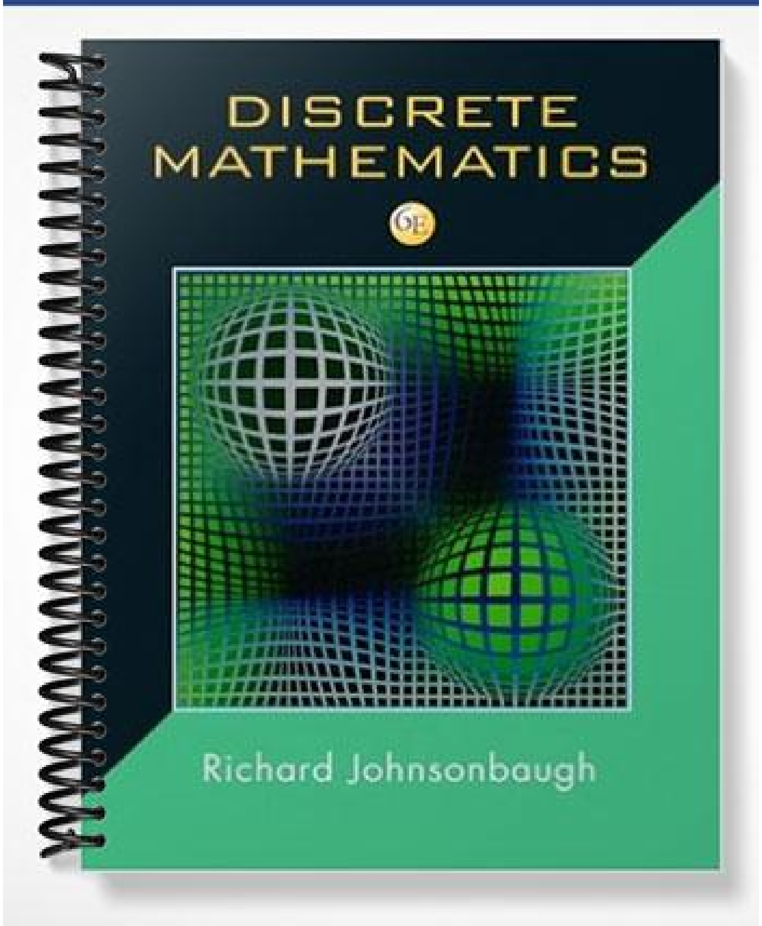
SOLUTIONS MANUAL



Instructor's Manual

DISCRETE MATHEMATICS

6E

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Table of Contents

Chapter 1

Solutions to Selected Exercises

Section 1.1

2. Not a proposition

3. Is a proposition. Negation: For every positive integer n, $19340 \neq n \cdot 17$.

5. Not a proposition

6. Is a proposition. Negation: The line "Play it again, Sam" does not occur in the movie *Casablanca*.

8. Not a proposition

10. No heads were obtained.

11. No heads or no tails were obtained.

14. True

15. True

17. False

18. False

20.

p	q	$(\neg p \vee \neg q) \vee p$
T	T	T
T	F	T
F	T	T
F	F	T

21.

p	q	$(p \vee q) \wedge \neg p$
T	T	F
T	F	F
F	T	T
F	F	F

23.

p	q	$(p \wedge q) \vee (\neg p \vee q)$
T	T	T
T	F	F
F	T	T
F	F	T

24.

p	q	r	$\neg(p \wedge q) \vee (r \wedge \neg p)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

26.

p	q	r	$\neg(p \wedge q) \vee (\neg q \vee r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

28. $\neg(p \wedge q)$. True.

29. $p \vee \neg(q \wedge r)$. True.

31. Lee takes computer science and mathematics.

32. Lee takes computer science or mathematics.

34. Lee takes computer science but not mathematics.

35. Lee takes neither computer science nor mathematics.

37. It is not Monday and either it is raining or it is hot.

38. It is not the case that (today is Monday or it is raining) and it is hot.

40. Today is Monday and either it is raining or it is hot, and it is hot or either it is raining or today is Monday.

42. $p \wedge q$

43. $p \wedge \neg q$

45. $p \vee q$

46. $(p \vee q) \wedge \neg p$

48. $p \wedge \neg r$

49. $p \wedge q \wedge r$

51. $\neg p \wedge \neg q \wedge r$

52. $\neg(p \vee q \vee \neg r)$

53.

p	q	p *exor* q
T	T	F
T	F	T
F	T	T
F	F	F

56. lung AND disease AND NOT cancer

57. minor AND league AND team AND illinois AND NOT midwest

Section 1.2

2. If Rosa has 160 quarter-hours of credits, then she may graduate.

3. If Fernando buys a computer, then he obtains \$2000.

5. If a better car is built, then Buick will build it.

6. If the chairperson gives the lecture, then the audience will go to sleep.

9. Contrapositive of Exercise 2: If Rosa does not graduate, then she does not have 160 quarter-hours of credits.

11. False 12. False 14. False 15. True 17. True

19. Unknown 20. Unknown 22. True 23. Unknown 25. Unknown

26. Unknown 29. $(p \wedge r) \to q$ 30. $\neg((r \wedge \neg q) \to r)$

33. If it is not raining, then it is hot and today is Monday.

34. If today is not Monday, then either it is raining or it is hot.

36. If today is Monday and either it is raining or it is hot, then either it is hot, it is raining, or today is Monday.

37. If today is Monday or (it is not Monday and it is not the case that (it is raining or it is hot)), then either today is Monday or it is not the case that (it is hot or it is raining).

39. Let p: $4 > 6$ and q: $9 > 12$. Given statement: $p \to q$; true. Converse: $q \to p$; if $9 > 12$, then $4 > 6$; true. Contrapositive: $\neg q \to \neg p$; if $9 \leq 12$, then $4 \leq 6$; true.

40. Let p: $|1| < 3$ and q: $-3 < 1 < 3$. Given statement: $q \to p$; true. Converse: $p \to q$; if $|1| < 3$, then $-3 < 1 < 3$; true. Contrapositive: $\neg p \to \neg q$; if $|1| \geq 3$, then either $-3 \geq 1$ or $1 \geq 3$; true.

43. $P \not\equiv Q$ 44. $P \equiv Q$ 46. $P \not\equiv Q$ 47. $P \equiv Q$ 49. $P \not\equiv Q$

50. $P \not\equiv Q$

53. (a) If p and q are both false, $(p \textit{ imp2 } q) \wedge (q \textit{ imp2 } p)$ is false, but $p \leftrightarrow q$ is true.

    (b) Making the suggested change does not alter the last line of the *imp2* table.

54.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Section 1.3

2. The statement is a command, not a propositional function.

3. The statement is a command, not a propositional function.

5. The statement is not a propositional function since it has no variables.

6. The statement is a propositional function. The domain of discourse is the set of real numbers.

8. 1 divides 77. True. 9. 3 divides 77. False. 11. For some n, n divides 77. True.

