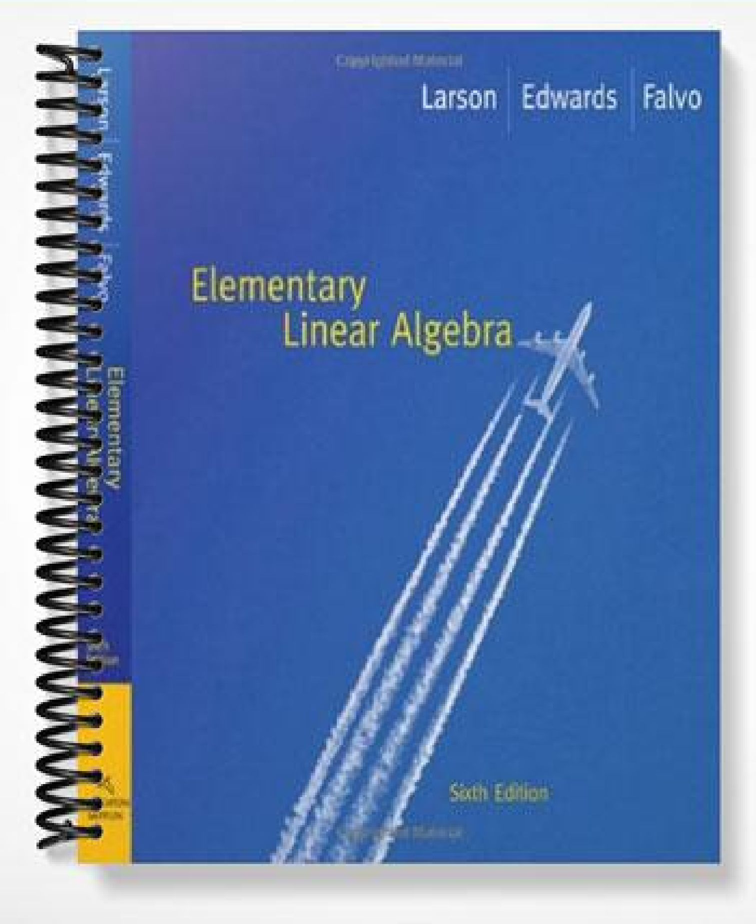
SOLUTIONS MANUAL



CHAPTER 2
Matrices

CHAPTER 2

Matrices

Section 2.1 Operations with Matrices

2. (a) $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1-3 & 2-2 \\ 2+4 & 1+2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+2 \\ 2-4 & 1-2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix}$

(c) $2A = 2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(2) & 2(1) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

(d) $2A - B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & -1 \\ 5 & \frac{5}{2} \end{bmatrix}$

4. (a) $A + B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2+2 & 1-3 & 1+4 \\ -1-3 & -1+1 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 5 \\ -4 & 0 & 2 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2-2 & 1+3 & 1-4 \\ -1+3 & -1-1 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -3 \\ 2 & -2 & 6 \end{bmatrix}$

(c) $2A = 2\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(1) & 2(1) \\ 2(-1) & 2(-1) & 2(4) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix}$

(d) $2A - B = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -2 \\ 1 & -3 & 10 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{9}{2} \\ -\frac{7}{2} & \frac{1}{2} & 0 \end{bmatrix}$

6. (a) $A + B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2+0 & 3+6 & 4+2 \\ 0+4 & 1+1 & -1+0 \\ 2+(-1) & 0+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 6 \\ 4 & 2 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2-0 & 3-6 & 4-2 \\ 0-4 & 1-1 & -1-0 \\ 2-(-1) & 0-2 & 1-4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 2 \\ -4 & 0 & -1 \\ 3 & -2 & -3 \end{bmatrix}$

(c) $2A = 2\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(3) & 2(4) \\ 2(0) & 2(1) & 2(-1) \\ 2(2) & 2(0) & 2(1) \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 2 & -2 \\ 4 & 0 & 2 \end{bmatrix}$

(d) $2A - B = 2\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 2 & -2 \\ 4 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 6 \\ -4 & 1 & -2 \\ 5 & -2 & -2 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 2 \\ 4 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{15}{2} & 4 \\ 4 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & \frac{9}{2} \end{bmatrix}$

8. (a) $c_{23} = 5a_{23} + 2b_{23} = 5(2) + 2(11) = 32$

(b) $c_{32} = 5a_{32} + 2b_{32} = 5(1) + 2(4) = 13$

10. Simplifying the right side of the equation produces

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 + 2y & 3 + 2w \\ 2 + 2z & -1 + 2x \end{bmatrix}$$

By setting corresponding entries equal to each other, you obtain four equations.

$$\begin{aligned} w &= -4 + 2y \\ x &= 3 + 2w \\ y &= 2 + 2z \\ x &= -1 + 2x \end{aligned} \Rightarrow \begin{cases} -2y + w = -4 \\ x - 2w = 3 \\ y - 2z = 2 \\ x = 1 \end{cases}$$

The solution to this linear system is: $x = 1$, $y = \frac{3}{2}$, $z = -\frac{1}{4}$, and $w = -1$.

12. (a) $$AB = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(2) + 7(1) & 1(1) + (-1)(1) + 7(-3) & 1(2) + (-1)(1) + 7(2) \\ 2(1) + (-1)(2) + 8(1) & 2(1) + (-1)(1) + 8(-3) & 2(2) + (-1)(1) + 8(2) \\ 3(1) + 1(2) + (-1)(1) & 3(1) + 1(1) + (-1)(-3) & 3(2) + 1(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix}$$

(b) $$BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) + 2(3) & 1(-1) + 1(-1)1 + 2(1) & 1(7) + 1(8) + 2(-1) \\ 2(1) + 1(2) + 1(3) & 2(-1) + 1(-1) + 1(1) & 2(7) + 1(8) + 1(-1) \\ 1(1) + (-3)(2) + 2(3) & 1(-1) + (-3)(-1) + 2(1) & 1(7) + (-3)(8) + 2(-1) \end{bmatrix} = \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}$$

14. (a) $$AB = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3(2) + 2(3) + 1(0) \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$$

(b) $$BA = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(2) & 2(1) \\ 3(3) & 3(2) & 3(1) \\ 0(3) & 0(2) & 0(1) \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ 9 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

16. (a) $$AB = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0(2) + (-1)(-3) + 0(1) \\ 4(2) + 0(-3) + 2(1) \\ 8(2) + (-1)(-3) + 7(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 26 \end{bmatrix}$$

(b) BA is not defined because B is 3×1 and A is 3×3.

18. (a) AB is not defined because A is 2×5 and B is 2×2.

(b) $$\begin{aligned} BA &= \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 6(6) & 1(0) + 6(13) & 1(3) + 6(8) & 1(-2) + 6(-17) & 1(4) + 6(20) \\ 4(1) + 2(6) & 4(0) + 2(13) & 4(3) + 2(8) & 4(-2) + 2(-17) & 4(4) + 2(20) \end{bmatrix} \\ &= \begin{bmatrix} 37 & 78 & 51 & -104 & 124 \\ 16 & 26 & 28 & -42 & 56 \end{bmatrix} \end{aligned}$$

20. Because A is 6×5 and B is 6×6, you have

(a) $2A + B$ is undefined.

(b) $3B - A$ is undefined.

(c) AB is undefined.

(d) Using a graphing utility or computer software program, you have

$$BA = \begin{bmatrix} 10 & 1 & 18 & 11 & 4 \\ -5 & 3 & -25 & 4 & 11 \\ -2 & 2 & 19 & -1 & -15 \\ 10 & -15 & 8 & 1 & 6 \\ -3 & -5 & -6 & -2 & -17 \\ -18 & 9 & 2 & -8 & -11 \end{bmatrix}.$$

22. $C + E$ is not defined because C and E have different sizes.

24. $-4A$ is defined and has size 3×4 because A has size 3×4.

26. BE is defined. Because B has size 3×4 and E has size 4×3, the size of BE is 3×3.

28. $2D + C$ is defined and has size 4×2 because $2D$ and C have size 4×2.

30. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$

32. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} -4 & 9 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 12 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} -4 & 9 & -13 \\ 1 & -3 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -23 \\ 0 & 1 & -\frac{35}{3} \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -23 \\ -\frac{35}{3} \end{bmatrix}.$

34. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$

36. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \\ 0 & -6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 40 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 4 & 17 \\ 1 & 3 & 0 & -11 \\ 0 & -6 & 5 & 40 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$

38. Expanding the left side of the equation produces

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} A &= \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 2a_{11} - a_{21} & 2a_{12} - a_{22} \\ 3a_{11} - 2a_{21} & 3a_{12} - 2a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and you obtain the system

$$\begin{array}{rrrrl} 2a_{11} & & - a_{21} & & = 1 \\ & 2a_{12} & & - a_{22} & = 0 \\ 3a_{11} & & - 2a_{21} & & = 0 \\ & 3a_{12} & & - 2a_{22} & = 1. \end{array}$$

Solving by Gauss-Jordan elimination yields

$a_{11} = 2$, $a_{12} = -1$, $a_{21} = 3$, and $a_{22} = -2$.

So, you have $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$

40. Expanding the left side of the matrix equation produces

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2a + 3b & a + b \\ 2c + 3d & c + d \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}.$$

You obtain two systems of linear equations (one involving a and b and the other involving c and d).

$$\begin{aligned} 2a + 3b &= 3 \\ a + b &= 17, \end{aligned}$$

and

$$\begin{aligned} 2c + 3d &= 4 \\ c + d &= -1. \end{aligned}$$

Solving by Gauss-Jordan elimination yields $a = 48$, $b = -31$, $c = -7$ and $d = 6$.

42. $$AB = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha(-\sin\beta) - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \sin\alpha(-\sin\beta) + \cos\alpha\cos\beta \end{bmatrix}$$

$$BA = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & \cos\beta(-\sin\alpha) - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & \sin\beta(-\sin\alpha) + \cos\beta\cos\alpha \end{bmatrix}$$

So, you see that $AB = BA = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}.$

44. $$AA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

46. $$AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 3(-7) + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + (-5)4 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$BA = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

48. (a) $$AB = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{bmatrix}$$

The ith row of B has been multiplied by a_{ii}, the ith diagonal entry of A.

(b) $$BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{22}b_{12} & a_{33}b_{13} \\ a_{11}b_{21} & a_{22}b_{22} & a_{33}b_{23} \\ a_{11}b_{31} & a_{22}b_{32} & a_{33}b_{33} \end{bmatrix}$$

The ith column of B has been multiplied by a_{ii}, the ith diagonal entry of A.

