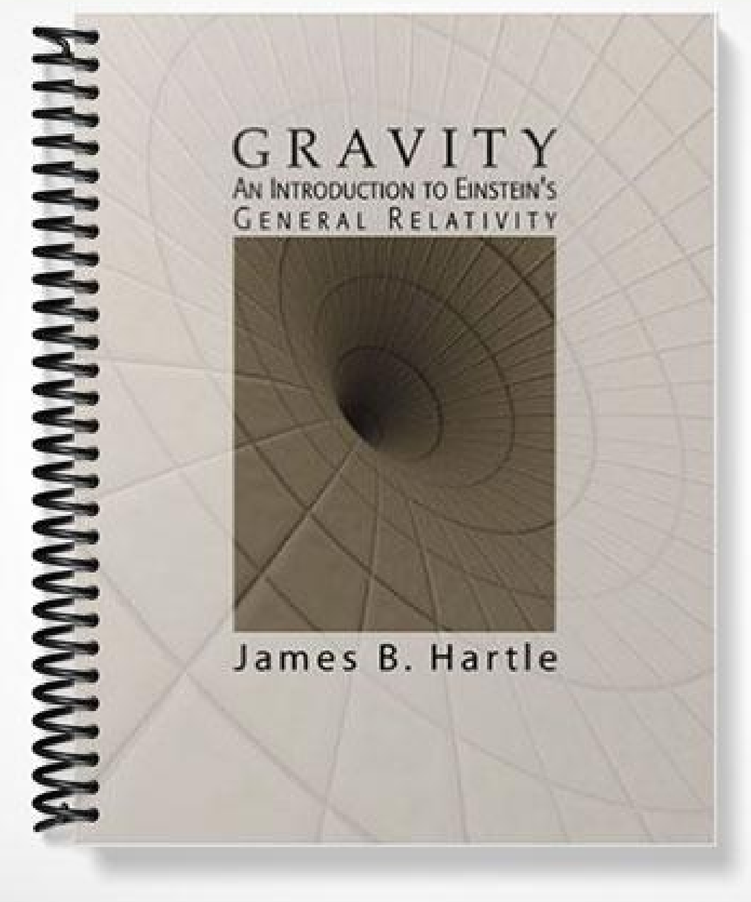
SOLUTIONS MANUAL



GRAVITY

An Introduction to Einstein's General Relativity

SOLUTIONS TO PROBLEMS

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Preface

This manual contains the author's solutions to the 392 problems in *Gravity: An Introduction to Einstein's General Relativity*. I have aimed at explaining the central ideas needed to solve each problem; I have not generally attempted to write out each step of the calculations involved. I hope that the solutions will be clear to instructors teaching from the text. Depending on their level, students may need further discussion or more details.

The text of the problems are provided for convenience. This may differ slightly from the published text because of copy editing changes or subsequent errata.

While a considerable effort has been made to check the solutions, it is inevitable that mistakes remain in a collection of problems of this size and occasional complexity. I invite suggestions for correction and improvement.

The manuscript was typed by Thea Howard who also drew all the line figures. The solutions were checked and corrected by Matt Hansen and Taro Sato. Thea, Matt, and Taro have my gratitude in these regards, as do the many students and teaching assistants who tried out these problems in various courses at Santa Barbara and helped to make them better.

James Hartle
November, 2002

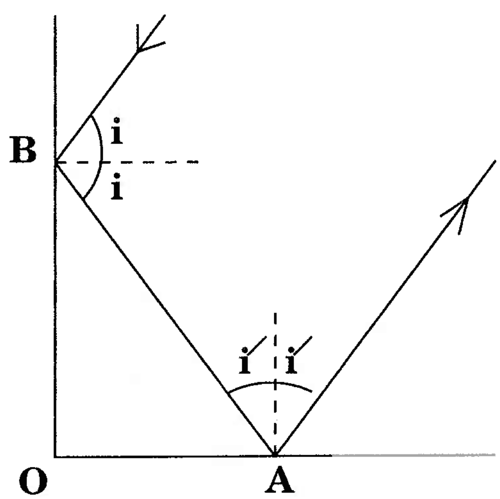
Chapter 2

Geometry as Physics

2-1. *[B] (a) In a plane, show that a light ray incident from any angle on a right angle corner reflector returns in the same direction from whence it came.*

(b) Show the same thing in three dimensions with a cubical corner reflector.

