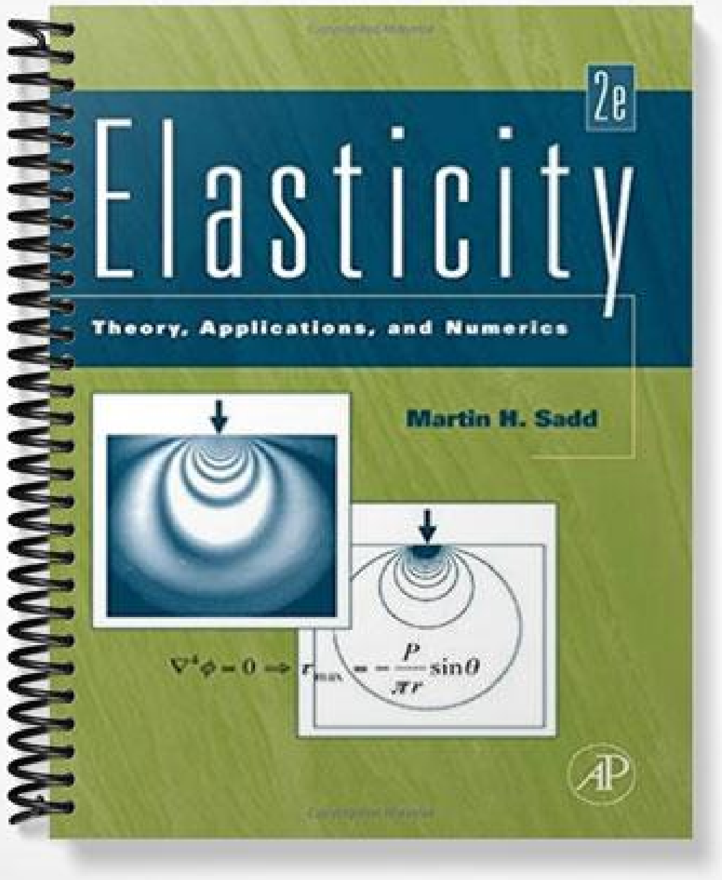
SOLUTIONS MANUAL



Solutions Manual

Elasticity: Theory, Applications and Numerics
Second Edition

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Foreword

Exercises found at the end of each chapter are an important ingredient of the text as they provide homework for student engagement, problems for examinations, and can be used in class to illustrate other features of the subject matter. This solutions manual is intended to aid the instructors in their own particular use of the exercises. Review of the solutions should help determine which problems would best serve the goals of homework, exams or be used in class.

The author is committed to continual improvement of engineering education and welcomes feedback from users of the text and solutions manual. Please feel free to send comments concerning suggested improvements or corrections to sadd@egr.uri.edu. Such feedback will be shared with the text user community via the publisher's web site.

Martin H. Sadd
January 2009

1-1.

(a) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 4 + 1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 0 + 2 + 0 + 0 + 0 + 0 + 0 + 4 = 7$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 0 + 4 = 5$ (scalar)

(b) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 2 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 4 + 0 + 0 + 4 + 1 + 0 + 16 + 4 = 30$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 4 + 4 + 0 + 0 + 2 + 1 + 0 + 4 + 2 = 17$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 1 + 1 = 6$ (scalar)

(c) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 0 + 4 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 1 + 0 + 4 + 0 + 1 + 16 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \text{(matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{(vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0 = 3$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 1 + 0 = 2$ (scalar)

1-2.

(a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

(b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

(c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

1-3.

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$$

From Exercise 1-2(a): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2(b): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2(c): $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

1-4.

$$\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ . & . & . \\ . & . & . \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

1-5.

$$\det(a_{ij}) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{312}a_{13}a_{21}a_{32}$$
$$+ \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{213}a_{12}a_{21}a_{33}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$
$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

1-6.

45° rotation about x_1 - axis $\Rightarrow$ $Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

From Exercise 1-1(a): $b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

From Exercise 1-1(b): $b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From Exercise 1-1(c): $b_i' = Q_{ij}b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

1-7.

