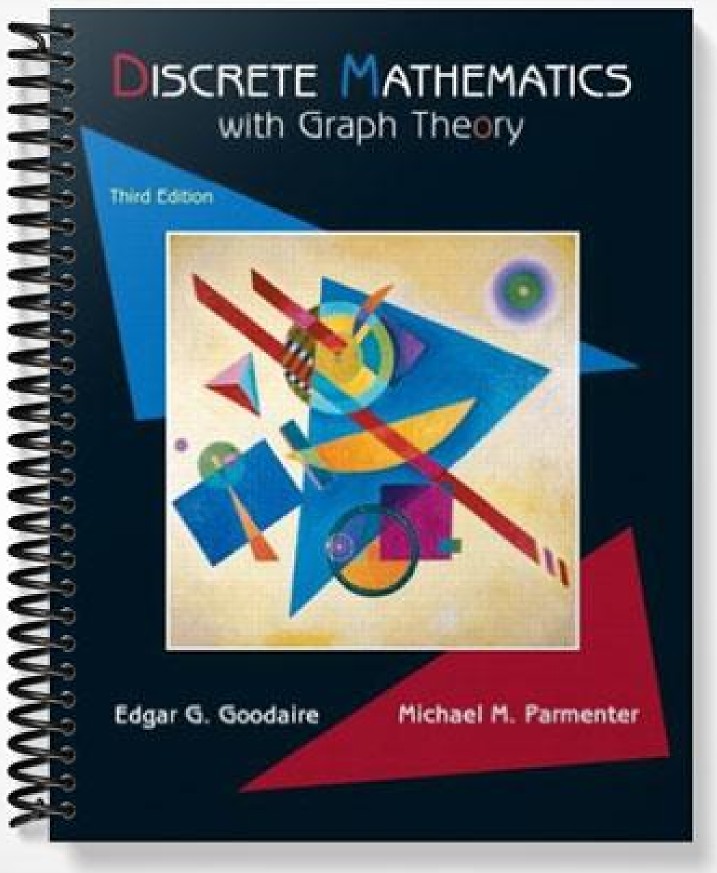
SOLUTIONS MANUAL



Instructor's Solutions Manual to accompany

Discrete Mathematics, 3e

By

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This manual contains complete solutions to all exercises in *Discrete Mathematics with Graph Theory*, Third Edition, by Edgar G. Goodaire and Michael M. Parmenter. It is intended solely for the use of instructors whom, we trust, will not make it available to students.

The solutions here have been read many times. We think that the number of errors is small, but are always grateful for help in improving accuracy. Our mailing address is below and our email addresses are `edgar@math.mun.ca` and `michael1@math.mun.ca`. Please feel free to contact either of us at any time about any aspect of our book or this solutions manual, both of which have been improved substantially by comments received since the first edition of this book appeared in 1998.

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Solutions to Exercises

Exercises 0.1

1. (a) [BB] True (b) [BB] Not valid (c) [BB] False (0^2 is not positive.) (d) Not valid (e) True (f) Not valid (g) False

2. (a) [BB] True, because both $4 = 2 + 2$ and $7 < \sqrt{50}$ are true statements.
   (b) False, because one of the two statements is false.
   (c) [BB] False, because 5 is not even.
   (d) True, because $16^{-1/4} = \frac{1}{2}$.
   (e) [BB] True, since $9 = 3^2$ is true (or because $3.14 < \pi$).
   (f) True, because $(-4)^2 = 16$ is true.
   (g) [BB] True, because both hypothesis and conclusion are true.
   (h) False, because the hypothesis is true but the conclusion is false.
   (i) [BB] Not a valid mathematical statement.
   (j) True, because both statements are true.
   (k) True, because this is an implication with false hypothesis.
   (l) False, because one of the statements is false while the other is true.
   (m) False, because the hypothesis is true but the conclusion is false.
   (n) [BB] False, because the area of a circle of radius r is not $2\pi r$ and its circumference is not πr^2.
   (o) False, because the hypothesis of this implication is true, but the conclusion is false.
   (p) [BB] This is true: The hypothesis is true only when $a \geq b$ and $b \geq a$, that is, when $a = b$, and then the conclusion is also true.
   (q) This implication is true because the hypothesis is always false.

3. (a) [BB] If $x > 0$, then $\dfrac{1}{x} > 0$.
   (b) If a and b are rational numbers, then ab is a rational number.
   (c) If f is a differentiable function, then f is continuous.
   (d) [BB] If $\mathcal{G}$ is a graph, then the sum of the degrees of the vertices of $\mathcal{G}$ is even.
   (e) [BB] If A is a matrix and $A \neq 0$, then A is invertible.
   (f) If P is a parallelogram, then the diagonals of P bisect each other.
   (g) If n is an even integer, then $n < 0$.
   (h) If two vectors are orthogonal, then their dot product is 0.
   (i) If n is an integer, then $\frac{n}{n+1}$ is not an integer.
   (j) If n is a natural number, then $n + 3 > 2$.

4. (a) [BB] True (the hypothesis is false).
   (b) True (hypothesis and conclusion are both true).

(c) [BB] True (the hypothesis is false).