(c) If $a_{11} = a_{22} = a_{33}$, then $AB = a_{11}B = BA.$

50. The trace is the sum of the elements on the main diagonal.

$1 + 1 + 1 = 3.$

52. The trace is the sum of the elements on the main diagonal.

$1 + 0 + 2 + (-3) = 0$

54. Let $AB = \left[c_{ij}\right]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then, $Tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$.

Similarly, if $BA = \left[d_{ij}\right]$, $d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$, then $Tr(BA) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} b_{ik}a_{ki}\right) = Tr(AB)$.

56. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Then the matrix equation $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This equation implies that

$$a_{11}b_{11} + a_{12}b_{21} - b_{11}a_{11} - b_{12}a_{21} = a_{12}b_{21} - b_{12}a_{21} = 1$$

$$a_{21}b_{12} + a_{22}b_{22} - b_{21}a_{12} - b_{22}a_{22} = a_{21}b_{12} - b_{21}a_{12} = 1$$

which is impossible. So, the original equation has no solution.

58. Assume that A is an $m \times n$ matrix and B is a $p \times q$ matrix. Because the product AB is defined, you know that $n = p$. Moreover, because AB is square, you know that $m = q$. Therefore, B must be of order $n \times m$, which implies that the product BA is defined.

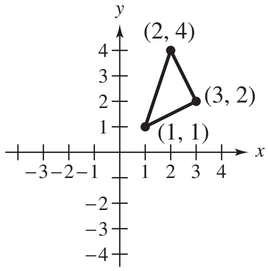
60. Let rows s and t be identical in the matrix A. So, $a_{sj} = a_{tj}$ for $j = 1, \ldots, n$. Let $AB = \left[c_{ij}\right]$, where

$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then, $c_{sj} = \sum_{k=1}^{n} a_{sk}b_{kj}$, and $c_{tj} = \sum_{k=1}^{n} a_{tk}b_{kj}$. Because $a_{sk} = a_{tk}$ for $k = 1, \ldots, n$, rows s and t of AB are the same.

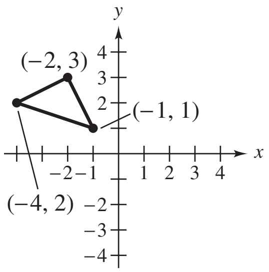
62. (a) $AT = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

$AAT = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -4 & -2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -1 & -4 & -2 \end{bmatrix}$

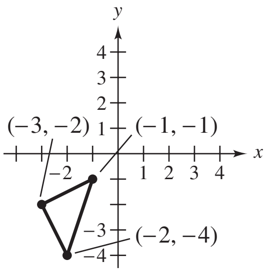
Triangle associated with T



Triangle associated with AT



Triangle associated with AAT



The transformation matrix A rotates the triangle about the origin in a counterclockwise direction through 90°.

(b) Given the triangle associated with AAT, the transformation that would produce the triangle associated with AT would be a rotation about the origin of 90° in a clockwise direction. Another such rotation would produce the triangle associated with T.

64. $1.1\begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix} = \begin{bmatrix} 110 & 99 & 77 & 33 \\ 44 & 22 & 66 & 66 \end{bmatrix}$

66. (a) Use *scalar multiplication* to find L.

$$L = \tfrac{2}{3}C = \tfrac{2}{3}\begin{bmatrix} 627 & 681 \\ 135 & 150 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}(627) & \frac{2}{3}(681) \\ \frac{2}{3}(135) & \frac{2}{3}(150) \end{bmatrix} = \begin{bmatrix} 418 & 454 \\ 90 & 100 \end{bmatrix}$$

(b) Use *matrix addition* to find M.

$$M = C - L = \begin{bmatrix} 627 & 681 \\ 135 & 150 \end{bmatrix} - \begin{bmatrix} 418 & 454 \\ 90 & 100 \end{bmatrix} = \begin{bmatrix} 627 - 418 & 681 - 454 \\ 135 - 90 & 150 - 100 \end{bmatrix} = \begin{bmatrix} 209 & 227 \\ 45 & 50 \end{bmatrix}$$

68. (a) True. The number of elements in a row of the first matrix must be equal to the number of elements in a column of the second matrix. See page 51 of the text.

(b) True. See page 53 of the text.

70. (a) Multiply the matrix for 2005 by $\frac{100}{288{,}131}$. This produces a matrix giving the information as percents of the total population.

$$A = \frac{100}{288{,}131}\begin{bmatrix} 12{,}607 & 34{,}418 & 6286 \\ 16{,}131 & 41{,}395 & 7177 \\ 26{,}728 & 63{,}911 & 11{,}689 \\ 5306 & 12{,}679 & 2020 \\ 12{,}524 & 30{,}741 & 4519 \end{bmatrix} \approx \begin{bmatrix} 4.38 & 11.95 & 2.18 \\ 5.60 & 14.37 & 2.49 \\ 9.28 & 22.18 & 4.06 \\ 1.84 & 4.40 & 0.70 \\ 4.35 & 10.67 & 1.57 \end{bmatrix}$$

Multiply the matrix for 2015 by $\frac{100}{321{,}609}$. This produces a matrix giving the information as percents of the total population.

$$B = \frac{100}{321{,}609}\begin{bmatrix} 12{,}441 & 35{,}289 & 8835 \\ 16{,}363 & 42{,}250 & 9955 \\ 29{,}373 & 73{,}496 & 17{,}572 \\ 5263 & 14{,}231 & 3337 \\ 12{,}826 & 33{,}292 & 7086 \end{bmatrix} \approx \begin{bmatrix} 3.87 & 10.97 & 2.75 \\ 5.09 & 13.14 & 3.10 \\ 9.13 & 22.85 & 5.46 \\ 1.64 & 4.42 & 1.04 \\ 3.99 & 10.35 & 2.20 \end{bmatrix}$$

(b) $$B - A = \begin{bmatrix} 3.87 & 10.97 & 2.75 \\ 5.09 & 13.14 & 3.10 \\ 9.13 & 22.85 & 5.46 \\ 1.64 & 4.42 & 1.04 \\ 3.99 & 10.35 & 2.20 \end{bmatrix} - \begin{bmatrix} 4.38 & 11.95 & 2.18 \\ 5.60 & 14.37 & 2.49 \\ 9.28 & 22.18 & 4.06 \\ 1.84 & 4.40 & 0.70 \\ 4.35 & 10.67 & 1.57 \end{bmatrix} = \begin{bmatrix} -0.51 & -0.98 & 0.57 \\ -0.51 & -1.23 & 0.61 \\ -0.15 & 0.67 & 1.40 \\ -0.20 & 0.02 & 0.34 \\ -0.36 & -0.32 & 0.63 \end{bmatrix}$$

(c) The 65+ age group is projected to show relative growth from 2005 to 2015 over all regions because its column in $B - A$ contains all positive percents.

72. $$AB = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{array}\right]$$

74. The augmented matrix row reduces as follows.

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ -1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are an infinite number of solutions. For example, $x_3 = 0,\ x_2 = 2,\ x_1 = -3$.

So, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$

76. The augmented matrix row reduces as follows.

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 10 \\ 0 & 9 & -18 \\ 0 & -4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So,

$$\begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix} = 4\begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}.$$

Section 2.2 Properties of Matrix Operations

2. $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

4. $(a + b)B = (3 + (-4))\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

6. $(ab)0 = (3)(-4)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (-12)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. (a) $X = 3A - 2B = \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 4 & 0 \\ -8 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}$

(b) $2X = 2A - B$

$$2X = \begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

$$2X = \begin{bmatrix} -4 & -5 \\ 0 & 0 \\ 10 & -7 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & -\frac{5}{2} \\ 0 & 0 \\ 5 & -\frac{7}{2} \end{bmatrix}$$

(c) $2X + 3A = B$

$$2X + \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

$$2X = \begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$$

(d) $2A + 4B = -2X$

$$\begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ 8 & 0 \\ -16 & -4 \end{bmatrix} = -2X$$

$$\begin{bmatrix} -4 & 10 \\ 10 & 0 \\ -10 & -12 \end{bmatrix} = -2X$$

$$\begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix} = X$$

10. $$C(BC) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & -1 \end{bmatrix}$$

12. $$B(C + O) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix}$$

14. $$B(cA) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\left((-2)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -10 & 0 \\ 2 & 0 & 10 \end{bmatrix}$$

16. $$AC = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 9 \\ 16 & -8 & 12 \\ 4 & -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = BC$$

But $A \neq B$.

18. $$AB = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

But $A \neq O$ and $B \neq O$.

20. $$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

So, $A^4 = \left(A^2\right)^2 = I_2^2 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

22. $$A + IA = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

24. (a) $$A^T = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix}$$

(b) $$A^TA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 11 & 21 \end{bmatrix}$$

(c) $$AA^T = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 25 & -8 \\ 2 & -8 & 4 \end{bmatrix}$$

26. (a) $$A^T = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix}$$

(b) $$A^TA = \begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix}\begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 101 & -95 & -62 \\ -95 & 131 & 126 \\ -62 & 126 & 154 \end{bmatrix}$$

(c) $$AA^T = \begin{bmatrix} -7 & 11 & 12 \\ 4 & -3 & 1 \\ 6 & -1 & 3 \end{bmatrix}\begin{bmatrix} -7 & 4 & 6 \\ 11 & -3 & -1 \\ 12 & 1 & 3 \end{bmatrix}\begin{bmatrix} 314 & -49 & -17 \\ -49 & 26 & 30 \\ -17 & 30 & 46 \end{bmatrix}$$

28. (a) $$A^T = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix}$$

(b) $$A^TA = \begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 252 & 8 & 168 & -104 \\ 8 & 77 & -70 & 50 \\ 168 & -70 & 294 & -139 \\ -104 & 50 & -139 & 98 \end{bmatrix}$$

(c) $$AA^T = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 30 & 86 & -10 \\ 30 & 126 & 169 & -47 \\ 86 & 169 & 425 & -28 \\ -10 & -47 & -28 & 141 \end{bmatrix}$$

