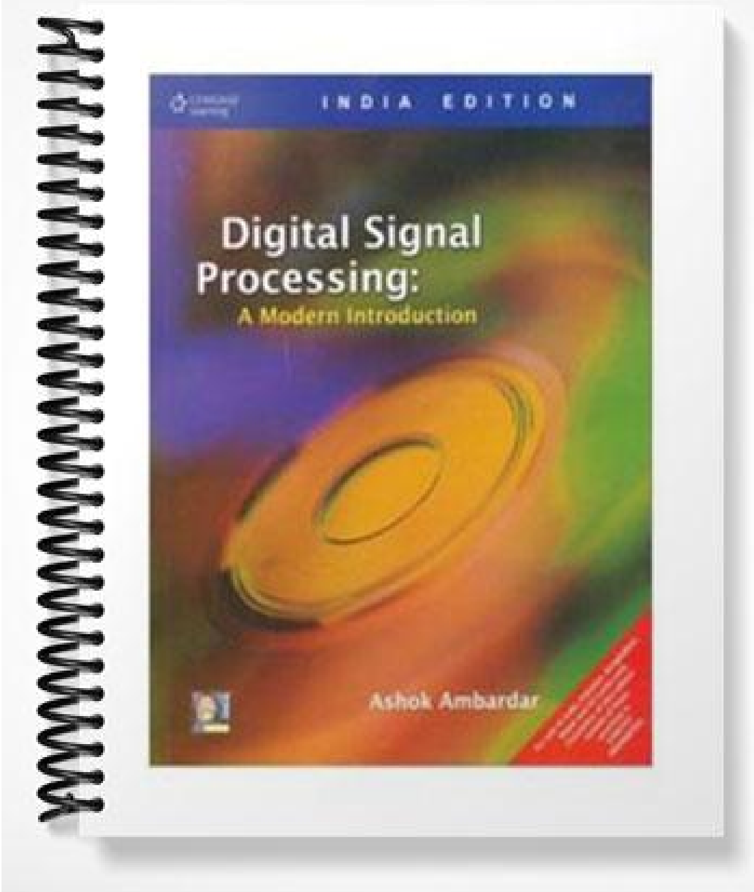
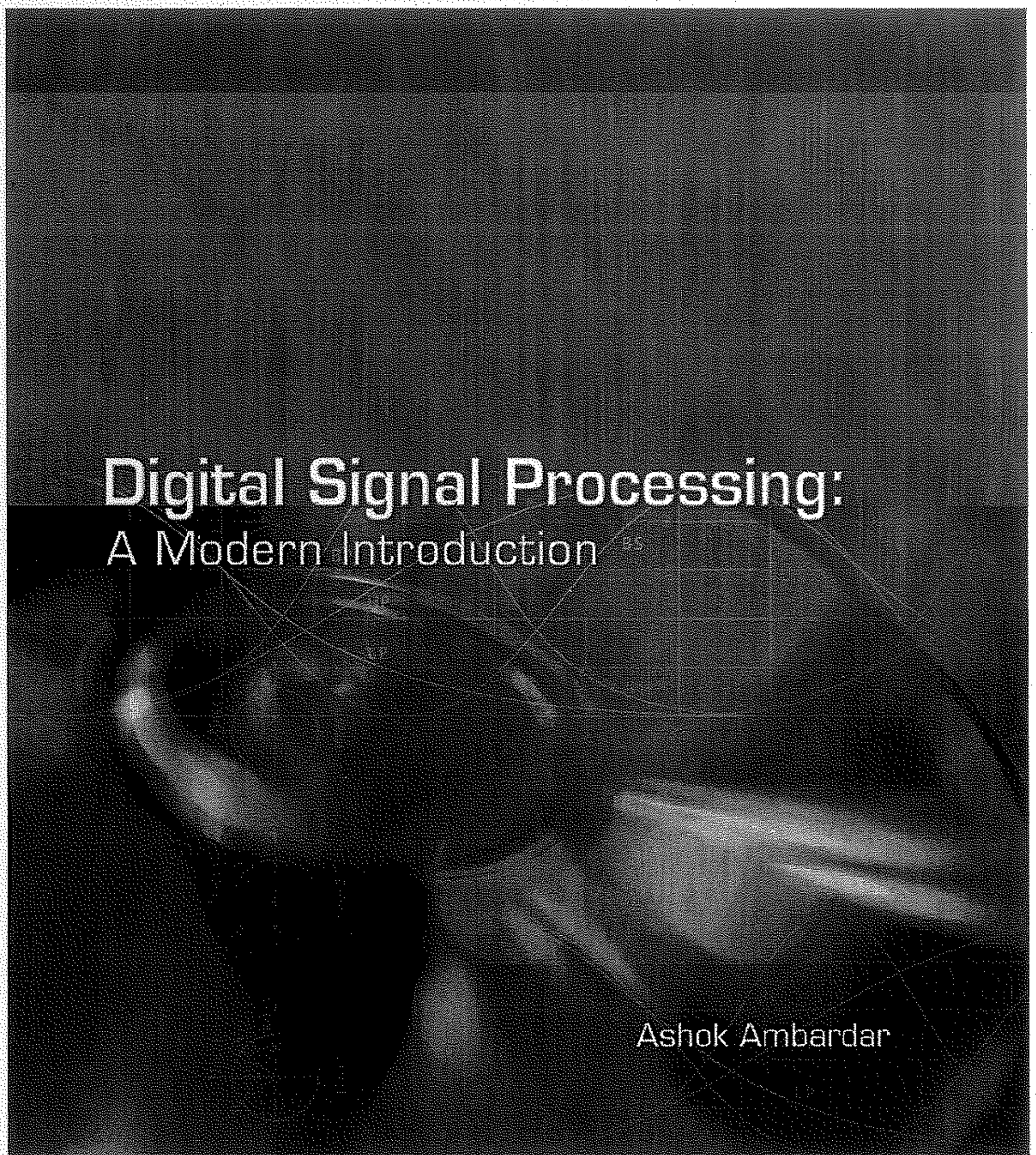
SOLUTIONS MANUAL



