \frac{2}{3}\alpha_3 = 0, -\frac{2}{3}\alpha_1 + \left(\frac{1}{3} - 1\right)\alpha_2 - \frac{2}{3}\alpha_3 = 0, -\frac{2}{3}\alpha_1 - \frac{2}{3}\alpha_2 + \left(\frac{1}{3} - 1\right)\alpha_3 = 0$$

All three equations lead to $\alpha_1 + \alpha_2 + \alpha_3 = 0 \rightarrow \alpha_3 = -\alpha_1 - \alpha_2$. Thus,

$\mathbf{n} = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}[\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 - (\alpha_1 + \alpha_2)\mathbf{e}_3]$, e.g., $\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3)$ etc. these vectors are all perpendicular to $\mathbf{n} = \pm(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ and thus parallel to the plane of reflection.

2.51 Given that tensors $\mathbf{R}$ and $\mathbf{S}$ have the same eigenvector $\mathbf{n}$ and corresponding eigenvalue r_1 and s_1 respectively. Find an eigenvalue and the corresponding eigenvector for the tensor $\mathbf{T} = \mathbf{RS}$.

Ans. We have, $\mathbf{Rn} = r_1\mathbf{n}$ and $\mathbf{Sn} = s_1\mathbf{n}$, thus, $\mathbf{Tn} = \mathbf{RSn} = \mathbf{R}s_1\mathbf{n} = s_1\mathbf{Rn} = r_1 s_1\mathbf{n}$. Thus, an eigenvalue for $\mathbf{T} = \mathbf{RS}$ is $r_1 s_1$ with eigenvector $\mathbf{n}$.

2.52 Show that if $\mathbf{n}$ is a real eigenvector of an antisymmetric tensor $\mathbf{T}$, then the corresponding eigenvalue vanishes.

Ans. $\mathbf{Tn} = \lambda\mathbf{n} \to \mathbf{n}\cdot\mathbf{Tn} = \lambda(\mathbf{n}\cdot\mathbf{n})$. Now, from the definition of transpose, we have $\mathbf{n}\cdot\mathbf{Tn} = \mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{n}$. But, since $\mathbf{T}$ is antisymmetric, i.e., $\mathbf{T}^{\mathrm{T}} = -\mathbf{T}$, therefore, $\mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{n} = -\mathbf{n}\cdot\mathbf{Tn}$. Thus, $\mathbf{n}\cdot\mathbf{Tn} = -\mathbf{n}\cdot\mathbf{Tn} \to 2\mathbf{n}\cdot\mathbf{Tn} = 0 \to \mathbf{n}\cdot\mathbf{Tn} = 0$. Thus, $\lambda(\mathbf{n}\cdot\mathbf{n}) = 0 \to \lambda = 0$.

2.53 (a) Show that $\mathbf{a}$ is an eigenvector for the dyadic product $\mathbf{ab}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ with eigenvalue $\mathbf{a}\cdot\mathbf{b}$, (b) find the first principal scalar invariant of the dyadic product $\mathbf{ab}$ and (c) show that the second and the third principal scalar invariants of the dyadic product $\mathbf{ab}$ vanish, and that zero is a double eigenvalue of $\mathbf{ab}$.

Ans. (a) From the definition of dyadic product, we have, $(\mathbf{ab})\mathbf{a} = \mathbf{a}(\mathbf{b}\cdot\mathbf{a})$, thus $\mathbf{a}$ is an eigenvector for the dyadic product $\mathbf{ab}$ with eigenvalue $\mathbf{a}\cdot\mathbf{b}$.

(b) Let $\mathbf{T} \equiv \mathbf{ab}$, then $T_{ij} = a_i b_j$ and the first scalar invariant of $\mathbf{ab}$ is $T_{ii} = a_i b_i = \mathbf{a}\cdot\mathbf{b}$.

(c) $$I_2 = \begin{vmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{vmatrix} + \begin{vmatrix} a_2b_2 & a_2b_3 \\ a_3b_2 & a_3b_3 \end{vmatrix} + \begin{vmatrix} a_1b_1 & a_1b_3 \\ a_3b_1 & a_3b_3 \end{vmatrix} = 0+0+0 = 0.$$

$$I_3 = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix} = a_1a_2a_3 \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

Thus, the characteristic equation is

$$\lambda^3 - I_1\lambda^2 = 0 \to (\lambda - I_1)\lambda^2 = 0 \to \lambda_1 = I_1,\ \ \lambda_2 = \lambda_3 = 0.$$

2.54 For any rotation tensor, a set of basis $\{\mathbf{e}_i'\}$ may be chosen with $\mathbf{e}_3'$ along the axis of rotation so that $\mathbf{Re}_1' = \cos\theta\mathbf{e}_1' + \sin\theta\mathbf{e}_2'$, $\mathbf{Re}_2' = -\sin\theta\mathbf{e}_1' + \cos\theta\mathbf{e}_2'$, $\mathbf{Re}_3' = \mathbf{e}_3'$, where θ is the angle of right hand rotation. (a) Find the antisymmetric part of $\mathbf{R}$ with respect to the basis $\{\mathbf{e}_i'\}$, i.e., find $[\mathbf{R}^{\mathrm{A}}]_{\mathbf{e}_i'}$. (b) Show that the dual vector of $\mathbf{R}^{\mathrm{A}}$ is given by $\mathbf{t}^{\mathrm{A}} = \sin\theta\mathbf{e}_3'$ and (c) show that the first scalar invariant of $\mathbf{R}$ is given by $1 + 2\cos\theta$. That is, for any given rotation tensor $\mathbf{R}$, its axis of rotation and the angle of rotation can be obtained from the dual vector of $\mathbf{R}^{\mathrm{A}}$ and the first scalar invariant of $\mathbf{R}$.

Ans. (a) From $\mathbf{Re}_1' = \cos\theta\mathbf{e}_1' + \sin\theta\mathbf{e}_2'$, $\mathbf{Re}_2' = -\sin\theta\mathbf{e}_1' + \cos\theta\mathbf{e}_2'$, $\mathbf{Re}_3' = \mathbf{e}_3'$, we have,

$$[\mathbf{R}]_{\mathbf{e}_i'} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i'} \to [\mathbf{R}^{\mathrm{A}}]_{\mathbf{e}_i'} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{e}_i'}$$

(b) the dual vector (or axial vector) of $\mathbf{R}^{\mathrm{A}}$ is given by

$\mathbf{t}^A = -(T'_{23}\mathbf{e}'_1 + T'_{31}\mathbf{e}'_2 + T'_{12}\mathbf{e}'_3) = -(0\mathbf{e}'_1 + 0\mathbf{e}'_2 - \sin\theta\mathbf{e}'_3) = \sin\theta\mathbf{e}'_3$.
(c) The first scalar invariant of $\mathbf{R}$ is $I_1 = \cos\theta + \cos\theta + 1 = 1 + 2\cos\theta$.

