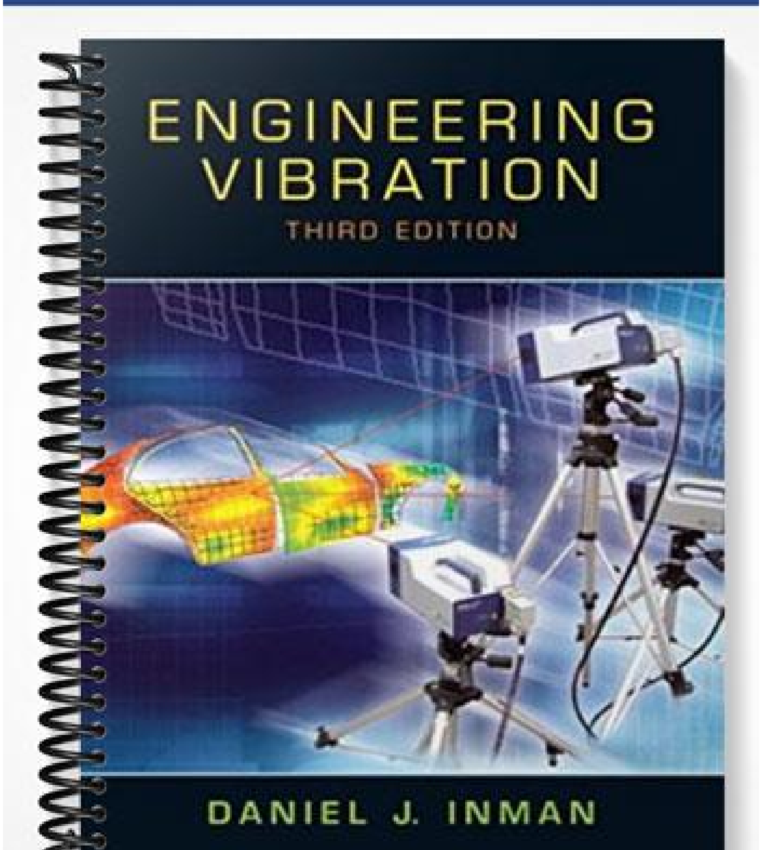
SOLUTIONS MANUAL



Problems and Solutions for Section 1.2 and Section 1.3 (1.20 to 1.51)

Problems and Solutions Section 1.2   (Numbers 1.20 through 1.30)

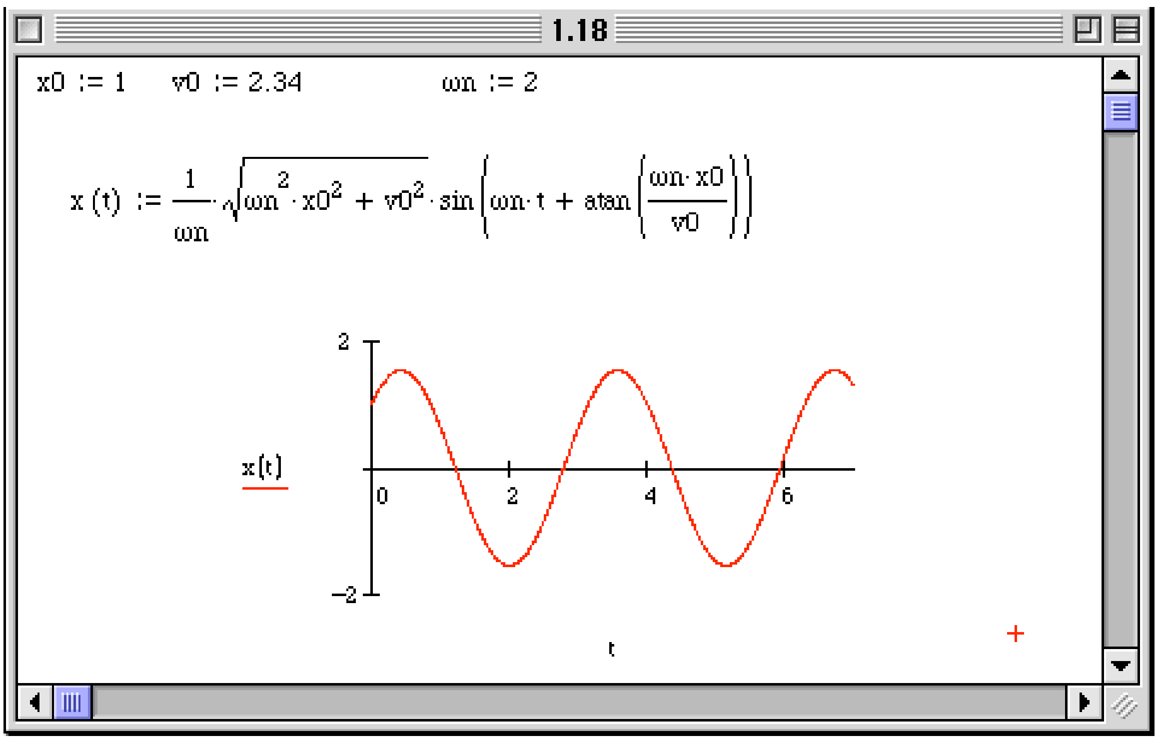
1.20* Plot the solution of a linear, spring and mass system with frequency $\omega_n = 2$ rad/s, $x_0 = 1$ mm and $v_0 =$ 2.34 mm/s, for at least two periods.

Solution: From Window 1.18, the plot can be formed by computing:

$$A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2} = 1.54 \text{ mm},\ \ \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = 40.52^\circ$$

$$x(t) = A\sin(\omega_n t + \phi)$$

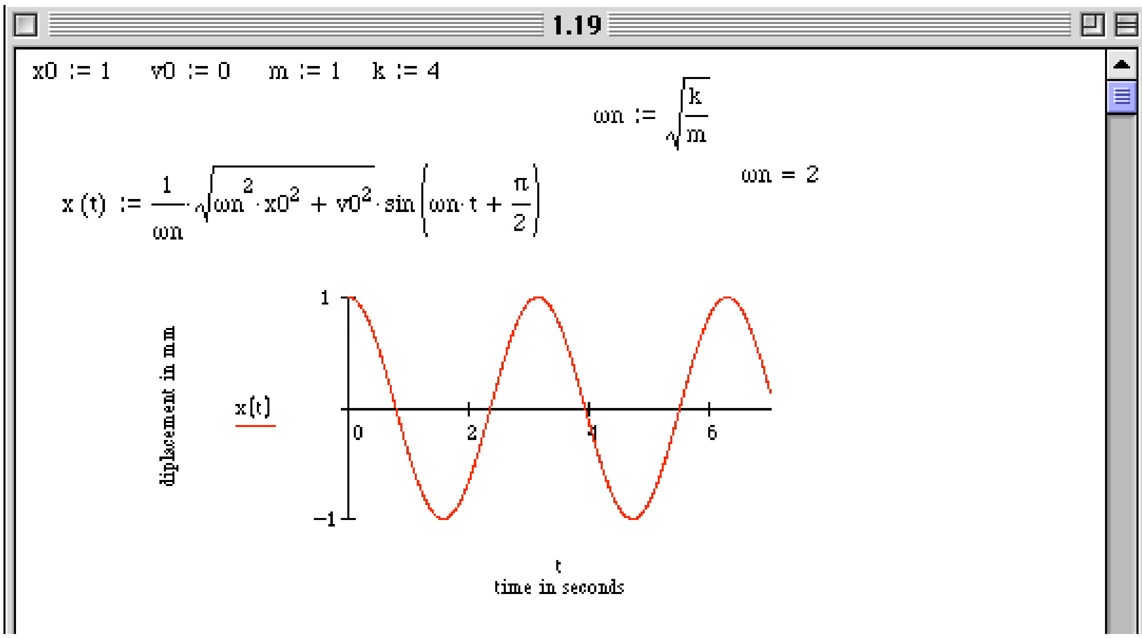
This can be plotted in any of the codes mentioned in the text. In Mathcad the program looks like.



