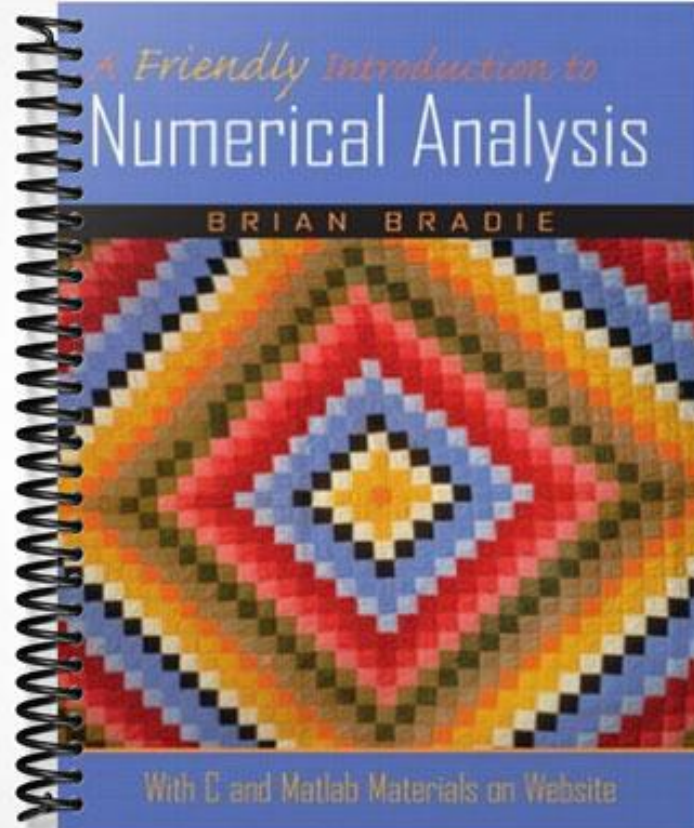


# SOLUTIONS MANUAL



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## 1.2 CONVERGENCE

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1. Compute each of the following limits and determine the corresponding rate of convergence.

- (a)  $\lim_{n \rightarrow \infty} \frac{n-1}{n^3+2}$   
 (b)  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$   
 (c)  $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$   
 (d)  $\lim_{n \rightarrow \infty} \frac{3n^2-1}{7n^2+n+2}$

(a) For  $n > 1$ ,

$$\left| \frac{n-1}{n^3+2} - 0 \right| = \frac{n-1}{n^3+2} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Thus,  $\frac{n-1}{n^3+2}$  converges to 0 with rate of convergence  $O(1/n^2)$ .

(b) Note that

$$\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

Because

$$|(\sqrt{n+1} - \sqrt{n}) - 0| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}},$$

it follows that  $\sqrt{n+1} - \sqrt{n}$  converges to 0 with rate of convergence  $O(1/\sqrt{n})$ .

(c) Since  $-1 \leq \sin n \leq 1$  for all  $n$ , it follows that

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

for all  $n$ . Then, by the squeeze theorem,  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$ . Moreover, because

$$\left| \frac{\sin n}{n} - 0 \right| \leq \frac{1}{n},$$

the rate of convergence is  $O(1/n)$ .

(d) For  $n > 13$ ,

$$\left| \frac{3n^2-1}{7n^2+n+2} - \frac{3}{7} \right| = \frac{3n+13}{7(7n^2+n+2)} < \frac{4n}{49n^2} < \frac{1}{10n}.$$

Therefore,  $\frac{3n^2-1}{7n^2+n+2}$  converges to  $\frac{3}{7}$  with rate of convergence  $O(1/n)$ .