2.55 The rotation of a rigid body is described by $\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = \mathbf{e}_3, \quad \mathbf{Re}_3 = \mathbf{e}_1$. Find the axis of rotation and the angle of rotation. Use the result of the previous problem.

Ans From the result of the previous problem, we have, the dual vector of $\mathbf{R}^A$ is given by $\mathbf{t}^A = \sin\theta\mathbf{e}'_3$, where $\mathbf{e}'_3$ is in the direction of axis of rotation and θ is the angle of rotation. Thus, we can obtain the direction of axis of rotation and the angle of rotation θ by obtaining the dual vector of $\mathbf{R}^A$. From $\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = \mathbf{e}_3, \quad \mathbf{Re}_3 = \mathbf{e}_1$, we have,

$$[\mathbf{R}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \left[\mathbf{R}^A\right] = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \to \mathbf{t}^A = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3). \text{ Thus ,}$$

$$\mathbf{t}^A = \frac{\sqrt{3}}{2}\frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{\sqrt{3}} = \frac{\sqrt{3}}{2}\mathbf{e}'_3, \text{ where } \mathbf{e}'_3 = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \text{ is in the direction of the axis of}$$

rotation and the angle of rotation is given by $\sin\theta = \sqrt{3}/2$, which gives $\theta = 60^o$ or 120^o . On the other hand, the first scalar invariant of $\mathbf{R}$ is 0. Thus, from the result in (c) of the previous problem, we have, $I_1 = 1 + 2\cos\theta = 0$, so that $\cos\theta = -1/2$ which gives $\theta = 120^o$or 240^o . We therefore conclude that $\theta = 120^o$.

2.56 Given the tensor $[\mathbf{Q}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Show that the given tensor is a rotation tensor. (b) Verify that the eigenvalues are $\lambda = 1$ and -1. (c) Find the direction for the axis of rotation (i.e., eigenvectors corresponding to $\lambda = 1$). (d) Find the eigenvectors corresponding $\lambda = -1$ and (e) obtain the angle of rotation using the formula $I_1 = 1 + 2\cos\theta$ (see Prob. 2.54), where I_1 is the first scalar invariant of the rotation tensor.

Ans. (a) $\det[\mathbf{Q}] = +1$, and $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{I}]$ therefore it is a rotation tensor.
(b) The principal scalar invariants are: $I_1 = -1, \quad I_2 = -1, \quad I_3 = 1 \to$ characteristic equation is
$\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)(\lambda^2 - 1) = 0 \to$ the eigenvalues are: $\lambda = -1, 1, 1$.
(c) For $\lambda = 1$, clearly, the eigenvector are: $\mathbf{n} = \pm\mathbf{e}_3$, which gives the axis of rotation.
(d) For $\lambda = -1$, with eigenvector $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, we have
$0\alpha_1 = 0, \quad 0\alpha_2 = 0, \quad 2\alpha_3 = 0$. Thus, $\alpha_1 =$ arbitrary, $\alpha_2 =$ arbitrary, $\alpha_3 = 0$. The eigenvectors are:
$\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2, \quad \alpha_1^2 + \alpha_2^2 = 1$. That is, all vectors perpendicular to the axis of rotation are eigenvectors.
(e) The first scalar invariant of $\mathbf{Q}$ is $I_1 = -1$. Thus, $1 + 2\cos\theta = -1 \to \cos\theta = -1 \to \theta = \pi$. (We note that for this problem, the antisymmetric part of $\mathbf{Q} = \mathbf{0}$, so that $\mathbf{t}^A = \mathbf{0} = \sin\theta\,\mathbf{n}$, of which $\theta = \pi$ is a solution).

2.57 Let $\mathbf{F}$ be an arbitrary tensor. (a) Show that $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ are both symmetric tensors. (b) If $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$, where $\mathbf{Q}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric, show that $\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ (c) If λ and $\mathbf{n}$ are eigenvalue and the corresponding eigenvector for $\mathbf{U}$, find the eigenvalue and eigenvector for $\mathbf{V}$. [note corrections for text]

Ans. (a) $(\mathbf{F}^{\mathrm{T}}\mathbf{F})^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}(\mathbf{F}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, thus $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is symmetric. Also $(\mathbf{F}\mathbf{F}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{F}^{\mathrm{T}})^{\mathrm{T}}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, therefore, $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ is also symmetric.

(b) $\mathbf{F} = \mathbf{QU} \rightarrow \mathbf{F}^{\mathrm{T}} = \mathbf{U}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} \rightarrow \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{QU} = \mathbf{U}^{\mathrm{T}}\mathbf{U} \rightarrow \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2$.

$\mathbf{F} = \mathbf{VQ} \rightarrow \mathbf{F}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \rightarrow \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{VQQ}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} = \mathbf{V}\mathbf{V}^{\mathrm{T}} \rightarrow \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2$.

(c) Since $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$, and $\mathbf{Un} = \lambda\mathbf{n}$, therefore, $\mathbf{VQn} = \mathbf{QUn} = \mathbf{Q}(\lambda\mathbf{n}) \rightarrow \mathbf{V}(\mathbf{Qn}) = \lambda(\mathbf{Qn})$, therefore, $\mathbf{Qn}$ is an eigenvector for $\mathbf{V}$ with the eigenvalue λ.

2.58 Verify that the second principal scalar invariant of a tensor $\mathbf{T}$ can be written: $I_2 = \left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)/2$.

Ans. $T_{ii}T_{jj} = (T_{11} + T_{22} + T_{33})^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11}$.

$T_{ij}T_{ji} = T_{1j}T_{j1} + T_{2j}T_{j2} + T_{3j}T_{j3} = T_{11}^2 + T_{12}T_{21} + T_{13}T_{31} + T_{21}T_{12} + T_{22}^2 + T_{23}T_{32} + T_{31}T_{13} + T_{32}T_{23} + T_{33}^2$.