Solution:



a) Snell's law of reflection is that the angle the incident ray makes with the normal to the surface is the same as the angle the reflected ray makes with the normal as shown above. Since the sum of the interior angles of the right triangle OAB is π, this implies

$$\imath' = \frac{\pi}{2} - \imath \ .$$

Equivalently the angle the incident ray makes with OB is $\pi/2$ minus the angle the reflected ray makes with OA. The reflected ray is thus parallel to the incident ray.

b) Snell's law may be stated vectorially as follows. Let $\vec{k}$ be a unit vector along a ray incident on a surface with normal $\vec{n}$. $\vec{k}$ can be divided into a component along $\vec{n}$ and a component perpendicular to $\vec{n}$ as follows

$$\vec{k} = \left(\vec{k}\cdot\vec{n}\right)\vec{n} + \left(\vec{k} - \left(\vec{k}\cdot\vec{n}\right)\vec{n}\right) \ .$$

On reflection the component along $\vec{n}$ changes sign while the perpendicular component remains unchanged. Thus, $\vec{k}'$ after reflection is

$$\begin{aligned} \vec{k}' &= -\left(\vec{k}\cdot\vec{n}\right)\vec{n} + \left(\vec{k} - \left(\vec{k}\cdot\vec{n}\right)\vec{n}\right) \qquad (1) \\ \vec{k}' &= \vec{k} - 2\left(\vec{k}\cdot\vec{n}\right)\vec{n} \ . \end{aligned}$$

Consider a ray which reflects off of all three faces of the corner reflector with orthogonal normals $\vec{n}_1, \vec{n}_2, \vec{n}_3$ respectively. Using (1) at each of the three reflections, the output of the previous reflection being the input to the next, one finds for the exiting ray $\vec{k}_{\mathrm{ex}}$ in terms of the incident ray $\vec{k}_{\mathrm{in}}$

$$\begin{aligned} \vec{k}_{\mathrm{ex}} = \vec{k}_{\mathrm{in}} &- 2\left(\vec{k}_{\mathrm{in}}\cdot\vec{n}_1\right)\vec{n}_1 - 2\left(\vec{k}_{\mathrm{in}}\cdot\vec{n}_2\right)\vec{n}_2 \\ &- 2\left(\vec{k}_{\mathrm{in}}\cdot\vec{n}_3\right)\vec{n}_3 \end{aligned}$$

But $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are three orthogonal vectors that form a basis. Thus,

$$\vec{k}_{\mathrm{ex}} = \vec{k}_{\mathrm{in}} - 2\vec{k}_{\mathrm{in}} = -\vec{k}_{\mathrm{in}} \ .$$

so the exit ray leaves in the direction opposite to the incident one.

2-2. *[S] The center of the Sun is much further way from a terrestrial measurement of angles than the center of the Earth is. But it is also much more massive. Using* (2.1), *estimate which would have the greatest effect on a measurement of angles such as is attributed to Gauss.*

Solution: For the Earth $GM_\oplus/R_\oplus c^2 \sim 10^{-9}$. The relevant ratio for the Sun is $GM_\odot/c^2 r_\oplus$ where $r_\oplus$ is the distance from the Sun to Earth. From the table of useful constants this is

$$\frac{GM_\odot}{c^2 r_\oplus} \sim \frac{1.48 \text{ km}}{1.4 \times 10^8 \text{ km}} \sim 10^{-8} \,.$$

The effect of the Sun is therefore potentially larger.

2-3. *[C] (a) Verify the relation* (2.4) *between the sum of the interior angles of a spherical triangle and its area when two of the angles are right angles.*

(b) Prove the relation generally.

Solution:

a) Such a triangle can be bounded by the equator and two lines of longitude differing by an angle α. The area A is $(\alpha/2\pi) \times$ (area of a hemisphere) $= \alpha a^2$.

$$\begin{pmatrix}\text{sum of the} \\ \text{interior angles}\end{pmatrix} = \pi + \alpha = \pi + \frac{(\alpha a^2)}{a^2} = \pi + \frac{A}{a^2} \,.$$

b) The three great circles that bound a spherical triangle divide the sphere up into eight triangles. Any two circles divide the sphere into wedges whose opening angle is one of the interior angles of a triangle, and whose area is the sum of the areas of two of the triangles. This gives a set of relations of the form

$$A + A' = \frac{1}{2\pi} \begin{pmatrix}\text{interior} \\ \text{angle}\end{pmatrix} \cdot 4\pi a^2$$

which could be solved for the areas of the triangles in terms of their interior angles.