13. Some student is taking a math course.

14. Every student is not taking a math course.

16. It is not the case that every student is taking a math course.

17. It is not the case that some student is taking a math course.

20. There is some person such that if the person is a professional athlete, then the person plays soccer. True.

21. Every soccer player is a professional athlete. False.

23. Every person is either a professional athlete or a soccer player. False.

24. Someone is either a professional athlete or a soccer player. True.

26. Someone is a professional athlete and a soccer player. True.

29. $\exists x(P(x) \wedge Q(x))$

30. $\forall x(Q(x) \rightarrow P(x))$

34. True 35. True 37. False 38. True

40. No. The suggested replacement returns false if $\neg P(d_1)$ is true, and true if $\neg P(d_1)$ is false.

42. Literal meaning: Every old thing does not covet a twenty-something. Intended meaning: Some old thing does not covet a twenty-something. Let $P(x)$ denote the statement "x is an old thing" and $Q(x)$ denote the statement "x covets a twenty-something." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

43. Literal meaning: Every hospital did not report every month. (Domain of discourse: the 74 hospitals.) Intended meaning (most likely): Some hospital did not report every month. Let $P(x)$ denote the statement "x is a hospital" and $Q(x)$ denote the statement "x reports every month." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

45. Literal meaning: Everyone does not have a degree. (Domain of discourse: People in Door County.) Intended meaning: Someone does not have a degree. Let $P(x)$ denote the statement "x has a degree." The intended statement is $\exists x \neg P(x)$.

46. Literal meaning: No lampshade can be cleaned. Intended meaning: Some lampshade cannot be cleaned. Let $P(x)$ denote the statement "x is a lampshade" and $Q(x)$ denote the statement "x can be cleaned." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

48. Literal meaning: No person can afford a home. Intended meaning: Some person cannot afford a home. Let $P(x)$ denote the statement "x is a person" and $Q(x)$ denote the statement "x can afford a home." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

49. Literal meaning: No circumstance is right for a formal investigation. Intended meaning: Some circumstance is not right for a formal investigation. Let $P(x)$ denote the statement "x is a circumstance" and $Q(x)$ denote the statement "x is right for a formal investigation." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

50. (a)

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

One of $p \to q$ or $q \to p$ is true since in each row, one of the last two entries is true.

(b) The statement, "All integers are positive or all positive numbers are integers," which is false, in symbols is

$$(\forall x(P(x) \to Q(x))) \vee (\forall x(Q(x) \to P(x))).$$

This is *not* the same as the given statement

$$\forall x((P(x) \to Q(x)) \vee (Q(x) \to P(x))),$$

which is true. The ambiguity results from attempting to distribute $\forall$ across the *or*.

Section 1.4

2. Everyone is taller than someone else. False.

3. Someone is taller than everyone else. True.

7. Everyone is taller than or the same height as someone. True.

8. Someone is taller than or the same height as everyone. True.

12. $\forall x \forall y L(x, y)$. False. 13. $\exists x \exists y L(x, y)$. True.

15. (Exercise 11) $\forall x \exists y \neg L(x, y)$. False. 17. True 18. False 22. True

23. False 25. False 26. False 28. False 29. False 31. True

32. True 34. True 35. False 37. True 38. True

40.
```
for i = 1 to n
    if (forall_dj(i))
        return true
return false

forall_dj(i) {
    for j = 1 to n
        if (¬P(d_i, d_j))
            return false
    return true
}
```

41.
```
for i = 1 to n
    for j = 1 to n
        if (P(d_i, d_j))
            return true
return false
```

43. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. If Farley chooses values that make $x \geq y$, say $x = y = 0$, whatever value you choose for z,

$$(z > x) \land (z < y)$$

is false. Since Farley can always win the game, the quantified propositional function is false.

44. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. Whatever values Farley chooses, you can choose z to be one less than the minimum of x and y; thus making

$$(z < x) \land (z < y)$$

true. Since you can always win the game, the quantified propositional function is true.

46. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. If Farley chooses values such that $x \geq y$, the proposition

$$(x < y) \to ((z > x) \land (z < y))$$

is true by default (i.e., it is true regardless of what value you choose for z). If Farley chooses values such that $x < y$, you can choose $z = (x + y)/2$ and again the proposition

$$(x < y) \to ((z > x) \land (z < y))$$

is true. Since you can always win the game, the quantified propositional function is true.

48. The proposition must be true. $P(x, y)$ is true for all x and y; therefore, no matter which value for x we choose, the proposition $\forall y P(x, y)$ is true.

49. The proposition must be true. Since $P(x, y)$ is true for all x and y, we may choose *any* values for x and y to make $P(x, y)$ true.

51. The proposition can be false. Let $P(x, y)$ be the statement $x \geq y$ and let the domain of discourse be the set of positive integers. Then $\forall x \exists y P(x, y)$ is true, but $\exists x \forall y P(x, y)$ is false.

52. The proposition must be true. Since $\forall x \exists y P(x, y)$ is true, if we choose any value for x whatsoever, there exists a value for y for which $P(x, y)$ is true. Therefore $\exists x \exists y P(x, y)$ is true.

54. The proposition can be false. Let the domain of discourse consist of the persons James James, Terry James, and Lee James, and let $P(x, y)$ be the statement "x's first name is the same as y's last name." Then $\exists x \forall y P(x, y)$ is true, but $\forall x \exists y P(x, y)$ is false.

55. The proposition must be true. Since $\exists x \forall y P(x, y)$ is true, there is some value for x for which $\forall y P(x, y)$ is true. Choosing any value for y whatsoever makes $P(x, y)$ true. Therefore $\exists x \exists y P(x, y)$ is true.

57. The proposition can be false. Let $P(x, y)$ be the statement $x > y$ and let the domain of discourse be the set of positive integers. Then $\exists x \exists y P(x, y)$ is true, but $\forall x \exists y P(x, y)$ is false.

58. The proposition can be false. Let $P(x, y)$ be the statement $x > y$ and let the domain of discourse be the set of positive integers. Then $\exists x \exists y P(x, y)$ is true, but $\exists x \forall y P(x, y)$ is false.

60. Not equivalent. Let $P(x, y)$ be the statement $x > y$ and let the domain of discourse be the set of positive integers. Then $\neg(\forall x \exists y P(x, y))$ is true, but $\forall x \neg(\exists y P(x, y))$ is false.

61. Equivalent by De Morgan's law

Section 1.5

2. For all x, for all y, $x + y = y + x$.

3. An *isosceles trapezoid* is a trapezoid with equal legs.

5. The medians of any triangle intersect at a single point.

6. If $0 < x < 1$ and $\varepsilon > 0$, there exists a positive integer n satisfying $x^n < \varepsilon$.

8. Let m and n be odd integers. Then there exist k_1 and k_2 such that $m = 2k_1 + 1$ and $n = 2k_2 + 1$. Now
$$m + n = (2k_1 + 1) + (2k_2 + 1) = 2(k_1 + k_2 + 1).$$
Therefore, $m + n$ is even.

9. Let m and n be even integers. Then there exist k_1 and k_2 such that $m = 2k_1$ and $n = 2k_2$. Now
$$mn = (2k_1)(2k_2) = 2(2k_1 k_2).$$
Therefore, mn is even.

11. Let m be an odd integer and n be an even integer. Then there exist k_1 and k_2 such that $m = 2k_1 + 1$ and $n = 2k_2$. Now
$$mn = (2k_1 + 1)(2k_2) = 2(2k_1 k_2 + k_2).$$
Therefore, mn is even.

12. From the definition of max, it follows that $d \geq d_1$ and $d \geq d_2$. From $x \geq d$ and $d \geq d_1$, we may derive $x \geq d_1$ from a previous theorem (the second theorem of Example 1.5.5). From $x \geq d$ and $d \geq d_2$, we may derive $x \geq d_2$ from the same previous theorem. Therefore, $x \geq d_1$ and $x \geq d_2$.