Thus, $T_{ii}T_{jj} - T_{ij}T_{ji} = (T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11})$

$-(T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}T_{21} + 2T_{13}T_{31} + 2T_{23}T_{32}) = 2(T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$.

Thus,

$$\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)/2 = (T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$$

$$= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = I_2.$$

2.59 A tensor has a matrix $[\mathbf{T}]$ given below. (a) Write the characteristic equation and find the principal values and their corresponding principal directions. (b) Find the principal scalar invariants. (c) If $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the principal directions, write $[\mathbf{T}]_{\mathbf{n}_i}$. (d) Could the following matrix $[\mathbf{S}]$ represent the same tensor $\mathbf{T}$ with respect to some basis.

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [\mathbf{S}] = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Ans.

(a) The characteristic equation is:

$$\begin{vmatrix} 5-\lambda & 4 & 0 \\ 4 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)\left[(5-\lambda)(-1-\lambda) - 16\right] = (3-\lambda)(\lambda^2 - 4\lambda - 21) = (3-\lambda)(\lambda+3)(\lambda-7) = 0$$

Thus, $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 7$.

For $\lambda_1 = 3$, clearly, $\mathbf{n}_1 = \pm\mathbf{e}_3$.

For $\lambda_2 = -3$

$(5+3)\alpha_1 + 4\alpha_2 = 0,\quad 4\alpha_1 + (-1+3)\alpha_2 = 0,\quad (3+3)\alpha_3 = 0,\quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\rightarrow 8\alpha_1 + 4\alpha_2 = 0,\quad 4\alpha_1 + 2\alpha_2 = 0,\quad 6\alpha_3 = 0 \rightarrow \alpha_2 = -2\alpha_1,\ \alpha_3 = 0. \rightarrow \mathbf{n}_2 = \pm(\mathbf{e}_1 - 2\mathbf{e}_2)/\sqrt{5}.$

For $\lambda_3 = 7$

$(5-7)\alpha_1 + 4\alpha_2 = 0,\quad 4\alpha_1 + (-1-7)\alpha_2 = 0,\quad (3-7)\alpha_3 = 0,\quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\rightarrow -2\alpha_1 + 4\alpha_2 = 0,\quad 4\alpha_1 - 8\alpha_2 = 0,\quad -4\alpha_3 = 0 \rightarrow \alpha_1 = 2\alpha_2,\quad \alpha_3 = 0. \rightarrow \mathbf{n}_3 = \pm(2\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{5}.$

(b) The principal scalar invariants are:

$I_1 = 5 - 1 + 3 = 7,\quad I_2 = (-5-16) + (-3-0) + (15-0) = -9,\quad I_3 = -15 - 48 = -63$. We note that

$\lambda^3 - 7\lambda^2 - 9\lambda + 63 = 0 \rightarrow (\lambda - 7)\lambda^2 - 9(\lambda - 7) = 0 \rightarrow (\lambda - 7)(\lambda^2 - 9) = 0$, same as obtained in (a)

(c) $[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}}$. (d) $\det\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -3 \neq -63$, therefore, the answer is NO. Or, clearly one of the eigenvalue for $[\mathbf{S}]$ is -1, which is not an eigenvalue for $[\mathbf{T}]$, therefore the answer is NO.

2.60 Do the previous problem for the following matrix:

$$[\mathbf{T}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

Ans. (a) The characteristic equation is:

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 0-\lambda & 4 \\ 0 & 4 & 0-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)(\lambda^2 - 16) = (3-\lambda)(\lambda - 4)(\lambda + 4) = 0$$

Thus, $\lambda_1 = 3,\quad \lambda_2 = 4,\quad \lambda_3 = -4$.

For $\lambda_1 = 3$, clearly, $\mathbf{n}_1 = \pm\mathbf{e}_1$, because $\mathbf{T}\mathbf{e}_1 = 3\mathbf{e}_1$.

For $\lambda_2 = 4$

$(3-4)\alpha_1 = 0,\quad (0-4)\alpha_2 + 4\alpha_3 = 0,\quad 4\alpha_2 + (0-4)\alpha_3 = 0,\quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$-\alpha_1 = 0,\quad -4\alpha_2 + 4\alpha_3 = 0,\ 4\alpha_2 - 4\alpha_3 = 0 \rightarrow \alpha_1 = 0,\quad \alpha_2 = \alpha_3, \rightarrow \mathbf{n}_2 = \pm(\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}.$

For $\lambda_3 = -4$

$(3+4)\alpha_1 = 0,\quad (0+4)\alpha_2 + 4\alpha_3 = 0,\quad 4\alpha_2 + (0+4)\alpha_3 = 0,\quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\rightarrow 7\alpha_1 = 0,\quad 4\alpha_2 + 4\alpha_3 = 0,\ 4\alpha_2 + 4\alpha_3 = 0 \rightarrow \alpha_1 = 0,\quad \alpha_2 = -\alpha_3, \rightarrow \mathbf{n}_3 = \pm(\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$

(b)

$I_1 = 3,\quad I_2 = (0-0) + (0-16) + (0-0) = -16,\ I_3 = -48$.

$\lambda^3 - 3\lambda^2 - 16\lambda + 48 = 0 \rightarrow (\lambda - 3)\lambda^2 - 16(\lambda - 3) = 0 \rightarrow (\lambda - 3)(\lambda^2 - 16) = 0$, same as in (a) .

(c) $[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}_{\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}}$

(d) $\det[\mathbf{S}] = \det\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -(7-4) = -3 \neq -48$, therefore, the answer is NO.

Or, clearly one of the eigenvalue for $[\mathbf{S}]$ is -1, which is not an eigenvalue for $[\mathbf{T}]$, therefore the answer is NO.

2.61 A tensor $\mathbf{T}$ has a matrix given below. Find the principal values and three mutually perpendicular principal directions: $[\mathbf{T}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Ans. The characteristic equation is:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \to (2-\lambda)\left[(1-\lambda)^2 - 1\right] = (2-\lambda)(-2\lambda+\lambda^2) = -\lambda(2-\lambda)^2 = 0 .$$

Thus, $\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 2$. That is, there is a double root $\lambda_2 = \lambda_3 = 2$.