30. In general, $AB \neq BA$ for matrices.

32. $$(AB)^T = \left(\begin{bmatrix}1 & 2\\0 & -2\end{bmatrix}\begin{bmatrix}-3 & -1\\2 & 1\end{bmatrix}\right)^T = \begin{bmatrix}1 & 1\\-4 & -2\end{bmatrix}^T = \begin{bmatrix}1 & -4\\1 & -2\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}-3 & -1\\2 & 1\end{bmatrix}^T\begin{bmatrix}1 & 2\\0 & -2\end{bmatrix}^T = \begin{bmatrix}-3 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\2 & -2\end{bmatrix} = \begin{bmatrix}1 & -4\\1 & -2\end{bmatrix}$$

34. $$(AB)^T = \left(\begin{bmatrix}2 & 1 & -1\\0 & 1 & 3\\4 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\2 & 1 & -2\\0 & 1 & 3\end{bmatrix}\right)^T = \begin{bmatrix}4 & 0 & -7\\2 & 4 & 7\\4 & 2 & 2\end{bmatrix}^T = \begin{bmatrix}4 & 2 & 4\\0 & 4 & 2\\-7 & 7 & 2\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}1 & 0 & -1\\2 & 1 & -2\\0 & 1 & 3\end{bmatrix}^T\begin{bmatrix}2 & 1 & -1\\0 & 1 & 3\\4 & 0 & 2\end{bmatrix}^T = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\-1 & -2 & 3\end{bmatrix}\begin{bmatrix}2 & 0 & 4\\1 & 1 & 0\\-1 & 3 & 2\end{bmatrix} = \begin{bmatrix}4 & 2 & 4\\0 & 4 & 2\\-7 & 7 & 2\end{bmatrix}$$

36. (a) False. In general, for $n \times n$ matrices A and B it is *not* true that $AB = BA$. For example, let

$$A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix},\ B = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}.$$

Then $AB = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} \neq \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = BA.$

(b) True. For any matrix A you have an additive inverse, namely $-A = (-1)A$. See Theorem 2.2(2) on page 621.

(c) False. Let

$A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix},\ B = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix},\ C = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$. Then

$AB = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} = AC$, but $B \neq C$.

(d) True. See Theorem 2.6(2) on page 68.

38. (a) $Z = aX + bY$

$$\begin{bmatrix}1\\4\\4\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}a + b\\2a\\3a + 2b\end{bmatrix}$$

Solving the linear system obtained from this matrix equation, you obtain $a = 2$ and $b = -1$. So, $Z = 2X - Y$.

(b) $W = aX + bY$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}a + b\\2a\\3a + 2b\end{bmatrix}$$

It follows from $a + b = 0$ and $2a = 0$ that *a, b* must both be zero, but this is impossible because $3a + 2b$ should be 1.

(c) $aX + bY + cW = O$

$$a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\0\\2\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This matrix equation yields the linear system

$$\begin{aligned} a + b \quad &= 0\\ 2a \quad &= 0\\ 3a + 2b + c &= 0 \end{aligned}$$

which has the unique solution $a = b = c = 0$.

(d) $$aX + bY + cZ = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\0\\2\end{bmatrix} + c\begin{bmatrix}1\\4\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This matrix equation yields the linear system

$$\begin{aligned} a + b + c &= 0\\ 2a \quad + 4c &= 0\\ 3a + 2b + 4c &= 0. \end{aligned}$$

Solving this system using Gauss-Jordan elimination, you find that there are an infinite number of solutions: $a = -2t,\ b = t$, and $c = t$. For instance, $a = -2,\ b = c = 1$.

40. $$A^{20} = \begin{bmatrix} (1)^{20} & 0 & 0 \\ 0 & (-1)^{20} & 0 \\ 0 & 0 & (1)^{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

42. Because $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (-1)^3 & 0 \\ 0 & 0 & (3)^3 \end{bmatrix}$, you have $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

44. $$\begin{aligned} f(A) &= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}^2 - 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

46. $$\begin{aligned} f(A) &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^3 - 2\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2 + 5\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 10 & 5 & -5 \\ 5 & 0 & 10 \\ -5 & 5 & 15 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix} - 2\begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{bmatrix} - \begin{bmatrix} 12 & 2 & -6 \\ 0 & 6 & 10 \\ -8 & 4 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & -12 \\ 3 & -11 & 21 \\ -15 & 9 & 25 \end{bmatrix} \end{aligned}$$

48. $(cd)A = (cd)[a_{ij}] = [(cd)a_{ij}] = [c(da_{ij})] = c[da_{ij}] = c(dA)$

50. $$\begin{aligned} (c + d)A &= (c + d)[a_{ij}] = [(c + d)a_{ij}] = [ca_{ij} + da_{ij}] \\ &= [ca_{ij}] + [da_{ij}] = c[a_{ij}] + d[a_{ij}] = cA + dA \end{aligned}$$

52. (a) To show that $A(BC) = (AB)C$, compare the ijth entries in matrices on both sides of this equality. Assume that A has size $n \times p$, B has size $p \times r$ and C has size $r \times m$. Then the entry in kth row and jth column of BC is $\sum_{l=1}^{r} b_{kl}c_{lj}$, Therefore the entry in ith row and jth column of $A(BC)$ is

$$\sum_{k=1}^{p} a_{ik} \sum_{l=1}^{r} b_{kl}c_{lj} = \sum_{k,\,l} a_{ik}b_{kl}c_{lj}.$$

The entry in the ith row and jth column of $(AB)C$ is $\sum_{l=1}^{r} d_{il}c_{lj}$, where d_{il} is the entry of AB in ith row and lth column. So, $d_{il} = \sum_{k=1}^{p} a_{ik}b_{kl}$ for each $l = 1, \ldots, r$. So, the ijth entry of $(AB)C$ is

$$\sum_{i=1}^{r}\sum_{k=1}^{p} a_{ik}b_{kl}c_{lj} = \sum_{k,\,l} a_{ik}b_{kl}c_{ij.}$$

Because all corresponding entries of $A(BC)$ and $(AB)C$ are equal and both matrices are of the same size $(n \times m)$ you conclude that $A(BC) = (AB)C$.

(b) The entry in the ith row and jth column of $(A + B)C$ is $(a_{il} + b_{il})c_{1j} + (a_{i2} + b_{i2})c_{2j} + \cdots + (a_{in} + b_{in})c_{nj}$, whereas the entry in the ith row and jth column of $AC + BC$ is $(a_{i1}c_{1j} + \cdots + a_{in}c_{nj}) + (b_{i1}c_{1j} + \cdots + b_{in}c_{nj})$, which are equal by the distributive law for real numbers.

(c) The entry in the ith row and jth column of $c(AB)$ is $c\left[a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}\right]$. The corresponding entry for $(cA)B$ is $(ca_{i1})b_{1j} + (ca_{i2})b_{2j} + \cdots + (ca_{in})b_{nj}$. And the corresponding entry for $A(cB)$ is $a_{i1}(cb_{1j}) + a_{i2}(cb_{2j}) + \cdots + a_{in}(cb_{nj})$. Because these three expressions are equal, you have shown that $c(AB) = (cA)B = A(cB)$.

54. (2) $(A + B)^T = \left(\left[a_{ij}\right] + \left[b_{ij}\right]\right)^T = \left[a_{ij} + b_{ij}\right]^T = \left[a_{ji} + b_{ji}\right] = \left[a_{ji}\right] + \left[b_{ji}\right] = A^T + B^T$

(3) $(cA)^T = \left(c\left[a_{ij}\right]\right)^T = \left[ca_{ij}\right]^T = \left[ca_{ji}\right] = c\left[a_{ji}\right] = c\left(A^T\right)$

(4) The entry in the ith row and jth column of $(AB)^T$ is $a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots a_{jn}b_{ni}$. On the other hand, the entry in the ith row and jth column of B^TA^T is $b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn}$, which is the same.

56. Many examples are possible. For instance, $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then, $(AB)^T = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, while $A^TB^T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, which are not equal.

58. Because $A = A^T$, this matrix is symmetric.

60. Because $-A = A^T$, this matrix is skew-symetric.

62. If $A^T = -A$ and $B^T = -B$, then $(A + B)^T = A^T + B^T = -A - B = -(A + B)$, which implies that $A + B$ is skew-symmetric.

64. $\left(A - A^T\right)^T = A^T - \left(A^T\right)^T = A^T - A = -\left(A - A^T\right)$, which implies that $A - A^T$ is skew-symmetric.

66. (a) An example of a 2×2 matrix of the given form is $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

An example of a 3×3 matrix of the given form is $A_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $A_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A_3^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A_3^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) The conjecture is that if A is a 4×4 matrix of the given form, then A^4 is the 4×4 zero matrix. A graphing utility shows this to be true.

(d) If A is an $n \times n$ matrix of the given form, then A^n is the $n \times n$ zero matrix.

Section 2.3 The Inverse of a Matrix

2. $$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. $$AB = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Use the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

So, the inverse is

$$A^{-1} = \frac{1}{(1)(-3) - (-2)(2)}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

8. Use the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

you see that $ad - bc = (-1)(-3) - (1)(3) = 0$. So, the matrix has no inverse.

10. Adjoin the identity matrix to form

$$\begin{bmatrix} A \vdots I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right].$$

Using elementary row operations, reduce the matrix as follows.

$$\begin{bmatrix} I \vdots A^{-1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

12. Adjoin the identity matrix to form

$$\begin{bmatrix} A \vdots I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right].$$

Using elementary row operations, reduce the matrix as follows.

$$\begin{bmatrix} I \vdots A^{-1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -4 & 27 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix}.$$

14. Adjoin the identity matrix to form

$$\begin{bmatrix} A \vdots I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right].$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

16. Adjoin the identity matrix to form

$$\left[I \vdots A^{-1}\right] = \begin{bmatrix} 2 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 3 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$\left[A \vdots I\right] = \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

18. Adjoin the identity matrix to form

$$\left[A \vdots I\right] = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 3 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 2 & 5 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

20. Adjoin the identity matrix to form

$$\left[A \vdots I\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$\left[I \vdots A^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

22. Adjoin the identity matrix to form

$$\left[A \vdots I\right] = \begin{bmatrix} 4 & 8 & -7 & 14 & \vdots & 1 & 0 & 0 & 0 \\ 2 & 5 & -4 & 6 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -7 & \vdots & 0 & 0 & 1 & 0 \\ 3 & 6 & -5 & 10 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$\left[A \vdots I\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 27 & -10 & 4 & -29 \\ 0 & 1 & 0 & 0 & \vdots & -16 & 5 & -2 & 18 \\ 0 & 0 & 1 & 0 & \vdots & -17 & 4 & -2 & 20 \\ 0 & 0 & 0 & 1 & \vdots & -7 & 2 & -1 & 8 \end{bmatrix}$$

Therefore the inverse is

$$A^{-1} = \begin{bmatrix} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & 8 \end{bmatrix}.$$

24. Adjoin the identity matrix to form

$$\left[A \vdots I\right] = \begin{bmatrix} 1 & 3 & -2 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$\left[I \vdots A^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & -1.5 & -4 & 2.6 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & 0.2 \end{bmatrix}.$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & -1.5 & -4 & 2.6 \\ 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}.$$

26. The coefficient matrix for each system is $A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$ and the formula for the inverse of a 2×2 matrix produces

$$A^{-1} = \frac{1}{2 + 2}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

The solution is: $x = 1$ and $y = 5$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

The solution is: $x = -1$ and $y = -1$.