2. Compute each of the following limits and determine the corresponding rate of convergence.

- (a)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$   
 (b)  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$   
 (c)  $\lim_{x \rightarrow 0} \frac{e^x - \cos x - x}{x^2}$   
 (d)  $\lim_{x \rightarrow 0} \frac{\cos x - 1 + x^2/2 - x^4/24}{x^6}$

- (a) From Taylor's Theorem,  $e^x = 1 + x + \frac{1}{2}x^2e^\xi$  for some  $\xi$  between 0 and  $x$ . Hence,

$$\frac{e^x - 1}{x} = 1 + \frac{1}{2}xe^\xi.$$

Because

$$\left| \frac{e^x - 1}{x} - 1 \right| = \frac{1}{2}|x|e^\xi < |x|$$

for all  $x$  satisfying  $|x| < \ln 2$ , it follows that

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \text{ with rate of convergence } O(x).$$

- (b) From Taylor's Theorem,  $\sin x = x - \frac{x^3}{6} \cos \xi$  for some  $\xi$  between 0 and  $x$ . Then,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} \cos \xi$$

and

$$\left| \frac{\sin x}{x} - 1 \right| = \frac{1}{6}|x^2 \cos \xi| \leq \frac{1}{6}x^2.$$

Finally,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ with rate of convergence } O(x^2).$$

- (c) From Taylor's Theorem, we have

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3e^{\xi_1}$$

and

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi_2$$

for some  $\xi_1$  and  $\xi_2$  between 0 and  $x$ . Then

$$\frac{e^x - \cos x - x}{x^2} = 1 + \frac{x}{6}(e^{\xi_1} - \sin \xi_2).$$

For sufficiently small  $x$ ,  $e^{\xi_1} < 2$ , so  $|e^{\xi_1} - \sin \xi_2| < 2 + 1 = 3$ . Thus,

$$\left| \frac{e^x - \cos x - x}{x^2} - 1 \right| = \frac{|x|}{6} |e^{\xi_1} - \sin \xi_2| \leq \frac{1}{2}|x|,$$

and

$$\lim_{x \rightarrow 0} \frac{e^x - \cos x - x}{x^2} = 1 \text{ with rate of convergence } O(x).$$

(d) From Taylor's Theorem, we have

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{8!}x^8 \cos \xi$$

for some  $\xi$  between 0 and  $x$ . Hence,

$$\frac{\cos x - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4}{x^6} = -\frac{1}{720} + \frac{1}{8!}x^2 \cos \xi,$$

and

$$\left| \frac{\cos x - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4}{x^6} + \frac{1}{720} \right| = \frac{1}{8!} |x^2 \cos \xi| \leq \frac{1}{8!} |x^2|.$$

It therefore follows that

$$\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4}{x^6} = -\frac{1}{720}$$

with rate of convergence  $O(x^2)$ .

3. Numerically determine which of the following sequences approaches 1 faster, and then confirm the numerical evidence by determining the rate of convergence of each sequence.

$$\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \quad \text{versus} \quad \lim_{x \rightarrow 0} \frac{(\sin x)^2}{x^2}.$$

The values in the following table suggest that  $\frac{\sin x^2}{x^2}$  converges toward 1 more rapidly than  $\frac{(\sin x)^2}{x^2}$ .

$x$	$\frac{\sin x^2}{x^2}$	$\frac{(\sin x)^2}{x^2}$
1.000	0.84147098480790	0.70807341827357
0.100	0.99998333341667	0.99667110793792
0.010	0.99999998333333	0.99996667111111
0.001	0.99999999999983	0.99999966666671

To confirm this conclusion, note that by Taylor's Theorem,

$$\sin u = u - \frac{1}{6}u^3 \cos \xi,$$

for some  $\xi$  between 0 and  $u$ . Using the substitution  $u = x^2$ , we find

$$\sin x^2 = x^2 - \frac{1}{6}x^6 \cos \xi$$

for some  $\xi$  between 0 and  $x^2$ . Consequently,

$$\left| \frac{\sin x^2}{x^2} - 1 \right| = \frac{1}{6}x^4 |\cos \xi| \leq \frac{1}{6}x^4.$$