14.

Step	*Justification*
1. $xy = 0, x \neq 0, y \neq 0$	Hypothesis
2. $x \cdot 0 = 0$	Exercise 7
3. $xy = x \cdot 0$	Two things equal to the same thing are equal to each other.
4. $y = 0$	Steps 1 and 3 and the property given in the statement of Exercise 8.

15. Suppose that every box contains less than 12 balls. Then each box contains at most 11 balls and the maximum number of balls contained by the nine boxes is $9 \cdot 11 = 99$. Contradiction.

17. Suppose that there does not exist i such that $s_i \geq A$. Then, for all i, $s_i < A$. Now

$$A = \frac{s_1 + s_2 + \cdots + s_n}{n} < \frac{A + A + \cdots + A}{n} = \frac{nA}{n} = A,$$

which is a contradiction.

18. The statement is false. A counterexample is $s_i = A$ for all i.

20. First assume that $x \geq 0$ and $y \geq 0$. Then $xy \geq 0$ and $|xy| = xy = |x||y|$. Next assume that $x < 0$ and $y \geq 0$. Then $xy \leq 0$ and $|xy| = -xy = (-x)(y) = |x||y|$. Next assume that $x \geq 0$ and $y < 0$. Then $xy \leq 0$ and $|xy| = -xy = (x)(-y) = |x||y|$. Finally assume that $x < 0$ and $y < 0$. Then $xy > 0$ and $|xy| = xy = (-x)(-y) = |x||y|$.

21. First, note that from Exercise 20, for all x,

$$|-x| = |(-1)x| = |-1||x| = |x|.$$

Example 1.5.14 states that for all x, $x \leq |x|$. Using these results, we consider two cases: $x + y \geq 0$ and $x + y < 0$. If $x + y \geq 0$, we have

$$|x + y| = x + y \leq |x| + |y|.$$

If $x + y < 0$, we have

$$|x + y| = -(x + y) = -x + -y \leq |-x| + |-y| = |x| + |y|.$$

23. Suppose that $xy > 0$. Then either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. If $x > 0$ and $y > 0$,

$$\text{sgn}(xy) = 1 = 1 \cdot 1 = \text{sgn}(x)\text{sgn}(y).$$

If $x < 0$ and $y < 0$,

$$\text{sgn}(xy) = 1 = -1 \cdot -1 = \text{sgn}(x)\text{sgn}(y).$$

Next, suppose that $xy = 0$. Then either $x = 0$ or $y = 0$. Thus either $\operatorname{sgn}(x) = 0$ or $\operatorname{sgn}(y) = 0$. In either case, $\operatorname{sgn}(x)\operatorname{sgn}(y) = 0$. Therefore

$$\operatorname{sgn}(xy) = 0 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

Finally, suppose that $xy < 0$. Then either $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$. If $x > 0$ and $y < 0$,

$$\operatorname{sgn}(xy) = -1 = 1 \cdot -1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

If $x < 0$ and $y > 0$,

$$\operatorname{sgn}(xy) = -1 = -1 \cdot 1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

24. $|xy| = \operatorname{sgn}(xy)xy = \operatorname{sgn}(x)\operatorname{sgn}(y)xy = [\operatorname{sgn}(x)x][\operatorname{sgn}(y)y] = |x||y|$

26. Suppose that $x \geq y$. Then

$$\max\{x, y\} = x \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\max\{x, y\} = x = \frac{2x}{2} = \frac{x + y + x - y}{2} = \frac{x + y + |x - y|}{2}.$$

The other case is $x < y$. Then

$$\max\{x, y\} = y \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\max\{x, y\} = y = \frac{2y}{2} = \frac{x + y + y - x}{2} = \frac{x + y + |x - y|}{2}.$$

27. Suppose that $x \geq y$. Then

$$\min\{x, y\} = y \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\min\{x, y\} = y = \frac{2y}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - |x - y|}{2}.$$

The other case is $x < y$. Then

$$\min\{x, y\} = x \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\min\{x, y\} = x = \frac{2x}{2} = \frac{x + y - (y - x)}{2} = \frac{x + y - |x - y|}{2}.$$

29. Let i be the greatest integer for which s_i is positive. Since s_1 is positive and the set of indexes $1, 2, \ldots, n$ is finite, such an i exists. Since s_n is negative, $i < n$. Now s_{i+1} is equal to either $s_i + 1$ or $s_i - 1$. If $s_{i+1} = s_i + 1$, then s_{i+1} is a positive integer (since s_i is a positive integer). This contradicts the fact that i is the *greatest* integer for which s_i is positive. Therefore, $s_{i+1} = s_i - 1$. Again, if $s_i - 1$ is a positive integer, we have a contradiction. Therefore, $s_{i+1} = s_i - 1 = 0$.

30. A counterexample is $n = 3$.

32. Invalid

$$\begin{array}{l} p \to q \\ \underline{\neg r \to \neg q} \\ \therefore r \end{array}$$

33. Valid

$$\begin{array}{l} p \leftrightarrow r \\ \underline{r \qquad} \\ \therefore p \end{array}$$

35. Valid

$$\begin{array}{l} p \to (q \vee r) \\ \underline{\neg q \wedge \neg r} \\ \therefore \neg p \end{array}$$

37. If 4 megabytes of memory is better than no memory at all, then either we will buy a new computer or we will buy more memory. If we will buy a new computer, then we will not buy more memory. Therefore if 4 megabytes of memory is better than no memory at all, then we will buy a new computer. Invalid.

38. If 4 megabytes of memory is better than no memory at all, then we will buy a new computer. If we will buy a new computer, then we will buy more memory. Therefore, we will buy more memory. Invalid.

40. If 4 megabytes of memory is better than no memory at all, then we will buy a new computer. If we will buy a new computer, then we will buy more memory. 4 megabytes of memory is better than no memory at all. Therefore we will buy more memory. Valid.

42. Valid 43. Valid 45. Valid

46. Suppose that $p_1, p_2, \ldots, p_n$ are all true. Since the argument $p_1, p_2 \ / \therefore p$ is valid, p is true. Since $p, p_3, \ldots, p_n$ are all true and the argument

$$p, p_3, \ldots, p_n \ / \therefore c$$

is valid, c is true. Therefore the argument

$$p_1, p_2, \ldots, p_n \ / \therefore c$$

is valid.

48. Let

$$\begin{aligned} p(x)&: \ x \text{ is good.} \\ q(x)&: \ x \text{ is too long.} \\ r(x)&: \ x \text{ is short enough.} \end{aligned}$$

The domain of discourse is the set of movies. The assertions are

$$\forall x(p(x) \to \neg q(x))$$
$$\forall x(\neg p(x) \to \neg r(x))$$
$$p(\text{“Love Actually”})$$
$$q(\text{“Love Actually”}).$$

By universal instantiation,

$$p(\text{“Love Actually”}) \to \neg q(\text{“Love Actually”}).$$

Since p(“Love Actually”) is true, then $\neg q$(“Love Actually”) is also true. But this contradicts, q(“Love Actually”).