(c) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

The solution is: $x = 4$ and $y = 2$.

28. The coefficient matrix for each system is

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Using the algorithm to invert a matrix, you find that the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}.$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 0$, and $x_3 = 1$.

30. Using a graphing utility or computer software program, you have

$A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 & -1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & -1 \\ 2 & 1 & 4 & 1 & -1 \\ 3 & 1 & 1 & -2 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -1 \\ 5 \end{bmatrix}.$$

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 0$, and $x_5 = 1$.

32. Using a graphing utility or computer software program, you have

$A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 & -5 & -1 \\ 3 & 6 & -5 & -6 & 3 & 3 \\ 2 & -3 & 1 & 3 & -1 & -2 \\ -1 & 4 & -4 & -6 & 2 & 4 \\ 3 & -1 & 5 & 2 & -3 & -5 \\ -2 & 3 & -4 & -6 & 1 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \text{ and}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -11 \\ 0 \\ -9 \\ 1 \\ -12 \end{bmatrix}.$$

The solution is:
$x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$, $x_5 = 0$, $x_6 = 1$.

34. (a) $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} = \frac{1}{77}\begin{bmatrix} -4 & 9 \\ -9 & 1 \end{bmatrix}$

(b) $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T = \begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$

(c) $(A)^{-2} = (A^{-1})^2 = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{bmatrix}$

(d) $(2A)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{14} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$

36. (a) $\left(AB\right)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 6 & 5 & -3 \\ -2 & 4 & -1 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -25 & 24 \\ -6 & 10 & 7 \\ 17 & 7 & 15 \end{bmatrix}$

(b) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(c) $A^{-2} = \left(A^{-1}\right)^2 = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -4 & -8 \\ 12 & 7 & 6 \\ 8 & -12 & 15 \end{bmatrix}$

(d) $\left(2A\right)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 2 & 1 & \frac{1}{2} \end{bmatrix}$

38. The inverse of A is given by

$$A^{-1} = \frac{1}{x-4}\begin{bmatrix} -2 & -x \\ 1 & 2 \end{bmatrix}.$$

Letting $A^{-1} = A$, you find that $\dfrac{1}{x-4} = -1$.

So, $x = 3$.

40. The matrix $\begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$ will be singular if

$ad - bc = (x)(4) - (-3)(2) = 0$, which implies that

$4x = -6$ or $x = -\frac{3}{2}$.

42. First find 4A.

$$4A = \left[(4A)^{-1}\right]^{-1} = \frac{1}{4+12}\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}.$$

Then, multiply by $\frac{1}{4}$ to obtain

$$A = \frac{1}{4}(4A) = \frac{1}{4}\begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{32} & -\frac{1}{16} \\ \frac{3}{64} & \frac{1}{32} \end{bmatrix}.$$

44. Using the formula for the inverse of a 2×2 matrix, you have

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{\sec^2\theta - \tan^2\theta}\begin{bmatrix} \sec\theta & -\tan\theta \\ -\tan\theta & \sec\theta \end{bmatrix} \\ &= \begin{bmatrix} \sec\theta & -\tan\theta \\ -\tan\theta & \sec\theta \end{bmatrix}. \end{aligned}$$

46. (a) True. If A_1, A_2, A_3, A_4 are invertible 7×7 matrices, then $B = A_1A_2A_3A_4$ is also an invertible 7×7 matrix with inverse $B^{-1} = A_4^{-1}A_3^{-1}A_2^{-1}A_1^{-1}$, by Theorem 2.9 on page 81 and induction.

(b) True. $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$ by Theorem 2.8(4) on page 79.

(c) False. For example consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ which is not invertible, but

$1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$.

(d) False. If A is a square matrix then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is a *nonsingular* matrix.

48. $A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I_n^T = I_n$, and

$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I_n^T = I_n$.

So, $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$.

50. Because C is invertible, you can multiply both sides of the equation $CA = CB$ by C^{-1} on the left to obtain the following.

$$\begin{aligned} C^{-1}(CA) &= C^{-1}(CB) \\ \left(C^{-1}C\right)A &= \left(C^{-1}C\right)B \\ IA &= IB \\ A &= B \end{aligned}$$

52. Because $ABC = I$, A is invertible and $A^{-1} = BC$.

So, $ABC\ A = A$ and $BC\ A = I$.

So, $B^{-1} = CA$.

54. Let $A^2 = A$ and suppose A is nonsingular. Then, A^{-1} exists, and you have the following.

$$\begin{aligned} A^{-1}\left(A^2\right) &= A^{-1}A \\ \left(A^{-1}A\right)A &= I \\ A &= I \end{aligned}$$

56. A has an inverse if $a_{ii} \neq 0$ for all $i = 1 \ldots n$ and

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}.$$

58. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(a) $A^2 - 2A + 5I = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A\left(\frac{1}{5}(2I - A)\right) = \frac{1}{5}\left(2A - A^2\right) = \frac{1}{5}(5I) = I$

Similarly, $\left(\frac{1}{5}(2I - A)\right)A = I.$ Or, $\frac{1}{5}(2I - A) = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = A^{-1}$ directly.

(c) The calculation in part (b) did not depend on the entries of A.

60. Let C be the inverse of $(I - AB)$, *i.e.*, $C = (I - AB)^{-1}$. Then $C(I - AB) = (I - AB)C = I$. Consider the matrix $I + BCA$. Claim that this matrix is the inverse of $I - BA$. To check this claim show that $(I + BCA)(I - BA) = (I - BA)(I + BCA) = I.$

First show

$$\begin{aligned} (I - BA)(I + BCA) &= I - BA + BCA - BABCA \\ &= I - BA + B(C - ABC)A \\ &= I - BA + B\underbrace{\left((I - AB)C\right)}_{I}A \\ &= 1 - BA + BA = 1 \end{aligned}$$

Similarly, show $(I + BCA)(I - BA) = I.$

62. Let A, D, P be $n \times n$ matrices. Suppose $P^{-1}AP = D$. Then $P\left(P^{-1}AP\right)P^{-1} = PDP^{-1}$, so $A = PDP^{-1}$. It is not necessary that $A = D$. For example, let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

It is easy to check that $AP = PD = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, so you have $P^{-1}AP = D$, but $A \neq D$.

Section 2.4 Elementary Matrices

2. This matrix *is* not elementary, because it is not square.

4. This matrix *is* elementary. It can be obtained by interchanging the two rows of I_2.

6. This matrix *is* elementary. It can be obtained by multiplying the first row of I_3 by 2, and adding the result to the third row.

8. This matrix is *not* elementary, because two elementary row operations are required to obtain it from I_4.

10. C is obtained by adding the third row of A to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. A is obtained by adding -1 times the third row of C to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14. To obtain the inverse matrix, reverse the elementary row operation that produced it. So,

$$E^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}.$$

16. To obtain the inverse matrix, Reverse the elementary row operation that produced it. So, add 3 times the second row to the third row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

18. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, divide the first row of I_3 by k to obtain

$$E^{-1} = \begin{bmatrix} \dfrac{1}{k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ k \neq 0.$$

20. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, add $-k$ times the third row to the second row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

22. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\tfrac{1}{2}\right)R_1 \to R_1 \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 - R_1 \to R_2 \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

24. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \left(\tfrac{1}{2}\right)R_2 \to R_2 \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_1 + 2R_3 \to R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_2 - \left(\tfrac{1}{2}\right)R_3 \to R_2 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$\begin{aligned} A^{-1} &= E_3E_2E_1 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

26. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is itself an elementary matrix, so the factorization is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

28. Reduce the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ as follows.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$	-2 times row one to row two	$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	-1 times row two	$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	-1 times row two to row one	$E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

30. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$ as follows.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$	-2 times row one to row two	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	-1 times row one to row three	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1 times row two to row three	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-3 times row three to row one	$E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-2 times row two to row one	$E_5 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 1\\ 0 & 0 & -1 & 2\\ 1 & 0 & 0 & -2\end{bmatrix} \quad \left(\tfrac{1}{4}\right)R_1 \to R_1 \qquad E_1 = \begin{bmatrix}\frac{1}{4} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 1\\ 0 & 0 & -1 & 2\\ 0 & 0 & 0 & -\frac{5}{2}\end{bmatrix} \quad R_4 - R_1 \to R_4 \qquad E_2 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 1\\ 0 & 0 & -1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} \quad \left(-\tfrac{2}{5}\right)R_4 \to R_4 \qquad E_3 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\frac{2}{5}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \quad -R_3 \to R_3 \qquad E_4 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \quad R_1 - \left(\tfrac{1}{2}\right)R_4 \to R_1 \qquad E_5 = \begin{bmatrix}1 & 0 & 0 & -\frac{1}{2}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \quad R_2 - R_4 \to R_2 \qquad E_6 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad R_3 + 2R_4 \to R_3 \qquad E_7 = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$$

$$= \begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\frac{5}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

34. (a) False. It is impossible to obtain the zero matrix by applying any elementary row operation to the identity matrix.

(b) True. See the definition of row equivalence on page 90.

(c) True. If $A = E_1E_2 \ldots E_k$, where each E_i is an elementary matrix, then A is invertible (because every elementary matrix is) and $A^{-1} = E_k^{-1} \ldots E_2^{-1}E_1^{-1}$.

(d) True. See equivalent conditions (2) and (3) of Theorem 2.15 on page 93.

36. (a) EA and A have the same rows, except that the corresponding row in A is multiplied by c.

(b) E^2 is obtained by multiplying the corresponding row in I by c^2.