Starting from  $f(x) = (\sin x)^2$ , we find

$$f'(x) = 2 \sin x \cos x = \sin 2x, \quad f''(x) = 2 \cos 2x, \quad f'''(x) = -4 \sin 2x,$$

and  $f^{(4)}(x) = -8 \cos 2x$ . Therefore,

$$(\sin x)^2 = x^2 - \frac{1}{3}x^4 \cos 2\xi$$

for some  $\xi$  between 0 and  $x$ , and

$$\left| \frac{(\sin x)^2}{x^2} - 1 \right| = \frac{1}{3}x^2 |\cos 2\xi| \leq \frac{1}{3}x^2.$$

Finally,

$$\frac{\sin x^2}{x^2} = 1 + O(x^4) \quad \text{and} \quad \frac{(\sin x)^2}{x^2} = 1 + O(x^2).$$

4. Suppose that  $0 < a < b$ .

(a) Show that if  $\alpha_n = \alpha + O(1/n^b)$ , then  $\alpha_n = \alpha + O(1/n^a)$ .

(b) Show that if  $f(x) = L + O(x^b)$ , then  $f(x) = L + O(x^a)$ .

(a) Suppose  $\alpha_n = \alpha + O(1/n^b)$ . Then, there exists a constant  $\lambda$  such that for sufficiently large  $n$ ,  $|\alpha_n - \alpha| \leq \lambda \frac{1}{n^b}$ . Because  $a < b$ , it follows that  $n^a < n^b$  and  $\frac{1}{n^a} > \frac{1}{n^b}$  for all  $n > 1$ . Thus,

$$|\alpha_n - \alpha| \leq \lambda \frac{1}{n^b} < \lambda \frac{1}{n^a},$$

and  $\alpha_n = \alpha + O(1/n^a)$ .

(b) Suppose  $f(x) = L + O(x^b)$ . Then, there exists a constant  $K$  such that for all sufficiently small  $x$ ,  $|f(x) - L| \leq K|x|^b$ . Because  $a < b$ , it follows that for all  $|x| \leq 1$ ,  $|x|^b \leq |x|^a$ . Thus, for sufficiently small  $x$ ,

$$|f(x) - L| \leq K|x|^b \leq K|x|^a,$$

and  $f(x) = L + O(x^a)$ .

5. Suppose that  $f_1(x) = L_1 + O(x^a)$  and  $f_2(x) = L_2 + O(x^b)$ . Show that

$$c_1 f_1(x) + c_2 f_2(x) = c_1 L_1 + c_2 L_2 + O(x^c),$$

where  $c = \min(a, b)$ .

Suppose  $f_1(x) = L_1 + O(x^a)$  and  $f_2(x) = L_2 + O(x^b)$ . Then, there exist constants  $K_1$  and  $K_2$  such that for all sufficiently small  $x$ ,  $|f_1(x) - L_1| \leq K_1|x^a|$  and  $|f_2(x) - L_2| \leq K_2|x^b|$ . Let  $c_1$  and  $c_2$  be any real numbers. Applying the triangle inequality, we find

$$\begin{aligned} |c_1f_1(x) + c_2f_2(x) - (c_1L_1 + c_2L_2)| &\leq |c_1||f_1(x) - L_1| + |c_2||f_2(x) - L_2| \\ &\leq |c_1|K_1|x^a| + |c_2|K_2|x^b|. \end{aligned}$$

Now, let  $c = \min(a, b)$ . Then, for  $|x| < 1$ ,

$$|c_1|K_1|x^a| + |c_2|K_2|x^b| < |c_1|K_1|x^c| + |c_2|K_2|x^c| = (|c_1|K_1 + |c_2|K_2)|x^c|.$$

Consequently,

$$c_1f_1(x) + c_2f_2(x) = c_1L_1 + c_2L_2 + O(x^c).$$

6. The table below lists the errors of successive iterates for three different methods for approximating  $\sqrt[3]{5}$ . Estimate the order of convergence of each method, and explain how you arrived at your conclusions.