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90^o - \theta) \\ \cos(90^o + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12} + a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11} - a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11} - a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12} + a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

1-8.

$$a'\delta_{ij}' = Q_{ip} Q_{jq} a \delta_{pq} = a Q_{ip} Q_{jp} = a\delta_{ij}$$

1-9.

$$\alpha'\delta_{ij}'\delta_{kl}' + \beta'\delta_{ik}'\delta_{jl}' + \gamma'\delta_{il}'\delta_{jk}' = Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np})$$

$$= \alpha Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{\ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

1-10.

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= \alpha\delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) = C_{klij}$$

1-11.

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

1-12.

(a) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1,\ II_a = -2,\ III_a = 0$

$\therefore$ Characteristic Eqn is $-\lambda^3 - \lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$

Roots $\Rightarrow \lambda_1 = -2\ ,\ \lambda_2 = 0,\ \lambda_3 = 1$

$\lambda_1 = -2$ Case :

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1 \end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2 = 0$ Case :

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1 \end{matrix} \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow \boldsymbol{n}^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 1$ Case :

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1 \end{matrix} \Rightarrow n_1 = n_2 = 0,\ n_3^{(3)} = 1 \Rightarrow \boldsymbol{n}^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1-12.

(b) $a_{ij}=\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\Rightarrow I_a=-4,\ II_a=3,\ III_a=0$

$\therefore$ Characteristic Eqn is $-\lambda^3-4\lambda^2-3\lambda=0\Rightarrow\lambda(\lambda^2+4\lambda+3)=0\Rightarrow\lambda(\lambda+3)(\lambda+1)=0$

Roots $\Rightarrow\lambda_1=-3$, $\lambda_2=-1$, $\lambda_3=0$

$\lambda_1=-3$ Case :

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}n_1^{(1)}\\ n_2^{(1)}\\ n_3^{(1)}\end{bmatrix}=0\Rightarrow\begin{matrix}n_1^{(1)}+n_2^{(1)}=0\\ n_3^{(1)}=0\\ n_1^{(1)2}+n_2^{(1)2}+n_3^{(1)2}=1\end{matrix}\Rightarrow n_1^{(1)}=-n_2^{(1)}=\pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)}=\pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2=-1$ Case :

$$\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0\Rightarrow\begin{matrix}-n_1^{(2)}+n_2^{(2)}=0\\ n_3^{(2)}=0\\ n_1^{(2)2}+n_2^{(2)2}+n_3^{(2)2}=1\end{matrix}\Rightarrow n_1=n_2=\pm\sqrt{2}/2\Rightarrow\boldsymbol{n}^{(2)}=\pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3=0$ Case :

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=0\Rightarrow\begin{matrix}-2n_1^{(3)}+n_2^{(3)}=0\\ n_1^{(3)}-2n_2^{(3)}=0\\ n_1^{(3)2}+n_2^{(3)2}+n_3^{(3)2}=1\end{matrix}\Rightarrow n_1=n_2=0,\ n_3^{(3)}=1\Rightarrow\boldsymbol{n}^{(3)}=\pm(0,0,1)$$

The rotation matrix is given by $Q_{ij}=\sqrt{2}/2\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}$ and

$$a'_{ij}=Q_{ip}Q_{jp}a_{pq}=\frac{1}{2}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}^T=\begin{bmatrix}-3 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

1-12.