50. Modus ponens
51. Disjunctive syllogism
52. Universal instantiation

54. Let p denote the proposition “there is gas in the car,” let q denote the proposition “I go to the store,” let r denote the proposition “I get a soda,” and let s denote the proposition “the car transmission is defective.” Then the hypotheses are:

$$p \to q, \quad q \to r, \quad \neg r.$$

From $p \to q$ and $q \to r$, we may use the hypothetical syllogism to conclude $p \to r$. From $p \to r$ and $\neg r$, we may use modus tollens to conclude $\neg p$. From $\neg p$, we may use addition to conclude $\neg p \vee s$. Since $\neg p \vee s$ represents the proposition “there is not gas in the car or the car transmission is defective,” we conclude that the conclusion does follow from the hypotheses.

55. Let p denote the proposition “Jill can sing,” let q denote the proposition “Dweezle can play,” let r denote the proposition “I'll buy the compact disk,” and let s denote the proposition “I'll buy the compact disk player.” Then the hypotheses are:

$$(p \vee q) \to r, \quad p, \quad s.$$

From p, we may use addition to conclude $p \vee q$. From $p \vee q$ and $(p \vee q) \to r$, we may use modus ponens to conclude r. From r and s, we may use conjunction to conclude $r \wedge s$. Since $r \wedge s$ represents the proposition “I'll buy the compact disk and the compact disk player,” we conclude that the conclusion does follow from the hypotheses.

57. Let $P(x)$ denote the propositional function “x is a member of the Titans,” let $Q(x)$ denote the propositional function “x can hit the ball a long way,” and let $R(x)$ denote the propositional function “x can make a lot of money.” The hypotheses are

$$P(\text{Ken}), \quad Q(\text{Ken}), \quad \forall x\, Q(x) \to R(x).$$

By universal instantiation, we have $Q(\text{Ken}) \to R(\text{Ken})$. From $Q(\text{Ken})$ and $Q(\text{Ken}) \to R(\text{Ken})$, we may use modus ponens to conclude $R(\text{Ken})$. From $P(\text{Ken})$ and $R(\text{Ken})$, we may use conjunction to conclude $P(\text{Ken}) \wedge R(\text{Ken})$. By existential generalization, we have $\exists x\, P(x) \wedge R(x)$ or, in words, someone is a member of the Titans and can make a lot of money. We conclude that the conclusion does follow from the hypotheses.

58. Let $P(x)$ denote the propositional function "x is in the discrete mathematics class," let $Q(x)$ denote the propositional function "x loves proofs," and let $R(x)$ denote the propositional function "x has taken calculus." The hypotheses are

$$\forall x\ P(x) \to Q(x),\ \ \exists x\ P(x) \wedge \neg R(x).$$

By existential instantiation, we have $P(d) \wedge \neg R(d)$ for some d in the domain of discourse. From $P(d) \wedge \neg R(d)$, we may use simplification to conclude $P(d)$ and $\neg R(d)$. By universal instantiation, we have $P(d) \to Q(d)$. From $P(d) \to Q(d)$ and $P(d)$, we may use modus ponens to conclude $Q(d)$. From $Q(d)$ and $\neg R(d)$, we may use conjunction to conclude $Q(d) \wedge \neg R(d)$. By existential generalization, we have $\exists\ Q(x) \wedge \neg R(x)$ or, in words, someone who loves proofs has never taken calculus. We conclude that the conclusion does follow from the hypotheses.

60. The truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

shows that whenever p is true, $p \vee q$ is also true. Therefore addition is a valid argument.

61. The truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

shows that whenever $p \wedge q$ is true, p is also true. Therefore simplification is a valid argument.

63. The truth table

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

shows that whenever $p \to q$ and $q \to r$ are true, $p \to r$ is also true. Therefore hypothetical syllogism is a valid argument.

64. The truth table

p	q	$p \vee q$	$\neg p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

shows that whenever $p \vee q$ and $\neg p$ are true, q is also true. Therefore disjunctive syllogism is a valid argument.

66. By definition, the proposition $\exists x \in D\ P(x)$ is true when $P(x)$ is true for some x in the domain of discourse. Taking x equal to a $d \in D$ for which $P(d)$ is true, we find that $P(d)$ is true for some $d \in D$.

67. By definition, the proposition $\exists x \in D\ P(x)$ is true when $P(x)$ is true for some x in the domain of discourse. Since $P(d)$ is true for some $d \in D$, $\exists x \in D\ P(x)$ is true.

Section 1.6

3.

1.	$\neg p \vee r$	
2.	$\neg r \vee q$	
3.	p	
4.	$\neg p \vee q$	from 1,2
5.	q	from 3,4

4.

1.	$\neg p \vee t$	
2.	$\neg q \vee s$	
3.	$\neg r \vee s$	
4.	$\neg r \vee t$	
5.	$p \vee q \vee r \vee u$	
6.	$t \vee q \vee r \vee u$	from 1,5
7.	$s \vee t \vee r \vee u$	from 2,6
8.	$s \vee t \vee u$	from 3,7

6. $(p \leftrightarrow r) \equiv (p \to r)(r \to p) \equiv (\neg p \vee r)(\neg r \vee p)$

1.	$\neg p \vee r$	
2.	$\neg r \vee p$	
3.	r	
4.	p	from 2,3

8.

1.	$a \vee \neg b$	
2.	$a \vee c$	
3.	$\neg a$	
4.	$\neg d$	
5.	b	negated conclusion
6.	$\neg b$	from 1,3

Now 5 and 6 combine to give a contradiction.

Section 1.7

In some of these solutions, the Basis Steps are omitted.

2. $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2)$
$$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

3. $1(1!) + 2(2!) + \cdots + n(n!) + (n+1)(n+1)!$
$$= (n+1)! - 1 + (n+1)(n+1)! = (n+2)! - 1$$

5. $1^2 - 2^2 + \cdots + (-1)^{n+1}n^2 + (-1)^{n+2}(n+1)^2$
$$= \frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{n+2}(n+1)^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

6. $1^3 + 2^3 + \cdots + n^3 + (n+1)^3$
$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

8. $\dfrac{1}{2 \cdot 4} + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \cdots + \dfrac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n+2)} + \dfrac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n+2)} + \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+3)}{2 \cdot 4 \cdots (2n+4)}$$

9. $\dfrac{1}{2^2-1} + \dfrac{1}{3^2-1} + \cdots + \dfrac{1}{(n+1)^2-1} + \dfrac{1}{(n+2)^2-1}$
$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} + \frac{1}{(n+2)^2-1}$$
$$= \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

11. The solution is similar to that for Exercise 10, which is given in the book.

13. First note that
$$\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n)(2n+2)} \le \frac{1}{\sqrt{n+1}} \frac{2n+1}{2n+2}.$$
The proof will be complete if we can show that
$$\frac{2n+1}{(2n+2)\sqrt{n+1}} \le \frac{1}{\sqrt{n+2}}.$$
This last inequality is successively equivalent to
$$\left(\frac{n+2}{n+1}\right)^{1/2} \le \frac{2n+2}{2n+1}$$

$$\begin{aligned}
\frac{n+2}{n+1} &\leq \frac{4(n+1)^2}{(2n+1)^2} \\
(n+2)(2n+1)^2 &\leq 4(n+1)^3 \\
4n^3 + 12n^2 + 9n + 2 &\leq 4n^3 + 12n^2 + 12n + 4 \\
-2 &\leq 3n.
\end{aligned}$$

This last inequality is true for all $n \geq 1$.