(d) False (hypothesis is true, conclusion is false).

(e) [BB] False (hypothesis is true, conclusion is false: $\sqrt{4} = 2$).

(f) True (g) [BB] True (h) True (i) [BB] True (the hypothesis is false: $\sqrt{x^2} = |x|$)
(j) True (k) [BB] False (l) True

5. (a) [BB] $a^2 \leq 0$ and a is a real number (more simply, $a = 0$).
   (b) x is not real or $x^2 + 1 \neq 0$ (more simply, x is any number, complex or real).
   (c) [BB] $x \neq 1$ and $x \neq -1$.
   (d) There exists an integer which is not divisible by a prime.
   (e) [BB] There exists a real number x such that $n \leq x$ for every integer n.
   (f) $(ab)c = a(bc)$ for all a, b, c.
   (g) [BB] Every planar graph can be colored with at most four colors.
   (h) Some Canadian is a fan of neither the Toronto Maple Leafs nor the Montreal Canadiens.
   (i) There exists $x > 0$ and some y such that $x^2 + y^2 \leq 0$.
   (j) $x \geq 2$ or $x \leq -2$.
   (k) [BB] There exist integers a and b such that for all integers q and r, $b \neq qa + r$.
   (l) [BB] For any infinite set, some proper subset is not finite.
   (m) For every real number x, there exists an integer n such that $x \leq n < x + 1$.
   (n) There exists an integer n such that $\frac{n}{n+1}$ is an integer.
   (o) $a > x$ or $a > y$ or $a > z$.
   (p) There exists a vector in the plane and there exists a normal to the plane such that the vector is not orthogonal to the normal.

6. (a) [BB] Converse: If $\frac{a}{c}$ is an integer, then $\frac{a}{b}$ and $\frac{b}{c}$ are also integers.
   Contrapositive: If $\frac{a}{c}$ is not an integer, then $\frac{a}{b}$ is not an integer or $\frac{b}{c}$ is not an integer.
   (b) Converse: $x = \pm 1 \rightarrow x^2 = 1$.
   Contrapositive: $x \neq 1$ and $x \neq -1 \rightarrow x^2 \neq 1$.
   (c) Converse: If $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$, then $x^2 = x + 1$.
   Contrapositive: If $x \neq 1 + \sqrt{5}$ and $x \neq 1 - \sqrt{5}$, then $x^2 \neq x + 1$.
   (d) Converse: If $n^2 + n - 2$ is an even integer, then n is an odd integer.
   Contrapositive: If $n^2 + 2 - 2$ is an odd integer, then n is an even integer.
   (e) [BB] Converse: A connected graph is Eulerian.
   Contrapositive: If a graph is not connected, then it is not Eulerian.
   (f) Converse: $a = 0$ or $b = 0 \rightarrow ab = 0$.
   Contrapositive: $a \neq 0$ and $b \neq 0 \rightarrow ab \neq 0$.
   (g) Converse: A four-sided figure is a square.
   Contrapositive: If a figure does not have four sides, then it is not a square.
   (h) [BB] Converse: If $a^2 = b^2 + c^2$, then $\triangle BAC$ is a right triangle.
   Contrapositive: If $a^2 \neq b^2 + c^2$, then $\triangle BAC$ is not a right triangle.

(i) Converse: If $p(x)$ is a polynomial with at least one real root, then $p(x)$ has odd degree.
Contrapositive: If $p(x)$ is a polynomial with no real roots, then $p(x)$ has even degree.

(j) Converse: A set of at most n vectors is linearly independent.
Contrapositive: A set of more than n vectors is not linearly independent.

(k) Converse: If f is not one-to-one, then, for all real numbers x and y, $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If f is one-to-one, then there exist real numbers x and y such that $x = y$ or $x^2 + xy + y^2 + x + y \neq 0$.

(l) [BB] Converse: If f is not one-to-one, then there exist real numbers x and y with $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If f is one-to-one, then for all real numbers x and y either $x = y$ or $x^2 + xy + y^2 + x + y = 0$.

7. (a) [BB] There exists a continuous function which is not differentiable.
(b) $2^x \geq 0$ for all real numbers x.
(c) [BB] For every real number x, there exists a real number y such that $y > x$.
(d) For every set of primes $p_1, p_2, \ldots, p_n$, there exists a prime not in this set.
(e) [BB] For every positive integer n, there exist primes $p_1, p_2, \ldots, p_t$ such that $n = p_1 p_2 \cdots p_t$.
(f) For every real number $x > 0$, there exists a real number a such that $a^2 = x$.
(g) [BB] For every integer n, there exists an integer m such that $m < n$.
(h) For every real number $x > 0$, there exists a real number $y > 0$ such that $y < x$.
(i) [BB] There exists a polynomial p such that for every real number x, $p(x) \neq 0$.
(j) For every pair of real numbers x and y with $x < y$, there exists a rational number a such that $x < a < y$.
(k) For every polynomial $p(x)$ of degree 3, there exists a real number x such that $p(x) = 0$.
(l) There exists a matrix $A \neq 0$ such that A is not invertible.
(m) There exists a real number x such that $x \geq 0$.
(n) For any integer n, n is not both even and odd.
(o) For all integers a, b, c, $a^3 + b^3 \neq c^3$.