In this plot the units are in mm rather than meters.

1.21* Compute the natural frequency and plot the solution of a spring-mass system with mass of 1 kg and stiffness of 4 N/m, and initial conditions of x_0 = 1 mm and v_0 = 0 mm/s, for at least two periods.

Solution: Working entirely in Mathcad, and using the units of mm yields:



Any of the other codes can be used as well.

1.22 To design a linear, spring-mass system it is often a matter of choosing a spring constant such that the resulting natural frequency has a specified value. Suppose that the mass of a system is 4 kg and the stiffness is 100 N/m. How much must the spring stiffness be changed in order to increase the natural frequency by 10%?

Solution: Given m =4 kg and k = 100 N/m the natural frequency is

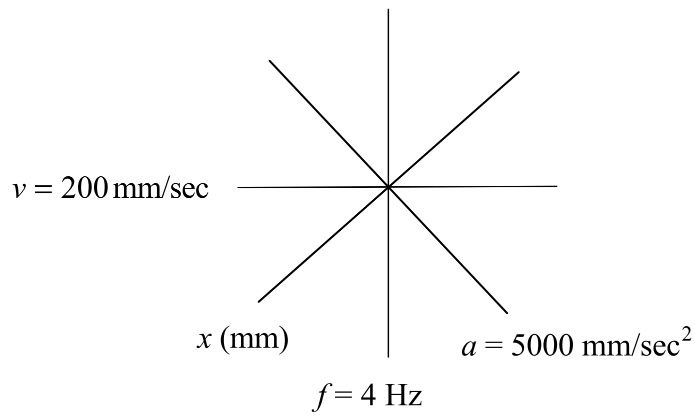
$$\omega_n = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}$$

Increasing this value by 10% requires the new frequency to be 5 x 1.1 = 5.5 rad/s. Solving for k given m and ω_n yields:

$$5.5 = \sqrt{\frac{k}{4}} \Rightarrow k = (5.5)^2(4) = 121 \text{ N/m}$$

Thus the stiffness k must be increased by about 20%.

1.23 Referring to Figure 1.8, if the maximum peak velocity of a vibrating system is 200 mm/s at 4 Hz and the maximum allowable peak acceleration is 5000 mm/s^2, what will the peak displacement be?



Solution:

Given: $v_{max} = 200$ mm/s @ $4\ Hz$

$a_{max} = 5000$ mm/s @ $4\ Hz$

$x_{max} = A$

$v_{max} = A\omega_n$

$a_{max} = A\omega_n{}^2$

$$\therefore x_{\max} = \frac{v_{\max}}{\omega_n} = \frac{v_{\max}}{2\pi f} = \frac{200}{8\pi} = 7.95\ mm$$

At the center point, the peak displacement will be **x = 7.95 mm**

1.24 Show that lines of constant displacement and acceleration in Figure 1.8 have slopes of +1 and –1, respectively. If rms values instead of peak values are used, how does this affect the slope?

Solution: Let

$$x = x_{\max}\sin\omega_n t$$
$$\dot{x} = x_{\max}\omega_n\cos\omega_n t$$
$$\ddot{x} = -x_{\max}\omega_n^{\ 2}\sin\omega_n t$$

Peak values:

$$\dot{x}_{\max} = x_{\max}\omega_n = 2\pi f x_{\max}$$
$$\ddot{x}_{\max} = x_{\max}\omega_n^2 = (2\pi f)^2 x_{\max}$$

Location:

$$\ln\dot{x}_{\max} = \ln x_{\max} + \ln 2\pi f$$
$$\ln\dot{x}_{\max} = \ln\ddot{x}_{\max} - \ln 2\pi f$$

Since x_{max} is constant, the plot of ln $\dot{x}_{\max}$ versus ln 2πf is a straight line of slope +1. If ln $\ddot{x}_{\max}$ is constant, the plot of ln $\dot{x}_{\max}$ versus ln 2πf is a straight line of slope –1. Calculate RMS values

Let

$$x(t) = A\sin\omega_n t$$
$$\dot{x}(t) = A\omega_n\cos\omega_n t$$
$$\ddot{x}(t) = -A\omega_n^{\ 2}\sin\omega_n t$$

Mean Square Value: $\overline{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\, dt$

$$\overline{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2 \sin^2 \omega_n t\, dt = \lim_{T\to\infty} \frac{A^2}{T}\int_0^T (1 - \cos 2\omega_n t)\, dt = \frac{A^2}{2}$$

$$\overline{\dot{x}}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\, \omega_n^2 \cos^2 \omega_n t\, dt = \lim_{T\to\infty} \frac{A^2\omega_n{}^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega_n t)\, dt = \frac{A^2\omega_n{}^2}{2}$$

$$\overline{\ddot{x}}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\, \omega_n^4 \sin^2 \omega_n t\, dt = \lim_{T\to\infty} \frac{A^2\omega_n{}^4}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega_n t)\, dt = \frac{A^2\omega_n{}^4}{2}$$

Therefore,

$$x_{rms} = \sqrt{\overline{x}^2} = \frac{\sqrt{2}}{2} A$$

$$\dot{x}_{rms} = \sqrt{\overline{\dot{x}}^2} = \frac{\sqrt{2}}{2} A\omega_n$$

$$\ddot{x}_{rms} = \sqrt{\overline{\ddot{x}}^2} = \frac{\sqrt{2}}{2} A\omega_n^2$$

The last two equations can be rewritten as:

$$\dot{x}_{rms} = x_{rms}\, \omega = 2\pi f\, x_{rms}$$

$$\ddot{x}_{rms} = x_{rms}\, \omega^2 = 2\pi f\, \dot{x}_{rms}$$

The logarithms are:

$$\ln \dot{x}_{\max} = \ln x_{\max} + \ln 2\pi f$$

$$\ln \ddot{x}_{\max} = \ln \dot{x}_{\max} + \ln 2\pi f$$

The plots of $\ln \dot{x}_{rms}$ versus $\ln 2\pi f$ is a straight line of slope +1 when x_{rms} is constant, and −1 when $\ddot{x}_{rms}$ is constant. Therefore **the slopes are unchanged.**

1.25 A foot pedal mechanism for a machine is crudely modeled as a pendulum connected to a spring as illustrated in Figure P1.25. The purpose of the spring is to keep the pedal roughly vertical. Compute the spring stiffness needed to keep the pendulum at 1° from the horizontal and then compute the corresponding natural frequency. Assume that the angular deflections are small, such that the spring deflection can be approximated by the arc length, that the pedal may be treated as a point mass and that pendulum rod has negligible mass. The values in the figure are $m = 0.5$ kg, $g = 9.8$ m/s^2, $\ell_1 = 0.2$ m and $\ell_2 = 0.3$ m.

