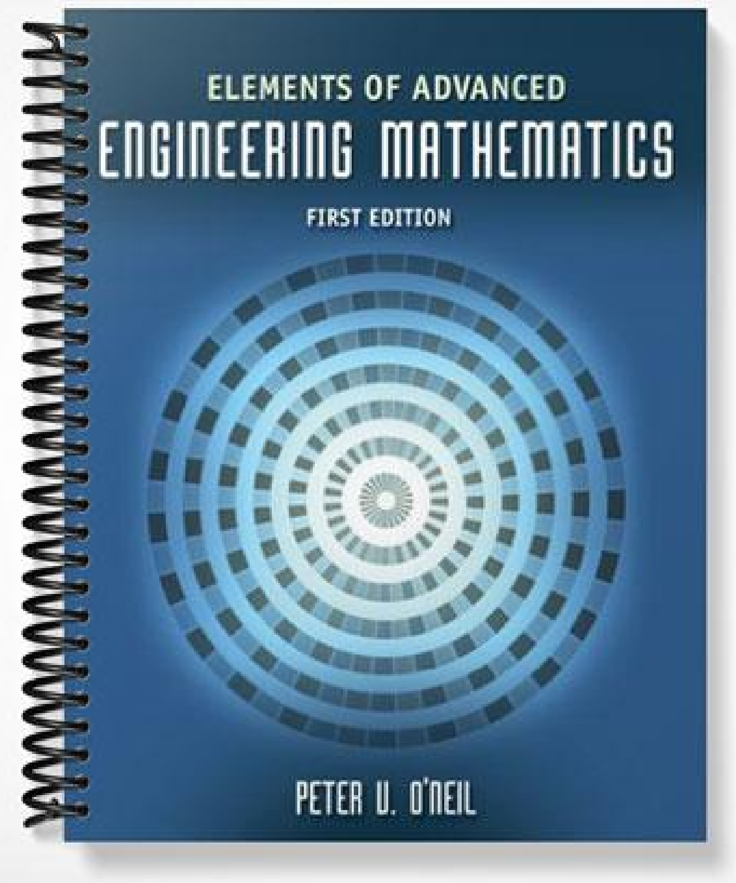
SOLUTIONS MANUAL



Chapter 2

Linear Second-Order Equations

2.1 Theory of the Linear Second-Order Equation

1. The general solution is $y(x) = c_1 \sin(6x) + c_2 \cos(6x)$. For the initial conditions, we need $y(0) = c_2 = -5$ and $y'(0) = 6c_1 = 2$. Then $c_1 = 1/3$ and the solution of the initial value problem is

$$y(x) = \frac{1}{3}\sin(6x) - 5\cos(6x).$$

2. The general solution is $y(x) = c_1 e^{4x} + c_2 e^{-4x}$. For the initial conditions, compute

$$y(0) = c_1 + c_2 = 12 \text{ and } y'(0) = 4c_1 - 4c_2 = 3.$$

Solve these algebraic equations to obtain $c_1 = 51/8$ and $c_2 = 45/8$. The solution of the initial value problem is

$$y(x) = \frac{51}{8}e^{4x} + \frac{45}{8}e^{-4x}.$$

3. The general solution is $y(x) = c_1 e^{-2x} + c_2 e^{-x}$. For the initial conditions, we have

$$y(0) = c_1 + c_2 = -3 \text{ and } y'(0) = -2c_1 - c_2 = -1.$$

Solve these to obtain $c_1 = 4$, $c_2 = -7$. The solution of the initial value problem is

$$y(x) = 4e^{-2x} - 7e^{-x}.$$

4. The general solution is $y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$. We will need

$$y'(x) = 3c_1 e^{3x}\cos(2x) - 2c_1 e^{3x}\sin(2x) + 3c_2 e^{3x}\sin(2x) + 2c_2 e^{3x}\cos(2x).$$

From the initial conditions,

$$y(0) = c_1 = -1 \text{ and } y'(0) = 3c_1 + 2c_2 = 1.$$

Then $c_2 = 2$ and the solution of the initial value problem is

$$y(x) = -e^{3x}\cos(2x) + 2e^{3x}\sin(2x).$$

5. The general solution is $y(x) = c_1e^x\cos(x) + c_2e^x\sin(x)$. Then $y(0) = c_1 = 6$. We find that $y'(0) = c_1 + c_2 = 1$, so $c_2 = -5$. The initial value problem has solution

$$y(x) = 6e^x\cos(x) - 5e^x\sin(x).$$

6.