8. If a given implication "$\mathcal{A} \rightarrow \mathcal{B}$" is false, then $\mathcal{A}$ is true and $\mathcal{B}$ is false. The converse, "$\mathcal{B} \rightarrow \mathcal{A}$" is then true because its hypothesis, $\mathcal{B}$, is false. It is **not** possible for both an implication and its converse to be false.

9. First we remember that x **and** y is true if x and y are both true and false otherwise. Now $p \leftrightarrow q$ means $p \rightarrow q$ **and** $q \rightarrow p$. Also

A. $p \rightarrow q$ is true if p is false or if p is true and q is true.
B. $q \rightarrow p$ is true if q is false or if q is true and p is true.

If p and q are both false, both statements A and B are true, so $p \leftrightarrow q$ is true. Similarly, if both p and q are true, then statements A and B are again true, so $p \leftrightarrow q$ is true. Thus $p \leftrightarrow q$ is true if p and q have the same truth values. Suppose p and q have different truth values. To be specific, say p is true and q is false. If p is true and q is false, we see that statement A is false, so A **and** B is false. Similarly, if p is false and q is true, then statement B is false, so A **and** B is false. This verifies statement $(*)$.

Exercises 0.2

1. (a) [BB] Hypothesis: a and b are positive numbers.
   Conclusion: $a + b$ is positive.

   (b) Hypothesis: T is a right angled triangle with hypotenuse of length c and the other sides of lengths a and b.
   Conclusion: $a^2 + b^2 = c^2$.

   (c) [BB] Hypothesis: p is a prime.
   Conclusion: p is even.

   (d) Hypothesis: $n > 1$ is an integer.
   Conclusion: n is the product of prime numbers.

   (e) Hypothesis: A graph is planar.
   Conclusion: The chromatic number is 3.

2. (a) [BB] a and b are positive is sufficient for $a + b$ to be positive; $a + b$ is positive is necessary for a and b to be positive.

   (b) A right angled triangle has sides of lengths a, b, c, c the hypotenuse, is sufficient for $a^2 + b^2 = c^2$; $a^2 + b^2 = c^2$ is necessary for a right angled triangle to have sides of lengths a, b, c, c the hypotenuse.

   (c) [BB] p is a prime is sufficient for p to be even; p is even is necessary for p to be prime.

   (d) $n > 1$ an integer is sufficient for n to be the product of primes; n a product of primes is necessary for n to be an integer bigger than 1.

   (e) A graph being planar is sufficient for its chromatic number to be 3. Chromatic number 3 is necessary for a graph to be planar.

3. (a) [BB] $x = -2$ (b) $a = b = -1$ (c) [BB] $x = 4$
   (d) $8, 9, 11, 12$ (e) $\sqrt{2}$ and $\dfrac{1}{\sqrt{2}}$ (f) $x = 5, y = 2$

4. $\mathcal{A}$ can easily be proven false with the counterexample 0. No single counterexample can disprove a statement claiming "there exists" so we prove $\mathcal{B}$ directly. $\mathcal{B}$ is false because the square of a real number is nonnegative.

5. [BB] This statement is true. Suppose the hypothesis, x is an even integer, is true. Then $x = 2k$ for some other integer k. Then $x + 2 = 2k + 2 = 2(k + 1)$ is also twice an integer. So $x + 2$ is even. The conclusion is also true.

6. The converse is "$x + 2$ is an even integer $\rightarrow x$ is an even integer." This is true, for suppose that the hypothesis, $x + 2$ is an even integer, is true. Then $x + 2 = 2k$ for some integer k, so $x = 2k - 2 = 2(k - 1)$ is also twice an integer. The conclusion is also true.

7. This is true. Let $\mathcal{A}$ be the statement "x is an even integer" and let $\mathcal{B}$ be the statement $x + 2$ is an even integer". In Exercise 5, we showed that $\mathcal{A} \rightarrow \mathcal{B}$ is true and, in Exercise 6, that the converse $\mathcal{B} \rightarrow \mathcal{A}$ is also true. Thus $\mathcal{A} \leftrightarrow \mathcal{B}$ is also true.

8. (a) $\mathcal{A}$ is false: $n = 0$ is a counterexample.

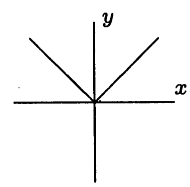
(b) Converse: If $\frac{n}{n+1}$ is not an integer, then n is an integer. This is false: $n = \frac{1}{2}$ is a counterexample ($\frac{n}{n+1} = \frac{1}{3}$).
Contrapositive: If $\frac{n}{n+1}$ is an integer, then n is not an integer. This is false: $n = 0$ is a counterexample.
Negation: There exists an integer n such that $\frac{n}{n+1}$ is an integer. This is true: Take $n = 0$.

