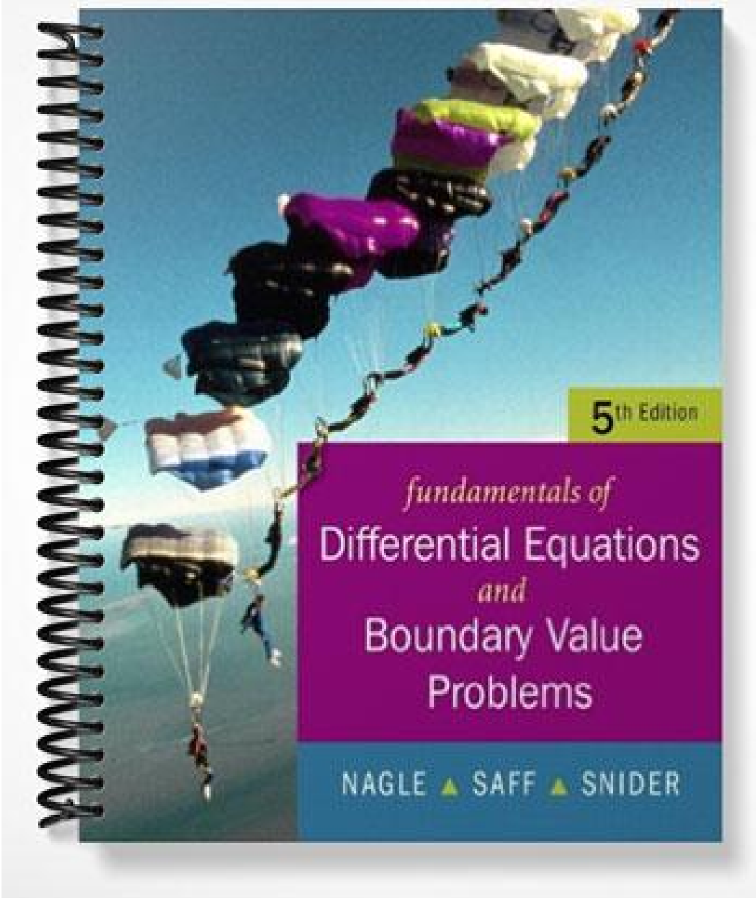
SOLUTIONS MANUAL



INSTRUCTOR'S SOLUTIONS MANUAL

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS

SEVENTH EDITION

AND

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

FIFTH EDITION

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PEARSON
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Contents

Notes to the Instructor

One goal in our writing has been to create flexible texts that afford the instructor a variety of topics and make available to the student an abundance of practice problems and projects. We recommend that the instructor read the discussion given in the preface in order to gain an overview of the prerequisites, topics of emphasis, and general philosophy of the text.

Software Supplements

Interactive Differential Equations CD-ROM: By Beverly West (Cornell University), Steven Strogatz (Cornell University), Jean Marie McDill (California Polytechnic State University – San Luis Obispo), John Cantwell (St. Louis University), and Hubert Hohn (Massachusetts College of Arts) is a popular software directly tied to the text that focuses on helping students visualize concepts. Applications are drawn from engineering, physics, chemistry, and biology. Runs on Windows or Macintosh and is included free with every book.

Instructor's MAPLE/MATHLAB/MATHEMATICA manual: By Thomas W. Polaski (Winthrop University), Bruno Welfert (Arizona State University), and Maurino Bautista (Rochester Institute of Technology). A collection of worksheets and projects to aid instructors in integrating computer algebra systems into their courses. Available via Addison-Wesley Instructor's Resource Center.

MATLAB Manual ISBN 13: 978-0-321-53015-8; ISBN 10: 0-321-53015-2
MAPLE Manual ISBN 13: 978-0-321-38842-1; ISBN 10: 0-321-38842-9
MATHEMATICA Manual ISBN 13: 978-0-321-52178-1; ISBN 10: 0-321-52178-1

Computer Labs

A computer lab in connection with a differential equations course can add a whole new dimension to the teaching and learning of differential equations. As more and more colleges and universities set up computer labs with software such as MAPLE, MATLAB, DERIVE, MATHEMATICA, PHASEPLANE, and MACMATH, there will be more opportunities to include a lab as part of the differential equations course. In our teaching and in our texts, we have tried to provide a variety of exercises, problems, and projects that encourage the student to use the computer to explore. Even one or two hours at a computer generating phase plane diagrams can provide the students with a feeling of how they will use technology together with the theory to investigate real world problems. Furthermore, our experience is that they thoroughly enjoy these activities. Of course, the software, provided free with the texts, is especially convenient for such labs.