Method 1	Method 2	Method 3
$4.0 \times 10^{-2}$	$3.7 \times 10^{-4}$	$4.3 \times 10^{-3}$
$9.1 \times 10^{-4}$	$1.2 \times 10^{-15}$	$1.8 \times 10^{-8}$
$4.8 \times 10^{-7}$	$1.5 \times 10^{-60}$	$1.4 \times 10^{-24}$

If a sequence converges of order  $\alpha$ , then the error in each term of the sequence is roughly the error in the previous term raised to the power  $\alpha$ . From the data for "Method 1," we see that each error is roughly the previous error squared; therefore, we estimate the order of convergence to be  $\alpha = 2$ . From the data for "Method 2," we see that each error is roughly the previous error raised to the fourth power; therefore, we estimate the order of convergence to be  $\alpha = 4$ . Finally, from the data for "Method 3," we see that each error is roughly the previous error raised to the third power; therefore, we estimate the order of convergence to be  $\alpha = 3$ .

7. Let  $\{p_n\}$  be a sequence which converges to the limit  $p$ .

(a) If

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = 0,$$

what can be said about the order of convergence of  $\{p_n\}$  to  $p$ ?

(b) If

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \rightarrow \infty,$$

what can be said about the order of convergence of  $\{p_n\}$  to  $p$ ?

(a) If

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = 0,$$

then the numerator approaches zero faster than the denominator. In order to achieve a nonzero limit, we must increase the power in the denominator. Therefore, the order of convergence must be greater than  $\alpha$ .

(b) If

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \rightarrow \infty,$$

then the denominator approaches zero faster than the numerator. In order to achieve a nonzero limit, we must decrease the power in the denominator. Therefore, the order of convergence must be less than  $\alpha$ .

8. Suppose theory indicates that the sequence  $\{p_n\}$  converges to  $p$  of order 1.5. Explain how you would numerically verify this order of convergence.

To numerically verify the order of convergence, calculate the ratio

$$\frac{|p_{n+1} - p|}{|p_n - p|^{1.5}}$$

for several successive values of  $n$ . If the order of convergence is  $\alpha = 1.5$ , these ratios should approach a constant, specifically the asymptotic error constant.

9. Theory indicates that the following sequence should converge to  $\sqrt{3}$  of order 1.618. Does the sequence actually achieve an order of convergence of 1.618? If not, what is the actual order?

$n$	$p_n$
0	2.0000000000000000
1	1.6666666666666667
2	1.727272727272727
3	1.732142857142857
4	1.732050680431722
5	1.732050807565499

Because the values in the third column of the following table appear to be approaching a constant, the evidence suggests that the sequence does, in fact, converge toward  $\sqrt{3}$  with order of convergence  $\alpha = 1.618$ .

$n$	$p_n$	$ p_n - \sqrt{3} / p_{n-1} - \sqrt{3} ^{1.618}$
1	2.0000000000000000	
2	1.6666666666666667	0.55066002953142
3	1.727272727272727	0.39429299851516
4	1.732142857142857	0.52358803162884
5	1.732050680431722	0.43100791441420
6	1.732050807565499	0.48525581579327

10. Theory indicates that the following sequence should converge to  $4/3$  of order 1.618. Does the sequence actually achieve an order of convergence of 1.618? If not, what is the actual order?

$n$	$p_n$
0	1.498664098580016
1	1.497353997792205
2	1.428801977335339
3	1.401092915389552
4	1.376493676051456
5	1.361345745573130
6	1.351034482500881
7	1.344479850695066

Because the values in the third column of the following table are increasing with  $n$ , the evidence suggests that the sequence does not have order of convergence  $\alpha = 1.618$ , but rather that the order of convergence is less than 1.618. Because the values in the fourth column appear to be approaching a constant, these values suggest that the sequence is converging to  $4/3$  with order of convergence  $\alpha = 1$ .