9. (a) n prime $\rightarrow 2^n - 1$ prime.
   (b) n prime is sufficient for $2^n - 1$ to be prime.
   (c) $\mathcal{A}$ is false. For example, $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23(89)$ is not. The integer $n = 11$ is a counterexample to $\mathcal{A}$.
   (d) $2^n - 1$ prime $\rightarrow n$ prime.
   (e) The converse of $\mathcal{A}$ is true. To show this, we establish the contrapositive. Thus, we assume n is not prime. Then there exists a pair of integers a and b such that $a > 1$, $b > 1$, and $n = ab$. Using the hint, we can factor $2^n - 1$ as

$$2^n - 1 = (2^a)^b - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1].$$

   Since $a > 1$ and $b > 1$, we have $2^a - 1 > 1$ and $(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1 > 1$, so $2^n - 1$ is the product of two integers both of which exceed one. Hence, $2^n - 1$ is not prime.

10. [BB] The converse is the statement, "A continuous function is differentiable." This is false. The absolute value function whose graph is shown to the right is continuous, but not differentiable at $x = 0$.



11. (a) $2(n-1)$ (b) n
    (c) $\mathcal{A}_n \rightarrow \mathcal{A}_{n-1}, \mathcal{A}_{n-1} \rightarrow \mathcal{A}_{n-2}, \ldots, \mathcal{A}_2 \rightarrow \mathcal{A}_1, \mathcal{A}_1 \rightarrow \mathcal{A}_n$.

12. [BB] $\mathcal{A}$ is true. It expresses the fact that every real number lies between two consecutive integers. Statement $\mathcal{B}$ is most definitely false. It asserts that there is a remarkable integer n with the property that every real number lies in the unit interval between n and $n + 1$.

13. $\mathcal{A}$ is false; $\mathcal{B}$ is true. There can be no y with the property described since y is not bigger than $y + 1$; $x = y + 1$ provides a counterexample. To prove $\mathcal{B}$, we note that for every real number x, we have $x + 1 > x$ and so $x + 1$ is a suitable y.

14. (a) This is false. Suppose such an n exists. Then $q = \dfrac{1}{n+1}$ is rational but nq is not an integer.
    (b) This is true. Given a rational number q, there exist integers m and n, $n \neq 0$, such that $q = \dfrac{m}{n}$. Then $nq = m$ is an integer.

15. (a) Since n is even, $n = 2k$ for some integer k. Thus $n^2 + 3n = 4k^2 + 6k = 2(2k^2 + 3k)$ is even too.
    (b) The converse is the statement $n^2 + 3n$ even $\rightarrow n$ even. This is false and $n = 1$ is a counterexample.

16. (a) [BB] **Case 1:** a is even. In this case, we have one of the desired conclusions.
    Case 2: a is odd. In this case, $a = 2m + 1$ for some integer m, so $a + 1 = 2m + 2 = 2(m + 1)$ is even, another desired result.
    (b) [BB] $n^2 + n = n(n + 1)$ is the product of consecutive integers one of which must be even; so $n^2 + n$ is even.

17. $n^2 - n + 5 = n(n-1) + 5$. Now either $n-1$ or n is even, since these integers are consecutive. So $n(n-1)$ is even. Since the sum of an even integer and the odd integer 5 is odd, the result follows.

18. [BB] $2x^2 - 4x + 3 = 2(x^2 - 2x) + 3 = 2[(x-1)^2 - 1] + 3 = 2(x-1)^2 + 1$ is the sum of 1 and a nonnegative number. So it is at least 1 and hence positive.

19. For $a^2 - b^2$ to be odd, it is necessary and sufficient for one of a or b to be even while the other is odd. Here's why.

    Case i: a, b even.

    In this case, $a = 2n$ and $b = 2m$ for some integers m and n, so $a^2 - b^2 = 4n^2 - 4m^2 = 4(n^2 - m^2)$ is even.

    Case ii: a, b odd.

    In this case, $a = 2n+1$ and $b = 2m+1$ for some integers m and n, so $a^2 - b^2 = (4n^2 + 4n + 1) - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m)$ is even.

    Case iii: a even, b odd.

    In this case, $a = 2n$ and $b = 2m+1$ for some integers n and m, so $a^2 - b^2 = 4n^2 - (4m^2 + 4m + 1) = 4(n^2 - m^2 - m) - 1$ is odd.

    Case iv: a odd, b even.

    This is similar to Case iii, and the result follows.

20. [BB] ($\longrightarrow$) To prove this direction, we establish the contrapositive, that is, we prove that n odd implies n^2 odd. For this, if n is odd, then $n = 2m + 1$ for some integer m. Thus $n^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ is odd.

    ($\longleftarrow$) Here we assume that n is even. Therefore, $n = 2m$ for some integer m. So $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$ which is even, as required.

21. We assert that $x + \frac{1}{x} \geq 2$ if and only if $x > 0$.

    Proof. ($\longrightarrow$) We offer a proof by contradiction. Suppose $x + \frac{1}{x} \geq 2$ but $x > 0$ is not true; thus $x \leq 0$. If $x = 0$, $\frac{1}{x}$ is not defined, so $x < 0$. In this case, however, $x + \frac{1}{x} < 0$, a contradiction.