(c) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2,\ II_a = 0,\ III_a = 0$

$\therefore$ Characteristic Eqn is $-\lambda^3 - 2\lambda^2 = 0$ or $\lambda^2(\lambda + 2) = 0$

Roots $\Rightarrow \lambda_1 = -2\ ,\ \lambda_2 = \lambda_3 = 0$

$\lambda_1 = -2$ Case:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ \boldsymbol{n}^{(1)} = \pm\sqrt{2}/2(-1,1,0)$$

$\lambda_2 = \lambda_3 = 0$ Case:

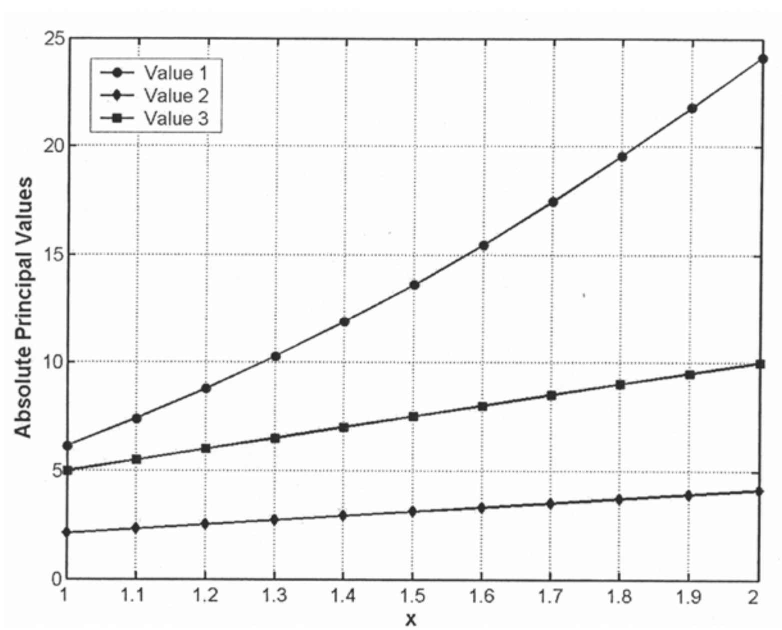
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1 + n_2 = 0 \\ n_1^{\,2} + n_2^{\,2} + n_3^{\,2} = 1 \end{matrix} \Rightarrow n_1 = n_2\ ,\ n_3^{\,2} = 1 - 2n_1^{\,2} \Rightarrow \boldsymbol{n} = \pm(k, k, \sqrt{1\text{-}2k^2})$$

for arbitrary k, and thus directions are not uniquely determined. For convenience we may choose $k = \sqrt{2}/2$ and 0 to get $\boldsymbol{n}^{(2)} = \pm\sqrt{2}/2(1,1,0)$ and $\boldsymbol{n}^{(3)} = \pm(0,0,1)$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1-13*.



1-14.

(a) $\boldsymbol{u} = x_1\boldsymbol{e_1} + x_1x_2\boldsymbol{e_2} + 2x_1x_2x_3\boldsymbol{e_3}$

$$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1x_2$$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1x_3\boldsymbol{e_1} - 2x_2x_3\boldsymbol{e_2} + x_2\boldsymbol{e_3}$$

$$\nabla^2\boldsymbol{u} = 0\boldsymbol{e_1} + 0\boldsymbol{e_2} + 0\boldsymbol{e_3} = 0$$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 1 + x_1 + 2x_1x_2$$

(b) $\boldsymbol{u} = x_1^2\boldsymbol{e_1} + 2x_1x_2\boldsymbol{e_2} + x_3^3\boldsymbol{e_3}$

$$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0\boldsymbol{e_1} - 0\boldsymbol{e_2} + 2x_2\boldsymbol{e_3}$$

$$\nabla^2\boldsymbol{u} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 6x_3\boldsymbol{e_3} = 0$$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 4x_1 + 3x_3^2$$

(c) $\boldsymbol{u} = x_2^2\boldsymbol{e_1} + 2x_2x_3\boldsymbol{e_2} + 4x_1^2\boldsymbol{e_3}$

$$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2\boldsymbol{e_1} - 8x_1\boldsymbol{e_2} - 2x_2\boldsymbol{e_3}$$

$$\nabla^2\boldsymbol{u} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 8\boldsymbol{e_3} = 0$$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 3x_3$$

1-15.

$$a_i = -\frac{1}{2}\varepsilon_{ijk}a_{jk}$$

$$\varepsilon_{imn}a_i = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn}a_{jk} = -\frac{1}{2}\begin{vmatrix}\delta_{ii} & \delta_{im} & \delta_{in}\\ \delta_{ji} & \delta_{jm} & \delta_{jn}\\ \delta_{ki} & \delta_{km} & \delta_{kn}\end{vmatrix}a_{jk} = -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})a_{jk}$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}$$

$$\therefore a_{jk} = -\varepsilon_{ijk}a_i$$

1-16.

