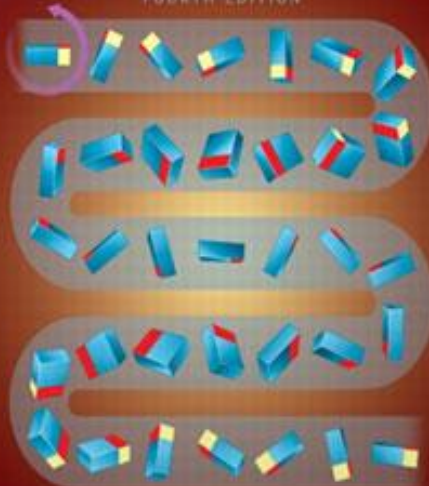


SOLUTIONS MANUAL



FUNDAMENTALS OF
Differential Equations
AND
Boundary Value Problems
FOURTH EDITION



Nagle Saff Snider

INSTRUCTOR'S RESOURCE GUIDE

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FUNDAMENTALS OF DIFFERENTIAL EQUATIONS SIXTH EDITION

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS FOURTH EDITION

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Notes to the Instructor

One goal in our writing has been to create flexible texts that afford the instructor a variety of topics and make available to the student an abundance of practice problems and projects. We recommend that the instructor read the discussion given in the preface in order to gain an overview of the prerequisites, topics of emphasis, and general philosophy of the texts.

An additional resource to accompany the texts is the on-line tool, *MyMathLab*. *MyMathLab* is ideal for lecture-based, lab-based, and on-line courses and provides students and instructors with a centralized point of access to the multimedia resources available with the texts. The pages of the actual text are loaded into *MyMathLab* and extensive course-management capabilities, including a host of communication tools for course participants, are provided to create a user-friendly and interactive on-line learning environment. Instructors can also remove, hide, or annotate Addison-Wesley preloaded content, add their own course documents, or change the order in which material is presented. A link to the Instructional Utility Software package is also included. For more information visit our Web site at www.mymathlab.com or contact your Addison-Wesley sales representative for a live demonstration.

INSTRUCTIONAL UTILITY SOFTWARE

Written specifically for the texts and available via *MyMathLab*, this utility has the following items:

GRAPHICAL METHODS

- Graph, tabulate or evaluate functions
- Direction field
- Phase plane diagram

MATRIX OPERATIONS

- Solve a system of linear equations
- Determinant and inverse of a matrix
- Matrix multiplication
- Eigenvalues and eigenvectors

NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS

- Euler's method subroutine
- Euler's method with tolerance
- Improved Euler's method subroutine
- Improved Euler's method with tolerance
- Fourth order Runge-Kutta method subroutine
- Fourth order Runge-Kutta method with tolerance

FOURTH ORDER RUNGE-KUTTA METHOD FOR SYSTEMS

- System with 2 equations
- System with 3 equations
- System with 4 equations

OTHER COMPUTATIONAL METHODS

- Roots of a polynomial equation
- Newton's method
- Simpson's Rule for definite integrals
- Simpson's Rule for linear differential equations

INTERACTIVE DIFFERENTIAL EQUATIONS CD-ROM

Written by Beverly West (Cornell University), Steven Strogatz (Cornell University), Jean Marie McDill (California Polytechnic State University—San Luis Obispo), John Cantwell (St. Louis University), and Hubert Hohn (Massachusetts College of Art), this CD-ROM is a revised version of a popular software directly tied to the text. It focuses on helping students visualize concepts with applications drawn from engineering, physics, chemistry, and biology. This software runs on supported Windows or Macintosh operating systems and is bundled *free* with every book.

INSTRUCTOR'S MAPLE/MATLAB/MATHEMATICA MANUAL

The Instructor's Maple/Matlab/Mathematica Manual, 0-321-17320-1, was written as an aid for anyone interested in coordinating the use of computer algebra systems with their course. This supplement, including sample worksheets, is available upon request from Addison Wesley and includes the following.

- specific instruction in the use of Maple, Matlab and Mathematica to obtain graphic (direction fields, solution curves, phase portraits), numeric (built-in and user defined), and symbolic information about differential equations;
- a sampling of techniques that can be used to ease the introduction of Maple into the differential equations classroom (including sample worksheets directly related to the text); and
- a collection of additional projects that are particularly amenable to solution using a computer algebra system.