However, it is not necessary to carry out this solution. Arguments of symmetry and some special cases are enough to find the result. The above relations show that the area of a spherical triangle are *linearly* related to the three interior angles: α, β, γ. Since, in a general triangle, no one of these angles is preferred over any other, the area must be related linearly and symmetrically to the angles by a relation of the form

$$A = c(\alpha + \beta + \gamma) + d$$

with constants c and d depending on a to be determined. The special case considered in (a) with $\beta = \gamma = \pi/2$ gives

$$A = c(\pi + \alpha) + d = \alpha a^2$$

holding for arbitrary α. Thus, $c = a^2$ and $d = -a^2\pi$, giving

$$A = a^2(\alpha + \beta + \gamma - \pi)$$

or, what is the same thing:

$$\alpha + \beta + \gamma = \pi + \frac{A}{a^2}\,.$$

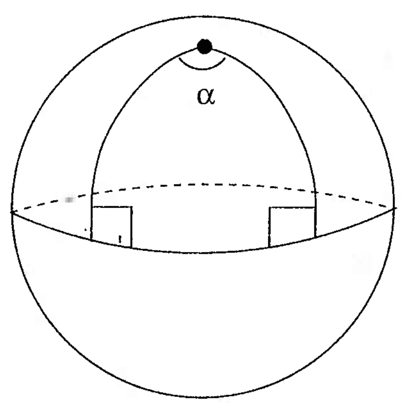
2-4. *Draw examples of a triangle on the surface of a sphere for which:*

a) *the sum of whose interior angles is just slightly greater than* π.

b) *the sum of whose angles is equal to* 2π.

c) *What is the maximum the sum of angles of a triangle on a sphere can be according to* (2.4)*? Can you exhibit a triangle where the sum achieves this value?*

Solution:



Consider the triangle contained within the equator and two lines of longitude differing by an angle α. That triangle has two right interior angles at the equator and the interior angle α at the pole. The sum of the interior angles is $\pi+\alpha$. By taking α near zero, one has a triangle whose sum of angles is slightly bigger than π. By taking $\alpha=\pi$, one has a triangle whose sum of angles is 2π.

From (2.4), the maximum sum of interior angles occurs when $A=4\pi R^2$ — the area of the whole sphere — and is 5π. A triangle which nearly realizes this bound is the *complement* of a small equilateral triangle. The three interior angles are each $(2\pi-\pi/3)$ and add up to 5π.

2-5. *Calculate the area of a circle of radius r (distance from center to circumference) in the two - dimensional geometry which is the surface of a sphere of radius a. Show that this reduces to πr^2 when $r \ll a$.*

Solution: Refer to Fig.2.6 for the geometry. Consider an element of area at (θ,ϕ) spanned by coordinate intervals $(d\theta, d\phi)$. The length of the edge of size $d\theta$ is $ad\theta$, the length of the edge of size $d\phi$ in $a\sin\theta d\phi$. Since the coordinate lines are orthogonal the area is

$$(ad\theta)(a\sin\theta d\phi)\ .$$

The circle of radius r lies at $\theta=r/a$. Integrating the element of area above

$$A=\int_0^{r/a} d\theta \int_0^{2\pi} d\phi\ a^2\sin\theta d\theta d\phi$$

gives the result:

$$A=2\pi a^2[1-\cos(r/a)]\ .$$

For small r/a

$$\begin{aligned}\cos\left(\frac{r}{a}\right) &= 1-\frac{1}{2}\left(\frac{r}{a}\right)^2+O\left(\frac{r}{a}\right)^4\\ A=\pi r^2 &+ O\left(\frac{r}{a}\right)^4\ .\end{aligned}$$

2-6. *[B] Consider a sphere of radius a and on it a segment of length s of a line of latitude that is a distance d from the north pole measured on the sphere. What is the angle between the lines of longitude that this segment spans? Is* *this angle greater or smaller than the angle the segment would subtend at the same distance on a flat plane?*

Solution: This problem is solved in the same way that the ratio of the circumference to radius of a "circle" on the sphere was calculated in (2.17) — (2.19). The answer is

$$s = \Delta\phi\, a \sin\left(\frac{d}{a}\right) .$$

The subtended angle is therefore

$$\Delta\phi = s\left[a \sin\left(\frac{d}{a}\right)\right]^{-1}$$

since $\sin x < x$ this is *more* than the angle $\Delta\phi = s/d$ that would be subtended geometry were flat.