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi = \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi\boldsymbol{u}) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot\boldsymbol{u} + \phi(\nabla\cdot\boldsymbol{u})$$

(b)

$$\nabla\times(\phi\boldsymbol{u}) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_k = \nabla\phi\times\boldsymbol{u} + \phi(\nabla\times\boldsymbol{u})$$

$$\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v}) = (\varepsilon_{ijk}u_jv_k)_{,i} = \varepsilon_{ijk}(u_jv_{k,i} + u_{j,i}v_k) = v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i} = \boldsymbol{v}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{v})$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0 \text{ because of symmetry and antisymmetry in } jk$$

$$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c)

$$\nabla\cdot(\nabla\times\boldsymbol{u}) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0, \text{ because of symmetry and antisymmetry in } ij$$

$$\nabla\times(\nabla\times\boldsymbol{u}) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$$

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_ju_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_ju_{n,m} = u_nu_{n,i} - u_mu_{i,m}$$

$$= \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

1-17.

$$\text{Cylindrical coordinates}:\ \xi^1 = r\,,\ \xi^2 = \theta\,,\ \xi^3 = z$$

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r\,,\ h_3 = 1$$

$$\hat{\boldsymbol{e}}_r = \cos\theta \boldsymbol{e_1} + \sin\theta \boldsymbol{e_2}\,,\ \hat{\boldsymbol{e}}_\theta = -\sin\theta \boldsymbol{e_1} + \cos\theta \boldsymbol{e_2}\,,\ \hat{\boldsymbol{e}}_z = \boldsymbol{e_3}$$

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial\theta} = \hat{\boldsymbol{e}}_\theta\ ,\ \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\hat{\boldsymbol{e}}_r\ ,\ \frac{\partial \hat{\boldsymbol{e}}_r}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial\theta} = \frac{\partial \hat{\boldsymbol{e}}_z}{\partial z} = 0$$

$$\nabla = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{\boldsymbol{e}}_r \frac{\partial f}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{e}}_z \frac{\partial f}{\partial z}$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla\times\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z$$

1-18.

Spherical coordinates : $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3\ ,\ x^2 = \xi^1 \sin\xi^2 \sin\xi^3\ ,\ x^3 = \xi^1 \cos\xi^2$

Scale factors :

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors :

$$\hat{\boldsymbol{e}}_{\boldsymbol{R}} = \cos\theta\sin\phi\boldsymbol{e}_{\boldsymbol{1}} + \sin\theta\sin\phi\boldsymbol{e}_{\boldsymbol{2}} + \cos\phi\boldsymbol{e}_{\boldsymbol{3}}$$

$$\hat{\boldsymbol{e}}_{\phi} = \cos\theta\cos\phi\boldsymbol{e}_{\boldsymbol{1}} + \sin\theta\cos\phi\boldsymbol{e}_{\boldsymbol{2}} - \sin\phi\boldsymbol{e}_{\boldsymbol{3}}$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{e}_{\boldsymbol{1}} + \cos\theta\boldsymbol{e}_{\boldsymbol{2}}$$

$$\frac{\partial\hat{\boldsymbol{e}}_{\boldsymbol{R}}}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_{\boldsymbol{R}}}{\partial\phi} = \hat{\boldsymbol{e}}_{\phi}\ ,\ \frac{\partial\hat{\boldsymbol{e}}_{\boldsymbol{R}}}{\partial\theta} = \sin\phi\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\boldsymbol{e}}_{\phi}}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_{\phi}}{\partial\phi} = -\hat{\boldsymbol{e}}_{r}\ ,\ \frac{\partial\hat{\boldsymbol{e}}_{\phi}}{\partial\theta} = \cos\phi\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\boldsymbol{e}}_{\theta}}{\partial R} = 0,\ \frac{\partial\hat{\boldsymbol{e}}_{\theta}}{\partial\phi} = 0\ ,\ \frac{\partial\hat{\boldsymbol{e}}_{\theta}}{\partial\theta} = -\cos\phi\hat{\boldsymbol{e}}_{\phi}$$