    ($\longleftarrow$) Conversely, assume that $x > 0$. Note that $(x-1)^2 \geq 0$ implies $x^2 - 2x + 1 \geq 0$, which in turn implies $x^2 + 1 \geq 2x$. Division by the positive number x gives $x + \frac{1}{x} \geq 2$ as required. ■

22. [BB] Since n is odd, $n = 2k + 1$ for some integer k.

    Case 1: k is even.

    In this case $k = 2m$ for some integer m, so $n = 2(2m) + 1 = 4m + 1$.

    Case 2: k is odd.

    In this case, $k = 2m + 1$ for some integer m, so $n = 2(2m+1) + 1 = 4m + 3$.

    Since each case leads to one of the desired conclusions, the result follows.

23. By Exercise 22, there exists an integer k such that $n = 4k + 1$ or $n = 4k + 3$.

Case 1: $n = 4k + 1$.

If k is even, there exists an integer m such that $k = 2m$, so $n = 4(2m) + 1 = 8m + 1$, and the desired conclusion is true. If k is odd, there exists an integer m such that $k = 2m + 1$, so $n = 4(2m + 1) + 1 = 8m + 5$, and the desired conclusion is true.

Case 2: $n = 4k + 3$.

If k is even, there exists an integer m such that $k = 2m$, so $n = 4(2m) + 3 = 8m + 3$, and the desired conclusion is true. If k is odd, there exists an integer m such that $k = 2m + 1$, so $n = 4(2m + 1) + 3 = 8m + 7$, and the desired conclusion is true. In all cases, the desired conclusion is true.

24. [BB] If the statement is false, then there does exist a smallest positive real number r. Since $\frac{1}{2}r$ is positive and smaller than r, we have reached an absurdity. So the statement must be true.

25. We give a proof by contradiction. If the result is false, then both $a > \sqrt{n}$ and $b > \sqrt{n}$. (Note that the negation of an "or" statement is an "and" statement.) But then $n = ab > \sqrt{n}\sqrt{n} = n$, which isn't true.

26. [BB] Since 0 is an eigenvalue of A, there is a nonzero vector x such that $A\mathsf{x} = 0$. Now suppose that A is invertible. Then $A^{-1}(A\mathsf{x}) = A^{-1}0 = 0$, so $\mathsf{x} = 0$, a contradiction.

27. (a) The given equation is equivalent to $(b - 5)\sqrt{2} = 3 - a$. If $b \neq 5$, then $\sqrt{2} = \frac{3-a}{b-5}$ is a rational number. This is false. Thus $b = 5$, so $a = 3$.

(b) Note that $(a + b\sqrt{2})^2 = (a^2 + 2b^2) + 2ab\sqrt{2}$. Thus, if $(a + b\sqrt{2})^2 = 3 + 5\sqrt{2}$, then $a^2 + 2b^2 = 3$ and $2ab = 5$, by part (a). The second equation says $a \neq 0$ and $b \neq 0$. Since a and b are integers, it follows that $a^2 \geq 1$ and $b^2 \geq 1$ and $a^2 + 2b^2 \geq 3$ with equality if and only $a = \pm 1 = b$. But then $2ab \neq 5$.

28. [BB] Observe that $(1+a)(1+b) = 1+a+b+ab = 1$. Thus $1+a$ and $1+b$ are integers whose product is 1. There are two possibilities: $1 + a = 1 + b = 1$, in which case $a = b = 0$, or $1 + a = 1 + b = -1$, in which case $a = b = -2$.

29. We offer a proof by contradiction. Suppose $\dfrac{1}{a}$ is not irrational. Then it is rational, so there exist integers m and n, $n \neq 0$, such that $\dfrac{1}{a} = \dfrac{m}{n}$. Since $\dfrac{1}{a} \neq 0$, we know also that $m \neq 0$. Now $\dfrac{1}{a} = \dfrac{m}{n}$ implies $a = \dfrac{n}{m}$ is a rational number, a contradiction.

30. We give a proof by contradiction. Assume that a is rational, b is irrational and $a + b$ is rational. Then $a + b = \frac{m}{n}$ for integers m and n, $n \neq 0$. Since a is rational, $a = \frac{k}{\ell}$ for integers k and ℓ, $\ell \neq 0$. Thus

$$b = \frac{m}{n} - a = \frac{m}{n} - \frac{k}{\ell} = \frac{m\ell - kn}{n\ell}$$

is the quotient of integers with nonzero denominator. This contradicts the fact that b is not rational.

31. [BB] We begin by assuming the negation of the desired conclusion; in other words, we assume that there exist real numbers x, y, z which simultaneously satisfy each of these three equations. Subtracting the second equation from the first we see that $x + 5y - 4z = -2$. Since the third equation we were given says $x + 5y - 4z = 0$, we have $x + 5y - 4z$ equal to both 0 and to -2. Thus, the original assumption has led us to a contradiction.

32. (a) [BB] False: $x = y = 0$ is a counterexample.

(b) False: $a = 6$ is a counterexample.

(c) [BB] False: $x = 0$ is a counterexample.

(d) False: $a = \sqrt{2}$, $b = -\sqrt{2}$ is a counterexample.