While this supplement is written for use with MAPLE, the general ideas can be adapted for use with MATHEMATICA, MATLAB, or any other sophisticated numerical and/or computer algebra software.

Computer Labs: A computer lab in connection with a differential equations course can add a whole new dimension to the teaching and learning of differential equations. As more and more colleges and universities set up computer labs with software such as MATLAB, MAPLE, DERIVE, MATHEMATICA, PHASEPLANE, and MACMATH, there will be more opportunities to include a lab as part of the differential equations course. In our teaching and in our texts, we have tried to provide a variety of exercises, problems, and projects that encourage the student to use the computer to explore. Even one or two hours at a computer generating phase plane diagrams can provide the students with a feeling of how they will use technology together with the theory to investigate real world problems. Furthermore, our experience is that they thoroughly enjoy these activities. Of course the software provided free with the texts is especially convenient for such labs.

Group Projects: Although the projects that appear at the end of the chapters in the texts can be worked out by the conscientious student working alone, making them *group* projects adds a social element that encourages discussion and interactions that simulate a professional work place atmosphere. Group sizes of 3 or 4 seem to be optimal. Moreover, requiring that each individual student separately write up the group's solution as a formal technical report for grading by the instructor also contributes to the professional flavor. Typically our students each work on 3 or 4 projects per semester. If class time permits, oral presentations by the groups can be scheduled and help to improve the communication skills of the students. The role of the instructor is, of course, to help the students solve these elaborate problems on their own and to recommend additional reference material when appropriate. Some additional Group Projects are presented in this guide (see page 8).

Technical Writing Exercises: The technical writing exercises at the end of most chapters invite students to make documented responses to questions dealing with the concepts in the chapter. This not only gives students an opportunity to improve their writing skills, but it helps them organize their thoughts and better understand the new concepts. Moreover, many questions deal with critical thinking skills that will be useful in their careers as engineers, scientists, or mathematicians.

Since most students have little experience with technical writing, it may be necessary to return *ungraded* the first few technical writing assignments with comments and have the students redo the exercise. This has worked well in our classes and is much appreciated by the students. Handing out a “model” technical writing response is also helpful for the students.

Student Presentations: It is not uncommon for an instructor to have students go to the board and present a solution to a problem. Differential equations is so rich in theory and applications that it is an excellent course to allow (require) a student to give a presentation on a special application (e.g. almost any topic from Chapters 3 and 5), on a new technique not covered in class (e.g. material from Section 2.6, Projects A, B or C in Chapter 4), or on additional theory (e.g. material from Chapter 6 which generalizes the results in Chapter 4). In addition to improving students' communication skills, these “special” topics are long remembered by the students. Here, too, working in groups of 3 or 4 and sharing the presentation responsibilities can add substantially to the interest and quality of the presentation. Students should also be encouraged to enliven their communication by building physical models, preparing part of their lectures on video cassette, etc.

Homework Assignments: We would like to share with you an obvious, nonoriginal, but effective method to encourage students to do homework problems.

An essential feature is that it requires little extra work on the part of the instructor or grader. We assign homework problems (about 10 of them) after each lecture. At the end of the week (Fridays), students are asked to turn in their homeworks (typically 3 sets) for that week. We then choose at random one problem from each assignment (typically a total of 3) that will be graded. (The point is that the student does not know in advance which problems will be chosen.) Full credit is given for any of the chosen problems for which there is evidence that the student has made an honest attempt at solving. The homework problem sets are returned to the students at the next meeting (Mondays) with grades like 0/3, 1/3, 2/3 or 3/3 indicating the proportion of problems for which the student received credit. The homework grades are tallied at the end of the semester and count as one test grade. Certainly there are variations on this theme. The point is that students are motivated to do their homework with little additional cost (= time) to the instructor.

Syllabus Suggestions: To serve as a guide in constructing a syllabus for a one-semester or two-semester course, the prefaces to the texts list sample outlines that emphasize methods, applications, theory, partial differential equations, phase plane analysis, or computation or combinations of these. As a further guide in making a choice of subject matter, we provide below a listing of text material dealing with some common areas of emphasis.