INSTRUCTOR'S SOLUTIONS MANUAL

to accompany



ISBN 0-534-40510-X

ASHOK AMBARDAR

ISBN 0-534-40510-X
9 780534 4051

THOMSON
ENGINEERING

THOMSON

ENGINEERING

For more information contact Nelson, 1120 Birchmount Road, Scarborough, Ontario M1K 5G4. Or you can visit our Internet site at www.nelson.com.

Table of Contents

Chapter 2

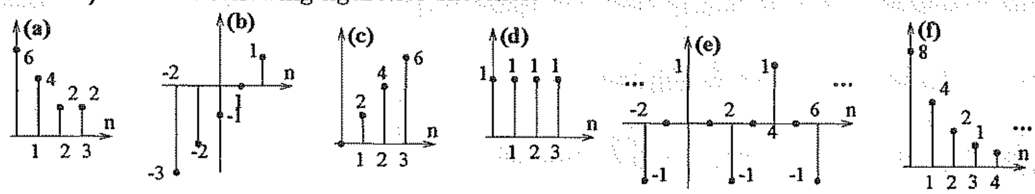
DISCRETE SIGNALS

2.1 (Discrete Signals) Sketch each signal and find its energy or power as appropriate.

(a) $x[n] = \{\overset{\Downarrow}{6},\ 4,\ 2,\ 2\}$ (b) $y[n] = \{-3,\ -2,\ \overset{\Downarrow}{-1},\ 0,\ 1\}$
(c) $f[n] = \{\overset{\Downarrow}{0},\ 2,\ 4,\ 6\}$ (d) $g[n] = u[n] - u[n-4]$
(e) $p[n] = \cos(n\pi/2)$ (f) $q[n] = 8(0.5)^n u[n]$

[**Hints and Suggestions:** Only $p[n]$ is a power signal. The rest have finite energy.]

(Solution) See the following figure for sketches.



(a) $x[n] = \{\overset{\Downarrow}{6},\ 4,\ 2,\ 2\}$ Energy signal. $E = \sum x^2[n] = 36 + 16 + 4 + 4 = 60$

(b) $y[n] = \{-3,\ -2,\ \overset{\Downarrow}{-1},\ 0,\ 1\}$ Energy signal. $E = \sum y^2[n] = 9 + 4 + 1 + 1 = 15$

(c) $f[n] = \{\overset{\Downarrow}{0},\ 2,\ 4,\ 6\}$ Energy signal. $E = \sum f^2[n] = 4 + 16 + 36 = 56$