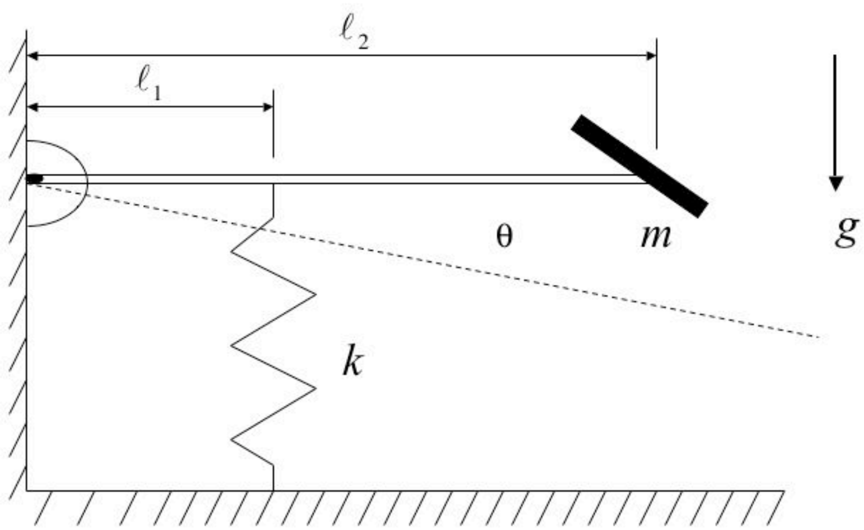
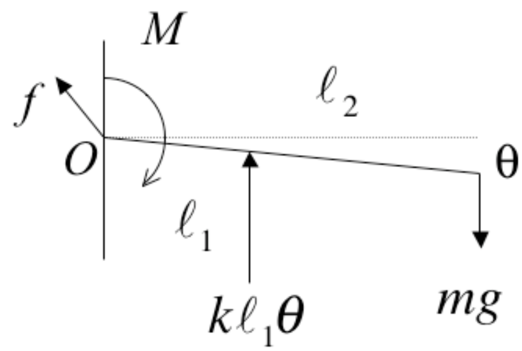


Figure P1.25

Solution: You may want to note to your students, that many systems with springs are often designed based on static deflections, to hold parts in specific positions as in this case, and yet allow some motion. The free-body diagram for the system is given in the figure.



For static equilibrium the sum of moments about point O yields (θ_1 is the static deflection):

$$\sum M_0 = -\ell_1\theta_1(\ell_1)k + mg\ell_2 = 0$$

$$\Rightarrow \ell_1^2\theta_1 k = mg\ell_2 \qquad (1)$$

$$\Rightarrow k = \frac{mg\ell_2}{\ell_1^2\theta_1} = \frac{0.5\cdot 0.3}{(0.2)^2\dfrac{\pi}{180}} = \underline{2106\text{ N/m}}$$

Again take moments about point O to get the dynamic equation of motion:

$$\sum M_O = J\ddot{\theta} = m\ell_2^2\ddot{\theta} = -\ell_1^2 k(\theta+\theta_1) + mg\ell_2 = -\ell_1^2 k\theta + \ell_1^2 k\theta_1 - mg\ell_2\theta$$

Next using equation (1) above for the static deflection yields:

$$m\ell_2^2\ddot{\theta} + \ell_1^2 k\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{\ell_1^2 k}{m\ell_2^2}\right)\theta = 0$$

$$\Rightarrow \omega_n = \frac{\ell_1}{\ell_2}\sqrt{\frac{k}{m}} = \frac{0.2}{0.3}\sqrt{\frac{2106}{0.5}} = \underline{43.27\ \text{ rad/s}}$$

1.26 An automobile is modeled as a 1000-kg mass supported by a spring of stiffness k = 400,000 N/m. When it oscillates it does so with a maximum deflection of 10 cm. When loaded with passengers, the mass increases to as much as 1300 kg. Calculate the change in frequency, velocity amplitude, and acceleration amplitude if the maximum deflection remains 10 cm.

Solution:

Given: $m_1 = 1000\ kg$

$m_2 = 1300\ kg$

$k = 400{,}000\ N/m$

$$x_{max} = A = 10 \text{ cm}$$

$$\omega_{n1} = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{400{,}000}{1000}} = 20\ rad/s$$

$$\omega_{n2} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{400{,}000}{1300}} = 17.54\ rad/s$$

$$\Delta\omega = 17.54 - 20 = -2.46\ rad/s$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \left|\frac{-2.46}{2\pi}\right| = 0.392\ Hz$$

$v_1 = A\omega_{n1} = 10\ cm \text{ x } 20\ rad/s = 200\ cm/s$

$v_2 = A\omega_{n2} = 10\ cm \text{ x } 17.54 \text{ rad/s} = 175.4\ cm/s$

$\boldsymbol{\Delta v}$ = 175.4 - 200 = -**24.6 cm/s**

$a_1 = A\omega_{n1}^2 = 10\ cm \text{ x } (20\ rad/s)^2 = 4000\ cm/s^2$

$a_2 = A\omega_{n2}^2 = 10\ cm \text{ x } (17.54 \text{ rad/s})^2 = 3077\ cm/s^2$

$\boldsymbol{\Delta a}$ = 3077 - 4000 = -**923 cm/s²**

1.27 The front suspension of some cars contains a torsion rod as illustrated in Figure P1.27 to improve the car's handling. (a) Compute the frequency of vibration of the wheel assembly given that the torsional stiffness is 2000 N m/rad and the wheel assembly has a mass of 38 kg. Take the distance $x = 0.26$ m. (b) Sometimes owners put different wheels and tires on a car to enhance the appearance or performance. Suppose a thinner tire is put on with a larger wheel raising the mass to 45 kg. What effect does this have on the frequency?

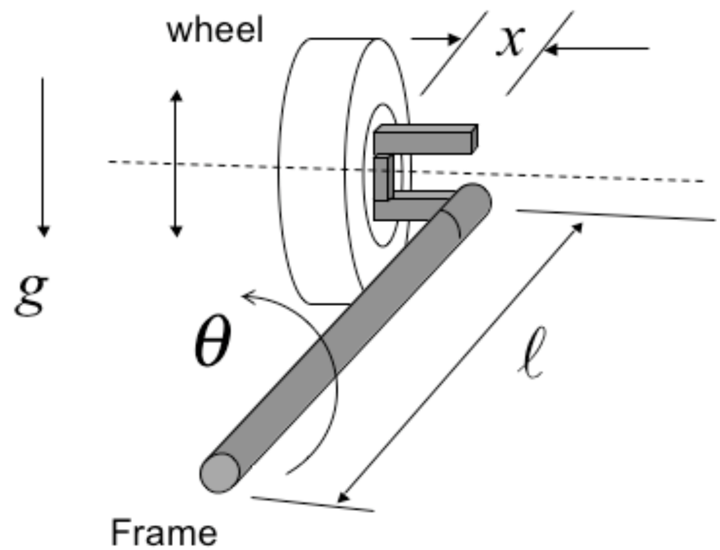


Figure P1.27

Solution: (a) Ignoring the moment of inertial of the rod, and computing the moment of inertia of the wheel as mx^2, the frequency of the shaft mass system is

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000\ \text{N}\cdot\text{m}}{38\cdot\text{kg}\ (0.26\ \text{m})^2}} = \underline{27.9\ \text{rad/s}}$$

(b) The same calculation with 45 kg will *reduce* the frequency to

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000\ \text{N}\cdot\text{m}}{45\cdot\text{kg}\ (0.26\ \text{m})^2}} = \underline{25.6\ \text{rad/s}}$$

This corresponds to about an 8% change in unsprung frequency and could influence wheel hop etc. You could also ask students to examine the effect of increasing *x*, as commonly done on some trucks to extend the wheels out for appearance sake.