$$y(x) = c_1\sin(6x) + c_2\cos(6x) + \frac{1}{36}(x-1)$$

7.

$$y(x) = c_1e^{4x} + c_2e^{-4x} - \frac{1}{4}x^2 + \frac{1}{2}$$

8.

$$y(x) = c_1e^{-2x} + c_2e^{-x} + \frac{15}{2}$$

9.

$$y(x) = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x) - 8e^x$$

10.

$$y(x) = c_1e^x\cos(x) + c_2e^x\sin(x) - \frac{5}{2}x^2 - 5x - 4$$

2.2 The Constant Coefficient Homogeneous Equation

1. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$, with roots $-2, 3$. The general solution is

$$y = c_1e^{-2x} + c_2e^{3x}.$$

2. The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$, with roots $1 \pm 3i$. The general solution is

$$y = c_1e^x\cos(3x) + c_2e^x\sin(3x).$$

3. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, with repeated root -3. The general solution is

$$y = c_1e^{-3x} + c_2xe^{-3x}.$$

4. The characteristic equation is $\lambda^2 - 3\lambda = 0$, with roots $0, 3$. The general solution is

$$y = c_1 + c_2e^{3x}.$$

5. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$. The general solution is

$$y = c_1 e^{-5x}\cos(x) + c_2 e^{-5x}\sin(x).$$

6. The characteristic equation is $\lambda^2 + 6\lambda - 40 = 0$, with roots $-10, 4$. The general solution is

$$y = c_1 e^{-10x} + c_2 e^{4x}.$$

7. The characteristic equation is $\lambda^2 + 3\lambda + 18 = 0$, with roots $-3/2 \pm 3\sqrt{7}i/2$. The general solution is

$$y = e^{-3x/2}\left[c_1 \cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2 \sin\left(\frac{3\sqrt{7}x}{2}\right)\right].$$

8. The characteristic equation is $\lambda^2 + 16\lambda + 64 = 0$, with repeated root -8. The general solution is

$$y = e^{-8x}(c_1 + c_2 x).$$

9. The characteristic equation is $\lambda^2 - 14\lambda + 49 = 0$, with repeated root 7. The general solution is

$$y = e^{7x}(c_1 + c_2 x).$$

10. The characteristic equation is $\lambda^2 - 6\lambda + 7 = 0$, with roots $3 \pm \sqrt{2}i$. The general solution is

$$y = e^{3x}[c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)].$$

In each of Problems 11 through 20, the solution is obtained by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions. We give only the final solution of the initial value problem.

11. $y = 5 - 2e^{-3x}$
12. $y = 4e^{x} + 2e^{-3x}$
13. $y = 0$ for all x
14. $y = e^{2x}(3 - x)$
15.

$$y = \frac{1}{7}[9e^{3(x-2)} + 5e^{-4(x-2)}]$$

16.

$$y = \frac{\sqrt{6}}{4}e^{x}\left[e^{\sqrt{6}x} - e^{-\sqrt{6}x}\right]$$

17. $y = e^{x-1}(29 - 17x)$
18.

$$y = -4(5 - \sqrt{23})e^{5(x-2)/7}\sin\left(\frac{\sqrt{23}}{2}(x-2)\right)$$

19.

$$y = e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right) + \frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$$

20.

$$y = ae^{(-1+\sqrt{5})x/2} + be^{(-1-\sqrt{5})x/2},$$

where

$$a = \frac{(9+7\sqrt{5})}{2\sqrt{5}}e^{-2+\sqrt{5}}$$

and

$$b = \frac{(7\sqrt{5}-9)}{2\sqrt{5}}e^{-2-\sqrt{5}}$$

2.3 Solutions of the Nonhomogeneous Equation

1. Two independent solutions of $y'' + y = 0$ are $y_1 = \cos(x)$ and $y_2 = \sin(x)$. Using variation of parameters, integrate to obtain