For $\lambda_1 = 0$,

$$(1-0)\alpha_1 + \alpha_2 = 0, \quad \alpha_1 + (1-0)\alpha_2 = 0, \quad (2-0)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$\to \alpha_1 + \alpha_2 = 0, \quad 2\alpha_3 = 0 \to \alpha_1 = -\alpha_2, \quad \alpha_3 = 0, \to \mathbf{n}_1 = \pm(\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}.$$

For $\lambda_2 = \lambda_3 = 2$, one eigenvector is clearly $\mathbf{n}_3$. There are infinitely many others all lie on the plane whose normal is $\mathbf{n}_1 = \pm(\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$. In fact, we have,

$$(1-2)\alpha_1 + \alpha_2 = 0, \quad \alpha_1 + (1-2)\alpha_2 = 0, \quad (2-2)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$\to -\alpha_1 + \alpha_2 = 0, \quad 0\alpha_3 = 0 \to \alpha_1 = \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{1-2\alpha^2} \to \mathbf{n} = \pm(\alpha\mathbf{e}_1 + \alpha\mathbf{e}_2 + \alpha_3\mathbf{e}_3),$$

which include the case where $\alpha = 0, \quad \alpha_3 = \pm 1 \to \mathbf{n} = \pm\mathbf{e}_3$.

CHAPTER 2, PART C

2.62 Prove the identity $\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}$, using the definition of derivative of a tensor.

Ans.

$$\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \lim_{\Delta t \to 0} \frac{\{\mathbf{T}(t+\Delta t) + \mathbf{S}(t+\Delta t)\} - \{\mathbf{T}(t) + \mathbf{S}(t)\}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)\} + \{\mathbf{S}(t+\Delta t) - \mathbf{S}(t)\}}{\Delta t} -$$

$$\lim_{\Delta t \to 0} \frac{\{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)\}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\{\{\mathbf{S}(t+\Delta t) - \mathbf{S}(t)\}}{\Delta t} = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}.$$

2.63 Prove the identity $\frac{d}{dt}(\mathbf{TS}) = \mathbf{T}\frac{d\mathbf{S}}{dt} + \frac{d\mathbf{T}}{dt}\mathbf{S}$ using the definition of derivative of a tensor.

Ans. $\frac{d}{dt}(\mathbf{TS}) = \lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)\mathbf{S}(t+\Delta t)-\mathbf{T}(t)\mathbf{S}(t)}{\Delta t}$

$= \lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)\mathbf{S}(t+\Delta t)-\mathbf{T}(t+\Delta t)\mathbf{S}(t)+\mathbf{T}(t+\Delta t)\mathbf{S}(t)-\mathbf{T}(t)\mathbf{S}(t)}{\Delta t}$

$= \lim_{\Delta t\to 0}\frac{\mathbf{T}(t+\Delta t)\{\mathbf{S}(t+\Delta t)-\mathbf{S}(t)\}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\{\mathbf{T}(t+\Delta t)-\mathbf{T}(t)\}\mathbf{S}(t)}{\Delta t}$

$= \mathbf{T}(t)\lim_{\Delta t\to 0}\frac{\{\mathbf{S}(t+\Delta t)-\mathbf{S}(t)\}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\{\mathbf{T}(t+\Delta t)-\mathbf{T}(t)\}}{\Delta t}\mathbf{S}(t) = \mathbf{T}\frac{d\mathbf{S}}{dt}+\frac{d\mathbf{T}}{dt}\mathbf{S}.$

2.64 Prove that $\frac{d\mathbf{T}^{\mathrm{T}}}{dt} = \left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}}$ by differentiating the definition $\mathbf{a}\cdot\mathbf{Tb} = \mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant arbitrary vectors.

Ans. $\mathbf{a}\cdot\mathbf{Tb} = \mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{a} \to \mathbf{a}\cdot(d\mathbf{T}/dt)\mathbf{b} = \mathbf{b}\cdot(d\mathbf{T}^{\mathrm{T}}/dt)\mathbf{a}$. Now, the definition of transpose also gives $\mathbf{a}\cdot(d\mathbf{T}/dt)\mathbf{b} = \mathbf{b}\cdot(d\mathbf{T}/dt)^{\mathrm{T}}\mathbf{a}$. Thus, $\mathbf{b}\cdot(d\mathbf{T}/dt)^{\mathrm{T}}\mathbf{a} = \mathbf{b}\cdot(d\mathbf{T}^{\mathrm{T}}/dt)\mathbf{a}$.

Since $\mathbf{a}$ and $\mathbf{b}$ arbitrary vectors, therefore, $\left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}} = \left(\frac{d\mathbf{T}^{\mathrm{T}}}{dt}\right)$.

2.65 Consider the scalar field $\phi = x_1^2 + 3x_1x_2 + 2x_3$. (a) Find the unit vector normal to the surface of constant ϕ at the origin (0,0,0) and at $(1,0,1)$. (b) what is the maximum value of the directional derivative of ϕ at the origin? At (1,0,1)? (c) Evaluate $d\phi/dr$ at the origin if $d\mathbf{r} = ds(\mathbf{e}_1+\mathbf{e}_3)$.

Ans. (a) $\nabla\phi = (2x_1+3x_2)\mathbf{e}_1 + 3x_1\mathbf{e}_2 + 2\mathbf{e}_3$,

at $(0,0,0)$, $\nabla\phi = 2\mathbf{e}_3 \to \mathbf{n} = \mathbf{e}_3$, at $(1,0,1)$, $\nabla\phi = 2\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3, \to \mathbf{n} = (2\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3)/\sqrt{17}$.

(b) At $(0,0,0)$, $(d\phi/dr)_{\max} = |\nabla\phi| = 2$ in the direction of $\mathbf{n}=\mathbf{e}_3$.

At $(1,0,1)$, $(d\phi/dr)_{\max} = |\nabla\phi| = \sqrt{17}$.

(c) At $(0,0,0)$, $d\phi/dr = (\nabla\phi)_{\mathrm{o}}\cdot d\mathbf{r}/dr = 2\mathbf{e}_3\cdot(\mathbf{e}_1+\mathbf{e}_3)/\sqrt{2} = \sqrt{2}$.

2.66 Consider the ellipsoidal surface defined by the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find the unit vector normal to the surface at a given point (x,y,z).