(d) $g[n] = u[n] - u[n-4]$ Energy signal. $E = \sum g^2[n] = 1 + 1 + 1 + 1 = 4$

(e) $p[n] = \cos(n\pi/2)$ Period $N = 4$. Power signal. $P = \frac{1}{N}\sum_{n=0}^{N-1} p^2[n] = \frac{1}{4}\sum_{n=0}^{3} p^2[n] = \frac{1}{4}(1+1) = 0.5$

(f) $q[n] = 8(0.5)^n u[n]$ Energy signal. $E = \sum_{n=0}^{\infty} q^2[n] = 6464\sum_{n=0}^{\infty}(0.25)^n = \dfrac{64}{1-0.25} = 85.3333$

2.2 (Signal Duration) Use examples to argue that the product of a right-sided and a left-sided discrete-time signal is always time-limited or identically zero.

[**Hints and Suggestions:** Select simple signals that either overlap or do not overlap.]

(Solution) The product of a right-sided and a left-sided discrete-time signal is always time-limited or identically zero.

Example 1: $u[n-3]$ and $u[-n]$. Their product is zero.

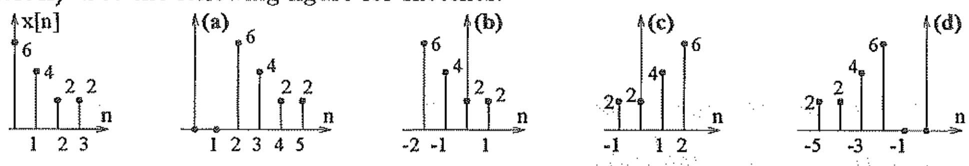
Example 2: $u[n+2]$ and $u[-n]$. Their product is $\{\ldots,0,0,1,1,\overset{\Downarrow}{1},0,0,\ldots\}$

2.3 (Operations) Let $x[n] = \{\overset{\Downarrow}{6},\ 4,\ 2,\ 2\}$. Sketch the following signals and find their signal energy.

(a) $y[n] = x[n-2]$ (b) $f[n] = x[n+2]$ (c) $g[n] = x[-n+2]$ (d) $h[n] = x[-n-2]$

[**Hints and Suggestions:** Note that $g[n]$ is a folded version of $f[n]$.]

(Solution) See the following figure for sketches.



With $x[n] = \{\overset{\Downarrow}{6},\ 4,\ 2,\ 2\}$, we find

(a) $y[n] = x[n-2] = \{\overset{\Downarrow}{0}, 0, 6, 4, 2, 2\}$ (shift right by 2).

(b) $f[n] = x[n+2] = \{6, 4, \overset{\Downarrow}{2}, 2\}$ (shift left by 2).

(c) $g[n] = x[-n+2] = \{2, \overset{\Downarrow}{2}, 4, 6\}$ (fold $f[n]$)

(d) $h[n] = x[-n-2] = \{2, 2, 4, 6, 0, \overset{\Downarrow}{0}\}$ (fold $y[n]$)

The energy in each signal is $E = 36 + 16 + 4 + 4 = 60$

2.4 (Operations) Let $x[n] = 8(0.5)^n(u[n+1] - u[n-3])$. Sketch the following signals.

(a) $y[n] = x[n-3]$ (b) $f[n] = x[n+1]$ (c) $g[n] = x[-n+4]$ (d) $h[n] = x[-n-2]$

[**Hints and Suggestions:** Note that $x[n]$ contains 5 samples (from $n = -1$ to $n = 3$). To display the marker for $y[n]$ (which starts at $n = 2$), we include two zeros at $n = 0$ (the marker) and $n = 1$.]

(Solution) Note that $x[n] = 8(0.5)^n(u[n+1] - u[n-3]) = \{16,\ \overset{\Downarrow}{8},\ 4,\ 2,\ 1\}$. Then

(a) $y[n] = x[n-3] = \{\overset{\Downarrow}{0},\ 0,\ 16,\ 8,\ 4,\ 2,\ 1\}$ (shift right by 3)

(b) $f[n] = x[n+1] = \{16, 8,\ \overset{\Downarrow}{4},\ 2,\ 1\}$ (shift left by 1)

(c) $g[n] = x[-n+4] = \{\overset{\Downarrow}{0},\ 1,\ 2,\ 4,\ 8,\ 16\}$ (shift left by 4, then fold)

(d) $h[n] = x[-n-2] = \{1,\ 2,\ 4,\ 8,\ 16,\ \overset{\Downarrow}{0}\}$ (shift right by 2, then fold)

2.5 (Energy and Power) Classify the following as energy signals, power signals, or neither and find the energy or power as appropriate.