14. $2(n+1) + 1 = (2n+1) + 2 \leq 2^n + 2 \leq 2^n + 2^n = 2^{n+1}$

16. By the inductive assumption,

$$(a_1 \cdots a_{2^n})^{1/2^n} \leq \frac{a_1 + \cdots + a_{2^n}}{2^n} \tag{1.1}$$

$$(a_{2^n+1} \cdots a_{2^{n+1}})^{1/2^n} \leq \frac{a_{2^n+1} + \cdots + a_{2^{n+1}}}{2^n}. \tag{1.2}$$

Let

$$\begin{aligned}
A &= \frac{a_1 + \cdots + a_{2^n}}{2^n} \\
B &= \frac{a_{2^n+1} + \cdots + a_{2^{n+1}}}{2^n}.
\end{aligned}$$

Multiplying inequalities (1.1) and (1.2), we have

$$(a_1 \cdots a_{2^{n+1}})^{1/2^n} \leq AB. \tag{1.3}$$

Applying the Basis Step to the numbers A and B, we have

$$(AB)^{1/2} \leq \frac{A+B}{2}$$

or, equivalently,

$$AB \leq \left[\frac{a_1 + \cdots + a_{2^{n+1}}}{2^{n+1}}\right]^2. \tag{1.4}$$

Combining inequalities (1.3) and (1.4), we have

$$(a_1 \cdots a_{2^{n+1}})^{1/2^n} \leq \left[\frac{a_1 + \cdots + a_{2^{n+1}}}{2^{n+1}}\right]^2.$$

Taking the square root of both sides of the last inequality gives the desired result.

17.
$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \\
&= 1 + nx + x + nx^2 \\
&\geq 1 + (n+1)x
\end{aligned}$$

19. If we sum the terms in the diagonal direction, we obtain one r, two r^2's, three r^3's, and so on; that is, we obtain the sum

$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n.$$

Multiplying the inequality of Exercise 18 by r yields

$$r^1 + r^2 + \cdots + r^{n+1} < \frac{r}{1-r} \qquad \text{for all } n \geq 0. \tag{1.5}$$

Thus, the sum of the entries in the first column is less than $r/(1-r)$. Similarly, the sum of the entries in the second column is less than $r^2/(1-r)$, and so on. It follows from the preceding discussion that

$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n < \frac{1}{1-r}(r^1 + r^2 + \cdots + r^n).$$

Using inequality (1.5), we obtain the desired result

$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n < \frac{1}{1-r}(r^1 + r^2 + \cdots + r^n) < \left(\frac{1}{1-r}\right)\left(\frac{r}{1-r}\right) = \frac{r}{(1-r)^2}.$$

20. Take $r = 1/2$ in Exercise 19.

22. Assume that $11^n - 6$ is divisible by 5.

$$11^{n+1} - 6 = 11^n \cdot 11 - 6 = 11^n(10+1) - 6 = 10 \cdot 11^n + 11^n - 6,$$

which is divisible by 5.

23. Suppose that 4 divides $6 \cdot 7^n - 2 \cdot 3^n$. Now

$$\begin{aligned} 6 \cdot 7^{n+1} - 2 \cdot 3^{n+1} &= 7 \cdot 6 \cdot 7^n - 3 \cdot 2 \cdot 3^n \\ &= 6 \cdot 7^n - 2 \cdot 3^n + 6 \cdot 6 \cdot 7^n - 2 \cdot 2 \cdot 3^n \\ &= 6 \cdot 7^n - 2 \cdot 3^n + 36 \cdot 7^n - 4 \cdot 3^n. \end{aligned}$$

Since 4 divides $6 \cdot 7^n - 2 \cdot 3^n$, $36 \cdot 7^n$, and $-4 \cdot 3^n$, it divides their sum, which is $6 \cdot 7^{n+1} - 2 \cdot 3^{n+1}$.

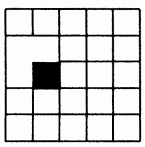
25. $\dfrac{n}{n+1}$

27. We use induction on n, the number of lines, to prove the result. If there is one line, the result is certainly true. Suppose that there are $n > 1$ lines. Remove one line. By the inductive hypothesis, the regions that result may be colored red and green so that no two regions that share an edge are the same color. Now restore the removed line. The regions above (or, if the line is vertical, to the left of) the restored line are colored red and green so that no two regions that share an edge are the same color, and the regions below (or, if the line is vertical, to the right of) the restored line are also colored red and green so that no two regions that share an edge are the same color. Now reverse the color of every region below (or, if the line is vertical, to the right of) the restored line. The regions below (or, if the line is vertical, to the right of) the restored line are still colored red and green so that no two regions that share an edge are the same color. Since the colors below the restored line have been reversed, regions that share an edge that is part of the restored line do not have the same color. Therefore the regions may be colored red and green so that no two regions that share an edge are the same color, and the inductive proof is complete.

28. We assume that we proceed around the circle in clockwise order. The proof is by induction on the number n of zeros with the Basis Step, as usual, omitted.

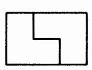
Suppose that the result is true for n zeros, and we are given $n+1$ zeros and $n+1$ ones distributed around a circle. Find a zero followed, in clockwise order, by a one. Temporarily remove these two numbers. By the inductive assumption, it is possible to start at some number and proceed around the circle to the original starting position in such a way that, at any point during the cycle, one has seen at least as many zeros as ones. Notice that this last statement remains true if we restore the removed zero and one.

30. A tromino can cover the square to the left of the missing square as shown

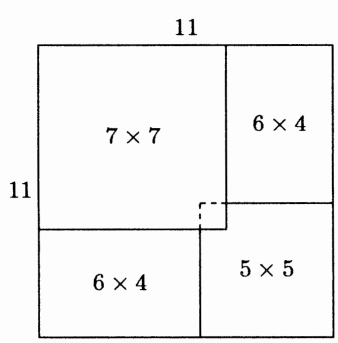


or in a symmetric fashion by reversing "up" and "down." In the first case, it is impossible to cover the two squares in the top row at the extreme left. In the second case, it is impossible to cover the two squares in the bottom row at the extreme left. Therefore, it is impossible to tile the board with trominoes.

31. Such a board can be tiled with ij 2×3 rectangles of the form

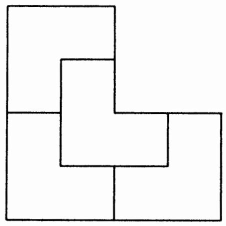


33. By symmetry, we may assume that the missing square is located in the 7×7 subboard shown in the following figure. Exercise 32 shows how to tile this subboard. Exercise 31 shows that the two 6×4 subboards can be tiled. Exercise 29 shows that the 5×5 subboard with a corner square can be tiled. Thus the deficient 11×11 board can be tiled with trominoes.



34. Basis Step ($n = 0$). In this case, the $2^n \times 2^n$ L-shape is a tromino and, so, it is tiled.

Inductive Step. Assume that we can tile a $2^{n-1} \times 2^{n-1}$ L-shape with trominoes. Given a $2^n \times 2^n$ L-shape, divide it into four $2^{n-1} \times 2^{n-1}$ L-shapes:

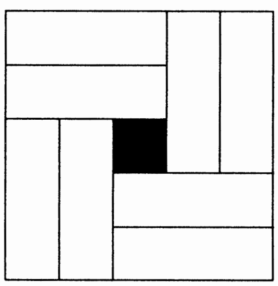


By the inductive assumption, we can tile each of the four $2^{n-1} \times 2^{n-1}$ L-shapes with trominoes. The inductive step is complete.