Ans. Let $f(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, then

$\frac{\partial f}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{b^2}, \quad \frac{\partial f}{\partial z} = \frac{2z}{c^2} \to \nabla f = \frac{2x}{a^2}\mathbf{e}_1 + \frac{2y}{b^2}\mathbf{e}_2 + \frac{2z}{c^2}\mathbf{e}_3$, thus,

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \left[\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2\right]^{-1/2}\left(\frac{2x}{a^2}\mathbf{e}_1 + \frac{2y}{b^2}\mathbf{e}_2 + \frac{2z}{c^2}\mathbf{e}_3\right).$$

2.67 Consider the temperature field given by: $\Theta = 3x_1x_2$. (a) Find the heat flux at the point $A(1,1,1)$, if $\mathbf{q} = -k\nabla\Theta$. (b) Find the heat flux at the same point if $\mathbf{q} = -\mathbf{K}\nabla\Theta$, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

Ans. $\Theta = 3x_1x_2 \to \nabla\Theta = 3(x_2\mathbf{e}_1 + x_1\mathbf{e}_2) \to (\nabla\Theta)_A = 3(\mathbf{e}_1 + \mathbf{e}_2)$.

(a) $\mathbf{q} = -k\nabla\Theta = -3k(\mathbf{e}_1 + \mathbf{e}_2)$.

(b) $[\mathbf{q}] = -[\mathbf{K}\nabla\Theta] = -\begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} 3k \\ 6k \\ 0 \end{bmatrix} \to \mathbf{q} = -(3k\mathbf{e}_1 + 6k\mathbf{e}_2)$.

2.68 Let $\phi(x_1,x_2,x_3)$ and $\psi(x_1,x_2,x_3)$ be scalar fields, and let $\mathbf{v}(x_1,x_2,x_3)$ and $\mathbf{w}(x_1,x_2,x_3)$ be vector fields. By writing the subscripted components form, verify the following identities.

(a) $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$, sample solution: $[\nabla(\phi+\psi)]_i = \dfrac{\partial(\phi+\psi)}{\partial x_i} = \dfrac{\partial\phi}{\partial x_i} + \dfrac{\partial\psi}{\partial x_i} = \nabla\phi + \nabla\psi$.

(b) $\text{div}(\mathbf{v}+\mathbf{w}) = \text{div}\mathbf{v} + \text{div}\mathbf{w}$, (c) $\text{div}(\phi\mathbf{v}) = (\nabla\phi)\cdot\mathbf{v} + \phi(\text{div}\mathbf{v})$ and (d) $\text{div}(\text{curl}\mathbf{v}) = 0$.

Ans. (b) $\text{div}(\mathbf{v}+\mathbf{w}) = \dfrac{\partial(v_i + w_i)}{\partial x_i} = \dfrac{\partial v_i}{\partial x_i} + \dfrac{\partial w_i}{\partial x_i} = \text{div}\mathbf{v} + \text{div}\mathbf{w}$.

(c) $\text{div}(\phi\mathbf{v}) = \dfrac{\partial(\phi v_i)}{\partial x_i} = \phi\dfrac{\partial v_i}{\partial x_i} + \dfrac{\partial\phi}{\partial x_i}v_i = \phi(\text{div}\mathbf{v}) + (\nabla\phi)\cdot\mathbf{v}$.

(d) $\text{curl}\mathbf{v} = -\varepsilon_{ijk}\dfrac{\partial v_j}{\partial x_k}\mathbf{e}_i = \varepsilon_{ijk}\dfrac{\partial v_k}{\partial x_j}\mathbf{e}_i \to \text{div}(\text{curl}\mathbf{v}) = \dfrac{\partial}{\partial x_i}\varepsilon_{ijk}\dfrac{\partial v_k}{\partial x_j} = \varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j}$.

By changing the dummy indices, ($i \to j, j \to i$) we have, $\varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j} = \varepsilon_{jik}\dfrac{\partial}{\partial x_j}\dfrac{\partial v_k}{\partial x_i}$. Thus,

$$\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_j} = -\varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_i} \to 2\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_j} = 0 \to \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}\right) = 0.$$ Thus, $\text{div}(\text{curl}\mathbf{v}) = 0$.

2.69 Consider the vector field $\mathbf{v} = x_1^2\mathbf{e}_1 + x_3^2\mathbf{e}_2 + x_2^2\mathbf{e}_3$. For the point $(1,1,0)$, find (a) $\nabla\mathbf{v}$, (b) $(\nabla\mathbf{v})\mathbf{v}$, (c) $\text{div}\,\mathbf{v}$ and $\text{curl}\,\mathbf{v}$ and (d) the differential $d\mathbf{v}$ for $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$.

Ans.(a) $[\nabla\mathbf{v}] = \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 0 & 2x_3 \\ 0 & 2x_2 & 0 \end{bmatrix} \to [\nabla\mathbf{v}]_{(1,1,0)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

(b) $[(\nabla\mathbf{v})\mathbf{v}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \to (\nabla\mathbf{v})\mathbf{v} = 2\mathbf{e}_1$.

(c) $\text{div}\,\mathbf{v} = 2x_1 + 0 + 0 = 2x_1 \to$ at $(1,1,0)$, $\text{div}\,\mathbf{v} = 2$.

$$\text{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 = 2(x_2 - x_3)\mathbf{e}_1.$$

At $(1,1,0)$, $\text{curl}\mathbf{v} = 2(1-0)\mathbf{e}_1 = 2\mathbf{e}_1$.

(d)

$$\text{At}(1,1,0),\ d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{r} \to [d\mathbf{v}] = [\nabla\mathbf{v}][d\mathbf{r}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} ds/\sqrt{3} \\ ds/\sqrt{3} \\ ds/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2ds/\sqrt{3} \\ 0 \\ 2ds/\sqrt{3} \end{bmatrix}.$$

$\to d\mathbf{v} = 2ds(\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{3}$

CHAPTER 2, PART D

2.70 Calculate div $\mathbf{u}$ for the following vector field in cylindrical coordinates:
(a) $u_r = u_\theta = 0,\quad u_z = A + Br^2$. (b) $u_r = \sin\theta / r,\ u_\theta = u_z = 0$, and
(c) $u_r = r^2\sin\theta/2,\quad u_\theta = r^2\cos\theta/2,\quad u_z = 0$.

Ans.(a) $u_r = u_\theta = 0,\quad u_z = A + Br^2 \to \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0+0+0+0 = 0$.

(b) $u_r = \sin\theta/r,\ u_\theta = u_z = 0 \to \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = -\sin\theta/r^2 + 0 + \sin\theta/r^2 + 0 = 0$

(c) $u_r = r^2\sin\theta/2,\quad u_\theta = r^2\cos\theta/2,\quad u_z = 0$

$\to \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = r\sin\theta - r\sin\theta/2 + r\sin\theta/2 + 0 = r\sin\theta$.