(a) $x[n] = 2^n u[-n]$ **(b)** $y[n] = 2^n u[-n-1]$ **(c)** $f[n] = \cos(n\pi)$
(d) $g[n] = \cos(n\pi/2)$ **(e)** $p[n] = \dfrac{1}{n}u[n-1]$ **(f)** $q[n] = \dfrac{1}{\sqrt{n}}u[n-1]$
(g) $r[n] = \dfrac{1}{n^2}u[n-1]$ **(h)** $s[n] = e^{jn\pi}$ **(i)** $d[n] = e^{jn\pi/2}$
(j) $t[n] = e^{(j+1)n\pi/4}$ **(k)** $v[n] = j^{n/4}$ **(l)** $w[n] = (\sqrt{j})^n + (\sqrt{j})^{-n}$

[**Hints and Suggestions:** For $x[n]$ and $y[n]$, $2^{2n} = 4^n = (0.25)^{-n}$. Sum this from $n = -\infty$ to $n = 0$ (or $n = -1$) using a change of variable ($n \to -n$) in the summation. For $p[n]$, sum $1/n^2$ over $n = 1$ to $n = \infty$ using tables. For $q[n]$, the sum of $1/n$ from $n = 1$ to $n = \infty$ does not converge! For $t[n]$, separate the exponentials. To compute the power for $s[n]$ and $d[n]$, note that $|s[n]| = |d[n]| = 1$. For $v[n]$, use $j = e^{j\pi/2}$. For $w[n]$, set $\sqrt{j} = e^{j\pi/4}$ and use Euler's relation to convert to a sinusoid.]

(Solution)

(a) $x^2[n] = 2^{2n}u[-n] = (0.25)^{-n}u[-n]$. So, $E = \sum_{n=-\infty}^{0}(0.25)^{-n} = \sum_{n=0}^{\infty}(0.25)^n = \dfrac{1}{1-0.25} = \dfrac{4}{3} = 1.3333$

(b) $y^2[n] = 2^{2n}u[-n-1] = (0.25)^{-n}u[-n-1]$. So, $E = \sum_{n=-\infty}^{-1}(0.25)^{-n} = \sum_{n=1}^{\infty}(0.25)^n = \dfrac{0.25}{1-0.25} = \dfrac{1}{3} = 0.3333$

(c) $f[n] = \cos(n\pi)$. Periodic, $F = 0.5$ and $N = 2$. $x[n] = \{1, \ -1\}$ for one period.

So, $P = \frac{1}{N}\sum_{n=0}^{1} f^2[n] = 0.5(1+1) = 1$

(d) $g[n] = \cos(n\pi/2)$. Periodic, $F = 0.25$ and $N = 4$. $g[n] = \{1, 0, -1, 0\}$ for one period.

So, $P = \frac{1}{N}\sum_{n=0}^{3} g^2[n] = 0.25(1+1) = 0.5$

(e) $p[n] = \dfrac{1}{n}u[n-1]$. So, $E = \sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$ (from tables)

(f) $q[n] = \dfrac{1}{\sqrt{n}}u[n-1]$. Neither a power signal nor an energy signal (because $q^2[n]$ does not decay faster than $1/n$ and $\sum q^2[n] = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ does not converge).

(g) $r[n] = \dfrac{1}{n^2}u[n-1]$. So, $E = \sum_{n=1}^{\infty}\dfrac{1}{n^4} = \dfrac{\pi^4}{90}$ (from tables)

(h) $s[n] = e^{jn\pi}$. Periodic, $F = 0.5$ and $N = 2$ and $|s[n]| = 1$.

So, $P = \frac{1}{N}\sum_{n=0}^{1} |s[n]|^2 = 0.5(1+1) = 1$

(i) $d[n] = e^{jn\pi/2}$. Periodic, $F = 0.25$ and $N = 4$ and $|d[n]| = 1$.

So, $P = \frac{1}{N}\sum_{n=0}^{3} d^2[n] = 0.25(1+1+1+1) = 1$

(j) $t[n] = e^{(j+1)n\pi/4} = e^{n\pi/4}e^{jn\pi/4}$. Neither power nor energy ($e^{n\pi/4} = (2.1933)^n$ is a growing exponential).