38. First factor A as a product of elementary matrices.

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$$

So, $A^{-1} = \left(E_1^{-1}E_2^{-1}E_3^{-1}\right)^{-1} = E_3E_2E_1.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a}{c} & -\frac{b}{c} & \frac{1}{c} \end{bmatrix}.$$

40. No. For example $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ is not elementary.

42. <u>Matrix</u> <u>Elementary Matrix</u>

$$\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = A$$

$$\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = U \qquad E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$E_1A = U \Rightarrow A = E_1^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = LU$$

44. <u>Matrix</u> <u>Elementary Matrix</u>

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 12 & 3 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = U \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = LU$$

46. (a)

Matrix	Elementary Matrix
$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = A$	
$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U$	$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$E_3E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

(b) $L\mathbf{y} = \mathbf{b}$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 15 \\ -1 \end{bmatrix}$

$y_1 = 4, -y_1 + y_2 = -4 \Rightarrow y_2 = 0,$
$3y_1 + 2y_2 + y_3 = 15 \Rightarrow y_3 = 3,$ and $y_4 = -1.$

(c) $U\mathbf{x} = \mathbf{y}$: $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ -1 \end{bmatrix}$

$x_4 = 1,\ x_3 = 1,\ x_2 - x_3 = 0 \Rightarrow x_2 = 1,$ and $x_1 = 2.$

So, the solution to the system $A\mathbf{x} = \mathbf{b}$ is: $x_1 = 2,\ x_2 = x_3 = x_4 = 1.$

48. (a) Suppose $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be + cf \end{bmatrix}.$

Because $0 = ad$, either $a = 0$ or $d = 0$.

If $a = 0$, then $ae = 0 = 1$, which is impossible.

If $d = 0$, then $bd = 0 = 1$, which is impossible.

(b) Consider the following LU-factorization.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} y & z \\ 0 & w \end{bmatrix} = LU$$

Then $y = a,\ xy = xa = c \Rightarrow x = \dfrac{c}{a}.$

Also, $z = b,$ and $xz + w = d \Rightarrow w = d - \left(\dfrac{c}{a}\right)b.$

The factorization is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ \dfrac{c}{a} & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d - \dfrac{cb}{a} \end{bmatrix},\ a \neq 0.$

50. $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A.$

Because $A^2 \neq A$, A is *not* idempotent.

52. $A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}.$

Because $A^2 \neq A$, A is *not* idempotent.

54. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Because $A^2 \neq A$, A is *not* idempotent.

56. $A^2 = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ ab + bc & c^2 \end{bmatrix}.$

In order for A^2 to equal A, you need $a = a^2$ and $c = c^2$. If $a = a^2$ there two cases to consider.

(i) $a = 0$: If $c = 0$, then $b = 0 \Rightarrow a = b = c = 0$ is a solution.
If $c = 1$, then b can be any number t $\Rightarrow a = 0,\ c = 1,\ b = t$ is a solution.

(ii) $a = 1$: If $c = 0$, then b can be any number t $\Rightarrow a = 1,\ c = 0,\ b = t$ is a solution.
If $c = 1$, then $b = 0 \Rightarrow a = c = 1$, $b = 0$ is a solution.

So, the possible matrices are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ t & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

58. Assume A is idempotent. Then

$$\begin{aligned} A^2 &= A \\ \left(A^2\right)^T &= A^T \\ \left(A^T A^T\right) &= A^T \end{aligned}$$

which means that A^T is idempotent.

Now assume A^T is idempotent. Then

$$\begin{aligned} A^T A^T &= A^T \\ \left(A^T A^T\right)^T &= \left(A^T\right)^T \\ AA &= A \end{aligned}$$

which means that A is idempotent.

60. If A is row equivalent to B, then

$A = E_k \cdots E_2 E_1 B$

where $E_1, \ldots, E_k$ are elementary matrices.

So,

$B = E_1^{-1}E_2^{-1}\cdots E_k^{-1}A$

which shows that B is row equivalent to A.

62. If B is row equivalent to A, then

$B = E_k \cdots E_2 E_1 A$

where $E_1, \ldots, E_k$ are elementary matrices. Because elementary matrices are nonsingular,

$B^{-1} = \left(E_k \cdots E_1 A\right)^{-1} = A^{-1}E_1^{-1} \cdots E_k^{-1}$

which shows that B is also nonsingular.

Section 2.5 Applications of Matrix Operations

2. This matrix is *not* stochastic because every entry in a stochastic matrix must satisfy $0 \leq a_{ij} \leq 1$.

4. This matrix *is* stochastic because each entry is between 0 and 1, and each column adds up to 1.

6. This matrix *is* stochastic because each entry is between 0 and 1, and each column adds up to 1.

8. Form the matrix representing the given transition probabilities. A represents infected mice and B noninfected.

$$\begin{array}{cc} & \overbrace{\begin{array}{cc} A & \;\;B \end{array}}^{\text{From}} \\ P = & \left[\begin{array}{cc} 0.2 & 0.1 \\ 0.8 & 0.9 \end{array}\right] \!\left.\begin{array}{c} A \\ B \end{array}\right\} \text{To} \end{array}$$

The state matrix representing the current population is

$$X = \begin{bmatrix} 100 \\ 900 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}.$$

The state matrix for next week is

$$PX = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 900 \end{bmatrix} = \begin{bmatrix} 110 \\ 890 \end{bmatrix}.$$

The state matrix for the week after next is

$$P(PX) = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 110 \\ 890 \end{bmatrix} = \begin{bmatrix} 111 \\ 889 \end{bmatrix}.$$

So, next week 110 will be infected, while in 2 weeks 111 will be infected.

10. Form the matrix representing the given transition probabilities. Let A represents users of brand A, B users of brand B, and N users of neither brands.

$$\begin{array}{cc} & \overbrace{\begin{array}{ccc} A & \;\;B & \;\;N \end{array}}^{\text{From}} \\ P = & \left[\begin{array}{ccc} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{array}\right] \!\left.\begin{array}{c} A \\ B \\ N \end{array}\right\} \text{To} \end{array}$$

The state matrix representing the current product usage is

$$X = \begin{bmatrix} 20{,}000 \\ 30{,}000 \\ 50{,}000 \end{bmatrix} \begin{matrix} A \\ B \\ N \end{matrix}$$

The state matrix for next month is

$$PX = \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{bmatrix} \begin{bmatrix} 20{,}000 \\ 30{,}000 \\ 50{,}000 \end{bmatrix} = \begin{bmatrix} 24{,}500 \\ 34{,}000 \\ 41{,}500 \end{bmatrix}.$$

Similarly, the state matrices for the following two months are

$$P(PX) = P\begin{bmatrix} 24{,}500 \\ 34{,}000 \\ 41{,}500 \end{bmatrix} = \begin{bmatrix} 27{,}625 \\ 36{,}625 \\ 35{,}750 \end{bmatrix} \text{ and}$$

$$P(P(PX)) = \begin{bmatrix} 29{,}788 \\ 38{,}356 \\ 31{,}856 \end{bmatrix}.$$

So, the next month's users will be grouped as follows: 24,500 for brand A, 34,000 brand B, and 41,500 neither. In two months the distribution will be 27,625 brand A, 36,625 brand B, and 35,750 neither. Finally, in three months the distribution will be 29,788 brand A, 38,356 brand B, and 31,856 neither.

12. First find

$$PX = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 800 \end{bmatrix} = \begin{bmatrix} 150 \\ 170 \\ 680 \end{bmatrix}, \text{ and}$$

$$P^2X = P(PX) = \begin{bmatrix} 175 \\ 217 \\ 608 \end{bmatrix}.$$

Continuing, you have $P^3X = \begin{bmatrix} 187.5 \\ 247.7 \\ 564.8 \end{bmatrix}$.

Finally, the steady state matrix for P is $\begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix}$ because

$$\begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix}.$$

14. Let

$$P = \begin{bmatrix} a & b \\ 1 - a & 1 - b \end{bmatrix}$$

be a 2×2 stochastic matrix, and consider the system of equations $PX = X$.

$$\begin{bmatrix} a & b \\ 1 - a & 1 - b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

you have

$$\begin{aligned} ax_1 + \qquad bx_2 &= x_1 \\ (1 - a)x_1 + (1 - b)x_2 &= x_2 \end{aligned}$$

or

$$\begin{aligned} (a - 1)x_1 + bx_2 &= 0 \\ (1 - a)x_1 - bx_2 &= 0. \end{aligned}$$

Letting $x_1 = b$ and $x_2 = 1 - a$, you have 2×1 state matrix X satisfying $PX = X$

$$X = \begin{bmatrix} b \\ 1 - a \end{bmatrix}.$$

16. Divide the message into groups of three and form the uncoded matrices.

P L E A S E _ S E N D _ M O N E Y _

$$[16\ \ 12\ \ 5]\quad[1\ \ 19\ \ 5]\quad[0\ \ 19\ \ 5]\quad[14\ \ 4\ \ 0]\quad[13\ \ 15\ \ 14]\quad[5\ \ 25\ \ 0]$$

Multiplying each uncoded row matrix on the right by A yields the following coded row matrices.

$$[16\ \ 12\ \ 5]A = [16\ \ 12\ \ 5]\begin{bmatrix}4 & 2 & 1\\ -3 & -3 & -1\\ 3 & 2 & 1\end{bmatrix} = [43\ \ 6\ \ 9]$$

$$\begin{aligned}
[1\ \ 19\ \ 5]A &= [-38\ \ -45\ \ -13]\\
[0\ \ 19\ \ 5]A &= [-42\ \ -47\ \ -14]\\
[14\ \ 4\ \ 0]A &= [44\ \ 16\ \ 10]\\
[13\ \ 15\ \ 14]A &= [49\ \ 9\ \ 12]\\
[5\ \ 25\ \ 0]A &= [-55\ \ -65\ \ -20]
\end{aligned}$$

So, the coded message is

43, 6, 9, −38, −45, −13, −42, 47 −14, 44, 16, 10, 49, 9, 12, −55, −65, −20.