$$u(x) = \sin(x) - \ln|\sec(x) + \tan(x)| \text{ and} v(x) = -\cos(x).$$

This yields the general solution

$$y = c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|.$$

2. Two independent solutions of the associated homogeneous equation are $y_1(x) = e^{3x}$ and $y_2(x) = e^x$. These have Wronskian $W(x) = -2e^{4x}$. Then

$$u(x) = \int e^{-3x}\cos(x+3)\,dx = -\frac{3}{10}e^{-3x}\cos(x+3) + \frac{1}{10}e^{-3x}\sin(x+3)$$

and

$$v(x) = \int e^{-x}\cos(x+3)\,dx = \frac{1}{2}e^{-x}\cos(x+3) - \frac{1}{2}e^{-x}\sin(x+3).$$

The general solution is

$$\begin{aligned} y(x) &= c_1e^{3x} + c_2e^x \\ &\quad - \frac{3}{10}\cos(x+3) + \frac{1}{10}\sin(x+3) \\ &\quad + \frac{1}{2}\cos(x+3) - \frac{1}{2}\sin(x+3). \end{aligned}$$

This can be written

$$\begin{aligned} y(x) &= c_1e^{3x} + c_2e^x \\ &\quad + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3). \end{aligned}$$

3. With details omitted, we obtain by variation of parameters that

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + 4x \sin(3x) + \frac{4}{3} \cos(3x) \ln|\cos(3x)|.$$

4. Use the identity $2\sin^2(x) = 1 - \cos(2x)$ to obtain

$$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3} + \frac{7}{65} \cos(2x) + \frac{4}{65} \sin(2x).$$

5.

$$y(x) = c_1 e^{x} + c_2 e^{2x} - e^{2x} \cos(e^{-x})$$

6. Use the identity $8\sin^2(4x) = 4\cos(8x) - 4$ to obtain

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{2}{3} + \frac{58}{1241} \cos(8x) + \frac{40}{1241} \sin(8x).$$

7. Two independent solutions of the associated homogeneous equation are e^{2x} and e^{-x}. For a particular solution, try $y_p(x) = Ax^2 + Bx + C$. This yields the general solution

$$y = c_1 e^{2x} + c_2 e^{-x} - x^2 + x - 4.$$

8. $y = c_1 e^{3x} + c_2 e^{-2x} - 2e^{2x}$
9. $y = e^{x}[c_1 \cos(3x) + c_2 \sin(3x)] + 2x^2 + x - 1$
10. $y = e^{2x}[c_1 \cos(x) + c_2 \sin(x)] + 21e^{2x}$
11. $y = c_1 e^{2x} + c_2 e^{4x} + e^{x}$
12. $y = e^{-3x}[c_1 + c_2 x] + \frac{1}{2} \sin(3x)$
13. $y = c_1 e^{x} + c_2 e^{2x} + 3\cos(x) + \sin(x)$
14. $y = c_1 + c_2 e^{-4x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{2}{3}e^{3x}$
15. $y = e^{2x}[c_1 \cos(3x) + c_2 \sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}$
16. $y = e^{x}[c_1 + c_2 x] + 3x + 6 + \frac{3}{2}\cos(3x) - 2\sin(3x)$

17. As usual, up to this point, we solve an initial value problem by finding the general solution of the differential equation and then using the initial conditions to solve for the constants. We obtain

$$y = \tfrac{7}{4}e^{2x} - \tfrac{3}{4}e^{-2x} - \tfrac{7}{4}xe^{2x} - \tfrac{1}{4}x.$$

18. $y = 3 + 2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x$
19. $y = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$
20. $y = \frac{1}{5} + e^{3x} - \frac{1}{5}e^{2x}[\cos(x) + 3\sin(x)]$
21. $y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}$
22. The general solution is

$$y = e^{x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1$$

To make it easier to fit the initial conditions specified at $x = 1$, we can also write this general solution as

$$y = e^{x/2}\left[d_1 \cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2 \sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

Now

$$y(1) = e^{1/2}d_1 + 1 = 4 \text{ and } y'(1) = \frac{1}{2}e^{1/2}d_1 + \frac{\sqrt{3}}{2}e^{1/2}d_2 = -2.$$

Solve these to get $d_1 = 3e^{-1/2}$ and $d_2 = -7e^{-1/2}/\sqrt{3}$. The solution is

$$y = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

23. $y = 4e^{-x} - \sin^2(x) - 2$
24. $y = 4\cos(x) + 4\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$

2.4 Spring Motion

1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is

$$y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)].$$

With initial conditions $y(0) = 0, y'(0) = 5$, we obtain

$$y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t).$$

Graphs of these solutions are shown in Figure 2.1.