(k) $v[n] = j^n/4 = e^{jn\pi/8}$. Periodic, $F = \frac{1}{16}$ and $N = 16$ and $|v[n]| = 1$.

So, $P = \frac{1}{N}\sum_{n=0}^{15} v^2[n] = \frac{1}{16}(16) = 1$

(l) $w[n] = (\sqrt{j})^n + (\sqrt{j})^{-n} = e^{jn\pi/2} + e^{-jn\pi/2} = 2\cos(n\pi/2)$. Periodic, $F = 0.25$ and $N = 4$ and $w[n] = \{2, 0, -2, 0\}$ for one period.

So, $P = \frac{1}{N}\sum_{n=0}^{3} w^2[n] = 0.25(4+4) = 2$

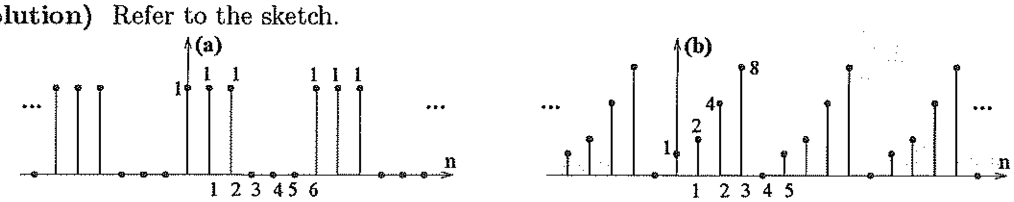
2.6 (Energy and Power) Sketch each of the following signals, classify as an energy signal or power signal, and find the energy or power as appropriate.

(a) $x[n] = \sum_{k=-\infty}^{\infty} y[n-kN]$, where $y[n] = u[n] - u[n-3]$ and $N = 6$

(b) $f[n] = \sum_{k=-\infty}^{\infty} (2)^{n-5k}(u[n-5k] - u[n-5k-4])$

[**Hints and Suggestions:** The period of $x[n]$ is $N = 6$. With $y[n] = u[n] - u[n-3]$, one period of $x[n]$ (starting at $n = 0$) is $\{1,\ 1,\ 1,\ 0,\ 0,\ 0\}$. The period of $f[n]$ is $N = 5$. Its one period (starting at $n = 0$) contains four samples from $2^n(u[n] - u[n-4])$ and one trailing zero.]

(Solution) Refer to the sketch.



(a) $x[n] = \sum_{k=-\infty}^{\infty} y[n-kN]$ where $y[n] = u[n] - u[n-3]$ and $N = 6$. Thus $x[n] = \{1, 1, 1, 0, 0, 0\}$ for one period.

So, $P = \frac{1}{6}\sum_{n=0}^{5} x^2[n] = \frac{1}{6}(1+1+1) = 0.5$

(b) $f[n] = \sum_{k=-\infty}^{\infty} (2)^{n-5k}(u[n-5k] - u[n-5k-4]) = \sum_{k=-\infty}^{\infty} g[n-kN]$

where $N = 5$ and $g[n] = 2^n(u[n] - u[n-4])$.

Thus, $f[n] = \{1, 2, 4, 8, 0\}$ for one period and $P = \frac{1}{5}\sum_{n=0}^{4} f^2[n] = \frac{1}{5}(1 + 4 + 16 + 64) = 17$

2.7 (Decimation and Interpolation) Let $x[n] = \{4,\ 0,\ \overset{\Downarrow}{2},\ -1,\ 3\}$. Find and sketch the following signals and compare their signal energy with the energy in $x[n]$.

(a) The decimated signal $d[n] = x[2n]$

(b) The *zero-interpolated* signal $f[n] = x[\frac{n}{2}]$

(c) The *step-interpolated* signal $g[n] = x[\frac{n}{2}]$

(d) The *linearly interpolated* signal $h[n] = x[\frac{n}{2}]$

[**Hints and Suggestions:** To get $d[n]$, retain the samples of $x[n]$ at $n = 0, \pm 2, \pm 4, \ldots$. Assuming that the interpolated signals will be twice the length of $x[n]$, the last sample will be 0 for $f[n]$, 3 for $g[n]$ and 1.5 (the linearly interpolated value with $x[n] = 0,\ n > 2$) for $h[n]$.]

(Solution) Let $x[n] = \{4,\ 0,\ \overset{\Downarrow}{2},\ -1,\ 3\}$. Find and sketch each of the following signals and compare their signal energy with the energy in $x[n]$.