18. Divide the message into groups of four and form the uncoded matrices.

H E L P _ I S _ C O M I N G _ _

$$[8\ \ 5\ \ 12\ \ 16]\quad[0\ \ 9\ \ 19\ \ 0]\quad[3\ \ 15\ \ 13\ \ 9]\quad[14\ \ 7\ \ 0\ \ 0]$$

Multiplying each uncoded row matrix on the right by A yields the coded row matrices

$$\begin{aligned}
[8\ \ 5\ \ 12\ \ 16]A &= [8\ \ 5\ \ 12\ \ 16]\begin{bmatrix}-2 & 3 & -1 & -1\\ -1 & 1 & 1 & 1\\ -1 & -1 & 1 & 2\\ 3 & 1 & -2 & -4\end{bmatrix}\\
&= [15\ \ 33\ \ -23\ \ -43]
\end{aligned}$$

$$\begin{aligned}
[0\ \ 9\ \ 19\ \ 0]A &= [-28\ \ -10\ \ 28\ \ 47]\\
[3\ \ 15\ \ 13\ \ 9]A &= [-7\ \ 20\ \ 7\ \ 2]\\
[14\ \ 7\ \ 0\ \ 0]A &= [-35\ \ 49\ \ -7\ \ -7].
\end{aligned}$$

So, the coded message is

15, 33, −23, −43, −28, −10, 28, 47, −7, 20, 7, 2, −35, 49, −7, −7.

20. Find $A^{-1} = \begin{bmatrix}-4 & 3\\ 3 & -2\end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[85\ \ 120]\begin{bmatrix}-4 & 3\\ 3 & -2\end{bmatrix} = [20\ \ 15] \Rightarrow \text{T, } O$$

$$\begin{aligned}
[6\ \ 8]A^{-1} &= [0\ \ 2] \Rightarrow \text{\_, B}\\
[10\ \ 15]A^{-1} &= [5\ \ 0] \Rightarrow \text{E, \_}\\
[84\ \ 117]A^{-1} &= [15\ \ 18] \Rightarrow \text{O, R}\\
[42\ \ 56]A^{-1} &= [0\ \ 14] \Rightarrow \text{\_, N}\\
[90\ \ 125]A^{-1} &= [15\ \ 20] \Rightarrow \text{O, T}\\
[60\ \ 80]A^{-1} &= [0\ \ 20] \Rightarrow \text{\_, T}\\
[30\ \ 45]A^{-1} &= [15\ \ 0] \Rightarrow \text{O, \_}\\
[19\ \ 26]A^{-1} &= [2\ \ 5] \Rightarrow \text{B, E}
\end{aligned}$$

So, the message is TO_BE_OR_NOT_TO_BE.

22. Find $A^{-1} = \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$\begin{bmatrix} 112 & -140 & 83 \end{bmatrix} A^{-1} = \begin{bmatrix} 112 & -140 & 83 \end{bmatrix} \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 22 \end{bmatrix} \Rightarrow \text{H, A, V}$$

$$\begin{array}{lcll}
\begin{bmatrix} 19 & -25 & 13 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 5 & 0 & 1 \end{bmatrix} & \Rightarrow \ \text{E, \_, A} \\
\begin{bmatrix} 72 & -76 & 61 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 0 & 7 & 18 \end{bmatrix} & \Rightarrow \ \text{\_, G, R} \\
\begin{bmatrix} 95 & -118 & 71 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 5 & 1 & 20 \end{bmatrix} & \Rightarrow \ \text{E, A, T} \\
\begin{bmatrix} 20 & 21 & 38 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 0 & 23 & 5 \end{bmatrix} & \Rightarrow \ \text{\_, W, E} \\
\begin{bmatrix} 35 & -23 & 36 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 5 & 11 & 5 \end{bmatrix} & \Rightarrow \ \text{E, K, E} \\
\begin{bmatrix} 42 & -48 & 32 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 14 & 4 & 0 \end{bmatrix} & \Rightarrow \ \text{N, D, \_}
\end{array}$$

The message is HAVE_A_GREAT_WEEKEND_.

24. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find that

$$\begin{bmatrix} -19 & -19 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \overset{\_\quad S}{\begin{bmatrix} 0 & 19 \end{bmatrix}}$$

$$\begin{bmatrix} 37 & 16 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \overset{U\quad E}{\begin{bmatrix} 21 & 5 \end{bmatrix}}.$$

This produces a system of 4 equations.

$$\begin{aligned}
-19a \qquad - 19c \qquad\quad &= 0 \\
- 19b \qquad - 19d &= 19 \\
37a \qquad + 16c \qquad\quad &= 21 \\
37b \qquad + 16d &= 5.
\end{aligned}$$

Solving this system, you find $a = 1$, $b = 1$, $c = -1$, and $d = -2$. So,

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

Multiply each coded row matrix on the right by A^{-1} to yield the uncoded row matrices.

$\begin{bmatrix} 3 & 1 \end{bmatrix}, \begin{bmatrix} 14 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 12 \end{bmatrix}, \begin{bmatrix} 0 & 15 \end{bmatrix}, \begin{bmatrix} 18 & 4 \end{bmatrix},$
$\begin{bmatrix} 5 & 18 \end{bmatrix}, \begin{bmatrix} 19 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 19 \end{bmatrix}, \begin{bmatrix} 21 & 5 \end{bmatrix}.$

This corresponds to the message CANCEL_ORDERS_SUE.

26. You have

$$\begin{bmatrix} 45 & -35 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 10 & 15 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 38 & -30 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 8 & 14 \end{bmatrix}.$$

So,
$$\begin{array}{lcl}
45w - 35y = 10 & \text{and} & 45x - 35z = 15 \\
38w - 30y = 8 & & 38x - 30z = 14.
\end{array}$$

Solving these two systems gives $w = y = 1$ and $x = -2$, $z = -3$. So,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}.$$

(b) Decoding you have:

$$\begin{array}{lcll}
\begin{bmatrix} 45 & -35 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 10 & 15 \end{bmatrix} & \Rightarrow \ \text{J, O} \\
\begin{bmatrix} 38 & -30 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 8 & 14 \end{bmatrix} & \Rightarrow \ \text{H, N} \\
\begin{bmatrix} 18 & -18 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 0 & 18 \end{bmatrix} & \Rightarrow \ \text{\_, R} \\
\begin{bmatrix} 35 & -30 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 5 & 20 \end{bmatrix} & \Rightarrow \ \text{E, T} \\
\begin{bmatrix} 81 & -60 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 21 & 18 \end{bmatrix} & \Rightarrow \ \text{U, R} \\
\begin{bmatrix} 42 & -28 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 14 & 0 \end{bmatrix} & \Rightarrow \ \text{N, \_} \\
\begin{bmatrix} 75 & -55 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 20 & 15 \end{bmatrix} & \Rightarrow \ \text{T, O} \\
\begin{bmatrix} 2 & -2 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 0 & 2 \end{bmatrix} & \Rightarrow \ \text{\_, B} \\
\begin{bmatrix} 22 & -21 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 1 & 19 \end{bmatrix} & \Rightarrow \ \text{A, S} \\
\begin{bmatrix} 15 & -10 \end{bmatrix} A^{-1} & = & \begin{bmatrix} 5 & 0 \end{bmatrix} & \Rightarrow \ \text{E, \_}
\end{array}$$

The message is JOHN_RETURN_TO_BASE_.

28. Use the given information to find D.

$$D = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} A & B & \\ \left[\begin{matrix} 0.3 & 0.4 \\ 0.4 & 0.2 \end{matrix}\right. & \left.\right] & \begin{matrix} A \\ B \end{matrix} \end{array}$$

The equation $X = DX + E$ may be written in the form $(I - D)X = E$; that is,

$$\begin{bmatrix} 0.7 & 0.4 \\ -0.4 & 0.8 \end{bmatrix} X = \begin{bmatrix} 50{,}000 \\ 30{,}000 \end{bmatrix}.$$

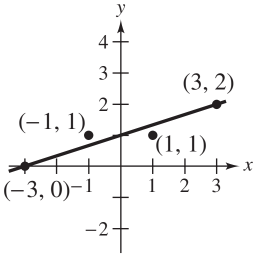
Solving the system, $X = \begin{bmatrix} 130{,}000 \\ 102{,}500 \end{bmatrix}$.

30. From the given matrix D, form the linear system $X = DX + E$, which can be written as $(I - D)X = E$

$$\begin{bmatrix} 0.8 & -0.4 & -0.4 \\ -0.4 & 0.8 & -0.2 \\ 0 & -0.2 & 0.8 \end{bmatrix} X = \begin{bmatrix} 5000 \\ 2000 \\ 8000 \end{bmatrix}.$$

Solving this system, $X = \begin{bmatrix} 21{,}875 \\ 17{,}000 \\ 14{,}250 \end{bmatrix}$.

32. (a) The line that best fits the given points is shown on the graph.



(b) Using the matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

you have $X^TX = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$ and $X^TY = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$A = (X^TX)^{-1}X^TY = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{10} \end{bmatrix}.$$

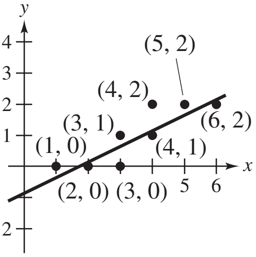
So, the least squares regression line is $y = \frac{3}{10}x + 1$.

(c) Solving $Y = XA + E$ for E, you have

$$E = Y - XA = \begin{bmatrix} -0.1 \\ 0.3 \\ -0.3 \\ 0.1 \end{bmatrix}.$$

So, the sum of the squares error is $E^TE = 0.2$.

34. (a) The line that best fits the given points is shown on the graph.



(b) Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

you have

$$X^TX = \begin{bmatrix} 8 & 28 \\ 28 & 116 \end{bmatrix} \quad X^TY = \begin{bmatrix} 8 \\ 37 \end{bmatrix}$$

$$A = (X^TX)^{-1}(X^TY) = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{1}{2}x - \frac{3}{4}$.

(c) Solving $Y = XA + E$ for E, you have

$$E = Y - XA = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}^T$$

and the sum of the squares error is $E^TE = 1.5$.

36. Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

you have

$$X^TX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$$

$$X^TY = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 39 \end{bmatrix}$$

$$A = (X^TX)^{-1}(X^TY) = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{3}{2}x - \frac{3}{2}$.

38. Using matrices

$$X = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 40 \end{bmatrix}, \; X^T Y = \begin{bmatrix} 8 \\ 32 \end{bmatrix}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{40} \end{bmatrix}\begin{bmatrix} 8 \\ 32 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.8 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.8x + 2$.

40. Using matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}, \; X^T Y = \begin{bmatrix} 7 \\ -13 \end{bmatrix}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix}\begin{bmatrix} 7 \\ -13 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{13}{20} \end{bmatrix}$$

So, the least squares regression line is $y = -0.65x + 1.75$.

42. Using matrices

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix}, \; Y = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 27 \\ 27 & 205 \end{bmatrix}$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -70 \end{bmatrix}.$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \frac{1}{296}\begin{bmatrix} 205 & -27 \\ -27 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ -70 \end{bmatrix}$$

$$= \frac{1}{296}\begin{bmatrix} 1890 \\ -350 \end{bmatrix}.$$

So, the least squares regression line is $y = -\frac{175}{148}x + \frac{945}{148}$.