(e) [BB] False: $a = b = \sqrt{2}$ is a counterexample.

(f) The roots of the polynomial $ax^2 + bx + c$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ is real and not 0, so the formula produces two distinct real numbers $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(g) False: $x = \frac{1}{2}$ is a counterexample.

(h) True: If n is a positive integer, then $n \geq 1$, so $n^2 = n(n) \geq n$.

33. The result is false. A square and a rectangle (which is not a square) have equal angles but not pairwise proportional sides.

34. (a) [BB] Since n^2+1 is even, n^2 is odd, so n must also be odd. Writing $n = 2k+1$, then $n^2+1 = 2m$ says $4k^2 + 4k + 2 = 2m$, so $m = 2k^2 + 2k + 1 = (k+1)^2 + k^2$ is the sum of two squares as required.

(b) [BB] We are given that $n^2+1 = 2m$ for $n = 4373$ and $m = 9561565$. Since $n = 2(2186)+1$, our solution to (a) shows that $m = k^2 + (k+1)^2$ where $k = 2186$. Thus, $9561565 = 2186^2 + 2187^2$.

35. (a) $2^{4n+2} + 1 = 4(2^{4n}) + 1 = 4(2^n)^4 + 1$. Applying the given identity with $x = 2^n$, we get

$$\begin{aligned} 2^{4n+2} + 1 &= (2 \cdot 2^{2n} + 2^{n+1} + 1)(2 \cdot 2^{2n} - 2^{n+1} + 1) \\ &= (2^{2n+1} + 2^{n+1} + 1)(2^{2n+1} - 2^{n+1} + 1). \end{aligned}$$

With $n = 4$, we get $2^{18} + 1 = (2^9 + 2^5 + 1)(2^9 - 2^5 + 1) = 545(481)$.

(b) $2^{36} - 1 = (2^{18} - 1)(2^{18} + 1) = (2^9 - 1)(2^9 + 1)(545)(481)$ (using the result of part (a)) $= 511(513)(545)(481)$.

36. If the result is false, then $f(n) = a_0 + a_1 n + \cdots + a_t n^t$ for some $t \geq 1$. Since $f(0) = a_0 = p$ is prime, $f(n) = p + ng(n)$ for $g(n) = a_1 + a_2 n + \cdots + a_t n^{t-1}$. Replacing n by pn, we have $f(pn) = p + npg(pn)$. The right hand side is divisible by the prime p, hence $f(pn)$ is divisible by p. But $f(pn)$ is prime, by hypothesis, so $f(pn) = p$. This means $g(pn) = 0$, contradicting the fact that a polynomial has only finitely many roots.

37. We offer a proof by contradiction. Suppose all the digits occur just a finite number of times. Then there is a number n_1 which has the property that after n_1 digits in the decimal expansion of π, the digit 1 no longer occurs. Similarly, there is a number n_2 such that after n_2 digits, the digit 2 no longer occurs, and so on. In general, for each $k = 1, 2, \ldots, 9$, there is a number n_k such that after n_k digits, the digit k no longer occurs. Let N be the largest of the numbers $n_1, n_2, \ldots, n_k$. Then after N digits in the decimal expansion of π, the only digit which can appear is 0. This contradicts the fact that the decimal expansion of π does not terminate.

38. We have proven in the text that $\sqrt{2}$ is irrational. Thus, if $\sqrt{2}^{\sqrt{2}}$ is rational, we are done (with $a = b = \sqrt{2}$). On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ in which case $a^b = \sqrt{2}^2 = 2$ is rational.

Chapter 0 Review

1. (a) This implication is true because the hypothesis is always false: $a - b > 0$ and $b - a > 0$ give $a > b$ and $b > a$, which never holds.
   (b) This implication is false: When $a = b$, the hypotheses are true while the conclusion is false.

2. (a) x is a real number and $x \leq 5$.
   (b) For every real number x, there exists an integer n such that $n \leq x$.
   (c) There exist positive integers x, y, z such that $x^3 + y^3 = z^3$.
   (d) There exists a graph with n vertices and $n + 1$ edges whose chromatic number is more than 3.
   (e) There exists an integer n such that for any rational number a, $a \neq n$.
   (f) $a \neq 0$ or $b \neq 0$.

3. (a) Converse: If ab is an integer, then a and b are integers.
   Contrapositive: If ab is not an integer, then either a or b is not an integer.
   Negation: There exist integers a and b such that ab is not an integer.
   (b) Converse: If x^2 is an even integer, then x is an even integer.
   Contrapositive: If x^2 is an odd integer, then x is an odd integer.
   Negation: There exists an even integer x such that x^2 is odd.
   (c) Converse: Every graph which can be colored with at most four colors is planar.
   Contrapositive: Every graph which cannot be colored with at most four colors is not planar.
   Negation: There exists a planar graph which cannot be colored with at most four colors.
   (d) Converse: A matrix which equals its transpose is symmetric.
   Contrapositive: If a matrix does not equal its transpose, then it is not symmetric.
   Negation: There exists a symmetric matrix which is not equal to its transpose.
   (e) Converse: A set of at least n vectors is a spanning set.
   Contrapositive: A set of less than n vectors is not a spanning set.
   Negation: There exists a spanning set containing less than n vectors.
   (f) Converse: If $x > -2$ and $x < 1$, then $x^2 + x - 2 < 0$.
   Contrapositive: If $x \leq -2$ or $x \geq 1$, then $x^2 + x - 2 \geq 0$.
   Negation: There exists an $x \leq -2$ or $x \geq 1$ such that $x^2 + x - 2 < 0$.