Group Projects

Although the projects that appear at the end of the chapters in the text can be worked out by the conscientious student working alone, making them *group* projects adds a social element that encourages discussion and interactions that simulate a professional work place atmosphere. Group sizes of 3 or 4 seem to be optimal. Moreover, requiring that each individual student separately write up the group's solution as a formal technical report for grading by the instructor also contributes to the professional flavor.

Typically, our students each work on 3 or 4 projects per semester. If class time permits, oral presentations by the groups can be scheduled and help to improve the communication skills of the students.

The role of the instructor is, of course, to help the students solve these elaborate problems on their own and to recommend additional reference material when appropriate.
Some additional Group Projects are presented in this guide (see page 9).

Technical Writing Exercises

The technical writing exercises at the end of most chapters invite students to make documented responses to questions dealing with the concepts in the chapter. This not only gives students an opportunity to improve their writing skills, but it helps them organize their thoughts and better understand the new concepts. Moreover, many questions deal with critical thinking skills that will be useful in their careers as engineers, scientists, or mathematicians.

Since most students have little experience with technical writing, it may be necessary to return *ungraded* the first few technical writing assignments with comments and have the students redo the the exercise. This has worked well in our classes and is much appreciated by the students. Handing out a "model" technical writing response is also helpful for the students.

Student Presentations

It is not uncommon for an instructor to have students go to the board and present a solution to a problem. Differential equations is so rich in theory and applications that it is an excellent course to allow (require) a student to give a presentation on a special application (e.g., almost any topic from Chapter 3 and 5), on a new technique not covered in class (e.g., material from Section 2.6, Projects A, B, or C in Chapter 4), or on additional theory (e.g., material from Chapter 6 which generalizes the results in Chapter 4). In addition to improving students' communication skills, these "special" topics are long remembered by the students. Here, too, working in groups of 3 or 4 and sharing the presentation responsibilities can add substantially to the interest and quality of the presentation. Students should also be encouraged to enliven their communication by building physical models, preparing part of their lectures on video cassette, etc.

Homework Assignments

We would like to share with you an obvious, non-original, but effective method to encourage students to do homework problems.

An essential feature is that it requires little extra work on the part of the instructor or grader. We assign homework problems (about 10 of them) after each lecture. At the end of the week (Fridays), students are asked to turn in their homework (typically, 3 sets) for that week. We then choose at random one problem from each assignment (typically, a total of 3) that will be graded. (The point is that the student does not know in advance which problems will be chosen.) Full credit is given for any of the chosen problems for which there is evidence that the student has made an honest attempt at solving. The homework problem sets are returned to the students at the next meeting (Mondays) with grades like 0/3, 1/3, 2/3, or 3/3 indicating the proportion of problems for which the student received credit. The homework grades are tallied at the end of the semester and count as one test grade. Certainly, there are variations on this theme. The point is that students are motivated to do their homework with little additional cost (= time) to the instructor.

Syllabus Suggestions

To serve as a guide in constructing a syllabus for a one-semester or two-semester course, the prefaces to the texts list sample outlines that emphasize methods, applications, theory, partial differential equations, phase plane analysis, computation, or combinations of these. As a further guide in making a choice of subject matter, we provide below a listing of text material dealing with some common areas of emphasis.

Numerical, Graphical, and Qualitative Methods

The sections and projects dealing with numerical, graphical, and qualitative techniques of solving differential equations include:

Section 1.3: *Direction Fields*

Section 1.4: *The Approximation Method of Euler*

Project A for Chapter 1: *Taylor Series*

Project B for Chapter 1: *Picard's Method*

Project D for Chapter 1: *The Phase Line*

Section 3.6: *Improved Euler's Method*, which includes step-by-step outlines of the improved Euler's method subroutine and improved Euler's method with tolerance. These outlines are easy for the student to translate into a computer program (cf. pages 135 and 136).

Section 3.7: *Higher-Order Numerical Methods*: *Taylor and Runge-Kutta*, which includes outlines for the Fourth Order Runge-Kutta subroutine and algorithm with tolerance (see pages 144 and 145).

Project H for Chapter 3: *Stability of Numerical Methods*

Project I for Chapter 3: *Period Doubling an Chaos*

Section 4.8: *Qualitative Considerations for Variable Coefficient and Nonlinear Equations*, which discusses the energy integral lemma, as well as the Airy, Bessel, Duffing, and van der Pol equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which describes the vectorized forms of Euler's method and the Fourth Order Runge-Kutta method, and discusses an application to population dynamics.