1.28 A machine oscillates in simple harmonic motion and appears to be well modeled by an undamped single-degree-of-freedom oscillation. Its acceleration is measured to have an amplitude of 10,000 mm/s^2 at 8 Hz. What is the machine's maximum displacement?

Solution:

Given: a_{max} = 10,000 mm/s^2 @ 8 Hz

The equations of motion for position and acceleration are:

$$x = A\sin(\omega_n t + \phi) \qquad (1.3)$$

$$\ddot{x} = -A\omega_n^2 \sin(\omega_n t + \phi) \qquad (1.5)$$

The amplitude of acceleration is $A\omega_n^2 = 10{,}000$ mm/s^2 and ω_n = 2πf = 2π(8) = 16π rad/s, from equation (1.12).

The machine's displacement is $A = \dfrac{10{,}000}{\omega_n^2} = \dfrac{10{,}000}{(16\pi)^2}$

A = 3.96 mm

1.29 A simple undamped spring-mass system is set into motion from rest by giving it an initial velocity of 100 mm/s. It oscillates with a maximum amplitude of 10 mm. What is its natural frequency?

Solution:

Given: $x_0 = 0$, v_0 = 100 mm/s, A = 10 mm

From equation (1.9), $A = \dfrac{v_0}{\omega_n}$ or $\omega_n = \dfrac{v_0}{A} = \dfrac{100}{10}$, so that: **$\omega_n$= 10 rad/s**

1.30 An automobile exhibits a vertical oscillating displacement of maximum amplitude 5 cm and a measured maximum acceleration of 2000 cm/s^2. Assuming that the automobile can be modeled as a single-degree-of-freedom system in the vertical direction, calculate the natural frequency of the automobile.

Solution:

Given: A = 5 cm. From equation (1.15)

$$|\ddot{x}| = A\omega_n^2 = 2000 \text{ cm/s}$$

Solving for ω_n yields:

$$\omega_n = \sqrt{\frac{2000}{A}} = \sqrt{\frac{2000}{5}}$$

$$\omega_n = \mathbf{20\,rad/s}$$

Problems Section 1.3 (Numbers 1.31 through 1.46)

1.31 Solve $\ddot{x}+4\dot{x}+x=0$ for x_0 = 1 mm, v_0 = 0 mm/s. Sketch your results and determine which root dominates.

Solution:

Given $\ddot{x}+4\dot{x}+x=0$ where $x_0 = 1$ mm, $v_0 = 0$

Let $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute these into the equation of motion to get:

$$ar^2e^{rt} + 4are^{rt} + ae^{rt} = 0$$

$$\Rightarrow r^2 + 4r + 1 = 0 \Rightarrow r_{1,2} = -2 \pm \sqrt{3}$$

So

$$x = a_1 e^{(-2+\sqrt{3})t} + a_2 e^{(-2-\sqrt{3})t}$$

$$\dot{x} = \left(-2+\sqrt{3}\right)a_1 e^{(-2+\sqrt{3})t} + \left(-2-\sqrt{3}\right)a_2 e^{(-2-\sqrt{3})t}$$

Applying initial conditions yields,

$$x_0 = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$

$$v_0 = \left(-2+\sqrt{3}\right)a_1 + \left(-2-\sqrt{3}\right)a_2 \qquad (2)$$

Substitute equation (1) into (2)

$$v_0 = \left(-2+\sqrt{3}\right)(x_0 - a_2) + \left(-2-\sqrt{3}\right)a_2$$

$$v_0 = \left(-2+\sqrt{3}\right)x_0 - 2\sqrt{3}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

Substituting the value of a_2 into equation (1), and solving for a_1 yields,

$$a_1 = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

$$\therefore\ x(t) = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}} e^{(-2+\sqrt{3})t} + \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}} e^{(-2-\sqrt{3})t}$$

The response is dominated by the root: $\underline{-2+\sqrt{3}}$ as the other root dies off very fast.

1.32 Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

Solution:

Given $\ddot{x} + 2\dot{x} + x = 0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute into the equation of motion to get

$$ar^2e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1e^{(-1+i)\,t} + \; c_2e^{(-1-i)\,t} \Rightarrow \dot{x} = (-1+i)c_1e^{(-1+i)\,t} + (-1-i)c_2e^{(-1-i)\,t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 \qquad (1)$$
$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \qquad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$
$$c_1 = -\frac{1}{2}i, \;\; c_2 = \frac{1}{2}i$$
$$x(t) = -\frac{1}{2}ie^{(-1+i)\,t} + \frac{1}{2}ie^{(-1-i)\,t} = -\frac{1}{2}ie^{-t}\left(e^{it} - e^{-it}\right)$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}\left(\cos t + i\sin t - (\cos t - i\sin t)\right)$$
$$\underline{x(t) = e^{-t}\sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.33 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the definition of the damping ratio.

Solution:

Equation (1.28): $\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$

Rewrite, $\lambda_{1,2} = -\left(\frac{c}{2\sqrt{m}\sqrt{m}}\right)\left(\frac{\sqrt{k}}{\sqrt{k}}\right) \pm \frac{1}{2\sqrt{m}\sqrt{m}}\left(\frac{\sqrt{k}}{\sqrt{k}}\right)\left(\frac{c}{c}\right)\sqrt{c^2 - \left(2\sqrt{km}^2\right)\left(\frac{c}{c}\right)^2}$

Rearrange, $\lambda_{1,2} = -\left(\frac{c}{2\sqrt{km}}\right)\left(\frac{\sqrt{k}}{\sqrt{m}}\right) \pm \frac{c}{2\sqrt{km}}\left(\frac{\sqrt{k}}{\sqrt{m}}\right)\left(\frac{1}{c}\right)\sqrt{c^2\left[1 - \left(\frac{2\sqrt{km}}{c}\right)^2\right]}$

Substitute:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{c}{2\sqrt{km}} \Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \zeta\omega_n\left(\frac{1}{c}\right)c\sqrt{1 - \left(\frac{1}{\zeta^2}\right)}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2\left[1 - \left(\frac{1}{\zeta^2}\right)\right]}$$

$$\Rightarrow \lambda_{1,2} = \underline{-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

1.34 Use the Euler formulas to derive equation (1.36) from equation (1.35) and to determine the relationships listed in Window 1.4.