4. (a) A is false: $a = u = b = 1, v = -1$ provides a counterexample.
   (b) Converse: Given four integers a, b, u, v with $u \neq 0$, $v \neq 0$, if $a = b = 0$, then $au + bv = 0$.
   Negation: There exist integers a, b, u, v, $u \neq 0$, $v \neq 0$, with $au + bv = 0$ and $a \neq 0$ or $b \neq 0$.
   Contrapositive: Given four integers a, b, u, v with $u \neq 0$ and $v \neq 0$, if $a \neq 0$ or $b \neq 0$, then $au + bv \neq 0$.

(c) The converse is certainly true since $0u + 0v = 0$.

(d) The negation is true: Take $a = u = b = 1$ and $v = -1$.
The contrapositive of A is false since A is false.

5. (a) There exists a countable set which is infinite.
   (b) For all positive integers n, $1 \leq n$.

6. (a) This is true. If x is positive, $x + 2$ is positive. In addition, if x is odd, $x + 2$ is odd.
   (b) This is false. When $x = -1$, $x + 2 = +1$ is a positive odd integer, while x is not.

7. (a) This statement expresses a well-known property of the real numbers. It is true.
   (b) This is false. The conclusion would have us believe that every two real numbers are equal.

8. The desired formula is $ab = \dfrac{(a+b)^2 - (a-b)^2}{4}$ which holds because $(a+b)^2 - (a-b)^2 = (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab$.

9. ($\longrightarrow$) Assume n^3 is odd and suppose, to the contrary, that n is even. Thus $n = 2x$ for some integer x. But then $n^3 = 8x^3 = 2(4x^3)$ is even, a contradiction. This means that n must be odd.

   ($\longleftarrow$) Assume n is odd. This means that $n = 2x + 1$ for some integer x. Then $n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$ is odd.

10. (a) $a^2 - 5a + 6 = (a-2)(a-3)$ is the product of two consecutive integers, one of which must be even.
    (b) The sum $(a^2 - 5b) + (b^2 - 5a)$ is $(a^2 - 5a) + (b^2 - 5b) = (a^2 - 5a + 6) + (b^2 - 5b + 6) - 12$ is the sum of three even integers [using the result of part (a)] and hence even. Thus $b^2 - 5a$ is the difference of even integers and hence even as well.

11. The sum of the angles of a triangle is 180°, so $\angle C = 45°$ and $\angle D = 75°$. Since triangles ABC and DEF are similar, $\dfrac{AC}{AB} = \dfrac{DE}{DF}$, so the length of AC is $|AB| \times \dfrac{DE}{DF} = 12 \times \dfrac{8}{6} = 16$.

12. The rectangle that remains has dimensions 1 by $\tau - 1$. These are in ratio

$$\frac{1}{\tau - 1} = \frac{\tau + 1}{(\tau - 1)(\tau + 1)} = \frac{\tau + 1}{\tau^2 - 1} = \frac{\tau + 1}{\tau} = \tau$$

using twice the fact that $\tau^2 = \tau + 1$.

13. If the result is false, then $x \geq -1$ and $x \leq 2$, so $x + 1 \geq 0$ and $x - 2 \leq 0$. But then $x^2 - x - 2 = (x+1)(x-2) \leq 0$, a contradiction.

14. Suppose, to the contrary, that $-\dfrac{x}{y}$ is the largest negative rational number, where x and y are positive integers. Then $\dfrac{x}{2y}$ is rational and $\dfrac{x}{2y} < \dfrac{x}{y}$, so $-\dfrac{x}{2y} > -\dfrac{x}{y}$. Since $-\dfrac{x}{2y}$ is a negative rational number, we have a contradiction.

15. We use a proof by contradiction which mimics that proof of the irrationality of $\sqrt{2}$ given in Problem 8. Thus, we suppose that $\sqrt{3} = \frac{a}{b}$ is rational and hence the quotient of integers a and b which have no factors in common. Squaring gives $a^2 = 3b^2$ and so $a = 3k$ is a multiple of 3. But then $9k^2 = 3b^2$, so $3k^2 = b^2$. This says that b is also a multiple of 3, contradicting our assumption that a and b have no factors in common.

16. Let the rational numbers be $\frac{a}{b}$ and $\frac{c}{d}$. We may assume that a, b, c, d are positive integers and that $\frac{a}{b} < \frac{c}{d}$. Thus $ad < bc$. The hint suggests that $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$, and this is the case: $\frac{a}{b} < \frac{a+c}{b+d}$ is equivalent to $a(b+d) < b(a+c)$ and $\frac{a+c}{b+d} < \frac{c}{d}$ is equivalent to $(a+c)d < (b+d)c$, both of which are true because $ad < bc$.