$n$	$p_n$	$ p_n - 4/3 / p_{n-1} - 4/3 ^{1.618}$	$ p_n - 4/3 / p_{n-1} - 4/3 $
1	1.49866409858002		
2	1.49735399779221	3.01718763541581	0.99207588021590
3	1.42880197733534	1.77891367138598	0.58205253781266
4	1.40109291538955	3.03079120639280	0.70975745769255
5	1.37649367605146	3.36181849329742	0.63696294174768
6	1.36134574557313	4.52671513900300	0.64903127444432
7	1.35103448250088	5.75689539760301	0.63190377951100
8	1.34447985069507	7.61855893491390	0.62970586012393

11. Show that the convergence of the sequence generated by the formula

$$x_{n+1} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a}$$

toward  $\sqrt{a}$  is third-order. What is the asymptotic error constant?

Note

$$\begin{aligned} x_{n+1} - \sqrt{a} &= \frac{x_n^3 + 3x_n a}{3x_n^2 + a} - \sqrt{a} = \frac{x_n^3 - 3x_n^2 \sqrt{a} + 3x_n a - a^{3/2}}{3x_n^2 + a} \\ &= \frac{(x_n - \sqrt{a})^3}{3x_n^2 + a}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - \sqrt{a}|}{|x_n - \sqrt{a}|^3} = \lim_{n \rightarrow \infty} \frac{1}{3x_n^2 + a} = \frac{1}{4a}.$$



Consequently,  $x_n \rightarrow \sqrt{a}$  with order of convergence  $\alpha = 3$  and asymptotic error constant  $\lambda = \frac{1}{4a}$ .

12. Let  $a$  be a non-zero real number. For any  $x_0$  satisfying  $0 < x_0 < 2/a$ , the recursive sequence defined by

$$x_{n+1} = x_n(2 - ax_n)$$

converges to  $1/a$ . What are the order of convergence and the asymptotic error constant?

Note

$$\begin{aligned} x_{n+1} - \frac{1}{a} &= x_n(2 - ax_n) - \frac{1}{a} = -ax_n^2 + 2x_n - \frac{1}{a} \\ &= -a \left( x_n^2 - \frac{2}{a}x_n + \frac{1}{a^2} \right) = -a \left( x_n - \frac{1}{a} \right)^2. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - \frac{1}{a}|}{|x_n - \frac{1}{a}|^2} = \lim_{n \rightarrow \infty} a = a.$$

Consequently,  $x_n \rightarrow \frac{1}{a}$  with order of convergence  $\alpha = 2$  and asymptotic error constant  $\lambda = a$ .

13. Suppose that the sequence  $\{p_n\}$  converges linearly to the limit  $p$  with asymptotic error constant  $\lambda$ . Further suppose that  $p_{n+1} - p$ ,  $p_n - p$  and  $p_{n-1} - p$  are all of the same sign. Show that

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} \approx \lambda.$$

Suppose the sequence  $\{p_n\}$  converges linearly to  $p$  with asymptotic error constant  $\lambda$ . Then

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda,$$

so, for sufficiently large  $n$ ,

$$|p_{n+1} - p| \approx \lambda |p_n - p|.$$

Moreover,

$$|p_n - p| \approx \lambda |p_{n-1} - p| \quad \text{or} \quad |p_{n-1} - p| \approx \frac{1}{\lambda} |p_n - p|.$$

Because we are given that  $p_{n+1} - p$ ,  $p_n - p$  and  $p_{n-1} - p$  are all of the same sign, we may drop the absolute values from the above expressions. Now,

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} = \frac{p_{n+1} - p - (p_n - p)}{p_n - p - (p_{n-1} - p)}$$

$$\begin{aligned} &\approx \frac{\lambda(p_n - p) - (p_n - p)}{p_n - p - \frac{1}{\lambda}(p_n - p)} \\ &= \frac{\lambda - 1}{1 - \frac{1}{\lambda}} = \lambda. \end{aligned}$$

14. A sequence  $\{p_n\}$  converges *superlinearly* to  $p$  provided

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

Show that if  $p_n \rightarrow p$  of order  $\alpha$  for  $\alpha > 1$ , then  $\{p_n\}$  converges superlinearly to  $p$ .