(a) $d[n] = x[2n] = \{4,\ \overset{\Downarrow}{2},\ 3\}$.

(b) *zero-interpolated* $f[n] = x[\frac{n}{2}] = \{4,\ 0,\ 0,\ 0,\ \overset{\Downarrow}{2},\ 0,\ -1,\ 0,\ 3,\ 0\}$.

(c) *step-interpolated* $g[n] = x[\frac{n}{2}] = \{4,\ 4,\ 0,\ 0,\ \overset{\Downarrow}{2},\ 2,\ -1,\ -1,\ 3,\ 3\}$.

(d) *linearly-interpolated* $h[n] = x[\frac{n}{2}] = \{4,\ 2,\ 0,\ 1,\ \overset{\Downarrow}{2},\ 0.5,\ -1,\ 1,\ 3,\ 1.5\}$ (last value interpolated assuming next sample is zero).

2.8 (Interpolation and Decimation) Let $x[n] = 4\,\text{tri}(n/4)$. Sketch the following signals and describe how they differ.

(a) $x[\frac{2}{3}n]$, using zero interpolation followed by decimation

(b) $x[\frac{2}{3}n]$, using step interpolation followed by decimation

(c) $x[\frac{2}{3}n]$, using decimation followed by zero interpolation

(d) $x[\frac{2}{3}n]$, using decimation followed by step interpolation

(Solution) $x[n] = 4\text{tri}(n/4) = \{0, 1, 2, 3, \overset{\Downarrow}{4}, 3, 2, 1, 0\}$

(a) $x[n/3] = \{0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0, \overset{\Downarrow}{4}, 0, 0, 3, 0, 0, 2, 0, 0, 1, 0, 0, 0, 0, 0\}$ (zero interpolation)

$x[2n/3] = \{0, 0, 0, 2, 0, 0, \overset{\Downarrow}{4}, 0, 0, 2, 0, 0, 0, 0\}$ (decimation)

(b) $x[n/3] = \{0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \overset{\Downarrow}{4}, 4, 4, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0, 0, 0\}$ (step interpolation)

$x[2n/3] = \{0, 0, 1, 2, 2, 3, \overset{\Downarrow}{4}, 4, 3, 2, 2, 1, 0, 0\}$ (decimation)

(c) $x[2n] = \{0, 2, \overset{\Downarrow}{4}, 2, 0\}$ (decimation)

$x[2n/3] = \{0, 0, 0, 2, 0, 0, \overset{\Downarrow}{4}, 0, 0, 2, 0, 0, 0, 0, 0\}$ (zero interpolation)

(d) $x[2n] = \{0, 2, \overset{\Downarrow}{4}, 2, 0\}$ (decimation)

$x[2n/3] = \{0, 0, 0, 2, 2, 2, \overset{\Downarrow}{4}, 4, 4, 2, 2, 2, 0, 0, 0\}$ (step interpolation).

2.9 (Fractional Delay) Starting with $x[n]$, we can generate the signal $x[n-2]$ (using a delay of 2) or $x[2n-3]$ (using a delay of 3 followed by decimation). However, to generate a *fractional delay* of the form $x[n-\frac{M}{N}]$ requires a delay, interpolation, and decimation!

(a) Describe the sequence of operations required to generate $x[n-\frac{2}{3}]$ from $x[n]$.

(b) Let $x[n] = \{\overset{\Downarrow}{1}, 4, 7, 10, 13\}$. Sketch $x[n]$ and $x[n-\frac{2}{3}]$. Use *linear* interpolation where required.

(c) Generalize the results of part (a) to generate $x[n-\frac{M}{N}]$ from $x[n]$. Any restrictions on M and N?

[**Hints and Suggestions:** In part (a) the sequence of operations requires interpolation, delay (by 2) and decimation. The interpolation and decimation factors are identical.]

(Solution)

(a) $x[n] \Longrightarrow$ (interpolate by 3) $x[\frac{n}{3}] \longrightarrow$ (delay by 2) $x[\frac{n-2}{3}] \longrightarrow$ (decimate by 3) $x[n-\frac{2}{3}]$.

(b) $x[n] = \{\overset{\Downarrow}{1}, 4, 7, 10, 13\}$.

$x[\frac{n}{3}] = \{\overset{\Downarrow}{1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \frac{26}{3}, \frac{13}{3}, 0\}$ (linear interpolation).