References to material in Chapters 11, 12, or 13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems*, 4th edition.

Numerical, Graphical, and Qualitative Methods

The sections and projects dealing with numerical, graphical, and qualitative techniques for solving differential equations include:

Section 1.3, *Direction Fields*

Section 1.4, *The Approximation Method of Euler*

Project A for Chapter 1, *Taylor Series Method*

Project B for Chapter 1, *Picard's Method*

Project D for Chapter 1, *The Phase Line*

Section 3.6, *Improved Euler's Method*, which includes step-by-step outlines of the improved Euler's method subroutine and improved Euler's method with tolerance. These outlines are easy for the student to translate into a computer program (cf. pages 129, 130).

Section 3.7, *Higher-Order Numerical Methods: Taylor and Runge-Kutta*, which includes outlines for the Fourth Order Runge-Kutta subroutine and algorithm with tolerance (cf. pages 138, 139).

Project G for Chapter 3, *Stability of Numerical Methods*

Project H for Chapter 3, *Period Doubling and Chaos*

Section 4.7, *Qualitative Considerations for Variable-Coefficient and Nonlinear Equations*, which discusses the energy integral lemma, as well as the Airy, Bessel, Duffing, and van der Pol equations.

Section 5.3, *Solving Systems and Higher-Order Equations Numerically*, which describes the vectorized forms of Euler's Method and the Fourth-Order Runge-Kutta method, and discusses an application to population dynamics.

Section 5.4, *Introduction to the Phase Plane*, which introduces the study of trajectories of autonomous systems, critical points, and stability.

Section 5.7, *Dynamical Systems, Poincaré Maps, and Chaos*, which discusses the use of numerical methods to approximate the Poincaré map and how to interpret the results.

Project B for Chapter 5, *Designing a Landing System for Interplanetary Travel*

Project C for Chapter 5, *Things That Bob*

Project F for Chapter 5, *Strange Behavior of Competing Species-Part I*

Project D for Chapter 9, *Strange Behavior of Competing Species-Part II*

Project D for Chapter 10, *Numerical Method for $\Delta u = f$ on a Rectangle*

Project D for Chapter 11, *Shooting Method*

Project E for Chapter 11, *Finite-Difference Method for Boundary Value Problems*

Project C for Chapter 12, *Computing Phase Plane Diagrams*

Project D for Chapter 12, *Ecosystem of Planet GLIA-2*

Appendix A, *Newton's Method*

Appendix B, *Simpson's Rule*

Appendix D, *Method of Least Squares*

Appendix E, *Runge-Kutta Procedure for n Equations*

The instructor who wishes to emphasize numerical methods should also note that the text contains an extensive chapter on series solutions of differential equations (Chapter 8).

Engineering/Physics Applications

Since Laplace Transforms is a subject vital to engineering, we have included a detailed chapter on this topic—see Chapter 7. Stability is also an important subject for engineers, so we have included an introduction to the subject in Section 5.4 along with an entire chapter addressing this topic—see Chapter 12. Further materials dealing with engineering/physics applications include:

Project C for Chapter 1, *Magnetic "Dipole"*

Project A for Chapter 2, *Torricelli's Law of Fluid Flow*

Section 3.1, *Mathematical Modeling*

Section 3.2, *Compartmental Analysis*, which contains a discussion of mixing problems and of population models.

Section 3.3, *Heating and Cooling of Buildings*, which discusses temperature variations in the presence of air conditioning or furnace heating.

Section 3.4, *Newtonian Mechanics*

Section 3.5, *Electrical Circuits*

Project B for Chapter 3, *Curve of Pursuit*

Project C for Chapter 3, *Aircraft Guidance in a Crosswind*

Project D for Chapter 3, *Feedback and the Op Amp*

Project E for Chapter 3, *Bang-Bang Controls*

Section 4.1, *Introduction: The Mass-Spring Oscillator*

Section 4.7, *Qualitative Considerations for Variable-Coefficient and Nonlinear Equations*

Section 4.8, *A Closer Look at Free Mechanical Vibrations*

Section 4.9, *A Closer Look at Forced Mechanical Vibrations*

Project F for Chapter 4, *Apollo Reentry*

Project G for Chapter 4, *Simple Pendulum*

Chapter 5, *Introduction to Systems and Phase Plane Analysis*, which includes sections on coupled mass-spring systems, electrical circuits, and phase plane analysis.