Suppose the sequence  $\{p_n\}$  converges  $p$  of order  $\alpha > 1$  with asymptotic error constant  $\lambda$ . Then

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p| \cdot |p_n - p|^{\alpha-1}}{|p_n - p|^\alpha} \\ &= \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot \lim_{n \rightarrow \infty} |p_n - p|^{\alpha-1} \\ &= \lambda \cdot 0 = 0. \end{aligned}$$

Therefore,  $\{p_n\}$  converges superlinearly to  $p$ .

15. Suppose that  $\{p_n\}$  converges superlinearly to  $p$  (see Exercise 14). Show that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

Note that

$$\frac{p_{n+1} - p_n}{p_n - p} = \frac{p_{n+1} - p - (p_n - p)}{p_n - p} = \frac{p_{n+1} - p}{p_n - p} - 1.$$

Because  $\{p_n\}$  converges superlinearly to  $p$ , it then follows that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \left| \lim_{n \rightarrow \infty} \left( \frac{p_{n+1} - p}{p_n - p} - 1 \right) \right| = |0 - 1| = 1.$$

16. (a) Determine the third-degree Taylor polynomial and associated remainder term for the function  $f(x) = \ln(1 - x)$ . Use  $x_0 = 0$ .

- (b) Using the results of part (a), approximate  $\ln(0.25)$  and compute the theoretical error bound associated with this approximation. Compare the theoretical error bound with the actual error.
- (c) Compute the following limit and determine the corresponding rate of convergence:

$$\lim_{x \rightarrow 0} \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3}.$$

- (a) Let  $f(x) = \ln(1-x)$ . Then

$$f'(x) = -\frac{1}{1-x}, \quad f''(x) = -\frac{1}{(1-x)^2}, \quad f'''(x) = -\frac{2}{(1-x)^3}, \quad \text{and } f^{(4)}(x) = -\frac{6}{(1-x)^4}.$$

Moreover,

$$f(0) = \ln 1 = 0, \quad f'(0) = -1, \quad f''(0) = -1, \quad f'''(0) = -2, \quad \text{and } f^{(4)}(\xi) = -\frac{6}{(1-\xi)^4}.$$

Finally,

$$\begin{aligned} \ln(1-x) &= P_3(x) + R_3(x) \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4(1-\xi)^4}, \end{aligned}$$

for some  $\xi$  between 0 and  $x$ .

- (b) Using the result of part (a),

$$\ln(0.25) \approx P_3(0.75) = -0.75 - \frac{0.75^2}{2} - \frac{0.75^3}{3} = -1.171875.$$

Because  $0 < \xi < 0.75$ ,  $(1-\xi)^{-4} \leq 4^4$  and

$$|\text{error}| = |R_3(0.75)| = \frac{0.75^4}{4(1-\xi)^4} \leq \frac{81}{4} = 20.25.$$

The actual error is  $|\ln(0.25) - P_3(0.75)| \approx 0.214419$ , which is significantly less than the theoretical error bound.

- (c) Once again using the result from part (a), we find

$$\frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} = -\frac{1}{3} - \frac{x}{4(1-\xi)^4}.$$

Moreover,

$$\left| \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} + \frac{1}{3} \right| = \frac{|x|}{4|1-\xi|^4} \leq |x|,$$

for all sufficiently small  $x$ . Therefore,

$$\lim_{x \rightarrow 0} \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} = -\frac{1}{3},$$

with rate of convergence  $O(x)$ .

17. (a) Determine the third-degree Taylor polynomial and associated remainder term for the function  $f(x) = \sqrt{1+x}$ . Use  $x_0 = 0$ .
- (b) Using the results of part (a), approximate  $\sqrt{1.5}$  and compute the theoretical error bound associated with this approximation. Compare the theoretical error bound with the actual error.
- (c) Compute the following limit and determine the corresponding rate of convergence:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}.$$

- (a) Let  $f(x) = \sqrt{1+x}$ . Then

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \quad f''(x) = -\frac{1}{4}(1+x)^{-3/2}, \quad f'''(x) = -\frac{3}{8}(1+x)^{-5/2},$$

and  $f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$ . Moreover,

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = -\frac{1}{4}, \quad f'''(0) = \frac{3}{8},$$

and  $f^{(4)}(\xi) = -\frac{15}{16}(1+\xi)^{-7/2}$ . Finally,

$$\begin{aligned} \sqrt{1+x} &= P_3(x) + R_3(x) \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4(1+\xi)^{-7/2}, \end{aligned}$$

for some  $\xi$  between 0 and  $x$ .