37. Arguing as in the solution to Exercise 36, the numberings

1	2	3	1	2
3	1	2	3	1
2	3	1	2	3
1	2	3	1	2
3	1	2	3	1

show that the only possibility for the missing square is the center square. This board can be tiled:



38. An argument like those in the solutions to Exercises 36 and 37 shows that the only board that can be tiled with straight trominoes is the one with the missing square in row 3, column 3 (and the three boards symmetric to it).

40. We show only the inductive step. There are two cases: $a[k] < val$ and $a[k] \geq val$. If $a[k] \geq val$, the value of h does not change. Thus, we still have $a[p] < val$, for all p, $i < p \leq h$. After k is incremented, for all p, $h < p < k$, $a[p] \geq val$.

    If $a[k] < val$, then h is incremented and $a[h]$ and $a[k]$ are swapped. Let h_{old} denote the original value of h, and h_{new} denote the new (incremented) value of h. The value at h_{new} is the original $a[k]$. Since this value is less than val, the value of $a[h_{new}]$ is less than val. Thus, for all p, $i < p \leq h_{new}$, $a[p] < val$. After the swap, the value at k becomes h_{new}. By the inductive assumption, this value is greater than or equal to val. Thus after k is incremented, for all p, $h_{new} < p < k$, $a[p] \geq val$.

41. The argument is essentially identical to that of Example 1.7.6 that shows that any $2^n \times 2^n$ deficient board can be tiled with trominoes.

42. Notice that

$$k^3 - 1 = (k-1)[(k-2)(k-4) + 7(k-1)].$$

Since 7 divides $k^3 - 1$, 7 divides $k - 1$ or $(k-2)(k-4) + 7(k-1)$. If 7 divides the latter expression, 7 also divides $(k-2)(k-4)$. If 7 divides $(k-2)(k-4)$, 7 divides either $k-2$ or $k-4$.

44. The Inductive Step fails if either a or b is 1. In this case, the inductive hypothesis is erroneously applied to the pair $a-1, b-1$, which includes a nonpositive integer.

45. To argue by contradiction, one must assume that the proposition fails *for some* $n \geq 2$. The alleged proof assumes that the proposition fails *for all* $n \geq 2$.

46. For $n = 2$, the inequality becomes $\frac{1}{2} + \frac{2}{3} < \frac{4}{3}$, which is true. Thus the Basis Step is true.

Assume that the given statement holds for n. Now

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \frac{n+1}{n+2} < \frac{n^2}{n+1} + \frac{n+1}{n+2}.$$

The Inductive Step will be proved provided

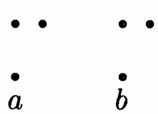
$$\frac{n^2}{n+1} + \frac{n+1}{n+2} < \frac{(n+1)^2}{n+2}.$$

If we multiply the last inequality by $(n+1)(n+2)$, we obtain

$$n^2(n+2) + (n+1)^2 < (n+1)^3,$$

which is readily verified as true.

48. In the following figure



a and b are both survivors.

49. Suppose that there are three persons. The two persons closest together throw at each other, and the third person throws at one of the two closest. Therefore the third person survives. This complete the Basis Step.

Suppose that the assertion is true for n, and consider $n+2$ persons. Again, the closest pair throws at each other. There are now two cases to consider. If the remaining n persons all throw at one another, by the inductive assumption, there is a survivor. If at least one of the remaining n persons throws at one of the closest pair, among the remaining n persons, at most $n-1$ pies are thrown at one another. In this case, someone survives because there are not enough pies to go around. The Inductive Step is complete.

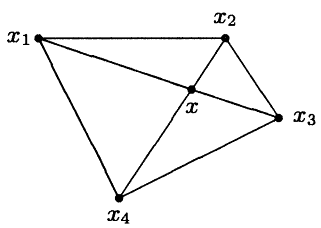
51. The statement is false. In the following figure

a throws a pie the greatest distance, but is not a survivor.

53. Let x_1 be a common point of X_2, X_3, X_4; let x_2 be a common point of X_1, X_3, X_4; let x_3 be a common point of X_1, X_2, X_4; and let x_4 be a common point of X_1, X_2, X_3. Since $x_1, x_2, x_3 \in X_4$, the triangle $x_1x_2x_3$ (perimeter and interior) is in X_4. Similarly, the triangle $x_1x_2x_4$ is in X_3; the triangle $x_1x_3x_4$ is in X_2; and the triangle $x_2x_3x_4$ is in X_1. We consider two cases:

CASE 1: One of the points x_1, x_2, x_3, x_4 is in the triangle whose vertices are the other three points. For example, suppose that x_1 is in triangle $x_2x_3x_4$. Since triangle $x_2x_3x_4$ is in X_1, $x_1 \in X_1$. By definition, $x_1 \in X_2 \cap X_3 \cap X_4$. Therefore, $x_1 \in X_1 \cap X_2 \cap X_3 \cap X_4$.

CASE 2: None of the points x_1, x_2, x_3, x_4 is in the triangle whose vertices are the other three points. In this case, x_1, x_2, x_3, x_4 are the vertices of a convex quadrilateral:



Now the intersection, x, of the diagonals of this quadrilateral belongs to each of the triangles and, thus, to each of X_1, X_2, X_3, X_4.

54. The proof is by induction on n. The Basis Step is $n = 4$, which is Exercise 53.

We turn to the Inductive Step. Assume that if $X_1, \ldots, X_n$ are convex sets, each three of which have a common point, then all n sets have a common point.

Let $X_1, \ldots, X_n, X_{n+1}$ be convex sets, each three of which have a common point. We must show that all $n + 1$ sets have a common point. By Exercise 52,

$$X_1, \ldots, X_{n-1}, X_n \cap X_{n+1} \tag{1.6}$$

are convex sets. We claim that any three of the sets in (1.6) have a common point. The claim is true by hypothesis if the three sets are any of $X_1, \ldots, X_{n-1}$. Consider $X_i, X_j, X_n \cap X_{n+1}$, $i < j \le n-1$. By hypothesis, any three of X_i, X_j, X_n, X_{n+1} have a common point. By Exercise 53, X_i, X_j, X_n, X_{n+1} have a common point. Therefore, $X_i, X_j, X_n \cap X_{n+1}$ have a common point. Thus, any three of the sets in (1.6) have a common point. By the inductive assumption, the sets in (1.6) have a common point. The Inductive Step is complete.

56. We first prove the result for $n = 3$. Let A_1, A_2, A_3 be open intervals such that each pair has a nonempty intersection. Choose $x_1 \in A_1 \cap A_2$, $x_2 \in A_1 \cap A_3$, $x_3 \in A_2 \cap A_3$. Note that if any pair (x_1, x_2 or x_1, x_3 or x_3, x_3) is equal, it is in $A_1 \cap A_2 \cap A_3$. We may assume $x_1 < x_2$. We consider three cases. First suppose that $x_3 < x_1$. Since $x_2, x_3 \in A_3$, $[x_3, x_2] \subseteq A_3$. ($[a, b]$ is the set of all x satisfying $a \le x \le b$.) Thus $x_1 \in A_3$. Therefore $x_1 \in A_1 \cap A_2 \cap A_3$.