2-7 *Consider the following coordinate transformation from familiar rectangular coordinates (x, y) labeling points in the plane to a new set of coordinates (μ, ν)*

$$\begin{aligned} x &= \mu\nu \\ y &= \frac{1}{2}\left(\mu^2 - \nu^2\right) \end{aligned}$$

a) *Sketch the curves of constant μ and constant ν in the (x, y) plane.*

b) *Transform the line element $dS^2 = dx^2 + dy^2$ into (μ, ν) coordinates.*

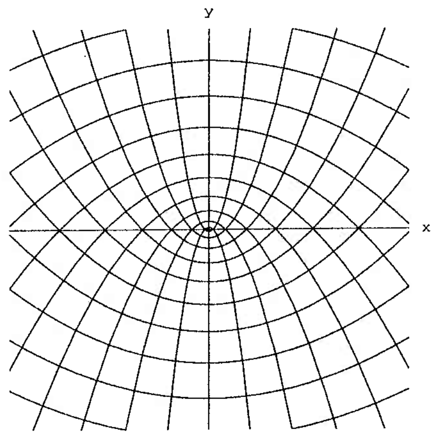
c) *Do the curves of constant μ and constant ν intersect at right angles?*

d) *Find the equation of a circle of radius r centered at the origin in terms of μ and ν.*

e) *Calculate the ratio of the circumference to the diameter of a circle using (μ, ν) coordinates. Do you get the correct answer?*

Solution:

a)



b)

$$\begin{aligned} dS^2 &= dx^2 + dy^2 \\ &= (\mu d\nu + \nu d\mu)^2 + (\mu d\mu - \nu d\nu)^2 \\ dS^2 &= (\mu^2 + \nu^2)\,(d\mu^2 + d\nu^2) \end{aligned}$$

c) The curves intersect at right angles because there are no cross terms $d\mu d\nu$ in the metric.

d) The equation of a circle is $x^2 + y^2 = r^2$ which becomes

$$\begin{aligned} \mu^2\nu^2 + \frac{1}{4}\left(\mu^2 - \nu^2\right)^2 &= r^2 \\ \frac{1}{4}\left(\mu^2 + \nu^2\right)^2 &= r^2 \\ \mu^2 + \nu^2 &= 2r \end{aligned}$$

e) The circumference C is

$$\begin{aligned} C = \oint dS &= \oint \left(\mu^2 + \nu^2\right)^{\frac{1}{2}} \left(d\mu^2 + d\nu^2\right)^{\frac{1}{2}} \\ C &= (2r)^{\frac{1}{2}} \oint d\mu \left(1 + \left(\frac{d\nu}{d\mu}\right)^2\right)^{\frac{1}{2}} \\ &= (2r)^{\frac{1}{2}} \int_{-\sqrt{2r}}^{+\sqrt{2r}} d\mu \left[1 + \frac{\mu^2}{(2r - \mu^2)}\right]^{\frac{1}{2}} \\ &= 2\pi r \end{aligned}$$

This is, of course, the correct answer. The key step in the above evaluation is recognizing that the whole circle is covered by the coordinate range $\mu = -\sqrt{2r}$ to $\mu = \sqrt{2r}$.

2-8. *[A] The surface of an egg is an axisymmetric geometry to a good approximation. In the line element for two-dimensional axisymmetric geometries* (2.21), *pick an* $f(\theta)$ *such that the resulting surface would resemble that of an egg. Calculate the ratio of the biggest circle around the axis to the distance from pole to pole.*

Solution: There are many solutions to this problem corresponding to the different choices of $f(\theta)$ that make a surface like an egg which is smaller near one pole than the other. A simple linear choice is

$$f(\theta) = 1 - \theta/2\pi$$

which varies between 1 at $\theta = 0$ (the larger end) and 1/2 at $\theta = \pi$ (the smaller end). The circumference of a circle around the axis at θ is

$$C(\theta) = 2\pi\, a\, f(\theta).$$

The maximum circumference occurs at $\theta = 0$: $C(0) = 2\pi\, a$. The distance d from pole to pole is

$$\begin{aligned} d &= \int_0^\pi d\theta\ a \\ &= \pi\, a \end{aligned}$$

The ratio C/d is thus 2 for this example.

2-9. *The surface of the Earth is not a perfect sphere. The polar radius of the Earth,* 6357 km, *is slightly less than the mean equatorial radius* 6378 km. *Suppose the surface of the Earth is modeled by an axisymmetric surface with a line element of the kind in* (2.21) *with*

$$f(\theta) = \sin\theta(1 + \epsilon\sin^2\theta)$$

for some small ϵ. *What values of* a *and* ϵ *would best fit reproduce the known polar and equatorial radii?*

Solution: The line element (2.21) with the given $f(\theta)$ depends on two parameters a and ϵ. We can determine these by fitting to the circumferences of the equator and a great circle through the polar axis. The circumference of the equator $\theta = \pi/2$ from (2.21) is

$$C_{\rm eq} = \int_0^{2\pi} af(\pi/2)d\phi = 2\pi af(\pi/2) = 2\pi(1+\epsilon)a \ .$$