2.71 Calculate $\nabla\mathbf{u}$ for the following vector field in cylindrical coordinate:
$u_r = A/r,\quad u_\theta = Br,\quad u_z = 0$.

$$\textit{Ans.}\ [\nabla\mathbf{u}] = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{A}{r^2} & -B & 0 \\ B & \frac{A}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.72 Calculate div $\mathbf{u}$ for the following vector field in spherical coordinates
$u_r = Ar + B/r^2,\quad u_\theta = u_\phi = 0$

Ans. $\to \text{div}\,\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(Ar+\frac{B}{r^2}\right)\right\}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^3+B\right)=3A.$$

2.73 Calculate $\nabla\mathbf{u}$ for the following vector field in spherical coordinates: $u_r = Ar + B/r^2,\quad u_\theta = u_\phi = 0$.

Ans.
$$[\nabla\mathbf{u}]=\begin{bmatrix}\frac{\partial u_r}{\partial r} & \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)\\ \frac{\partial u_\theta}{\partial r} & \left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cot\theta}{r}\right)\\ \frac{\partial u_\phi}{\partial r} & \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} & \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\partial u_r/\partial r & 0 & 0\\ 0 & u_r/r & 0\\ 0 & 0 & u_r/r\end{bmatrix}=\begin{bmatrix}A-2B/r^3 & 0 & 0\\ 0 & A+B/r^3 & 0\\ 0 & 0 & A+B/r^3\end{bmatrix}.$$

2.74 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v}=\nabla(\text{div }\mathbf{v})-\text{curl}\,\text{curl }\mathbf{v}$, derive the following results in cylindrical coordinates:

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right)\text{ and}$$

$$\left(\nabla^2\mathbf{v}\right)_\theta=\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}.$$

Ans. Let $\mathbf{v}(r)$ be a vector field. The Laplacian of $\mathbf{v}$ is $\nabla^2\mathbf{v}=\nabla(\text{div }\mathbf{v})-\text{curl}\,\text{curl }\mathbf{v}$. Now,

$$\text{div }\mathbf{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\text{, so that}$$

$$\nabla(\text{div }\mathbf{v})=\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_\theta$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_z=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)\mathbf{e}_r$$

$$+\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)\mathbf{e}_\theta+\left(\frac{\partial^2 v_r}{\partial z\partial r}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial z\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial z}+\frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{e}_z.$$

Next,

$$\text{curl }\mathbf{v}=\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta+\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_z\text{, so that}$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_r = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)$$

$$=\left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right)-\left(\frac{\partial^2 v_r}{\partial z^2}-\frac{\partial^2 v_z}{\partial z\partial r}\right),$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_\theta = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_z = \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)$$

Thus,

$$\left(\nabla^2\mathbf{v}\right)_r = \left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)$$

$$-\left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right)+\left(\frac{\partial^2 v_r}{\partial z^2}-\frac{\partial^2 v_z}{\partial z\partial r}\right)=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right).$$

$$\left(\nabla^2\mathbf{v}\right)_\theta = \left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)$$

$$-\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)=\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)$$

$$+\left(-\frac{1}{r}\frac{\partial^2 v_z}{\partial z\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}-\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right)$$

$$=\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}.$$

2.75 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v}=\nabla(\text{div}\,\mathbf{v})-\text{curlcurl}\,\mathbf{v}$, derive the following result in spherical coordinates:

$$\left(\nabla^2\mathbf{v}\right)_r = \left(\frac{1}{r^2}\frac{\partial^2 r^2 v_r}{\partial r^2}-\frac{2}{r^3}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

Ans.

From $\frac{1}{r^2}\frac{\partial\left(r^2 v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$, we have,

$$\nabla(\text{div}\mathbf{v})=\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\theta$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}++\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\phi,$$ that is

$$\nabla(\text{div}\mathbf{v})=\left(\frac{1}{r^2}\frac{\partial^2\left(r^2 v_r\right)}{\partial r^2}-\frac{2}{r^3}\frac{\partial\left(r^2 v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\theta\sin\theta}{\partial r\partial\theta}-\frac{1}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_r$$

$$+\left(\frac{1}{r^3}\frac{\partial^2\left(r^2v_r\right)}{\partial\theta\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial\theta^2}+\frac{1}{r^2\sin\theta}\frac{\partial^2 v_\theta\sin\theta}{\partial\theta^2}-\frac{1}{r^2}\left(\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\theta$$

$$+\left(\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial\left(r^2v_r\right)}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial\phi\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial^2\phi}\right)\mathbf{e}_\phi$$. Also,

$$\text{curl }\mathbf{v}=\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)\mathbf{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)\mathbf{e}_\theta+\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_\phi$$

so that

$$\text{curl}\,\text{curl }\mathbf{v}=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\sin\theta-\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)\right\}\mathbf{e}_r$$

$$+\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)-\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\right\}\mathbf{e}_\theta$$

$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)\right\}\mathbf{e}_\phi$$

i.e.,

$\text{curl}\,\text{curl }\mathbf{v}=$

$$\left\{\frac{1}{r^2}\left(\frac{\partial^2 rv_\theta}{\partial\theta\partial r}-\frac{\partial^2 v_r}{\partial\theta^2}\right)+\frac{\cot\theta}{r}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial^2 rv_\phi}{\partial\phi\partial r}\right)\right\}\mathbf{e}_r$$

$$+\left\{\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 v_\phi\sin\theta}{\partial\phi\partial\theta}-\frac{\partial^2 v_\theta}{\partial\phi^2}\right)-\frac{1}{r^2}\left(\frac{\partial rv_\theta}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r}\frac{\partial^2 rv_\theta}{\partial r^2}-\frac{1}{r^2}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right)\right\}\mathbf{e}_\theta$$

$$+\left\{\begin{array}{l}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial^2 rv_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial rv_\phi}{\partial r}\right)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{\partial rv_\phi}{\partial r}\right)\\ -\frac{1}{r^2}\frac{1}{\sin\theta}\left(-v_\phi\sin\theta+\sin\theta\frac{\partial^2 v_\phi}{\partial\theta^2}-\frac{\partial^2 v_\theta}{\partial\theta\partial\phi}\right)+\frac{\cos\theta}{r^2\sin^2\theta}\left(\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{\partial v_\theta}{\partial\phi}\right)\end{array}\right\}\mathbf{e}_\phi$$