- (b) Using the result of part (a),

$$\sqrt{1.5} \approx P_3(0.5) = 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 = 1.2265625.$$

Because  $0 < \xi < 0.5$ ,  $(1+\xi)^{-7/2} \leq 1$  and

$$|\text{error}| = |R_3(0.5)| \leq \frac{5}{128}(0.5)^4 \approx 2.44 \times 10^{-3}.$$

The actual error is  $|\sqrt{1.5} - P_3(0.5)| \approx 1.82 \times 10^{-3}$ , which is less than the theoretical error bound.

- (c) Once again using the result from part (a), we find

$$\frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = -\frac{1}{8} - \frac{x}{16}(1+\xi)^{-5/2}.$$

Moreover,

$$\left| \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} + \frac{1}{8} \right| = \frac{|x|}{16} |1 - \xi|^{-5/2} \leq \frac{1}{16}|x|,$$

for all sufficiently small  $x$ . Therefore,

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = -\frac{1}{8},$$

with rate of convergence  $O(x)$ .

In Exercises 18 - 21, verify that Taylor's theorem produces the indicated formula, where  $\xi$  is between 0 and  $x$ .

18.

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^\xi$$

Let  $f(x) = e^x$ . Then  $f^{(n)}(x) = e^x$  and  $f^{(n)}(0) = 1$  for all  $n$ . Therefore, by Taylor's Theorem with  $x_0 = 0$ ,

$$\begin{aligned} e^x &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \\ &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^\xi, \end{aligned}$$

for some  $\xi$  between 0 and  $x$ .

19.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cos \xi$$

Let  $f(x) = \sin x$ . Then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x.$$

Moreover,

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad \text{and} \quad f'''(0) = -1.$$

As higher-order derivatives are calculated, this cycle of four values repeats indefinitely. In particular, we find

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n.$$

Therefore, by Taylor's Theorem with  $x_0 = 0$ ,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cos \xi,$$

for some  $\xi$  between 0 and  $x$ .

20.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cos \xi$$

Let  $f(x) = \cos x$ . Then

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad \text{and} \quad f'''(x) = \sin x.$$

Moreover,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad \text{and} \quad f'''(0) = 0.$$

As higher-order derivatives are calculated, this cycle of four values repeats indefinitely. In particular, we find

$$f^{(2n)}(0) = (-1)^n \quad \text{and} \quad f^{(2n+1)}(0) = 0.$$

Therefore, by Taylor's Theorem with  $x_0 = 0$ ,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cos \xi,$$

for some  $\xi$  between 0 and  $x$ .

21.

$$\frac{1}{1+x} = 1 - x + x^2 - + \cdots + (-1)^n x^n + \frac{(-1)^{n+1} x^{n+1}}{(1+\xi)^{n+2}}$$

Let  $f(x) = \frac{1}{1+x} = (1+x)^{-1}$ . Then,

$$f'(x) = -1 \cdot (1+x)^{-2}, \quad f''(x) = 1 \cdot 2 \cdot (1+x)^{-3}, \quad f'''(x) = -1 \cdot 2 \cdot 3 \cdot (1+x)^{-4},$$

and, in general,  $f^{(n)}(x) = (-1)^n \cdot n! \cdot (1+x)^{-n-1}$ . Therefore, by Taylor's Theorem with  $x_0 = 0$ ,

$$\frac{1}{1+x} = 1 - x + x^2 - + \cdots + (-1)^n x^n + \frac{(-1)^{n+1} x^{n+1}}{(1+\xi)^{n+2}},$$

for some  $\xi$  between 0 and  $x$ .