Solution:

Equation (1.35): $x(t) = e^{-\zeta\omega_n t}\left(a_1 e\right)^{j\omega_n\sqrt{1-\zeta^2}t} - a_2 e^{-j\omega_n\sqrt{1-\zeta^2}t}$

From Euler,

$$x(t) = e^{-\zeta\omega_n t}(a_1 \cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + a_1 j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$$

$$+ a_2 \cos\left(\omega_n\sqrt{1-\zeta^2}t\right) - a_2 j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right))$$

$$= e^{-\zeta\omega_n t}(a_1 + a_2)\cos\omega_d t + j(a_1 - a_2)\sin\omega_d t$$

Let: $A_1 = (a_1 + a_2)$, $A_2 = (a_1 - a_2)$, then this last expression becomes

$$x(t) = e^{-\zeta\omega_n t} A_1 \cos\omega_d t + A_2 \sin\omega_d t$$

Next use the trig identity:

$$A = \sqrt{A_1 + A_2}, \quad \phi = \tan^{-1}\frac{A_1}{A_2}$$

to get: $\underline{x(t) = e^{-\zeta\omega_n t} A\sin(\omega_d t + \phi)}$

1.35 Using equation (1.35) as the form of the solution of the underdamped system, calculate the values for the constants a_1 and a_2 in terms of the initial conditions x_0 and v_0.

Solution:

Equation (1.35):

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_n\sqrt{1-\zeta^2}t} + a_2 e^{-j\omega_n\sqrt{1-\zeta^2}t}\right)$$

$$\dot{x}(t) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})a_1 e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)t} + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})a_2 e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t}$$

Initial conditions

$$x_0 = x(0) = a_1 + a_2 \Rightarrow a_1 = x_0 - a_2 \qquad (1)$$

$$v_0 = \dot{x}(0) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})a_1 + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)(x_0 - a_2) + \left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)x_0 - 2j\omega_n\sqrt{1-\zeta^2}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

1.36 Calculate the constants A and ϕ in terms of the initial conditions and thus verify equation (1.38) for the underdamped case.

Solution:

From Equation (1.36),

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Applying initial conditions ($t = 0$) yields,

$$x_0 = A\sin\phi \qquad (1)$$

$$v_0 = \dot{x}_0 = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi \qquad (2)$$

Next solve these two simultaneous equations for the two unknowns A and ϕ.

From (1),

$$A = \frac{x_0}{\sin\phi} \qquad (3)$$

Substituting (3) into (1) yields

$$v_0 = -\zeta\omega_n x_0 + \frac{\omega_d x_0}{\tan\phi} \quad \Rightarrow \quad \tan\phi = \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}.$$

Hence,

$$\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] \qquad (4)$$

From (3), $\sin\phi = \dfrac{x_0}{A}$ (5)

and From (4), $\cos\phi = \dfrac{v_0 + \zeta\omega_n x_0}{(x_0\omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2}$ (6)

Substituting (5) and (6) into (2) yields,

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

which are the same as equation (1.38)

1.37 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equations (1.42) and (1.43) for the overdamped case.

Solution: From Equation (1.41)

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{\omega_n \sqrt{\zeta^2-1}\,t} + a_2 e^{-\omega_n \sqrt{\zeta^2-1}\,t}\right)$$

taking the time derivative yields:

$$\dot{x}(t) = (-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})a_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + (-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})a_2\, e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

Applying initial conditions yields,

$$x_0 = x(0) = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$

$$v_0 = \dot{x}(0) = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)a_1 + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)(x_0 - a_2) + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)x_0 - 2\omega_n\sqrt{\zeta^2-1}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\;x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\;x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

1.38 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equation (1.46) for the critically damped case.

Solution:

From Equation (1.45),

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$

$$\Rightarrow \dot{x}_0 = -\omega_n a_1 e^{-\omega_n t} - \omega_n a_2 t e^{-\omega_n t} + a_2 e^{-\omega_n t}$$

Applying the initial conditions yields:

$$x_0 = a_1 \qquad (1)$$

and

$$v_0 = \dot{x}(0) = a_2 - \omega_n a_1 \qquad (2)$$

solving these two simultaneous equations for the two unknowns a_1 and a_2. Substituting (1) into (2) yields,

$$a_1 = x_0$$

$$a_2 = v_0 + \omega_n x_0$$

which are the same as equation (1.46).

1.39 Using the definition of the damping ratio and the undamped natural frequency, derive equitation (1.48) from (1.47).

Solution:

$\omega_n = \sqrt{\frac{k}{m}}$ thus, $\frac{k}{m} = \omega_n^{\ 2}$

$\zeta = \frac{c}{2\sqrt{km}}$ thus, $\frac{c}{m} = \frac{2\zeta\sqrt{km}}{m} = 2\zeta\omega_n$

Therefore, $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

becomes,

$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^{\ 2}x(t) = 0$

1.40 For a damped system, m, c, and k are known to be m = 1 kg, c = 2 kg/s, k = 10 N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

Solution:

Given: m = 1 kg, c = 2 kg/s, k = 10 N/m

Natural frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16\ rad/s$

Damping ratio: $\zeta = \frac{c}{2\omega_n m} = \frac{2}{2(3.16)(1)} = 0.316$

Damped natural frequency: $\omega_d = \sqrt{10}\sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 3.0$ rad/s

Since $0 < \zeta < 1$, the system is **underdamped**.

1.41 Plot $x(t)$ for a damped system of natural frequency ω_n = 2 rad/s and initial conditions x_0 = 1 mm, v_0 = 1 mm, for the following values of the damping ratio: $\zeta = 0.01$, $\zeta = 0.2$, $\zeta = 0.1$, $\zeta = 0.4$, and $\zeta = 0.8$.

Solution:

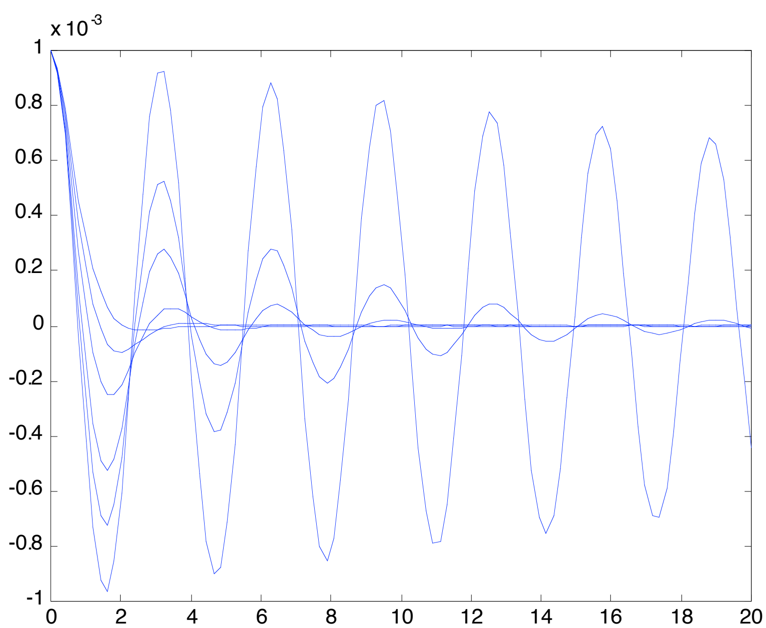
Given: ω_n = 2 rad/s, x_0 = 1 mm, v_0 = 1 mm, ζ_i = [0.01; 0.2; 0.1; 0.4; 0.8]
Underdamped cases:

$$\therefore \omega_{di} = \omega_n \sqrt{1 - \zeta_i^2}$$

From equation 1.38,

$$A_i = \sqrt{\frac{(v_0 + \zeta_i \omega_n x_0)^2 + (x_0 \omega_{di})^2}{\omega_{di}^2}} \qquad \phi_i = \tan^{-1} \frac{x_0 \omega_{di}}{v_0 + \zeta_i \omega_n x_0}$$

The response is plotted for each value of the damping ratio in the following using Matlab:



1.42 Plot the response $x(t)$ of an underdamped system with $\omega_n = 2$ rad/s, $\zeta = 0.1$, and $v_0 = 0$ for the following initial displacements: $x_0 = 10$ mm and $x_0 = 100$ mm.