Project B for Chapter 5, *Designing a Landing System for Interplanetary Travel*

Project C for Chapter 5, *Things that Bob*

Project E for Chapter 5, *Hamiltonian Systems*

Project B for Chapter 6, *Transverse Vibrations of a Beam*

Chapter 7, *Laplace Transforms*, which in addition to basic material includes discussions of transfer functions, the Dirac delta function, and frequency response modeling.

Projects for Chapter 8, which deal with Schrödinger's equation, buckling of a tower, and aging springs.

Project B for Chapter 9, *Matrix Laplace Transform Method*

Project C for Chapter 9, *Undamped Second-Order Systems*

Chapter 10, *Partial Differential Equations*, which includes sections on Fourier series, the heat equation, wave equation, and Laplace's equation.

Project A for Chapter 10, *Steady-State Temperature Distribution in a Circular Cylinder*

Project B for Chapter 10, *A Laplace Transform Solution of the Wave Equation*

Project A for Chapter 11, *Hermite Polynomials and the Harmonic Oscillator*

Section 12.4, *Energy Methods*, which addresses both conservative and nonconservative autonomous mechanical systems.

Project A for Chapter 12, *Solitons and Korteweg-de Vries Equation*

Project B for Chapter 12, *Burger's Equation*

Students of engineering and physics would also find Chapter 8 on series solutions particularly useful, especially Section 8.8 on Special Functions.

Biology/Ecology Applications

Project D for Chapter 1, *The Phase Line*, which discusses the logistic population model and bifurcation diagrams for population control.

Problem 32 in Exercises 2.2, which concerns radioisotopes and cancer detection.

Problem 37 in Exercises 2.3, which involves a simple model for the amount of a hormone in the blood during a 24-hour cycle.

Section 3.1, *Mathematical Modeling*

Section 3.2, *Compartmental Analysis*, which contains a discussion of mixing problems and population models.

Problem 19 in Exercises 3.6 which considers a generalization of the logistic model.

Problem 20 in Exercises 3.7, which involves chemical reaction rates.

Project A for Chapter 3, *Aquaculture*, which deals with a model for raising and harvesting catfish.

Section 5.1, *Interconnected Fluid Tanks*, which introduces systems of differential equations.

Section 5.3, *Solving Systems and Higher-Order Equations Numerically*, which contains an application to population dynamics.

Section 5.4, *Introduction to the Phase Plane*, which contains exercises dealing with the spread of a disease through a population (epidemic model).

Project A for Chapter 5, *The Growth of a Tumor*

Project D for Chapter 5, *Periodic Solutions to Volterra-Lotka Systems*

Project F for Chapter 5, *Strange Behavior of Competing Species-Part I*

Project G for Chapter 5, *Cleaning Up the Great Lakes*

Project D for Chapter 9, *Strange Behavior of Competing Species-Part II*

Problem 19 in Exercises 10.5, which involves chemical diffusion through a thin layer.

Project D for Chapter 12, *Ecosystem on Planet GLIA-2*

The basic content of the remainder of this instructor's manual consists of supplemental Group Projects along with the answers to the even numbered problems. These answers are for the most part not available any place else since the text only provides answers to odd numbered problems, and the Student Solution Manual contains only a handful of worked solutions to even numbered problems.

We would appreciate any comments you may have concerning the answers in this manual. These comments can be sent to the authors' email addresses given below. We also would encourage sharing with us (= the authors and users of the texts) any of your favorite Group Projects.

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Supplemental Group Projects

Group Project for Chapter 2

Designing a Solar Collector

You want to design a solar collector that will concentrate the sun's rays at a point. By symmetry this surface will have a shape that is a surface of revolution obtained by revolving a curve about an axis. Without loss of generality, you can assume that this axis is the x -axis and the rays parallel to this axis are focused at the origin (see Figure GP-1). To derive the equation for the curve, proceed as follows:

- The law of reflection says that the angles γ and δ are equal. Use this and results from geometry to show that $\beta = 2\alpha$.