$x[\frac{n-2}{3}] = \{\overset{\Downarrow}{0}, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \frac{26}{3}, \frac{13}{3}, 0\}$

$x[n-\frac{2}{3}] = \{\overset{\Downarrow}{0}, 2, 5, 8, 11, \frac{26}{3}\}$

(c) $x[n] \Longrightarrow$ (interpolate by N) $x[\frac{n}{N}] \longrightarrow$ (delay by M) $x[\frac{n-M}{N}] \longrightarrow$ (decimate by N) $x[n-\frac{M}{N}]$.
Restriction: M and N must be integers.

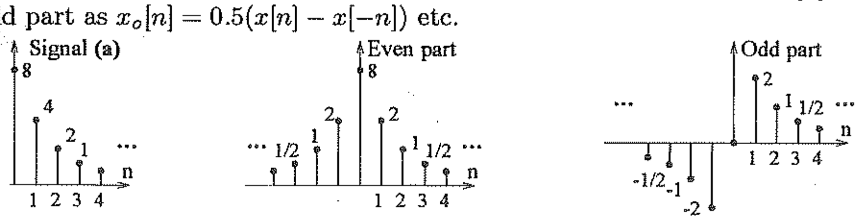
2.10 (Symmetry) Sketch each signal and its even and odd parts.

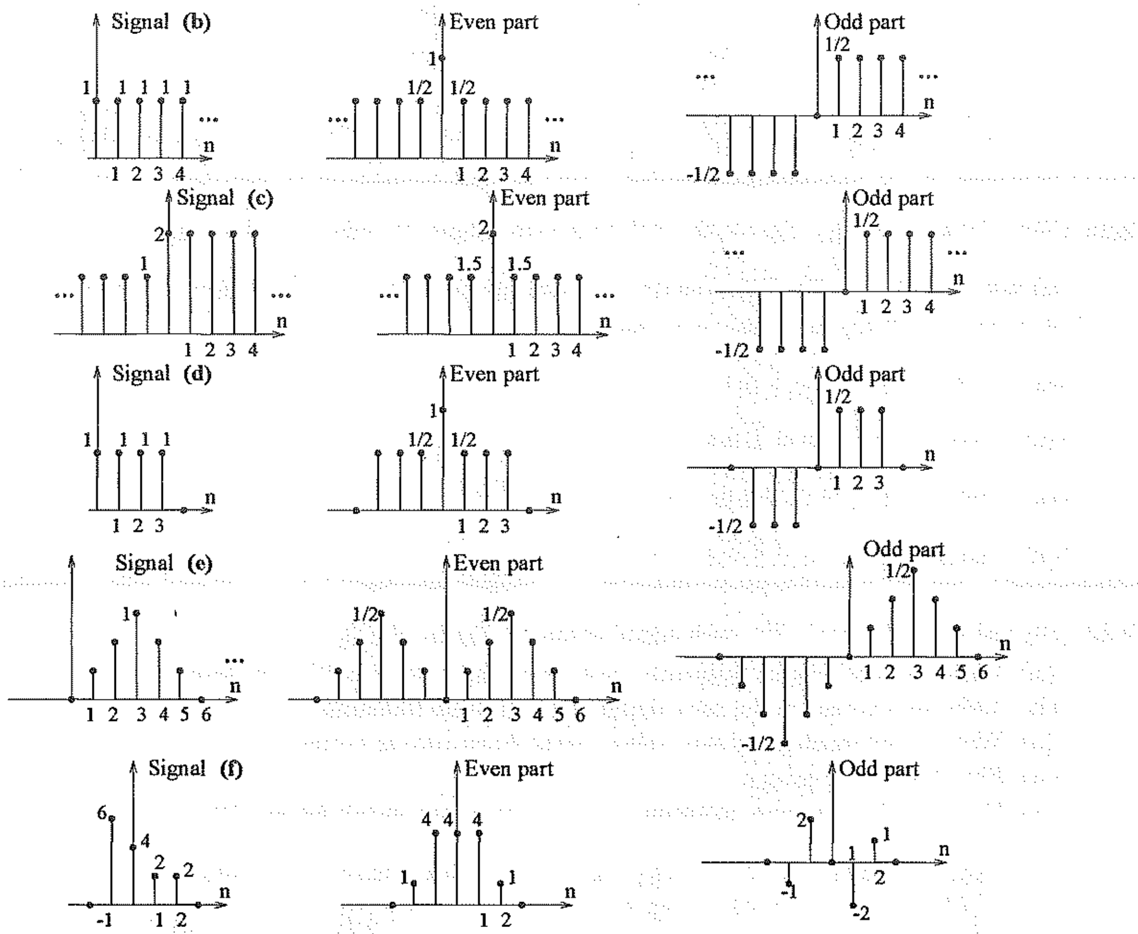
(a) $x[n] = 8(0.5)^n u[n]$ (b) $y[n] = u[n]$ (c) $f[n] = 1 + u[n]$