Section 5.4: *Introduction to the Phase Plane*, which introduces the study of trajectories of autonomous systems, critical points, and stability.

Section 5.8: *Dynamical Systems, Poincarè Maps, and Chaos*, which discusses the use of numerical methods to approximate the Poincarè map and how to interpret the results.

Project A for Chapter 5: *Designing a Landing System for Interplanetary Travel*

Project B for Chapter 5: *Things That Bob*

Project D for Chapter 5: *Strange Behavior of Competing Species – Part I*

Project D for Chapter 9: *Strange Behavior of Competing Species – Part II*

Project D for Chapter 10: *Numerical Method for* $\Delta u = f$ *on a Rectangle*

Project D for Chapter 11: *Shooting Method*

Project E for Chapter 11: *Finite-Difference Method for Boundary Value Problems*

Project C for Chapter 12: *Computing Phase Plane Diagrams*

Project D for Chapter 12: *Ecosystem of Planet GLIA-2*

Appendix A: *Newton's Method*

Appendix B: *Simpson's Rule*

Appendix D: *Method of Least Squares*

Appendix E: *Runge-Kutta Procedure for Equations*

The instructor who wishes to emphasize numerical methods should also note that the text contains an extensive chapter of series solutions of differential equations (Chapter 8).

Engineering/Physics Applications

Since Laplace transforms is a subject vital to engineering, we have included a detailed chapter on this topic – see Chapter 7. Stability is also an important subject for engineers, so we have included an introduction to the subject in Chapter 5.4 along with an entire chapter addressing this topic – see Chapter 12. Further material dealing with engineering/physic applications include:

Project C for Chapter 1: *Magnetic "Dipole"*

Project B for Chapter 2: *Torricelli's Law of Fluid Flow*

Section 3.1: *Mathematical Modeling*

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and of population models.

Section 3.3: *Heating and Cooling Buildings*, which discusses temperature variations in the presence of air conditioning or furnace heating.

Section 3.4: *Newtonian Mechanics*

Section 3.5: *Electrical Circuits*

Project C for Chapter 3: *Curve of Pursuit*

Project D for Chapter 3: *Aircraft Guidance in a Crosswind*

Project E for Chapter 3: *Feedback and the Op Amp*

Project F for Chapter 3: *Band-Bang Controls*

Section 4.1: *Introduction: Mass-Spring Oscillator*

Section 4.8: *Qualitative Considerations for Variable-Coefficient and Nonlinear Equations*

Section 4.9: *A Closer Look at Free Mechanical Vibrations*

Section 4.10: *A Closer Look at Forced Mechanical Vibrations*

Project B for Chapter 4: *Apollo Reentry*

Project C for Chapter 4: *Simple Pendulum*

Chapter 5: *Introduction to Systems and Phase Plane Analysis*, which includes sections on coupled mass-spring systems, electrical circuits, and phase plane analysis.

Project A for Chapter 5: *Designing a Landing System for Interplanetary Travel*

Project B for Chapter 5: *Things that Bob*

Project C for Chapter 5: *Hamiltonian Systems*

Project D for Chapter 5: *Transverse Vibrations of a Beam*

Chapter 7: *Laplace Transforms*, which in addition to basic material includes discussions of transfer functions, the Dirac delta function, and frequency response modeling.

Projects for Chapter 8, dealing with Schrödinger's equation, bucking of a tower, and again springs.

Project B for Chapter 9: *Matrix Laplace Transform Method*

Project C for Chapter 9: *Undamped Second-Order Systems*

Chapter 10: *Partial Differential Equations*, which includes sections on Fourier series, the heat equation, wave equation, and Laplace's equation.

Project A for Chapter 10: *Steady-State Temperature Distribution in a Circular Cylinder*

Project B for Chapter 10: *A Laplace Transform Solution of the Wave Equation*

Project A for Chapter 11: *Hermite Polynomials and the Harmonic Oscillator*

Section 12.4: *Energy Methods*, which addresses both conservative and nonconservative autonomous mechanical systems.

Project A for Chapter 12: *Solitons and Korteweg-de Vries Equation*

Project B for Chapter 12: *Burger's Equation*

Students of engineering and physics would also find Chapter 8 on series solutions particularly useful, especially Section 8.8 on special functions.

Biology/Ecology Applications

Project D for Chapter 1: *The Phase Plane*, which discusses the logistic population model and bifurcation diagrams for population control.

Project A for Chapter 2: *Differential Equations in Clinical Medicine*

Section 3.1: *Mathematical Modeling*

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and population models.