Solution:

Given: $\omega_n = 2$ rad/s, $\zeta = 0.1$, $v_0 = 0$, $x_0 = 10$ mm and $x_0 = 100$ mm.

Underdamped case:

$$\therefore \omega_d = \omega_n\sqrt{1-\zeta_i^2} = 2\sqrt{1-0.1^2} = 1.99 \text{ rad/s}$$

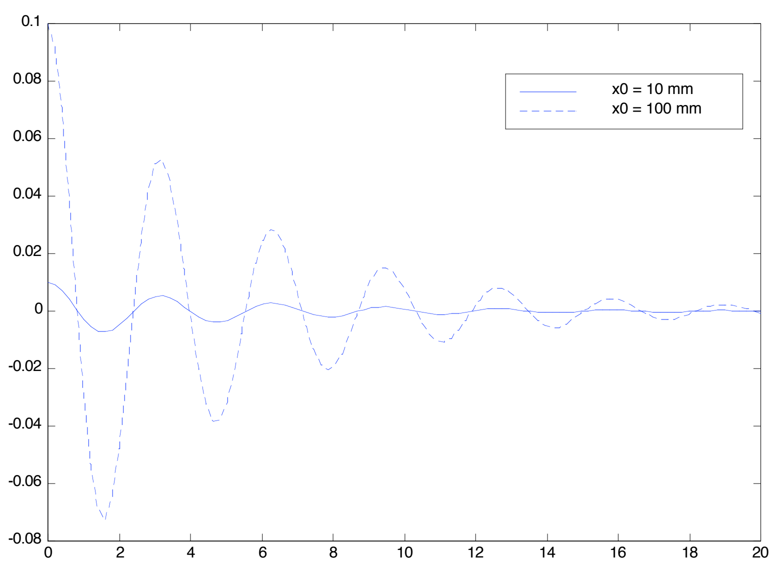
$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}} = 1.01\,x_0$$

$$\phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = 1.47 \text{ rad}$$

where

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

The following is a plot from Matlab.



1.43 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \text{ where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t) =$$

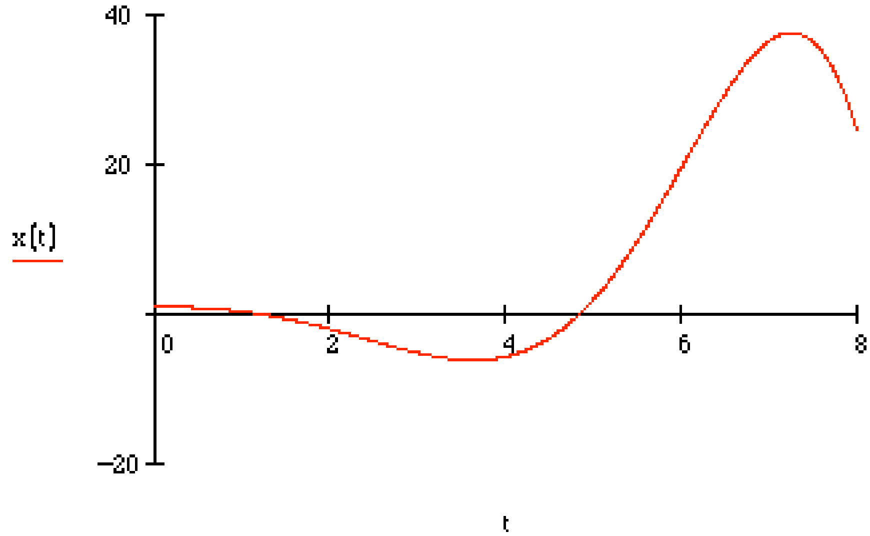
$$\frac{1}{2}e^0(A_1 \cos\frac{\sqrt{3}}{2}0 + A_2 \sin\frac{\sqrt{3}}{2}0) + e^0\frac{\sqrt{3}}{2}(-A_1 \sin\frac{\sqrt{3}}{2}0 + A_2 \cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

$$x(t) := e^{0.5\cdot t}\left[\cos\left(\frac{\sqrt{3}}{2}\cdot t\right) - \frac{1}{\sqrt{3}}\cdot\sin\left(\frac{\sqrt{3}}{2}\cdot t\right)\right]$$



1.44 A spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000 \text{ N/m}}{100 \text{ kg}}} = 5.477 \text{ rad/s}$$

$$\zeta = \frac{\text{c}}{\text{c}_{\text{cr}}} = \frac{300}{2\sqrt{km}} = \frac{300}{2\sqrt{(3000)(100)}} = 0.274$$

Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 5.27$ rad/s.

1.45 A sketch of a valve and rocker arm system for an internal combustion engine is give in Figure P1.45. Model the system as a pendulum attached to a spring and a mass and assume the oil provides viscous damping in the range of $\zeta = 0.01$. Determine the equations of motion and calculate an expression for the natural frequency and the damped natural frequency. Here J is the rotational inertia of the rocker arm about its pivot point, k is the stiffness of the valve spring and m is the mass of the valve and stem. Ignore the mass of the spring.

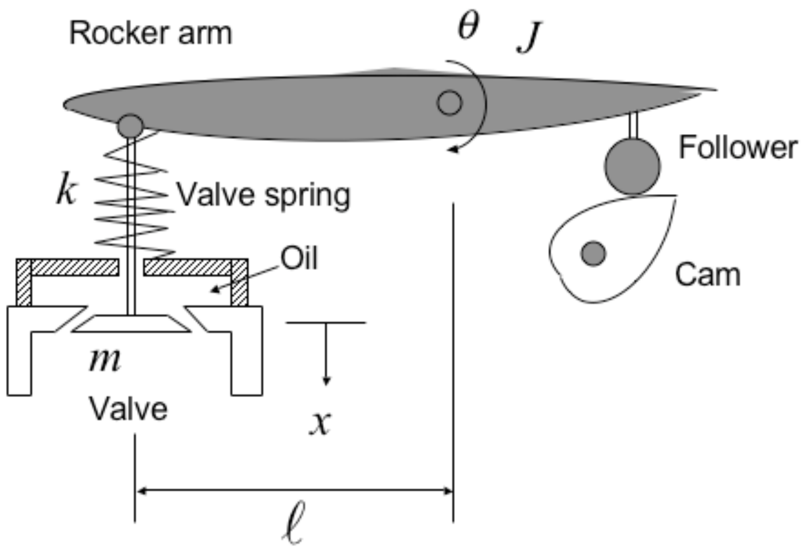
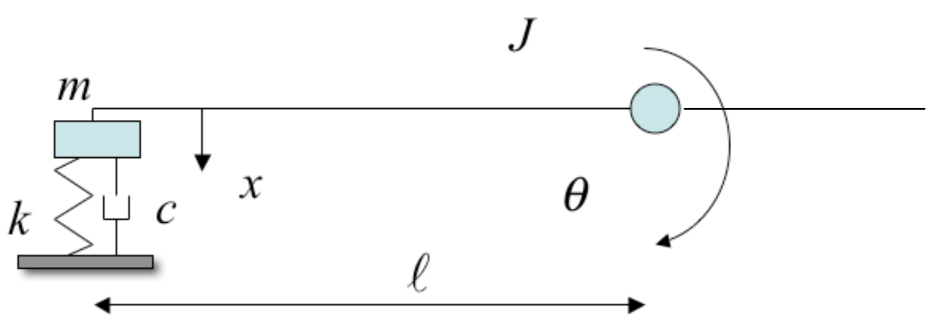


Figure P1.45

Solution: The model is of the form given in the figure. You may wish to give this figure as a hint as it may not be obvious to all students.