2. With $y(0) = 5$ and $y' = 0$, $y_1(t) = 5e^{-2t}(1 + 2t)$; with $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = 5te^{-2t}$. Graphs are given in Figure 2.2.

3. With $y(0) = 5$ and $y' = 0$,

$$y_1(t) = \frac{5}{2}e^{-t}[2\cos(2t) + \sin(2t)].$$

With $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = \frac{5}{2}e^{-t}\sin(2t)$. Graphs are given in Figure 2.3.

4. The solution is

$$y(t) = Ae^{-t}[\cosh(\sqrt{2}t) + \sqrt{(2)}\sinh(\sqrt{2}t)].$$

Graphs for $A = 1, 3, 6, 10, -4$ and -7 are given in Figure 2.4.

5. The solution is

$$y(t) = \frac{A}{\sqrt{2}}e^{-2t}\sinh(\sqrt{(2)}t)$$

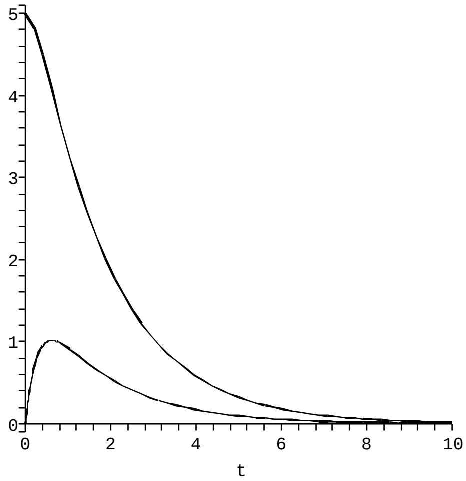


Figure 2.1: Problem 1, Section 2.4.

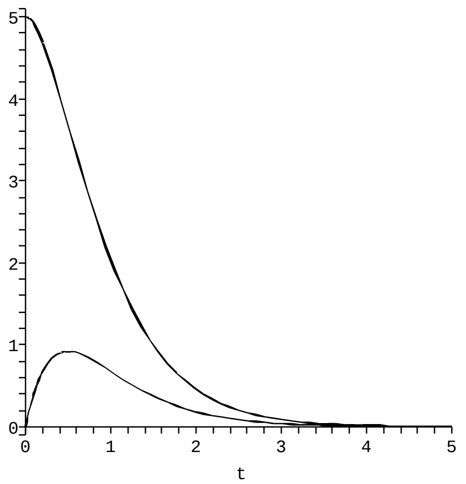


Figure 2.2: Problem 2, Section 2.4.

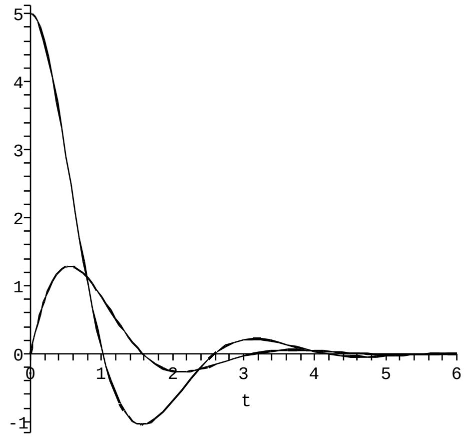


Figure 2.3: Problem 3, Section 2.4.

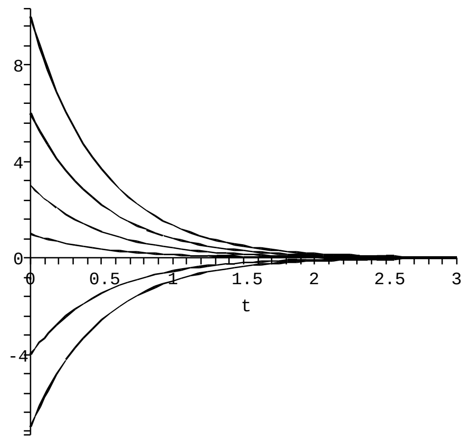


Figure 2.4: Problem 4, Section 2.4.