Thus, $\nabla^2\mathbf{v}=\nabla\left(\text{div}\mathbf{v}\right)-\text{curl}\,\text{curl }\mathbf{v}$ gives:

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{1}{r^2}\frac{\partial^2\left(r^2v_r\right)}{\partial r^2}-\frac{2}{r^3}\frac{\partial\left(r^2v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial r\partial\theta}-\frac{1}{r^2\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$-\left\{\frac{1}{r^2}\left(\frac{\partial^2 rv_\theta}{\partial\theta\partial r}-\frac{\partial^2 v_r}{\partial\theta^2}\right)+\frac{\cot\theta}{r}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial^2 rv_\phi}{\partial\phi\partial r}\right)\right\}$$

i.e.,

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{1}{r^2}\frac{\partial^2 r^2v_r}{\partial r^2}-\frac{2}{r^3}\frac{\partial r^2v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

2.76 From the equation $(\text{div}\mathbf{T})\cdot\mathbf{a} = \text{div}(\mathbf{T}^{\text{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a})$ [See Eq. 2.29.3)] verify that in polar coordinates, the θ-component of the vector $(\text{div}\mathbf{T})$ is: $(\text{div}\mathbf{T})_\theta = \dfrac{\partial T_{\theta r}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\theta}}{\partial\theta} + \dfrac{T_{r\theta}+T_{\theta r}}{r}$.

Ans. $(\text{div}\mathbf{T})\cdot\mathbf{a} = \text{div}(\mathbf{T}^{\text{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a}) \to (\text{div}\mathbf{T})\cdot\mathbf{e}_\theta = \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta)$

Now,

$$\mathbf{Te}_{\text{r}} = T_{rr}\mathbf{e}_{\text{r}} + T_{\theta r}\mathbf{e}_\theta, \mathbf{Te}_\theta = T_{r\theta}\mathbf{e}_{\text{r}} + T_{\theta\theta}\mathbf{e}_\theta \to \mathbf{e}_{\text{r}}\cdot\mathbf{T}^{\text{T}}\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{Te}_{\text{r}} = T_{\theta r},\ \ \mathbf{e}_\theta\cdot\mathbf{T}^{\text{T}}\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{Te}_\theta = T_{\theta\theta}$$

i.e., $\mathbf{T}^{\text{T}}\mathbf{e}_\theta = T_{\theta r}\mathbf{e}_{\text{r}} + T_{\theta\theta}\mathbf{e}_\theta \to \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) = \dfrac{\partial T_{\theta r}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{\theta\theta}}{\partial\theta} + \dfrac{T_{\theta r}}{r}$. Also

$$\mathbf{e}_\theta = 0\mathbf{e}_{\text{r}} + 1\mathbf{e}_\theta \to [\nabla\mathbf{e}_\theta] = \begin{bmatrix} 0 & -1/r \\ 0 & 0 \end{bmatrix} \to \left[\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta\right] = \begin{bmatrix} T_{rr} & T_{\theta r} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix}\begin{bmatrix} 0 & -1/r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -T_{rr}/r \\ 0 & -T_{r\theta}/r \end{bmatrix}$$

Thus,

$$(\text{div}\mathbf{T})_\theta = \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{\theta r}}{r} - (0 - T_{r\theta}/r) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{\theta r}+T_{r\theta}}{r}.$$

2.77 Calculate $\text{div}\,\mathbf{T}$ for the following tensor field in cylindrical coordinates;

$$T_{rr} = A + \frac{B}{r^2},\quad T_{\theta\theta} = A - \frac{B}{r^2},\quad T_{zz} = \text{constant},\quad T_{r\theta} = T_{\theta r} = T_{rz} = T_{zr} = T_{\theta z} = T_{z\theta} = 0$$

Ans. $(\text{div}\mathbf{T})_r = \dfrac{\partial T_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial T_{r\theta}}{\partial\theta} + \dfrac{T_{rr}-T_{\theta\theta}}{r} + \dfrac{\partial T_{rz}}{\partial z} = -\dfrac{2B}{r^3} + \dfrac{2B}{r^3} = 0$.

$$(\text{div}\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{r\theta}+T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} = 0.$$

$$(\text{div}\mathbf{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} = 0.$$

2.78 Calculate $\text{div}\mathbf{T}$ for the following tensor field in cylindrical coordinates;

$$T_{rr} = \frac{Az}{R^3} - \frac{3Br^2z}{R^5},\quad T_{\theta\theta} = \frac{Az}{R^3},\quad T_{zz} = -\left(\frac{Az}{R^3} + \frac{3Bz^3}{R^5}\right),\quad T_{rz} = T_{zr} = -\left(\frac{Ar}{R^3} + \frac{3Brz^2}{R^5}\right)$$

$$T_{r\theta} = T_{\theta r} = T_{\theta z} = T_{z\theta} = 0,\ \ R^2 = r^2 + z^2.$$

Ans.

$$(\text{div}\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} + \frac{T_{rr}-T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = \frac{\partial}{\partial r}\left(\frac{Az}{R^3} - \frac{3Br^2z}{R^5}\right) - \frac{3Brz}{R^5} - \frac{\partial}{\partial z}\left(\frac{Ar}{R^3} + \frac{3Brz^2}{R^5}\right)$$

$$= \left(Az\frac{\partial}{\partial r}\frac{1}{R^3} - 3Br^2z\frac{\partial}{\partial r}\frac{1}{R^5} - \frac{3Bz}{R^5}\frac{\partial}{\partial r}r^2\right) - \frac{3Brz}{R^5} - \left(Ar\frac{\partial}{\partial z}\frac{1}{R^3} + \frac{3Br}{R^5}\frac{\partial}{\partial z}z^2 + 3Brz^2\frac{\partial}{\partial z}\frac{1}{R^5}\right)$$

$$= \left(-\frac{3Az}{R^4}\frac{\partial R}{\partial r} + \frac{15Br^2z}{R^6}\frac{\partial R}{\partial r} - \frac{6Brz}{R^5}\right) - \frac{3Brz}{R^5} - \left(-\frac{3Ar}{R^4}\frac{\partial R}{\partial z} + \frac{6Bzr}{R^5} - \frac{15Brz^2}{R^6}\frac{\partial R}{\partial z}\right)$$