44. (a) Using the matrices

$$X = \begin{bmatrix} 1 & 25 \\ 1 & 30 \\ 1 & 35 \\ 1 & 40 \end{bmatrix}, \; Y = \begin{bmatrix} 82 \\ 75 \\ 67 \\ 55 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 4 & 130 \\ 130 & 4350 \end{bmatrix}, \; X^T Y = \begin{bmatrix} 279 \\ 8845 \end{bmatrix}$$

and

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} 127.6 \\ -1.78 \end{bmatrix}.$$

So, the least squares regression line is $y = -1.78x + 127.6$

(b) When $x = 32.95$,

$$y = -1.78(32.95) + 127.6 = 68.95 \approx 69.$$

46. (a) Using the matrices

$$X = \begin{bmatrix}1 & 100\\ 1 & 120\\ 1 & 140\end{bmatrix}, \; Y = \begin{bmatrix}75\\ 68\\ 55\end{bmatrix}$$

you have

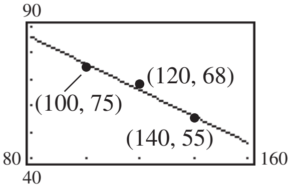
$$X^T X = \begin{bmatrix}1 & 1 & 1\\ 100 & 120 & 140\end{bmatrix}\begin{bmatrix}1 & 100\\ 1 & 120\\ 1 & 140\end{bmatrix} = \begin{bmatrix}3 & 360\\ 360 & 44{,}000\end{bmatrix}$$

$$X^T Y = \begin{bmatrix}198\\ 23{,}360\end{bmatrix}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \tfrac{1}{2400}\begin{bmatrix}44{,}000 & -360\\ -360 & 3\end{bmatrix}\begin{bmatrix}198\\ 23{,}360\end{bmatrix} = \tfrac{1}{2400}\begin{bmatrix}302{,}400\\ -1200\end{bmatrix} = \begin{bmatrix}126\\ -0.5\end{bmatrix}.$$

So, the least squares regression line is $y = -0.5x + 126$.

(b)



(c)

Number (x)	100	120	140
Percent (y)	75	68	55
Model percent (y)	76	66	56

(d) When $x = 170$, $y = -0.5(170) + 126 = 41\%$.

(e) When $y = 40\%$, you have $40 = -0.5x + 126$ and, therefore, $x = 172$.

Review Exercises for Chapter 2

2. $$-2\begin{bmatrix}1 & 2\\ 5 & -4\\ 6 & 0\end{bmatrix} + 8\begin{bmatrix}7 & 1\\ 1 & 2\\ 1 & 4\end{bmatrix} = \begin{bmatrix}-2 & -4\\ -10 & 8\\ -12 & 0\end{bmatrix} + \begin{bmatrix}56 & 8\\ 8 & 16\\ 8 & 32\end{bmatrix} = \begin{bmatrix}54 & 4\\ -2 & 24\\ -4 & 32\end{bmatrix}$$

4. $$\begin{bmatrix}1 & 5\\ 2 & -4\end{bmatrix}\begin{bmatrix}6 & -2 & 8\\ 4 & 0 & 0\end{bmatrix} = \begin{bmatrix}1(6) + 5(4) & 1(-2) + 5(0) & 1(8) + 5(0)\\ 2(6) - 4(4) & 2(-2) - 4(0) & 2(8) - 4(0)\end{bmatrix} = \begin{bmatrix}26 & -2 & 8\\ -4 & -4 & 16\end{bmatrix}$$

6. $$\begin{bmatrix}2 & 1\\ 6 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\ -3 & 1\end{bmatrix} + \begin{bmatrix}-2 & 4\\ 0 & 4\end{bmatrix} = \begin{bmatrix}5 & 5\\ 24 & 12\end{bmatrix} + \begin{bmatrix}-2 & 4\\ 0 & 4\end{bmatrix} = \begin{bmatrix}3 & 9\\ 24 & 16\end{bmatrix}$$

8. Multiplying the left side of the equation yields

$$\begin{bmatrix}2x - y\\ 3x + 4y\end{bmatrix} = \begin{bmatrix}5\\ -2\end{bmatrix}.$$

So, the corresponding system of linear equations is

$$\begin{aligned} 2x - y &= 5\\ 3x + 4y &= -2. \end{aligned}$$

10. Multiplying the left side of the equation yields

$$\begin{bmatrix}y + 2z\\ 3x + 2y + z\\ 4x - 3y + 4z\end{bmatrix} = \begin{bmatrix}0\\ -1\\ -7\end{bmatrix}.$$

So, the corresponding system of linear equations is

$$\begin{aligned} y + 2z &= 0\\ 3x + 2y + z &= -1\\ 4x - 3y + 4z &= -7. \end{aligned}$$

12. Letting

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} -8 \\ -4 \end{bmatrix},$$

the given system can be written in matrix form

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}.$$

14. Letting

$$A = \begin{bmatrix} -3 & -1 & 1 \\ 2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

the given system can be written in matrix form

$A\mathbf{x} = \mathbf{b}$.

16. $A^T = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$

18. $A^T = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

20. From the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

you see that $ad - bc = 4(2) - (-1)(-8) = 0$, and so the matrix has no inverse.

22. Begin by adjoining the identity matrix to the given matrix.

$$\begin{bmatrix} A \ \vdots \ I \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

This matrix reduces to

$$\begin{bmatrix} I \ \vdots \ A^{-1} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So, the inverse matrix is

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

24.

$$\overset{A}{\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}}\overset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{\mathbf{b}}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

Because $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$, solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

26.

$$\overset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}}\overset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{\mathbf{b}}{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}$$

Using Gauss-Jordan elimination, you find that

$$A^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}.$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

28. Because $(2A)^{-1} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, you can use the formula for the inverse of a 2×2 matrix to obtain

$$2A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2 - 0}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$$

So, $A = \frac{1}{4}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$.

30. The matrix $\begin{bmatrix} 2 & x \\ 1 & 4 \end{bmatrix}$ will be nonsingular if

$ad - bc = (2)(4) - (1)(x) \neq 0$, which implies that $x \neq 8$.

32. Because the given matrix represents 6 times the second row, the inverse will be $\frac{1}{6}$ times the second row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For Exercises 34 and 36, answers will vary. Sample answers are shown below.

34. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & -4 \\ -3 & 13 \end{bmatrix}$	Interchange 2 rows.	$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$	Add 3 times row 1 to row 2.	$E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Add 4 times row 2 to row 1.	$E_3 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Then, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

36. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$	$\frac{1}{3}$ times row one.	$E_1 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -1 times row one to row three.	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -2 times row three to row one.	$E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}$ times row two.	$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38. Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, you have

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, many answers are possible.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, etc.

40. There are many possible answers.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

But, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O.$

42. If $aX + bY + cZ = O$, then

$$a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \end{bmatrix} + c\begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which yields the system of equations

$$\begin{aligned} a - b + 3c &= 0 \\ 2a + 4c &= 0 \\ 3b - c &= 0 \\ a + 2b + 2c &= 0. \end{aligned}$$

Solving this homogeneous system, the only solution is $a = b = c = 0$.

44. No, this is not true. For example, let

$$A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then, $AC = \begin{bmatrix} 3 & -4 \\ 1 & -8 \end{bmatrix} = CB$, but $A \neq B$.

46.

Matrix	Elementary Matrix
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = A$	
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$E_3E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

48. Matrix — Elementary Matrix

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U \qquad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$EA = U \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$L\mathbf{y} = \mathbf{b}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 8 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y}: \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

50. (a) False. The product of a 2×3 matrix and a 3×5 matrix is a 2×5 matrix.

(b) True. See Theorem 2.6(4) on page 68.

52. (a) True. $\left(ABA^{-1}\right)^2 = \left(ABA^{-1}\right)\left(ABA^{-1}\right) = AB\left(A^{-1}A\right)BA^{-1} = ABIBA^{-1} = AB^2A^{-1}$.

(b) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A + B$ is a *singular* matrix, while both A and B are *nonsingular* matrices.

54. (a) $AB = \begin{bmatrix} 40 & 64 & 52 \\ 60 & 82 & 76 \\ 76 & 96 & 84 \end{bmatrix}\begin{bmatrix} 3.32 & 1.32 \\ 3.22 & 1.07 \\ 3.12 & 0.92 \end{bmatrix} = \begin{bmatrix} 501.12 & 169.12 \\ 700.36 & 236.86 \\ 823.52 & 280.32 \end{bmatrix}$

This matrix shows the total sales of milk each day in the first column and total profit each day in the second column.

(b) The profit for Friday through Sunday is the sum of the elements in the second column of AB, $169.12 + 236.86 + 280.32 = \$686.30$.

56. (a) In matrix B, grading system 1 counts each midterm as 25% of the grade and the final exam as 50% of the grade. Grading system 2 counts each midterm as 20% of the grade and the final exam as 60% of the grade.

(b) $$AB = \begin{bmatrix} 78 & 82 & 80 \\ 84 & 88 & 85 \\ 92 & 93 & 90 \\ 88 & 86 & 90 \\ 74 & 78 & 80 \\ 96 & 95 & 98 \end{bmatrix} \begin{bmatrix} 0.25 & 0.20 \\ 0.25 & 0.20 \\ 0.50 & 0.60 \end{bmatrix} = \begin{bmatrix} 80 & 80 \\ 85.5 & 85.4 \\ 91.25 & 91 \\ 88.5 & 88.8 \\ 78 & 78.4 \\ 96.75 & 97 \end{bmatrix}$$

(c) $$\begin{bmatrix} \text{B} & \text{B} \\ \text{B} & \text{B} \\ \text{A} & \text{A} \\ \text{B} & \text{B} \\ \text{C} & \text{C} \\ \text{A} & \text{A} \end{bmatrix}$$

58. This matrix *is* stochastic because $0 \le a_{ij} \le 1$ and each column adds up to 1.

60. $$PX = \begin{bmatrix} 0.6 & 0.2 & 0.0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 800 \\ 1000 \\ 1200 \end{bmatrix};$$

$$P^2X = P\begin{bmatrix} 800 \\ 1000 \\ 1200 \end{bmatrix} = \begin{bmatrix} 680 \\ 980 \\ 1340 \end{bmatrix};$$

$$P^3X = P\begin{bmatrix} 680 \\ 980 \\ 1340 \end{bmatrix} = \begin{bmatrix} 604 \\ 956 \\ 1440 \end{bmatrix}.$$

62. If you continue the computation in Exercise 61, you find that the steady state is

$$X = \begin{bmatrix} 140{,}000 \\ 100{,}000 \\ 60{,}000 \end{bmatrix}$$

which can be verified by calculating

$PX = X.$

64. The uncoded row matrices are

B E A M _ M E _ U P _ S C O T T Y _

$[2 \quad 5 \quad 1]$ $[13 \quad 0 \quad 13]$ $[5 \quad 0 \quad 21]$ $[16 \quad 0 \quad 19]$ $[3 \quad 15 \quad 20]$ $[20 \quad 25 \quad 0]$

Multiplying each 1×3 matrix on the right by A yields the coded row matrices.