Project A for Chapter 3: *Dynamics for HIV Infection*

Project B for Chapter 3: *Aquaculture*, which deals with a model of raising and harvesting catfish.

Section 5.1: *Interconnected Fluid Tanks*, which introduces systems of equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which contains an application to population dynamics.

Section 5.5: *Applications to Biomathematics: Epidemic and Tumor Growth Models*

Project D for Chapter 5: *Strange Behavior of Competing Species – Part I*

Project E for Chapter 5: *Cleaning Up the Great Lakes*

Project D for Chapter 9: *Strange Behavior of Competing Species – Part II*

Problem 19 in Exercises 10.5 , which involves chemical diffusion through a thin layer.

Project D for Chapter 12: *Ecosystem on Planet GLIA-2*

The basic content of the remainder of this instructor's manual consists of supplemental group projects, answers to the even-numbered problems, and detailed solutions to the even-numbered problems in Chapters 1, 2, 4, and 7 as well as Sections 3.2, 3.3, and 3.4. The answers are, for the most part, not available any place else since the text only provides answers to odd-numbered problems, and the *Student's Solutions Manual* contains only a handful of worked solutions to even-numbered problems.

We would appreciate any comments you may have concerning the answers in this manual. These comments can be sent to the authors' email addresses below. We also would encourage sharing with us (= the authors and users of the texts) any of your favorite group projects.

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Group Projects for Chapter 3

Delay Differential Equations

In our discussion of mixing problems in Section 3.2, we encountered the initial value problem

$$x'(t) = 6 - \frac{3}{500}\, x\left(t - t_0\right), \qquad (0.1)$$
$$x(t) = 0 \quad \text{for} \quad x \in [-t_0, 0]\,,$$

where t_0 is a positive constant. The equation in (0.1) is an example of a **delay differential equation**. These equations differ from the usual differential equations by the presence of the shift $(t - t_0)$ in the argument of the unknown function $x(t)$. In general, these equations are more difficult to work with than are regular differential equations, but quite a bit is known about them.[1]

(a) Show that the simple linear delay differential equation

$$x' = ax(t - b), \qquad (0.2)$$

where a, b are constants, has a solution of the form $x(t) = Ce^{st}$ for any constant C, provided s satisfies the transcendental equation $s = ae^{-bs}$.

(b) A solution to (0.2) for $t > 0$ can also be found using the **method of steps**. Assume that $x(t) = f(t)$ for $-b \le t \le 0$. For $0 \le t \le b$, equation (0.2) becomes

$$x'(t) = ax(t - b) = af(t - b),$$

and so

$$x(t) = \int_0^t af(\nu - b)d\nu + x(0).$$

Now that we know $x(t)$ on $[0, b]$, we can repeat this procedure to obtain

$$x(t) = \int_b^t ax(\nu - b)d\nu + x(b)$$

for $b \le x \le 2b$. This process can be continued indefinitely.

[1] See, for example, *Differential–Difference Equations*, by R. Bellman and K. L. Cooke, Academic Press, New York, 1963, or *Ordinary and Delay Differential Equations*, by R. D. Driver, Springer–Verlag, New York, 1977

Use the method of steps to show that the solution to the initial value problem

$$x'(t) = -x(t-1), \qquad x(t) = 1 \quad \text{on} \quad [-1, 0],$$

is given by

$$x(t) = \sum_{k=0}^{n} (-1)^k \frac{[t-(k-1)]^k}{k!}, \qquad \text{for} \quad n-1 \le t \le n,$$

where n is a nonnegative integer. (This problem can also be solved using the Laplace transform method of Chapter 7.)

(c) Use the method of steps to compute the solution to the initial value problem given in (0.1) on the interval $0 \le t \le 15$ for $t_0 = 3$.

Extrapolation

When precise information about the *form* of the error in an approximation is known, a technique called **extrapolation** can be used to improve the rate of convergence.

Suppose the approximation method converges with rate $O(h^p)$ as $h \to 0$ (cf. Section 3.6). From theoretical considerations, assume we know, more precisely, that

$$y(x;h) = \phi(x) + h^p a_p(x) + O\left(h^{p+1}\right), \tag{0.3}$$

where $y(x;h)$ is the approximation to $\phi(x)$ using step size h and $a_p(x)$ is some function that is independent of h (typically, we do not know a formula for $a_p(x)$, only that it exists). Our goal is to obtain approximations that converge at the faster rate $O(h^{p+1})$.