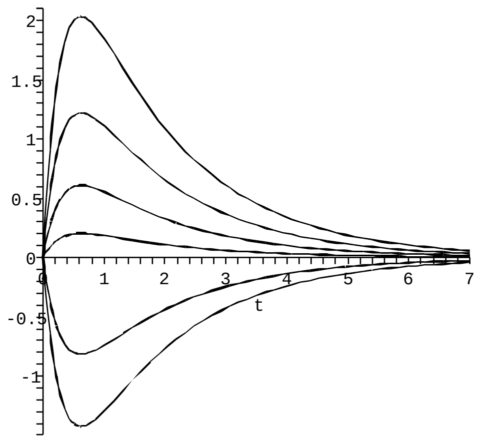


Figure 2.5: Problem 5, Section 2.4.

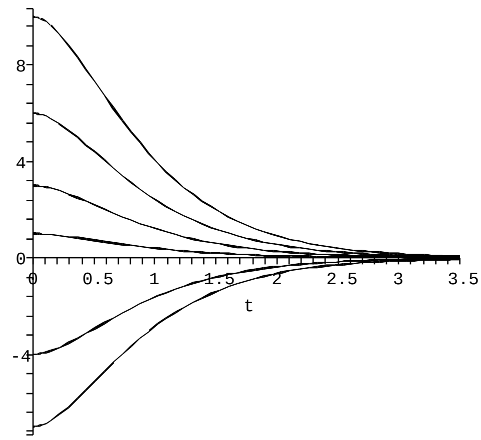


Figure 2.6: Problem 6, Section 2.4.

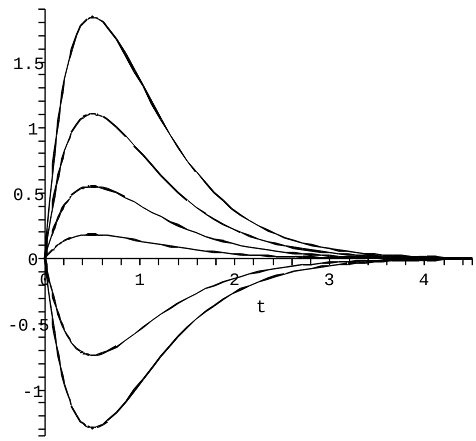


Figure 2.7: Problem 7, Section 2.4.

and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.5.

6. The solution is $y(t) = Ae^{-2t}(1 + 2t)$ and is graphed for $A = 1, 3, 6, 10, -4, -7$ in Figure 2.6.

7. The solution is $y(t) = Ate^{-2t}$, graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.7.

8. The solution is

$$y(t) = \frac{A}{2}e^{-t}[2\cos(2t) + \sin(2t)],$$

graphed in Figure 2.8 for $A = 1, 3, 6, 10, -4$ and -7.

9. The solution is

$$y(t) = \frac{A}{2}e^{-t}\sin(2t)$$

and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.9.

10. From Newton's second law of motion,

$$y'' = \text{ sum of external forces } = -29y - 10y'$$

so the motion is described by the solution of

$$y'' + 10y' + 29y = 0; y(0) = 3, y'(0) = -1.$$

The solution in this underdamped problem is

$$y(t) = e^{-5t}[3\cos(2t) + 7\sin(2t)].$$

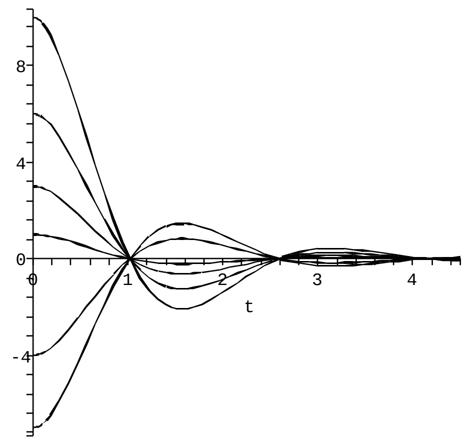


Figure 2.8: Problem 8, Section 2.4.