$[17 \quad 6 \quad 20]$ $[0 \quad 0 \quad 13]$ $[-32 \quad -16 \quad -43]$ $[-6 \quad -3 \quad 7]$ $[11 \quad -2 \quad -3]$ $[115 \quad 45 \quad 155]$

So, the coded message is

17, 6, 20, 0, 0, 13, −32, −16, −43, −6, −3, 7, 11, −2, −3, 115, 45, 155.

66. Find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

and the coded row matrices are

$[11\ \ 52], [-8\ \ -9], [-13\ \ -39], [5\ \ 20], [12\ \ 56], [5\ \ 20], [-2\ \ 7], [9\ \ 41], [25\ \ 100].$

Multiplying each coded row matrix on the right by A^{-1} yields the uncoded row matrices.

$$\begin{array}{cccccccccc} \text{S H} & \text{O W} & \text{\_ M} & \text{E \_} & \text{T H} & \text{E \_} & \text{M O} & \text{N E} & \text{Y \_} \\ [19\ \ 8] & [15\ \ 23] & [0\ \ 13] & [5\ \ 0] & [20\ \ 8] & [5\ \ 0] & [13\ \ 15] & [14\ \ 5] & [25\ \ 0] \end{array}$$

So, the message is SHOW_ME_THE_MONEY_.

68. Find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

and the coded row matrices are

$[23\ \ 20\ \ 132], [54\ \ 128\ \ 102], [32\ \ 21\ \ 203], [6\ \ 10\ \ 23], [21\ \ 15\ \ 129], [36\ \ 46\ \ 173], [29\ \ 72\ \ 45].$

Multiplying each coded row matrix on the right by A^{-1} yields the uncoded row matrices.

$$\begin{array}{ccccccc} \text{\_ D O} & \text{N T \_} & \text{H A V} & \text{E \_ A} & \text{\_ C O} & \text{W \_ M} & \text{A N \_} \\ [0\ \ 4\ \ 15] & [14\ \ 20\ \ 0] & [8\ \ 1\ \ 22] & [5\ \ 0\ \ 1] & [0\ \ 3\ \ 15] & [23\ \ 0\ \ 15] & [1\ \ 14\ \ 0] \end{array}$$

So, the message is _DONT_HAVE_A_COW_MAN_.

70. Find $A^{-1} = \begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix},$

and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[66\ \ 27\ \ -31]A^{-1} = [66\ \ 27\ \ -31]\begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix} = [25\ \ 1\ \ 14] \Rightarrow \text{Y, A, N}$$

$$\begin{aligned} [37\ \ 5\ \ -9]A^{-1} &= [11\ \ 5\ \ 5] \Rightarrow \text{K, E, E} \\ [61\ \ 46\ \ -73]A^{-1} &= [19\ \ 0\ \ 23] \Rightarrow \text{S, \_, W} \\ [46\ \ -14\ \ 9]A^{-1} &= [9\ \ 14\ \ 0] \Rightarrow \text{I, N, \_} \\ [94\ \ 21\ \ -49]A^{-1} &= [23\ \ 15\ \ 18] \Rightarrow \text{W, O, R} \\ [32\ \ -4\ \ 12]A^{-1} &= [12\ \ 4\ \ 0] \Rightarrow \text{L, D, \_} \\ [66\ \ 31\ \ -53]A^{-1} &= [19\ \ 5\ \ 18] \Rightarrow \text{S, E, R} \\ [47\ \ 33\ \ -67]A^{-1} &= [9\ \ 5\ \ 19] \Rightarrow \text{I, E, S} \end{aligned}$$

The message is YANKEES_WIN_WORLD_SERIES.

72. Solve the equation $X = DX + E$ for X to obtain $(I - D)X = E$, which corresponds to solving the augmented matrix.

$$\left[\begin{array}{rrr:r} 0.9 & -0.3 & -0.2 & 3000 \\ 0 & 0.8 & -0.3 & 3500 \\ -0.4 & -0.1 & 0.9 & 8500 \end{array}\right]$$

The solution to this system is

$$X = \begin{bmatrix} 10000 \\ 10000 \\ 15000 \end{bmatrix}.$$

74. Using the matrices

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 5 & 20 \\ 20 & 90 \end{bmatrix}, \ X^T Y = \begin{bmatrix} 14 \\ 63 \end{bmatrix}$$

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} 1.8 & -0.4 \\ -0.4 & 0.1 \end{bmatrix}\begin{bmatrix} 14 \\ 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.7x$.

76. Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}, \ X^T Y = \begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} 2 & -1.5 \\ -1.5 & 1.25 \end{bmatrix}\begin{bmatrix} 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}.$$

So, the least squares regression line is $y = 2.5x$.

78. (a) Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 177.1 \\ 179.9 \\ 184.0 \\ 188.9 \\ 195.3 \end{bmatrix}$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

and

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} 177.1 \\ 179.9 \\ 184.0 \\ 188.9 \\ 195.3 \end{bmatrix} = \begin{bmatrix} 925.2 \\ 2821 \end{bmatrix}.$$

Now, using $\left(X^T X\right)^{-1}$ to find the coefficient matrix A, you have

$$\begin{aligned} A = \left(X^T X\right)^{-1} X^T Y &= \tfrac{1}{50}\begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix}\begin{bmatrix} 925.2 \\ 2821 \end{bmatrix} \\ &= \tfrac{1}{50}\begin{bmatrix} 8571 \\ 227 \end{bmatrix} \\ &= \begin{bmatrix} 171.42 \\ 4.54 \end{bmatrix}. \end{aligned}$$

So, the least squares regression line is $y = 4.54x + 171.42$.

(b) The CPI in 2010 is $y = 4.54(10) + 171.42 = 216.82$

The CPI in 2015 is $y = 454(15) + 171.42 = 239.52$.

80. (a) Using the matrices $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} 109.5 \\ 128.3 \\ 140.8 \\ 158.7 \\ 182.1 \\ 207.9 \end{bmatrix}$ you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ 15 & 55 \end{bmatrix} \text{ and } X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 109.5 \\ 128.3 \\ 140.8 \\ 158.7 \\ 182.1 \\ 207.9 \end{bmatrix} = \begin{bmatrix} 927.3 \\ 2653.9 \end{bmatrix}.$$

Now, using $(X^T X)^{-1}$ to find the coefficient matrix A, you have

$$A = (X^T X)^{-1} X^T Y = \tfrac{1}{105}\begin{bmatrix} 55 & -15 \\ -15 & 6 \end{bmatrix} \begin{bmatrix} 927.3 \\ 2653.9 \end{bmatrix} = \tfrac{1}{105}\begin{bmatrix} 11{,}193 \\ 2013.9 \end{bmatrix} = \begin{bmatrix} 106.6 \\ 19.18 \end{bmatrix}.$$

So, the least squares regression line is $y = 19.18x + 106.6$.

(b) Using a graphing utility with $L_1 = \{0, 1, 2, 3, 4, 5\}$ and $L_2 = \{109.5, 128.3, 140.8, 158.7, 182.1, 207.9\}$ gives the same least squares regression line: $y = 19.18x + 106.6$.

(c)

Year	2000	2001	2002	2003	2004	2005
Actual	109.5	128.3	140.8	158.7	182.1	207.9
Estimated	106.6	125.8	145.0	164.1	183.3	202.5

The estimated values are close to the actual values.

(d) The number of subscribers in 2010 is $y = 19.18(10) + 106.6 = 298.4$ million.

(e)
$$\begin{aligned} 260 &= 19.18x + 106.6 \\ 153.4 &= 19.18x \\ 8 &\approx x \end{aligned}$$

The number of subscribers will be 260 million in 2008.

Project Solutions for Chapter 2

1 Exploring Matrix Multiplication

1. Test 1 seems to be the more difficult. The averages were:

Test 1 average $= 75$

Test 2 average $= 85.5$

2. Anna, David, Chris, Bruce

3. $M\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents scores on first test. $M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents scores on second test.

4. $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}M$ represents Anna's scores.

$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}M$ represents Chris's scores.

5. $M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents the sum of the test scores for each student, and $\frac{1}{2}M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents each students' average.

6. $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}M$ represents the sum of scores on each test; $\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}M$ represents the average on each test.

7. $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ represents the overall points total for all students on all tests.

8. $\frac{1}{8}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} M \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = 80.25$

9. $M\begin{bmatrix}1.1\\ \vdots\\ 1.0\end{bmatrix}$

2 Nilpotent Matrices

1. $A^2 \neq 0$ and $A^3 = 0$, the index is 3.

2. (a) Nilpotent of index 2
(b) Not nilpotent
(c) Nilpotent of index 2
(d) Not nilpotent
(e) Nilpotent of index 2
(f) Nilpotent of index 3

3. $\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ index 2; $\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ index 3

4. $\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ index 2; $\begin{bmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ index 3;

$\begin{bmatrix}0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ index 4

5. $\begin{bmatrix}0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

6. No. If A is nilpotent and invertible, then $A^k = O$ for some k and $A^{k-1} \neq O$. So,

$A^{-1}A = I \Rightarrow O = A^{-1}A^k = \left(A^{-1}A\right)A^{k-1} = IA^{k-1} \neq O$

which is impossible.

7. If A is nilpotent then $\left(A^k\right)^T = \left(A^T\right)^k = O$, But $\left(A^T\right)^{k-1} = \left(A^{k-1}\right)^T \neq O$ which shows that A^T is nilpotent with the same index.

8. Let A be nilpotent of index k. Then,

$\left(I - A\right)\left(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I\right) = I - A^k = I$

which shows that

$\left(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I\right)$

is the inverse of $I - A$.