We start by replacing h by $h/2$ in (0.3) to get

$$y\left(x; \frac{h}{2}\right) = \phi(x) + \frac{h^p}{2^p} a_p(x) + O\left(h^{p+1}\right).$$

If we multiply both sides by 2^p and subtract equation (0.3), we find

$$2^p y\left(x; \frac{h}{2}\right) - y(x;h) = (2^p - 1)\,\phi(x) + O\left(h^{p+1}\right).$$

Solving for $\phi(x)$ yields

$$\phi(x) = \frac{2^p y\,(x; h/2) - y(x;h)}{2^p - 1} + O\left(h^{p+1}\right).$$

Hence,

$$y^*\left(x; \frac{h}{2}\right) := \frac{2^p y\,(x; h/2) - y(x;h)}{2^p - 1}$$

has a rate of convergence of $O(h^{p+1})$.

(a) Assuming

$$y^*\left(x; \frac{h}{2}\right) = \phi(x) + h^{p+1}a_{p+1}(x) + O\left(h^{p+2}\right),$$

show that

$$y^{**}\left(x; \frac{h}{4}\right) := \frac{2^{p+1}y^*\left(x; h/4\right) - y^*(x; h/2)}{2^{p+1} - 1}$$

has a rate of convergence of $O\left(h^{p+2}\right)$.

(b) Assuming

$$y^{**}\left(x; \frac{h}{4}\right) = \phi(x) + h^{p+2}a_{p+2}(x) + O\left(h^{p+3}\right),$$

show that

$$y^{***}\left(x; \frac{h}{8}\right) := \frac{2^{p+2}y^{**}\left(x; h/8\right) - y^{**}(x; h/4)}{2^{p+2} - 1}$$

has a rate of convergence of $O\left(h^{p+3}\right)$.

(c) The results of using Euler's method (with $h = 1$, $1/2$, $1/4$, $1/8$) to approximate the solution to the initial value problem

$$y' = y, \quad y(0) = 1$$

at $x = 1$ are given in Table 1.2, page 27. For Euler's method, the extrapolation procedure applies with $p = 1$. Use the results in Table 1.2 to find an approximation to $e = y(1)$ by computing $y^{***}(1; 1/8)$. [Hint: Compute $y^*\left(1; 1/2\right)$, $y^*\left(1; 1/4\right)$, and $y^*\left(1; 1/8\right)$; then compute $y^{**}\left(1; 1/4\right)$ and $y^{**}\left(1; 1/8\right)$.]

(d) Table 1.2 also contains Euler's approximation for $y(1)$ when $h = 1/16$. Use this additional information to compute the next step in the extrapolation procedure; that is, compute $y^{****}(1; 1/16)$.

Group Projects for Chapter 5

Effects of Hunting on Predator–Prey Systems

As discussed in Section 5.3 (page 277), cyclic variations in the population of predators and their prey have been studied using the Volterra-Lotka predator–prey model

$$\frac{dx}{dt} = Ax - Bxy, \tag{0.4}$$

$$\frac{dy}{dt} = -Cy + Dxy, \tag{0.5}$$

where A, B, C, and D are positive constants, $x(t)$ is the population of prey at time t, and $y(t)$ is the population of predators. It can be shown that such a system has a periodic solution (see Project D). That is, there exists some constant T such that $x(t) = x(t+T)$ and $y(t) = y(t+T)$ for all t. The periodic or cyclic variation in the population has been observed in various systems such as sharks–food fish, lynx–rabbits, and ladybird beetles–cottony cushion scale. Because of this periodic behavior, it is useful to consider the average population $\overline{x}$ and $\overline{y}$ defined by

$$\overline{x} := \frac{1}{T}\int_0^t x(t)dt\,, \qquad \overline{y} := \frac{1}{T}\int_0^t y(t)dt\,.$$

(a) Show that $\overline{x} = C/D$ and $\overline{y} = A/B$. [Hint: Use equation (0.4) and the fact that $x(0) = x(T)$ to show that

$$\int_0^T [A - By(t)]\,dt = \int_0^T \frac{x'(t)}{x(t)}\,\frac{d=}{dt}0.\,]$$

(b) To determine the effect of indiscriminate hunting on the population, assume hunting reduces the rate of change in a population by a constant times the population. Then the predator–prey system satisfies the new set of equations

$$\frac{dx}{dt} = Ax - Bxy - \varepsilon x = (A - \varepsilon)x - Bxy\,, \tag{0.6}$$

$$\frac{dy}{dt} = -Cy + Dxy - \delta y = -(C + \delta)y + Dxy\,, \tag{0.7}$$

where ε and δ are positive constants with $\varepsilon < A$. What effect does this have on the average population of prey? On the average population of predators?