This must be $2\pi(6378)$ km. The circumference of the great circle $\phi = 0$ from (2.21) is

$$C_{\rm polar} = 2\int_0^{\pi} ad\theta = 2\pi a \ .$$

This must be $2\pi(6357)$ km. Thus

$$a = 6357 \text{ km}, \qquad \epsilon = .003 \ .$$

2-10. *[B] (Equal Area Projections.) An equal area map projection is one for which there is a constant proportionality between areas on the map and areas on the surface of the globe. Given $x = L\phi/2\pi$, what function $y(\lambda)$ would make an equal area map? [Hint: If an infinitesimal area $dxdy$ has the same constant of proportionality to the corresponding infinitesimal area on the sphere wherever it is located, bigger areas will be also proportional.]*

Solution: The metric on the sphere (2.24) can be written in the form (2.28) in terms of $x = (L\phi)/2\pi$ and arbitrary $y = y(\lambda)$. The area on the sphere bounded by a small rectangle of coordinate length dx and height dy is thus

$$\left(a\left(\frac{2\pi}{L}\right)\cos\lambda(y)\,dx\right)\left(a\frac{d\lambda}{dy}\,dy\right)$$

If the area $dx\,dy$ on the map is to be proportional to this, then the coefficient of $dx\,dy$ above must be constant. Choosing a convenient constant if proportionally, we have

$$\frac{d\lambda}{dy}\cos\lambda = 1 \ .$$

Integrating this and choosing $y = 0$ to be the equator $\lambda = 0$, we find

$$y(\lambda) = \sin\lambda$$

or

$$\lambda(y) = \sin^{-1}(\lambda)$$

2-11. *[B] (Conical Projections.) Conical projections map points on the globe into polar coordinates (r, ψ) in the plane of the map. (We use ψ to avoid confusion with the coordinate ϕ on the sphere.) Thus, in general $r = r(\lambda, \phi)$ and $\psi = \psi(\lambda, \phi)$. A particularly simple class of conical projections uses the north pole as the origin of the polar coordinates and has $r = r(\lambda)$ and $\psi = \phi$. For this simple class*

a) *express the line element on the sphere in terms of r and ψ.*

b) *find the function $r(\lambda)$ which makes this an equal area projection in which there is a constant proportionality between each area on map and the corresponding area on the sphere. [Hint: See the hint for the previous problem.]*

Solution:

(a)

$$\begin{aligned} dS^2 &= a^2\left(d\lambda^2 + \cos^2\lambda d\phi^2\right) \\ &= a^2\left[\left(\frac{d\lambda}{dr}\right)^2 dr^2 + \cos^2\lambda d\psi^2\right] . \end{aligned}$$

(b) The length on the sphere of a line of coordinate length dr extending in the r direction is $a(d\lambda/dr)dr$. The length of a line of coordinate length $d\psi$ extending in the ψ direction is $a\cos\lambda d\psi$. Since r and ψ are orthogonal coordinates the area spanned by dr and $d\psi$ is

$$\left[a\left(\frac{d\lambda}{dr}\right)dr\right]\left[a\cos\lambda d\psi\right] .$$

The area of the corresponding element in the plane is

$$(dr)(rd\psi) .$$

If these are related by a constant of proportionality L, we must have

$$a^2 \frac{d\lambda}{dr} \cos\lambda = -Lr \ .$$

(The constant of proportionality must be negative since latitude decreases as r increases from the north pole.) Integrating both sides and choosing the constant so $r = 0$ is $\lambda = \pi/2$ (the north pole) we find

$$a^2 (\sin\lambda - 1) = -\frac{1}{2} L r^2$$

$$r(\lambda) = \sqrt{\frac{2a^2}{L}(1 - \sin\lambda)}.$$

2-12. *[B,N]Your Personal World Map The maps in Box 2.3 were made with the Mathematica program* `WorldPlot`. *Make your own projection centered on your home city that uses a radial coordinate that represents your view of the importance of the rest of the world.*

Solution: The world map below is a projection which emphasizes the US and, in fact, the neighborhood of New York over other places. It was constructed using the projection

$$\begin{aligned} x &= \phi'/(1 + |\phi'|/6000)^{1.2} \\ y &= \lambda'/(1 + |\lambda'|/1000)^{1.2} \end{aligned}$$

with $\phi' = \phi + 74°$, and $\lambda' = \lambda - 41°$ expressed in minutes of arc.