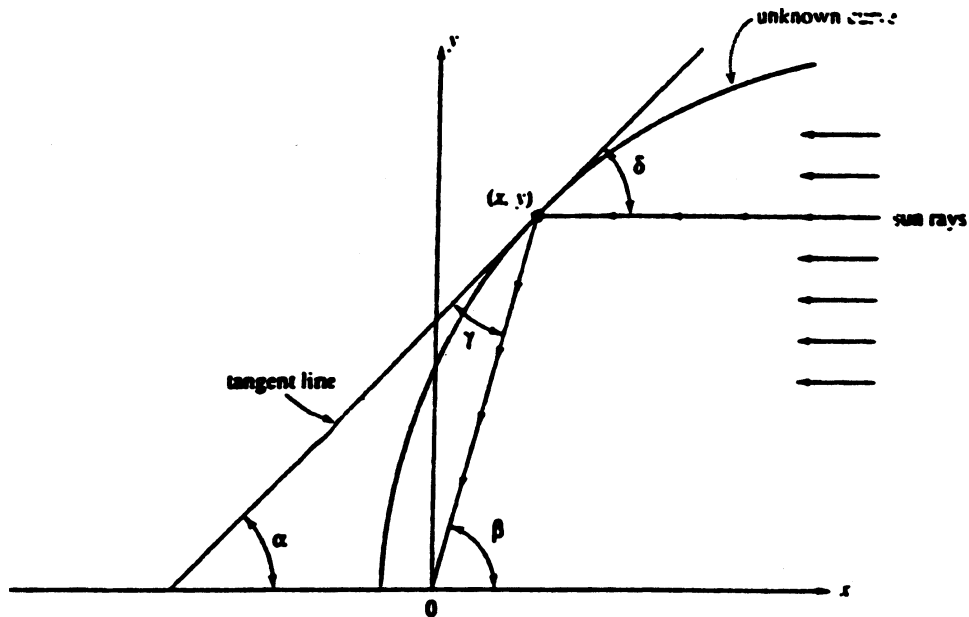


Figure GP-1: Curve that generates a solar collector

- From calculus recall that $\frac{dy}{dx} = \tan \alpha$. Use this, the fact that $\frac{y}{x} = \tan \beta$, and the double angle formula to show that

$$\frac{y}{x} = \frac{2 \frac{dy}{dx}}{1 - \left(\frac{dy}{dx}\right)^2}.$$

- Now show that the curve satisfies the differential equation.

$$(1) \quad \frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y}$$

- Solve equation (1).
- Describe the solutions and identify the type of collector obtained.

Group Projects for Chapter 3

Delay Differential Equations

In our discussion of mixing problems in Section 3.2, we encountered the initial value problem

$$(1) \quad \begin{aligned} x'(t) &= 6 - \frac{3}{500}x(t-t_0), \\ x(t) &= 0 \quad \text{for } t \in [-t_0, 0] \end{aligned}$$

where t_0 is a positive constant. The equation in (1) is an example of a **delay differential equation**. These equations differ from the usual differential equations by the presence of the shift $(t-t_0)$ in the argument of the unknown function $x(t)$. In general, these equations are more difficult to work with than are regular differential equations, but quite a bit is known about them.†

- a. Show that the simple linear delay differential equation

$$(2) \quad x'(t) = ax(t-b),$$

where a and b are constants, has a solution of the form $x(t) = Ce^{st}$, for any constant C , provided s satisfies the transcendental equation $s = ae^{-bs}$.

- b. A solution to (2) for $t > 0$ can also be found using the **method of steps**. Assume that $x(t) = f(t)$ for $-b \leq t \leq 0$. For $0 \leq t \leq b$, equation (2) becomes

$$x'(t) = ax(t-b) = af(t-b),$$

and so

$$x(t) = \int_0^1 af(v-b)dv + x(0).$$

Now that we know $x(t)$ on $[0, b]$, we can repeat this procedure to obtain

$$x(t) = \int_b^1 ax(v-b)dv + x(b)$$

for $b \leq t \leq 2b$. This process can be continued indefinitely.