(c) Assume the hunting was done selectively, as in shooting only rabbits (or shooting only lynx). Then we have $\varepsilon > 0$ and $\delta = 0$ (or $\varepsilon = 0$ and $\delta > 0$) in (0.6)–(0.7). What effect does this have on the average populations of predator and prey?

(d) In a rural county, foxes prey mainly on rabbits but occasionally include a chicken in their diet. The farmers decide to put a stop to the chicken killing by hunting the foxes. What do you predict will happen? What will happen to the farmers' gardens?

Limit Cycles

In the study of triode vacuum tubes, one encounters the van der Pol equation[2]

$$y'' - \mu \left(1 - y^2\right) y' + y = 0\,,$$

where the constant μ is regarded as a parameter. In Section 4.8 (page 224), we used the mass-spring oscillator analogy to argue that the nonzero solutions to the van der Pol equation with $\mu = 1$ should approach a periodic limit cycle. The same argument applies for any positive value of μ.

(a) Recast the van der Pol equation as a system in normal form and use software to plot some typical trajectories for $\mu = 0.1$, 1, and 10. Re-scale the plots if necessary until you can discern the limit cycle trajectory; find trajectories that spiral in, and ones that spiral out, to the limit cycle.

(b) Now let $\mu = -0.1$, -1, and -10. Try to predict the nature of the solutions using the mass-spring analogy. Then use the software to check your predictions. Are there limit cycles? Do the neighboring trajectories spiral into, or spiral out from, the limit cycles?

(c) Repeat parts (a) and (b) for the Rayleigh equation

$$y'' - \mu \left[1 - (y')^2\right] y' + y = 0\,.$$

Group Project for Chapter 13

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Satellite Altitude Stability

In this problem, we determine the orientation at which a satellite in a circular orbit of radius r can maintain a relatively constant facing with respect to a spherical primary (e.g., a planet) of mass M. The torque of gravity on the asymmetric satellite maintains the orientation.

[2] *Historical Footnote*: Experimental research by **E. V. Appleton** and **B. van der Pol** in 1921 on the oscillation of an electrical circuit containing a triode generator (vacuum tube) led to the nonlinear equation now called **van der Pol's equation**. Methods of solution were developed by van der Pol in 1926–1927. **Mary L. Cartwright** continued research into nonlinear oscillation theory and together with **J. E. Littlewood** obtained existence results for forced oscillations in nonlinear systems in 1945.

Suppose (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ refer to coordinates in two systems that have a common origin at the satellite's center of mass. Fix the xyz-axes in the satellite as principal axes; then let the $\overline{z}$-axis point toward the primary and let the $\overline{x}$-axis point in the direction of the satellite's velocity. The xyz-axes may be rotated to coincide with the $\overline{xyz}$-axes by a rotation ϕ about the x-axis (roll), followed by a rotation θ about the resulting y-axis (pitch), and a rotation ψ about the final z-axis (yaw). Euler's equations from physics (with high terms omitted[3] to obtain approximate solutions valid near $(\phi, \theta, \psi) = (0, 0, 0)$) show that the equations for the rotational motion due to gravity acting on the satellite are

$$\begin{aligned} I_x \phi'' &= -4\omega_0^2 (I_z - I_y)\, \phi - \omega_0 (I_y - I_z - I_x)\, \psi' \\ I_y \theta'' &= -3\omega_0^2 (I_x - I_z)\, \theta \\ I_z \psi'' &= -4\omega_0^2 (I_y - I_x)\, \psi + \omega_0 (I_y - I_z - I_x)\, \phi' \,, \end{aligned}$$

where $\omega_0 = \sqrt{(GM)/r^3}$ is the angular frequency of the orbit and the positive constants I_x, I_y, I_z are the moments of inertia of the satellite about the x, y, and z-axes.

(a) Find constants $c_1, \ldots, c_5$ such tha these equations can be written as two systems

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ \phi' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & 0 & 0 & c_2 \\ 0 & c_3 & c_4 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \phi' \\ \psi' \end{bmatrix}$$

and

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_5 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \theta' \end{bmatrix}.$$

(b) Show that the origin is asymptotically stable for the first system in (a) if

$$\begin{aligned} &(c_2 c_4 + c_3 + c_1)^2 - 4c_1 c_3 > 0\,, \\ &c_1 c_3 > 0\,, \\ &c_2 c_4 + c_3 + c_1 > 0 \end{aligned}$$

and hence deduce that $I_y > I_x > I_z$ yields an asymptotically stable origin. Are there other conditions on the moments of inertia by which the origin is stable?