If the condition on $y'(0)$ is $y'(0) = A$, this solution is

$$y(t) = e^{-5t}\left[3\cos(2t) + \left(\frac{A+15}{2}\right)\sin(2t)\right].$$

Graphs of this solution are shown in Figure 2.10 for $A = -1, -2, -4, 7, -12$ cm/sec (recall that down is the positive direction).

11. For overdamped motion the displacement is given by $y(t) = e^{-\alpha t}(A + Be^{\beta t})$, where α is the smaller of the roots of the characteristic equation and is positive, and β equals the larger root minus the smaller root. The factor $A + Be^{\beta t}$ can be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by the initial conditions. In fact, if $y_0 = y(0)$ and $v_0 = y'(0)$, we have

$$A + B = y_0 \text{ and } -\alpha(A+B) + \beta B = v_0.$$

We find from these that

$$-\frac{A}{B} = 1 - \frac{\beta y_0}{v_0 + \alpha y_0}.$$

No condition on only y_0 will ensure that $-A/B \leq 1$. If we also specify that $v_0 > -\alpha y_0$, we ensure that the overdamped bob will never pass through the equilibrium point.

12. For critically damped motion the displacement has the form $y(t) = e^{-\alpha t}(A + Bt)$ with $\alpha > 0$ and A and B determined by the initial conditions. From the linear factor, the bob can pass through the equilibrium at most once,

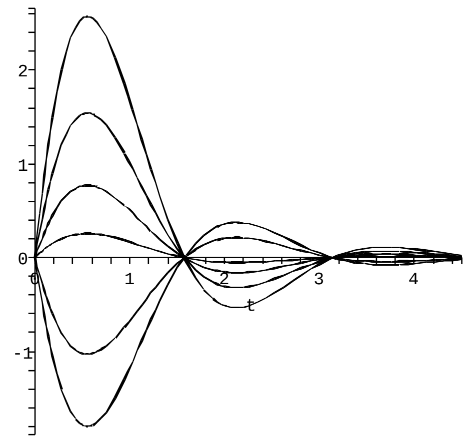


Figure 2.9: Problem 9, Section 2.4.

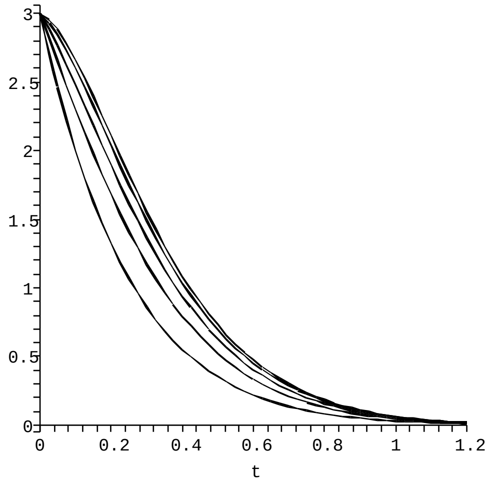


Figure 2.10: Problem 10, Section 2.4.

and will do this for some $t > 0$ if and only if $B \neq 0$ and $AB < 0$. Now note that $y_0 = A$ and $v_0 = y'(0) = -\alpha A + B$. Thus to ensure that the bob never passes through equilibrium we need $AB > 0$, which becomes the condition $(v_0 + \alpha y_0)y_0 > 0$. No condition on y_0 alone can ensure this. We would also need to specify $v_0 > -\alpha y_0$, and this will ensure that the critically damped bob never passes through the equilibrium point.

13. For underdamped motion, the solution has the appearance

$$y(t) = e^{-ct/2m}[c_1 \cos(\sqrt{4km - c^2}t/2m) + c_2 \sin(\sqrt{4km - c^2}t/2m)]$$

having frequency

$$\omega = \frac{\sqrt{4km - c^2}}{2m}.$$

Thus increasing c decreases the frequency of the the motion, and decreasing c increases the frequency.