Use the method of steps to show that the solution to the initial value problem

$$x'(t) = -x(t-1), \quad x(t) = 1 \quad \text{on } [-1, 0]$$

is given by

$$x(t) = \sum_{k=0}^n (-1)^k \frac{[t-(k-1)]^k}{k!}, \quad \text{for } n-1 \leq t \leq n,$$

where n is a nonnegative integer. (This problem can also be solved using the Laplace transform method of Chapter 7.)

- c. Use the method of steps to compute the solution to the initial value problem given in (1) on the interval $0 \leq t \leq 15$ for $t_0 = 3$.

†See, for example, *Differential-Difference Equations*, by R. Bellman and K. L. Cooke, Academic Press, New York, 1963, or *Ordinary and Delay Differential Equations*, by R. D. Driver, Springer-Verlag, New York, 1977.

Extrapolation

When precise information about the *form* of the error in an approximation is known, a technique called **extrapolation** can be used to improve the rate of convergence.

Suppose the approximation method converges with rate $O(h^p)$ as $h \rightarrow 0$ (cf. Section 3.6). From theoretical considerations assume we know, more precisely, that

$$(1) \quad y(x; h) = \phi(x) + h^p a_p(x) + O(h^{p+1}),$$

where $y(x; h)$ is the approximation to $\phi(x)$ using step size h and $a_p(x)$ is some function that is independent of h (typically we do not know a formula for $a_p(x)$, only that it exists). Our goal is to obtain approximations that converge at the *faster* rate $O(h^{p+1})$.

We start by replacing h by $\frac{h}{2}$ in (1) to get

$$y\left(x; \frac{h}{2}\right) = \phi(x) + \frac{h^p}{2^p} a_p(x) + O(h^{p+1}).$$

If we multiply both sides by 2^p and subtract equation (1), we find

$$2^p y\left(x; \frac{h}{2}\right) - y(x; h) = (2^p - 1)\phi(x) + O(h^{p+1}).$$

Solving for $\phi(x)$ yields

$$\phi(x) = \frac{2^p y\left(x; \frac{h}{2}\right) - y(x; h)}{2^p - 1} + O(h^{p+1}).$$

Hence

$$y^*\left(x; \frac{h}{2}\right) := \frac{2^p y\left(x; \frac{h}{2}\right) - y(x; h)}{2^p - 1}$$

has a rate of convergence of $O(h^{p+1})$.

a. Assuming

$$y^*\left(x; \frac{h}{2}\right) = \phi(x) + h^{p+1} a_{p+1}(x) + O(h^{p+2}),$$

show that

$$y^{**}\left(x; \frac{h}{4}\right) := \frac{2^{p+1} y^*\left(x; \frac{h}{4}\right) - y^*\left(x; \frac{h}{2}\right)}{2^{p+1} - 1} \text{ has a}$$

rate of convergence of $O(h^{p+2})$.

b. Assuming

$$y^{**}\left(x; \frac{h}{4}\right) = \phi(x) + h^{p+2} a_{p+2}(x) + O(h^{p+3}),$$

show that

$$y^{***}\left(x; \frac{h}{8}\right) := \frac{2^{p+2} y^{**}\left(x; \frac{h}{8}\right) - y^{**}\left(x; \frac{h}{4}\right)}{2^{p+2} - 1}$$

has a rate of convergence of $O(h^{p+3})$.

c. The results of using Euler's method

(with $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$) to approximate the

solution to the initial value problem $y' = y$, $y(0) = 1$, at $x = 1$ are given in Table 1.2, page 27. For Euler's method, the extrapolation procedure applies with $p = 1$. Use the results in Table 1.2 to find an approximation to $e = y(1)$

by computing $y^{***}\left(1; \frac{1}{8}\right)$. [Hint: Compute

$y^*\left(1; \frac{1}{2}\right)$, $y^*\left(1; \frac{1}{4}\right)$, and $y^*\left(1; \frac{1}{8}\right)$; then

compute $y^{**}\left(1; \frac{1}{4}\right)$, and $y^{**}\left(1; \frac{1}{8}\right)$.]

d. Table 1.2 also contains Euler's approximation for $y(1)$ when $h = \frac{1}{16}$. Use this additional

information to compute the next step in the extrapolation procedure; that is, compute

$y^{****}\left(1; \frac{1}{16}\right)$.