[3]The derivation of these equations is found in *Attitude Stabilization and Control of Earth Satellites*, by O. H. Gerlach, Space Science Reviews, #4 (1965), 541–566

(c) Show that, for the asymptotically stable configuration in (b), the second system in (a) becomes a harmonic oscillator problem, and find the frequency of oscillation in terms of I_x, I_y, I_z, and ω_0. Phobos maintains $I_y > I_x > I_z$ in its orientation with respect to Mars, and has angular frequency of orbit $\omega_0 = 0.82$ rad/hr. If $(I_x - I_z)/I_y = 0.23$, show that the period of the libration for Phobos (the period with which the side of Phobos facing Mars shakes back and forth) is about 9 hours.

CHAPTER 1: Introduction

EXERCISES 1.1: Background

2. This equation is an ODE because it contains no partial derivatives. Since the highest order derivative is d^2y/dx^2, the equation is a second order equation. This same term also shows us that the independent variable is x and the dependent variable is y. This equation is linear.

4. This equation is a PDE of the second order because it contains second partial derivatives. x and y are independent variables, and u is the dependent variable.

6. This equation is an ODE of the first order with the independent variable t and the dependent variable x. It is nonlinear.

8. ODE of the second order with the independent variable x and the dependent variable y, nonlinear.

10. ODE of the fourth order with the independent variable x and the dependent variable y, linear.

12. ODE of the second order with the independent variable x and the dependent variable y, nonlinear.

14. The velocity at time t is the rate of change of the position function $x(t)$, i.e., x'. Thus,

$$\frac{dx}{dt} = kx^4,$$

where k is the proportionality constant.

16. The equation is

$$\frac{dA}{dt} = kA^2,$$

where k is the proportionality constant.

Chapter 1

EXERCISES 1.2: Solutions and Initial Value Problems

2. (a) Writing the given equation in the form $y^2 = 3 - x$, we see that it defines two functions of x on $x \le 3$, $y = \pm\sqrt{3-x}$. Differentiation yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\pm\sqrt{3-x}\right) = \pm\frac{d}{dx}\left[(3-x)^{1/2}\right] \\ &= \pm\frac{1}{2}(3-x)^{-1/2}(-1) = -\frac{1}{\pm 2\sqrt{3-x}} = -\frac{1}{2y}\,.\end{aligned}$$

(b) Solving for y yields

$$\begin{aligned} y^3(x - x\sin x) = 1 \qquad &\Rightarrow \qquad y^3 = \frac{1}{x(1-\sin x)} \\ \Rightarrow \qquad y = \frac{1}{\sqrt[3]{x(1-\sin x)}} &= [x(1-\sin x)]^{-1/3}\,.\end{aligned}$$

The domain of this function is $x \ne 0$ and

$$\sin x \ne 1 \qquad \Rightarrow \qquad x \ne \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots\,.$$

For $0 < x < \pi/2$, one has

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left\{[x(1-\sin x)]^{-1/3}\right\} = -\frac{1}{3}[x(1-\sin x)]^{-1/3-1}\frac{d}{dx}[x(1-\sin x)] \\ &= -\frac{1}{3}[x(1-\sin x)]^{-1}[x(1-\sin x)]^{-1/3}[(1-\sin x) + x(-\cos x)] \\ &= \frac{(x\cos x + \sin x - 1)y}{3x(1-\sin x)}.\end{aligned}$$

We also remark that the given relation is an implicit solution on any interval not containing points $x = 0, \pi/2 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

4. Differentiating the function $x = 2\cos t - 3\sin t$ twice, we obtain

$$x' = -2\sin t - 3\cos t, \quad x'' = -2\cos t + 3\sin t.$$

Thus,

$$x'' + x = (-2\cos t + 3\sin t) + (2\cos t - 3\sin t) = 0$$

for any t on $(-\infty, \infty)$.

6. Substituting $x = \cos 2t$ and $x' = -2\sin 2t$ into the given equation yields

$$(-2\sin 2t) + t\cos 2t = \sin 2t \qquad \Leftrightarrow \qquad t\cos 2t = 3\sin 2t\,.$$

Clearly, this is not an identity and, therefore, the function $x = \cos 2t$ is not a solution.