Using (1.9.12) - (1.9.16) $\Rightarrow$

$$\nabla = \hat{\boldsymbol{e}}_R\frac{\partial}{\partial R} + \hat{\boldsymbol{e}}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{e}}_\theta\frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}$$

$$\nabla f = \hat{\boldsymbol{e}}_R\frac{\partial f}{\partial R} + \hat{\boldsymbol{e}}_\phi\frac{1}{R}\frac{\partial f}{\partial\phi} + \hat{\boldsymbol{e}}_z\frac{1}{R\sin\phi}\frac{\partial f}{\partial\theta}$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(R\sin\phi u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}(Ru_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial f}{\partial\phi}) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta})$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial f}{\partial\phi}) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

1-18. Continued

$$\nabla\times\boldsymbol{u}=\left(\frac{1}{R^2\sin\phi}[\frac{\partial}{\partial\phi}(R\sin\phi u_\theta)-\frac{\partial}{\partial\theta}(Ru_\phi)]\right)\hat{\boldsymbol{e}}_R+\left(\frac{1}{R\sin\phi}[\frac{\partial}{\partial\theta}(u_R)-\frac{\partial}{\partial R}(R\sin\phi u_\theta)]\right)\hat{\boldsymbol{e}}_\phi$$

$$+\left(\frac{1}{R}\frac{\partial}{\partial R}[(Ru_\phi)-\frac{\partial}{\partial\phi}(u_R)\right)\hat{\boldsymbol{e}}_\theta$$

$$=\left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi u_\theta)-\frac{\partial u_\phi}{\partial\theta}\right)\right]\hat{\boldsymbol{e}}_{\boldsymbol{R}}+\left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\theta}-\frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta)\right]\hat{\boldsymbol{e}}_{\boldsymbol{\varphi}}$$

$$+\left[\frac{1}{R}\left(\frac{\partial}{\partial R}(Ru_\phi)-\frac{\partial u_R}{\partial\phi}\right)\right]\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$$

2-1.

(a) $u = Axy$, $v = Bxz^2$, $w = C(x^2 + y^2)$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} Ay & \frac{1}{2}(Ax + Bz^2) & Cx \\ \cdot & 0 & Bxz + Cy \\ \cdot & \cdot & 0 \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Ax - Bz^2) & -Cx \\ -\frac{1}{2}(Ax - Bz^2) & 0 & Bxz - Cy \\ Cx & -Bxz + Cy & 0 \end{bmatrix}$$

(b) $u = Ax^2$, $v = Bxy$, $w = Cxyz$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} 2Ax & \frac{1}{2}By & \frac{1}{2}Cyz \\ \cdot & Bx & \frac{1}{2}Cxz \\ \cdot & \cdot & Cxy \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & -\frac{1}{2}By & -\frac{1}{2}Cyz \\ \frac{1}{2}By & 0 & -\frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & 0 \end{bmatrix}$$

(c) $u = Ayz^3$, $v = Bxy^2$, $w = C(x^2 + z^2)$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Az^3 + By^2) & \frac{1}{2}(3Ayz^2 + 2Cx) \\ \cdot & 2Bxy & 0 \\ \cdot & \cdot & 2Cz \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \begin{bmatrix} 0 & \frac{1}{2}(Az^3 - By^2) & \frac{1}{2}(3Ayz^2 - 2Cx) \\ -\frac{1}{2}(Az^3 - By^2)) & 0 & 0 \\ -\frac{1}{2}(3Ayz^2 - 2Cx) & 0 & 0 \end{bmatrix}$$