17. On a standard checker board, there are 32 squares of one color and 32 of another. Since squares in opposite corners have the same color, the hint shows that our defective board has 32 squares of one color and 30 of the other. Since each domino covers one square of each color, the result follows.

18. (a) We leave the primality checking of $f(1), \ldots, f(39)$ to the reader, but note that $f(40) = 41^2$.

 (b) $f(k^2 + 40) = 40^2 + 80k^2 + k^4 + 40 + k^2 + 41 = k^4 + 81k^2 + 41^2 = (k^2 + 41)^2 - k^2 = (k^2 + 41 + k)(k^2 + 41 - k)$.

19. The answer is no, since $333333331 = 19607843 \times 17$.

Exercises 1.1

1. (a) [BB]

p	q	$\neg q$	$(\neg q) \vee p$	$p \wedge ((\neg q) \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

 (b)

p	q	$\neg p$	$(\neg p) \to q$	$p \wedge q$	$(p \wedge q) \vee ((\neg p) \to q)$
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	F

 (c)

p	q	$q \vee p$	$p \wedge (q \vee p)$	$\neg (p \wedge (q \vee p))$	$\neg (p \wedge (q \vee p)) \leftrightarrow p$
T	T	T	T	F	F
T	F	T	T	F	F
F	T	T	F	T	F
F	F	F	F	T	F

 (d) [BB]

p	q	r	$\neg q$	$p \vee (\neg q)$	$\neg (p \vee (\neg q))$	$\neg p$	$(\neg p) \vee r$	$(\neg (p \vee (\neg q))) \wedge ((\neg p) \vee r)$
T	T	T	F	T	F	F	T	F
T	F	T	T	T	F	F	T	F
F	T	T	F	F	T	T	T	T
F	F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	F	F
T	F	F	T	T	F	F	F	F
F	T	F	F	F	T	T	T	T
F	F	F	T	T	F	T	T	F

(e)

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \vee r$	$(p \to (q \to r)) \to ((p \wedge q) \vee r)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	T	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	F	F	T	F	F	F
F	F	F	T	T	F	F	F

2. (a) If $p \to q$ is false, then necessarily p is true and q is false. (This is the only situation in which $p \to q$ is false.) We construct the relevant row of the truth table for $(p \wedge (\neg q)) \vee ((\neg p) \to q)$.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$(p \wedge (\neg q)) \vee ((\neg p) \to q)$
T	F	T	T	F	T	T

(b) [BB] There are three situations in which $p \to q$ is true. The question then is whether or not the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \to q)$ is the same in each of these cases. We construct a partial truth table.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$(p \wedge (\neg q)) \vee ((\neg p) \to q)$
T	T	F	F	F	T	T
F	F	T	F	T	F	F

As shown, $(p \wedge (\neg q)) \vee ((\neg p) \to q)$ has different truth values on two occasions where $p \to q$ is true, so it is **not** possible to answer the question in this case.

3. [BB]

p	q	r	s	$\neg r$	$q \wedge (\neg r)$	$p \to [q \wedge (\neg r)]$	$\neg s$	$(\neg s) \vee q$	$r \leftrightarrow [(\neg s) \vee q]$	$[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$
T	T	T	T	F	F	F	F	T	T	T

4.

p	q	r	s	$\neg r$	$q \wedge (\neg r)$	$p \to [q \wedge (\neg r)]$	$\neg s$	$(\neg s) \vee q$	$r \leftrightarrow [(\neg s) \vee q]$	$[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$
F	F	F	F	T	F	T	T	T	F	T

5. (a) [BB]

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

The final column shows that $(p \wedge q) \to (p \vee q)$ is true for all values of p and q, so this statement is a tautology.

(b) [BB]

p	q	$\neg p$	$(\neg p) \wedge q$	$\neg q$	$p \vee (\neg q)$	$((\neg p) \wedge q) \wedge (p \vee (\neg q))$
T	T	F	F	F	T	F
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	F	T	F	T	T	F

The final column shows that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is false for all values of p and q, so this statement is a contradiction.

6. (a)

p	q	$p \to q$	$q \to (p \to q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Since $q \to (p \to q)$ is true for all values of p and q, this statement is a tautology.

(b)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$((\neg p) \vee (\neg q))$	$(p \wedge q) \wedge [(\neg p) \vee (\neg q)]$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

Since $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$ is false for all values of p and q, this statement is a contradiction.

7. (a) [BB]

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to [p \to r]$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

Since $[(p \to q) \wedge (q \to r)] \to [p \to r]$ is true for all values of p, q, and r, this statement is a tautology.

(b) [BB] If p implies q which, in turn, implies r, then certainly p implies r.

8. We must show that the given "or" statement can be both true and false. We construct truth tables for each part of the "or" and show that certain identical values for the variables make both parts T (so that the "or" is true) and other certain identical values for the variables make both parts F (so that the "or" is false).

p	r	s	$\neg r$	$\neg s$	$(\neg r) \to (\neg s)$	$p \vee [(\neg r) \to (\neg s)]$
T	T	T	F	F	T	T
F	F	T	T	F	F	F