8. Using the chain rule, we have

$$\begin{aligned} y &= 3\sin 2x + e^{-x}, \\ y' &= 3(\cos 2x)(2x)' + e^{-x}(-x)' = 6\cos 2x - e^{-x}, \\ y'' &= 6(-\sin 2x)(2x)' - e^{-x}(-x)' = -12\sin 2x + e^{-x}. \end{aligned}$$

Therefore,

$$y'' + 4y = \left(-12\sin 2x + e^{-x}\right) + 4\left(3\sin 2x + e^{-x}\right) = 5e^{-x},$$

which is the right-hand side of the given equation. So, $y = 3\sin 2x + e^{-x}$ is a solution.

10. Taking derivatives of both sides of the given relation with respect to x yields

$$\begin{aligned} &\frac{d}{dx}(y - \ln y) = \frac{d}{dx}\left(x^2+1\right) \quad \Rightarrow \quad \frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 2x \\ \Rightarrow \quad &\frac{dy}{dx}\left(1 - \frac{1}{y}\right) = 2x \quad \Rightarrow \quad \frac{dy}{dx}\frac{y-1}{y} = 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{y-1}. \end{aligned}$$

Thus, the relation $y - \ln y = x^2+1$ is an implicit solution to the equation $y' = 2xy/(y-1)$.

12. To find dy/dx, we use implicit differentiation.

$$\begin{aligned} &\frac{d}{dx}\left[x^2 - \sin(x+y)\right] = \frac{d}{dx}(1) = 0 \quad \Rightarrow \quad 2x - \cos(x+y)\frac{d}{dx}(x+y) = 0 \\ \Rightarrow \quad &2x - \cos(x+y)\left(1 + \frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{\cos(x+y)} - 1 = 2x\sec(x+y) - 1, \end{aligned}$$

and so the given differential equation is satisfied.

14. Assuming that C_1 and C_2 are constants, we differentiate the function $\phi(x)$ twice to get

$$\phi'(x) = C_1\cos x - C_2\sin x, \quad \phi''(x) = -C_1\sin x - C_2\cos x.$$

Therefore,

$$\phi'' + \phi = (-C_1\sin x - C_2\cos x) + (C_1\sin x + C_2\cos x) = 0.$$

Thus, $\phi(x)$ is a solution with any choice of constants C_1 and C_2.

16. Differentiating both sides, we obtain

$$\frac{d}{dx}\left(x^2 + Cy^2\right) = \frac{d}{dx}(1) = 0 \quad \Rightarrow \quad 2x + 2Cy\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{Cy}.$$

Since, from the given relation, $Cy^2 = 1 - x^2$, we have

$$-\frac{x}{Cy} = \frac{xy}{-Cy^2} = \frac{xy}{x^2-1}\,.$$

So,

$$\frac{dy}{dx} = \frac{xy}{x^2-1}\,.$$

Writing $Cy^2 = 1 - x^2$ in the form

$$x^2 + \frac{y^2}{\left(1/\sqrt{C}\right)^2} = 1,$$

we see that the curves defined by the given relation are ellipses with semi-axes 1 and $1/\sqrt{C}$ and so the integral curves are half-ellipses located in the upper/lower half plane.

18. The function $\phi(x)$ is defined and differentiable for all values of x except those satisfying

$$c^2 - x^2 = 0 \qquad \Rightarrow \qquad x = \pm c.$$

In particular, this function is differentiable on $(-c, c)$.

Clearly, $\phi(x)$ satisfies the initial condition:

$$\phi(0) = \frac{1}{c^2 - 0^2} = \frac{1}{c^2}\,.$$

Next, for any x in $(-c, c)$,

$$\frac{d\phi}{dx} = \frac{d}{dx}\left[\left(c^2-x^2\right)^{-1}\right] = (-1)\left(c^2-x^2\right)^{-2}\left(c^2-x^2\right)' = 2x\left[\left(c^2-x^2\right)^{-1}\right]^2 = 2x\phi(x)^2.$$

Therefore, $\phi(x)$ is a solution to the equation $y' = 2xy^2$ on $(-c, c)$.

Several integral curves are shown in Fig. **1–A** on page 29.

20. **(a)** Substituting $\phi(x) = e^{mx}$ into the given equation yields

$$\left(e^{mx}\right)'' + 6\left(e^{mx}\right)' + 5\left(e^{mx}\right) = 0 \qquad \Rightarrow \qquad e^{mx}\left(m^2 + 6m + 5\right) = 0.$$

Since $e^{mx} \neq 0$ for any x, $\phi(x)$ satisfies the given equation if and only if

$$m^2 + 6m + 5 = 0 \quad \Leftrightarrow \quad m = -1, -5.$$