$$= \left(-\frac{3Arz}{R^5} + \frac{15Br^3z}{R^7} - \frac{6Brz}{R^5}\right) - \frac{3Brz}{R^5} + \left(\frac{3Arz}{R^5} - \frac{6Bzr}{R^5} + \frac{15Brz^3}{R^7}\right)$$

$$=B\left(\frac{15r^3z}{R^7}-\frac{15rz}{R^5}+\frac{15rz^3}{R^7}\right)=B\left(\frac{15rz}{R^7}\left(r^2+z^2\right)-\frac{15rz}{R^5}\right)=\left(\frac{15rz}{R^5}-\frac{15rz}{R^5}\right)=0.$$

$$(\text{div}\mathbf{T})_\theta=\frac{\partial T_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{T_{r\theta}+T_{\theta r}}{r}+\frac{\partial T_{\theta z}}{\partial z}=0+0+0+0=0$$

$$(\text{div}\mathbf{T})_z=\frac{\partial T_{zr}}{\partial r}+\frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta}+\frac{\partial T_{zz}}{\partial z}+\frac{T_{zr}}{r}=-\frac{\partial}{\partial r}\left(\frac{A}{R^3}+\frac{3Brz^2}{R^5}\right)-\frac{\partial}{\partial z}\left(\frac{Az}{R^3}+\frac{3Bz^3}{R^5}\right)-\left(\frac{A}{R^3}+\frac{3Bz^2}{R^5}\right)$$

$$=-\left(\frac{A}{R^3}-\frac{3Ar^2}{R^5}+\frac{3Bz^2}{R^5}-\frac{15Br^2z^2}{R^7}\right)-\left(\frac{A}{R^3}-\frac{3Az^2}{R^5}+\frac{9Bz^2}{R^5}-\frac{15Bz^4}{R^7}\right)-\left(\frac{A}{R^3}+\frac{3Bz^2}{R^5}\right)$$

$$=\left(-\frac{3A}{R^3}+\frac{3A}{R^5}\left(r^2+z^2\right)-\frac{15Bz^2}{R^5}+\frac{15Bz^2}{R^7}\left(r^2+z^2\right)\right)==\left(-\frac{3A}{R^3}+\frac{3A}{R^3}-\frac{15Bz^2}{R^5}+\frac{15Bz^2}{R^5}\right)=0.$$

2.79 Calculate $\text{div}\,\mathbf{T}$ for the following tensor field in spherical coordinates;

$$T_{rr}=A-\frac{2B}{r^3},\quad T_{\theta\theta}=T_{\phi\phi}=A+\frac{B}{r^3},\quad T_{r\theta}=T_{\theta r}=T_{\theta\phi}=T_{\phi\theta}=T_{r\phi}=T_{\phi r}=0$$

Ans. $(\text{div}\mathbf{T})_r=\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}$

$$=\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^2-\frac{2B}{r}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)$$

$$=\frac{1}{r^2}\left(2Ar+\frac{2B}{r^2}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)=\left(\frac{2A}{r}+\frac{2B}{r^4}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)=0.$$

$$(\text{div}\mathbf{T})_\theta=\frac{1}{r^3}\frac{\partial\left(r^3T_{\theta r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\theta\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}+\frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r}$$

$$=\frac{T_{\theta\theta}\cot\theta}{r}+\frac{-T_{\phi\phi}\cot\theta}{r}=0.$$

$$(\text{div}\mathbf{T})_\phi=\frac{1}{r^3}\frac{\partial\left(r^3T_{\phi r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\phi\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{r\phi}-T_{\phi r}+T_{\theta\phi}\cot\theta}{r}=0.$$

2.80 From the equation $(\text{div}\mathbf{T})\cdot\mathbf{a}=\text{div}(\mathbf{T}^{\text{T}}\mathbf{a})-\text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a})$ [See Eq. 2.29.3)] verify that in spherical coordinates, the θ-component of the vector $(\text{div}\mathbf{T})$ is:

$$(\text{div}\mathbf{T})_\theta=\frac{1}{r^3}\frac{\partial(r^3T_{\theta r})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta}++\frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}+\frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r}.$$

Ans. $(\text{div}\mathbf{T})\cdot\mathbf{a}=\text{div}(\mathbf{T}^{\text{T}}\mathbf{a})-\text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a})\rightarrow(\text{div}\mathbf{T})\cdot\mathbf{e}_\theta=\text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta)-\text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta)$. Now,

$$\mathbf{T}^{\mathrm{T}}\mathbf{e}_{\theta} = T_{\theta r}\mathbf{e}_{\mathrm{r}} + T_{\theta\theta}\mathbf{e}_{\theta} + T_{\theta\phi}\mathbf{e}_{\phi} \rightarrow \mathrm{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_{\theta}) = \frac{1}{r^2}\frac{\partial(r^2 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}.$$ Also,

$$\mathbf{e}_{\theta} = 0\mathbf{e}_{\mathrm{r}} + 1\mathbf{e}_{\theta} + 0\mathbf{e}_{\phi} \rightarrow [\nabla\mathbf{e}_{\theta}] = \begin{bmatrix} 0 & -1/r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cot\theta/r \end{bmatrix}$$

$$\rightarrow \left[\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_{\theta}\right] = \begin{bmatrix} T_{rr} & T_{\theta r} & T_{\phi r} \\ T_{r\theta} & T_{\theta\theta} & T_{\phi\theta} \\ T_{r\phi} & T_{\theta\phi} & T_{\phi\phi} \end{bmatrix} \begin{bmatrix} 0 & -1/r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cot\theta/r \end{bmatrix} = \begin{bmatrix} 0 & -T_{rr}/r & T_{\phi r}\cot\theta/r \\ 0 & -T_{r\theta}/r & T_{\phi\theta}\cot\theta/r \\ 0 & -T_{r\phi}/r & T_{\phi\phi}\cot\theta/r \end{bmatrix}$$

$$\rightarrow \mathrm{tr}\left[\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_{\theta}\right] = -\frac{T_{r\theta}}{r} + \frac{T_{\phi\phi}\cot\theta}{r}.$$ Thus,

$$(\mathrm{div}\mathbf{T})_{\theta} = \mathrm{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_{\theta}) - \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_{\theta})$$

$$= \frac{1}{r^2}\frac{\partial(r^2 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}}{r} - \frac{T_{\phi\phi}\cot\theta}{r}$$

$$= \frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} - \frac{T_{\theta r}}{r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}}{r} - \frac{T_{\phi\phi}\cot\theta}{r}.$$