Taking moments about the pivot point yields:

$$(J + m\ell^2)\ddot{\theta}(t) = -kx\ell - c\dot{x}\ell = -k\ell^2\theta - c\ell^2\dot{\theta}$$

$$\Rightarrow \underline{(J + m\ell^2)\ddot{\theta}(t) + c\ell^2\dot{\theta} + k\ell^2\theta = 0}$$

Next divide by the leading coefficient to get;

$$\ddot{\theta}(t) + \left(\frac{c\ell^2}{J + m\ell^2}\right)\dot{\theta}(t) + \frac{k\ell^2}{J + m\ell^2}\theta(t) = 0$$

From the coefficient of q, the undamped natural frequency is

$$\underline{\omega_n = \sqrt{\frac{k\ell^2}{J + m\ell^2}} \text{ rad/s}}$$

From equation (1.37), the damped natural frequency becomes

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \underline{0.99995\sqrt{\frac{k\ell^2}{J + m\ell^2}}} \sim \sqrt{\frac{k\ell^2}{J + m\ell^2}}$$

This is effectively the same as the undamped frequency for any reasonable accuracy. However, it is important to point out that the resulting response will still decay, even though the frequency of oscillation is unchanged. So even though the numerical value seems to have a negligible effect on the frequency of oscillation, the small value of damping still makes a substantial difference in the response.

1.46 A spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{150 \text{ kg}}} = 3.162 \text{ rad/s}$$

$$\zeta = \frac{\text{c}}{\text{c}_{\text{cr}}} = \frac{200}{2\sqrt{km}} = \frac{200}{2\sqrt{(1500)(150)}} = 0.211$$

This last expression follows from the equation following equation (1.29). Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.091$ rad/s, which follows from equation (1.37).

1.47* The system of Problem 1.44 is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the form of the response and plot it for as long as it takes to die out.

Solution: Working from equation (1.38) and using Mathcad the solution is:

1.42

$m := 100 \quad k := 3000 \quad c := 300$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2}$$

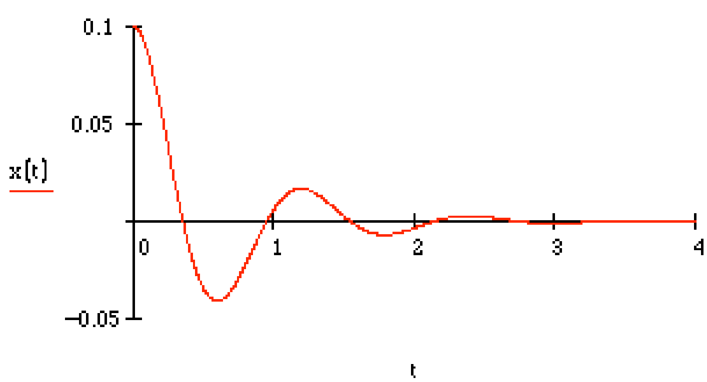
$v0 := 0 \quad x0 := 0.1$

$$A := \frac{1}{\omega d} \cdot \sqrt{(v0 + \zeta \cdot x0 \cdot \omega n)^2 + (x0 \cdot \omega d)^2} \qquad \phi := \operatorname{atan}\left(\frac{\omega d \cdot x0}{v0 + \zeta \cdot \omega n \cdot x0}\right)$$

$A = 0.104 \qquad \phi = 1.293 \qquad \omega n = 5.477 \qquad \omega d = 5.268$

$\zeta = 0.274$

$$x(t) := A \cdot \sin(\omega d \cdot t + \phi) \cdot e^{-\zeta \cdot \omega n \cdot t}$$



1.48* The system of Problem 1.46 is given an initial velocity of 10 mm/s and an initial displacement of -5 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Working from equation (1.38), the form of the response is programmed in Mathcad and is given by:

1.43

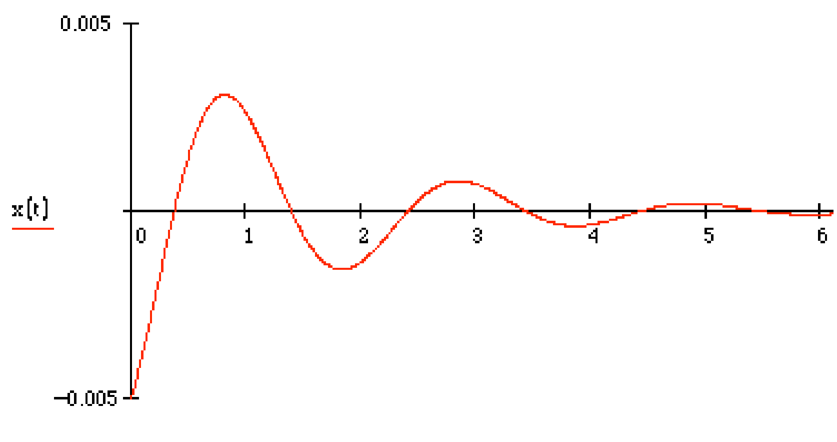
$k := 1500 \quad m := 150 \quad x0 := -0.005 \quad v0 := 0.010 \quad c := 200$

all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \zeta = 0.211 \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2} \qquad \omega d = 3.091$$

$$A := \frac{1}{\omega d}\cdot\sqrt{(v0+\zeta\cdot x0\cdot\omega n)^2+(x0\cdot\omega d)^2} \qquad A = 5.445\cdot 10^{-3} \qquad T := \frac{2\cdot\pi}{\omega d}$$

$$x(t) := A\cdot\sin(\omega d\cdot t+\phi)\cdot e^{-\zeta\cdot\omega n\cdot t} \quad \text{in meters} \qquad \phi := \operatorname{atan}\left(\frac{\omega d\cdot x0}{v0+\zeta\cdot\omega n\cdot x0}\right)$$



It appears to take a little over 6 to 8 seconds to die out. This can also be plotted in Matlab, Mathematica or by using the toolbox.

1.49* Choose the damping coefficient of a spring-mass-damper system with mass of 150 kg and stiffness of 2000 N/m such that it's response will die out after about 2 s, given a zero initial position and an initial velocity of 10 mm/s.