Next suppose that $x_1 < x_3 < x_2$. Since $x_1, x_2 \in A_1$, $[x_1, x_2] \subseteq A_1$. Thus $x_3 \in A_1$. Therefore $x_3 \in A_1 \cap A_2 \cap A_3$.

Finally suppose that $x_1 < x_2 < x_3$. Since $x_1, x_3 \in A_2$, $[x_1, x_3] \subseteq A_2$. Thus $x_2 \in A_2$. Therefore $x_2 \in A_1 \cap A_2 \cap A_3$. We have shown that if A_1, A_2, A_3 are open intervals such that each pair has a nonempty intersection, then $A_1 \cap A_2 \cap A_3$ is nonempty.

We now prove that given statement using induction on n. The Basis Step ($n = 2$) is trivial.

Assume that if $I_1, \ldots, I_n$ is a set of open intervals such that each pair has a nonempty intersection, then $I_1 \cap \cdots \cap I_n$ is nonempty. Let $I_1, \ldots, I_{n+1}$ be a set of open intervals such that each pair has a nonempty intersection. Since $I_n \cap I_{n+1}$ is nonempty, it is an open interval. We claim that

$$I_1, \ldots, I_{n-1}, I_n \cap I_{n+1}$$

is a set of open intervals such that each pair has a nonempty intersection. This is certainly true for pairs of the form I_i, I_j, $1 \le i < j \le n-1$. Consider a pair of the form I_i, $i \le n-1$, and $I_n \cap I_{n+1}$. Since each pair among I_i, I_n, I_{n+1} has nonempty intersection, by the case $n = 3$ proved previously, $I_i \cap I_n \cap I_{n+1}$ is nonempty. Therefore,

$$I_1, \ldots, I_{n-1}, I_n \cap I_{n+1}$$

is a set of open intervals such that each pair has a nonempty intersection. By the inductive assumption

$$I_i \cap \cdots \cap I_{n-1} \cap (I_n \cap I_{n+1})$$

is nonempty. The inductive step is complete.

58. 5

59. 5

61. After j rounds, $2, 4, \ldots, 2j$ have been eliminated. At this point, there are 2^i persons. This is exactly the Josephus problem when the number of persons is a power of 2, except that the round begins with person $2j+1$, rather than with person 1. By Exercise 60, person $2j+1$ is the survivor.

62. 977

65. $\Delta a_n = a_{n+1} - a_n = (n+1)^2 - n^2 = 2n+1$. Let $b_n = \Delta a_n$. Then

$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + \cdots + (2n+1) \\ &= 2(1 + 2 + \cdots + n) + (1 + 1 + \cdots + 1) \\ &= 2(1 + 2 + \cdots + n) + n. \end{aligned}$$

By Exercise 64,

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1 = (n+1)^2 - 1^2 = n^2 + 2n.$$

Combining the previous equations, we obtain

$$n^2 + 2n = 2(1 + 2 + \cdots + n) + n.$$

Solving for $1 + 2 + \cdots + n$, we obtain

$$1 + 2 + \cdots + n = \frac{n^2 + 2n - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

66. Let $a_n = n!$. Then

$$\Delta a_n = a_{n+1} - a_n = (n+1)! - n! = n![(n+1) - 1] = n(n!).$$

Let $b_n = \Delta a_n$. Then

$$b_1 + b_2 + \cdots + b_n = 1(1!) + 2(2!) + \cdots + n(n!).$$

By Exercise 64,

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1 = (n+1)! - 1!.$$

Combining the previous equations, we obtain

$$1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1.$$

68. Since p is divisible by k, there exists t_1 such that $p = t_1 k$. Since q is divisible by k, there exists t_2 such that $p = t_2 k$. Now

$$p + q = t_1 k + t_2 k = (t_1 + t_2)k.$$

Therefore, $p + q$ is divisible by k.

Problem-Solving Corner: Mathematical Induction

1. The Basis Step ($n = 0$) is $H_1 \leq 1 + 0$. Since $H_1 = 1$, the Basis Step is true.

   Now assume that $H_{2^n} \leq 1 + n$. Then

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &\leq 1 + n + \frac{1}{2^n+1} + \cdots + \frac{1}{2^n+1} \\ &= 1 + n + \frac{2^n}{2^n+1} \leq 1 + (n+1). \end{aligned}$$

   The Inductive Step is complete.

2. The Basis Step ($n = 1$) is $H_1 = 2H_1 - 1$. Since $H_1 = 1$, the Basis Step is true.

   Now assume that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

   Then

$$\begin{aligned} H_1 + H_2 + \cdots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)\left(H_{n+1} - \frac{1}{n+1}\right) \qquad \text{by Exercise 3} \\ &\quad - n + H_{n+1} \\ &= (n+2)H_{n+1} - (n+1). \end{aligned}$$

   The Inductive Step is complete.

3. $H_{n+1} - \dfrac{1}{n+1} = \left(\dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n} + \dfrac{1}{n+1}\right) - \dfrac{1}{n+1} = \dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n} = H_n$

4. We prove the assertion by induction. The Basis Step is $n = 1$:

$$1 \cdot H_1 = 1 = \frac{3}{2} - \frac{1}{2} = \frac{1 \cdot 2}{2} H_2 - \frac{1 \cdot 2}{4}.$$

For the Inductive Step, assume the assertion if true for n. Now

$$\begin{aligned}
1 \cdot H_1 + \cdots + nH_n + (n+1)H_{n+1} &= \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4} + (n+1)H_{n+1} \\
&= (n+1)H_{n+1}\left[\frac{n}{2} + 1\right] - \frac{n(n+1)}{4} \\
&= H_{n+1}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{n(n+1)}{4} \\
&= \left[H_{n+2} - \frac{1}{n+2}\right]\left[\frac{(n+1)(n+2)}{2}\right] \qquad \text{by Exercise 3} \\
&\quad - \frac{n(n+1)}{4} \\
&= H_{n+2}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{n+1}{2} - \frac{n(n+1)}{4} \\
&= H_{n+2}\left[\frac{(n+1)(n+2)}{2}\right] - \frac{(n+1)(n+2)}{4}.
\end{aligned}$$

Section 1.8

2. Verify directly the cases $n = 24, \ldots, 28$. Assume that the statement is true for postage i satisfying $24 \le i < n$. We must show that we can make n cents postage using only 5-cent and 7-cent stamps. We may assume that $n > 28$. Then $n > n - 5 > 23$. By the inductive assumption, we can make $n - 5$ cents postage using 5-cent and 7-cent stamps. Add a 5-cent stamp to obtain n cents postage.

4. The Basis Step ($n = 6$) is proved by using three 2-cent stamps. Now assume that we can make postage for n cents. If there is at least one 7-cent stamp, replace it by four 2-cent stamps to make $n + 1$ cents postage. If there are no 7-cent stamps, there are at least three 2-cent stamps (because $n \ge 6$). Replace three 2-cent stamps by one 7-cent stamp to make $n+1$ cents postage. The Inductive Step is complete.