14. For critical damping,

$$y(t) = e^{-ct/2m}(A + Bt).$$

For the maximum displacement at time t^* we need $y'(t^*) = 0$. This gives us

$$t^* = \frac{2mB - cA}{Bc}.$$

Now $y(0) = A$ and $y'(0) = B - Ac/2m$. Since we are given that $y(0) = y'(0) \neq 0$, we find that

$$t^* = \frac{4m^2}{2mc + c^2}$$

and this is independent of $y(0)$. The maximum displacement is

$$y(t^*) = \frac{y(0)}{c}(2m + c)e^{-2m/(2m+c)}.$$

15. The general solution of the overdamped problem

$$y'' + 6y' + 2y = 4\cos(3t)$$

is

$$y(t) = e^{-3t}[c_1 \cosh(\sqrt{7}t) + c_2 \sinh(\sqrt{7}t)] \\ - \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us

$$c_1 = \frac{2266}{373} \text{ and } c_2 = \frac{6582}{373\sqrt{7}}.$$

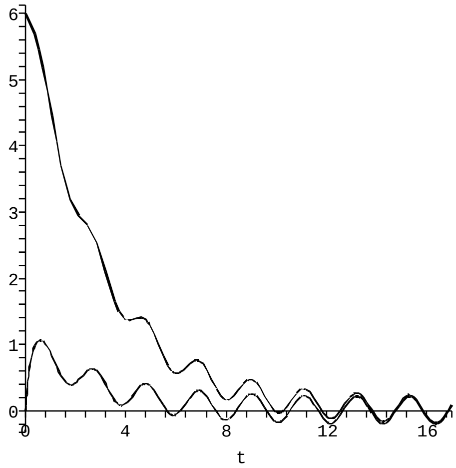


Figure 2.11: Problem 15, Section 2.4.

Now the solution is

$$y_a(t) = \frac{1}{373}[e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us $c_1 = 28/373$ and $c_2 = 2106/373$ and the unique solution

$$y_b(t) = \frac{1}{373}[e^{-3t}[29\cosh(\sqrt{7}t) + \tfrac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

These solutions are graphed in Figure 2.11.

16. The general solution of the critically damped problem

$$y'' + 4y' + 4y = 4\cos(3t)$$

is

$$y(t) = e^{-2t}[c_1 + c_2 t] - \frac{20}{169}\cos(3t) + \frac{48}{169}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us the unique solution

$$y_a(t) = \frac{1}{169}[e^{-2t}[1034 + 1924t] - 20\cos(3t) + 48\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us the unique solution

$$y_b(t) = \frac{1}{169}[e^{-2t}[20 + 910t] - 20\cos(3t) + 48\sin(3t)].$$

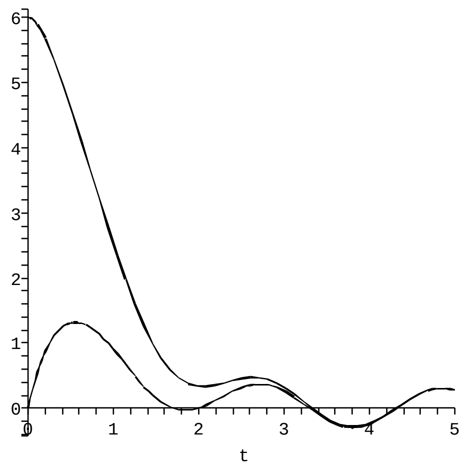


Figure 2.12: Problem 16, Section 2.4.

These solutions are graphed in Figure 2.12.

17. The general solution of the underdamped problem

$$y''(t) + y' + 3y = 4\cos(3t)$$

is

$$y(t) = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{11}t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ yield the unique solution

$$y_a(t) = \frac{1}{15}\left[e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ yield the unique solution

$$y_b(t) = \frac{1}{15}\left[e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

These solutions are graphed in Figure 2.13.

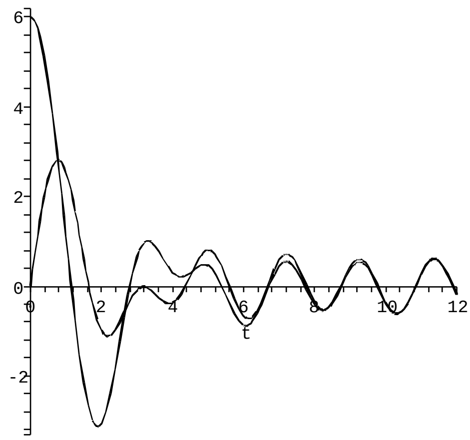


Figure 2.13: Problem 17, Section 2.4.