Solution: Working in Mathcad, the response is plotted and the value of c is changed until the desired decay rate is meet:

$c := 800 \qquad k := 2000 \qquad v0 := 0.010 \qquad x0 := 0$

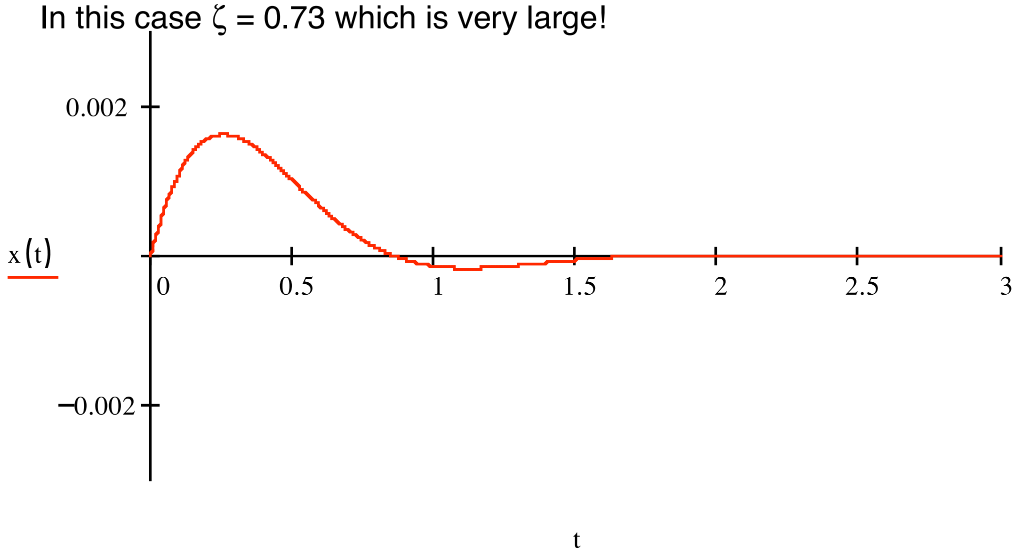
$m := 150$

$$\zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega n := \sqrt{\frac{k}{m}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$$

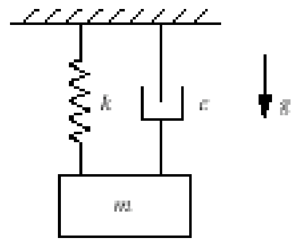
$$x(t) := A\cdot\sin(\omega n\cdot t + \phi)\cdot e^{-\zeta\cdot\omega n\cdot t}$$

$$\phi := \operatorname{atan}\left(\frac{\omega d\cdot x0}{v0 + \zeta\cdot\omega n\cdot x0}\right)$$

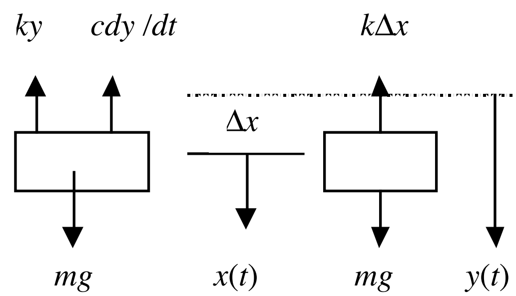
In this case $\zeta = 0.73$ which is very large!



1.50 Derive the equation of motion of the system in Figure P1.50 and discuss the effect of gravity on the natural frequency and the damping ratio.



Solution: This requires two free body diagrams. One for the dynamic case and one to show static equilibrium.



(a) (b)

From the free-body diagram of static equilibrium (b) we have that $mg = k\Delta x$, where Δx represents the static deflection. From the free-body diagram of the dynamic case given in (a) the equation of motion is:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) - mg = 0$$

From the diagram, $y(t) = x(t) + \Delta x$. Since Δx is a constant, differentiating and substitution into the equation of motion yields:

$$\dot{y}(t) = \dot{x}(t) \text{ and } \ddot{y}(t) = \ddot{x}(t) \Rightarrow$$

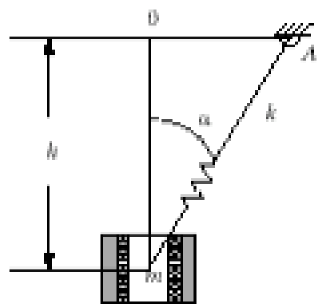
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \underbrace{(k\Delta x - mg)}_{=0} = 0$$

where the last term is zero from the relation resulting from static equilibrium. Dividing by the mass yields the standard form

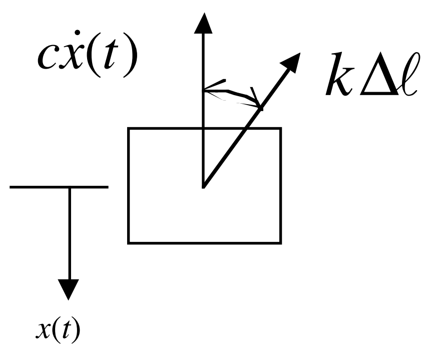
$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

It is clear that gravity has no effect on the damping ratio ζ or the natural frequency ω_n. Not that the damping force is not present in the static case because the velocity is zero.

1.51 Derive the equation of motion of the system in Figure P1.46 and discuss the effect of gravity on the natural frequency and the damping ratio. You may have to make some approximations of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. (From the A. Diaz-Jimenez, *South African Mechanical Engineer*, Vol. 26, pp. 65-69, 1976)



Solution: First consider a free-body diagram of the system:



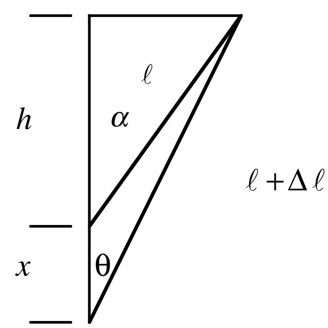
Let α be the angel between the damping and stiffness force. The equation of motion becomes

$$m\ddot{x}(t) = -c\dot{x}(t) - k(\Delta\ell + \delta_s)\cos\alpha$$

From static equilibrium, the free-body diagram (above with $c = 0$ and stiffness force $k\delta_s$) yields: $\sum F_x = 0 = mg - k\delta_s \cos\alpha$. Thus the equation of motion becomes

$$m\ddot{x} + c\dot{x} + k\Delta\ell\cos\alpha = 0 \tag{1}$$

Next consider the geometry of the dynamic state:



From the definition of cosine applied to the two different triangles:

$$\cos\alpha = \frac{h}{\ell} \quad \text{and} \quad \cos\theta = \frac{h+x}{\ell + \Delta\ell}$$

Next assume small deflections so that the angles are nearly the same cos α = cos θ, so that

$$\frac{h}{\ell} \approx \frac{h+x}{\ell + \Delta\ell} \Rightarrow \Delta\ell \approx x\frac{\ell}{h} \Rightarrow \Delta\ell \approx \frac{x}{\cos\alpha}$$

For small motion, then this last expression can be substituted into the equation of motion (1) above to yield:

$$m\ddot{x} + c\dot{x} + kx = 0\text{, } \alpha \text{ and } x \text{ small}$$

Thus the frequency and damping ratio have the standard values and are not effected by gravity. If the small angle assumption is not made, the frequency can be approximated as

$$\omega_n = \sqrt{\frac{k}{m}\cos^2\alpha + \frac{g}{h}\sin^2\alpha}, \quad \zeta = \frac{c}{2m\omega_n}$$

as detailed in the reference above. For a small angle these reduce to the normal values of

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

as derived here.