5. The Basis Step ($n = 24$) is proved by using two 5-cent stamps and two 7-cent stamps. Now assume that we can make postage for n cents. If there are at least two 7-cent stamps, replace two 7-cent stamps with three 5-cent stamps to make $n+1$ cents postage. If there is exactly one 7-cent stamp, then there are at least four 5-cent stamps (because $n \ge 24$). Replace one 7-cent stamp and four 5-cent stamps with four 7-cent stamps to make $n+1$ cents postage. If there are no 7-cent stamps, then there are at least five 5-cent stamps (again because $n \ge 24$). Replace five 5-cent stamps with three 7-cent stamps and one 5-cent to make $n + 1$ cents postage. The Inductive Step is complete.

7. We omit the Basis Step. For the Inductive Step, we have

$$c_n = c_{\lfloor n/2 \rfloor} + n^2 < 4\left\lfloor \frac{n}{2} \right\rfloor^2 + n^2 \le 4\left(\frac{n}{2}\right)^2 + n^2 = 2n^2 < 4n^2.$$

9. We omit the Basis Step. For the Inductive Step, we have

$$\begin{aligned} c_n = 4c_{\lfloor n/2 \rfloor} + n &\le 4[4(\lfloor n/2 \rfloor - 1)^2] + n \\ &\le 4[4(n/2 - 1)^2] + n \\ &= 4n^2 - 15n + 16 \\ &\le 4(n-1)^2. \end{aligned}$$

The last inequality reduces to $12 \le 7n$, which is true since $n > 1$.

10. We omit the Basis Steps ($n = 2, 3$). We turn to the Inductive Step. Assume that $n \ge 4$. Then $n/2 \ge 2$, so $\lfloor n/2 \rfloor \ge 2$. Then

$$\begin{aligned} c_n = 4c_{\lfloor n/2 \rfloor} + n &> 4(\lfloor n/2 \rfloor + 1)^2/8 + n \\ &\ge 4[(n-1)/2 + 1]^2/8 + n \\ &= (n+1)^2/8 + n \\ &> (n+1)^2/8. \end{aligned}$$

We used the fact that $\lfloor n/2 \rfloor \ge (n-1)/2$ for all n.

13. $q = -6$, $r = 7$

14. $q = 0$, $r = 7$

16. $q = 0$, $r = 0$

17. $q = 1$, $r = 0$

19. If

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

where $n_1 < n_2 < \ldots < n_k$, another representation is

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k + 1} + \frac{1}{n_k(n_k + 1)}$$

20. (b) Since $p/q < 1$, $n > 1$. Since n is the smallest positive integer satisfying $1/n \le p/q$ and $n - 1$ is a positive integer less than n, $p/q < 1/(n-1)$.

(d) We have

$$\frac{p_1}{q_1} = \frac{np - q}{nq} = \frac{p}{q} - \frac{1}{n}. \tag{1.7}$$

Since $1/n < p/q$, equation (1.7) shows that

$$0 < \frac{p_1}{q_1}.$$

Since

$$\frac{p}{q} < \frac{1}{n-1},$$

we have

$$np - p < q$$

or

$$p_1 = np - q < p.$$

The third inequality is established.

Now

$$\frac{p_1}{q_1} < \frac{p}{q_1} = \frac{p}{nq} = \frac{1}{n}\frac{p}{q} < \frac{1}{n} \cdot 1 = \frac{1}{n}. \tag{1.8}$$

In particular,

$$\frac{p_1}{q_1} < 1.$$

We have established the second inequality.

By the inductive assumption, p_1/q_1 can be expressed in Egyptian form. The last equation follows.

(e) See (1.8).

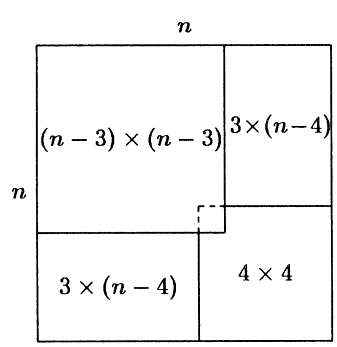
(f) The equation is true because of (d). For any $i = 1, \ldots, k$,

$$\frac{1}{n_i} \leq \frac{1}{n_1} + \cdots + \frac{1}{n_k} = \frac{p_1}{q_1} < \frac{1}{n}.$$

It follows that $n, n_1, \ldots, n_k$ are distinct.

21. $\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$, $\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}$, $\frac{13}{19} = \frac{1}{2} + \frac{1}{6} + \frac{1}{57}$

24. Enclose the missing square in a corner $(n-3) \times (n-3)$ subboard as shown in the following figure. Since 3 divides $n^2 - 1$, 3 also divides $(n-3)^2 - 1$. Now $n - 3$ is odd, $n - 3 > 5$, and 3 divides $(n-3)^2 - 1$, so by Exercise 23, we may tile this subboard. Tile the two $3 \times (n-4)$ subboards using the result of Exercise 31, Section 1.7. Tile the deficient 4×4 subboard using Example 1.7.6. The $n \times n$ board is tiled.



25. If $n = 0$, $d \cdot 1 = d > 0$, and 1 is in X. If $n > 0$, $d(2n) = n(2d) > n$; thus $2n$ is in X. In either case X is nonempty. Since $d > 0$ and $n \geq 0$, X contains only positive integers. By the Well-Ordering Property, X contains a least element $q' > 0$. Then $dq' > n$. Let $q = q' - 1$. We cannot have $dq > n$ (for then q' would not be the *least* element in X); therefore, $dq \leq n$. Let $r = n - dq$. Then $r \geq 0$. Also

$$r = n - dq = n - d(q' - 1) < dq' - d(q' - 1) = d.$$

Therefore, we have found q and r satisfying

$$n = dq + r \qquad 0 \leq r < d.$$

26. We first prove Theorem 1.8.5 for $n > 0$. The Basis Step is $n = 1$. If $d = 1$, we have $n = dq + r$, where $q = n$ and $r = 0$, $0 \leq r < d$. If $d > 1$, we have $n = dq + r$, where $q = 0$ and $r = 1$, $0 \leq r < d$. Thus Theorem 1.8.5 is true for $n = 1$.

Assume that Theorem 1.8.5 holds for n. Then there exists q' and r' such that

$$n = dq' + r' \qquad 0 \leq r' < d.$$

Now

$$n + 1 = dq' + (r' + 1).$$

If $r' < d - 1$, then $r' + 1 < d$. In this case, if we take $q = q'$ and $r = r' + 1$, we have

$$n + 1 = dq + r \qquad 0 \leq r < d.$$

If $r' = d - 1$, we have

$$n + 1 = d(q' + 1).$$

In this case, if we take $q = q' + 1$ and $r = 0$, we have

$$n + 1 = dq + r \qquad 0 \leq r < d.$$

The Inductive Step is complete. Therefore, Theorem 1.8.5 is true for all $n > 0$.

If $n = 0$, we may write

$$n = dq + r,$$

where $q = r = 0$. Therefore, Theorem 1.8.5 is true for $n = 0$.

Finally, suppose that $n < 0$. Then $-n > 0$, so by the first part of the proof, there exist q' and r' such that

$$-n = dq' + r' \qquad 0 \leq r' < d.$$

If $r' = 0$, we may take $q = -q'$ and $r = 0$ to obtain

$$n = dq + 0.$$

If $r' > 0$, we take $q = -q' - 1$ and $r = d - r'$. Then $0 < r < d$ and

$$n = d(-q') - r' = d(q + 1) + (r - d) = dq + r.$$

Therefore, Theorem 1.8.5 is true for $n < 0$.