(d) $g[n] = u[n] - u[n-4]$ (e) $p[n] = \text{tri}(\frac{n-3}{3})$ (f) $q[n] = \{6, \overset{\Downarrow}{4}, 2, 2\}$

[**Hints and Suggestions:** Confirm the appropriate symmetry for each even part and each odd part. For each even part, the sample at $n = 0$ must equal the original sample value. For each odd part, the sample at $n = 0$ must equal zero.]

(Solution) See the following figures (not to scale). We find the even part as $x_e[n] = 0.5(x[n] + x[-n])$ and the odd part as $x_o[n] = 0.5(x[n] - x[-n])$ etc.





2.11 (Sketching Discrete Signals) Sketch each of the following signals:

(a) $x[n] = r[n+2] - r[n-2] - 4u[n-6]$ **(b)** $y[n] = \text{rect}(\frac{n}{6})$
(c) $f[n] = \text{rect}(\frac{n-2}{4})$ **(d)** $g[n] = 6\,\text{tri}(\frac{n-4}{3})$

[**Hints and Suggestions:** Note that $f[n]$ is a rectangular pulse centered at $n = 2$ with 5 samples. Also, $g[n]$ is a triangular pulse centered at $n = 4$ with 7 samples (including the zero-valued end samples).]

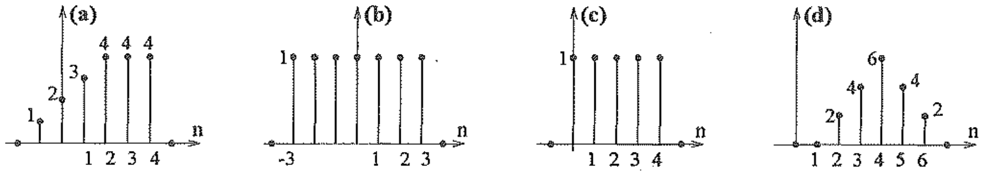
(Solution) See the following figure for sketches. We note that

(a) $x[n] = r[n+2] - r[n-2] - 4u[n-6]$ is easily sketched as a sum of steps and ramps.

(b) $y[n] = \text{rect}(\frac{n}{6})$ is a 7-sample rectangular pulse from $n = -3$ to $n = 3$

(c) $f[n] = \text{rect}(\frac{n-2}{4})$ is a 5-sample rectangular pulse centered at $n = 2$.

(d) $g[n] = 6\text{tri}(\frac{n-4}{3})$ is a 7-sample triangular pulse centered at $n = 4$ (with end values of zero).



2.12 (Sketching Signals) Sketch the following signals and describe how they are related.

(a) $x[n] = \delta[n]$ (b) $f[n] = \text{rect}(n)$ (c) $g[n] = \text{tri}(n)$ (d) $h[n] = \text{sinc}(n)$

(Solution) All represent the same signal because

$$x[n] = \delta[n] = \{\ldots,0,0,\overset{\Downarrow}{1},0,0,\ldots\}$$

$$f[n] = \text{rect}[n] = \{\ldots,0,0,\overset{\Downarrow}{1},0,0,\ldots\}$$

$$g[n] = \text{tri}[n] = \{\ldots,0,0,\overset{\Downarrow}{1},0,0,\ldots\}$$

$$h[n] = \text{sinc}[n] = \{\ldots,0,0,\overset{\Downarrow}{1},0,0,\ldots\}$$

2.13 (Signal Description) For each signal shown in Figure P2.13,

(a) Write out the numeric sequence, and mark the index $n = 0$ by an arrow.
(b) Write an expression for each signal using impulse functions.
(c) Write an expression for each signal using steps and/or ramps.
(d) Find the signal energy.
(e) Find the signal power, assuming that the sequence shown repeats itself.

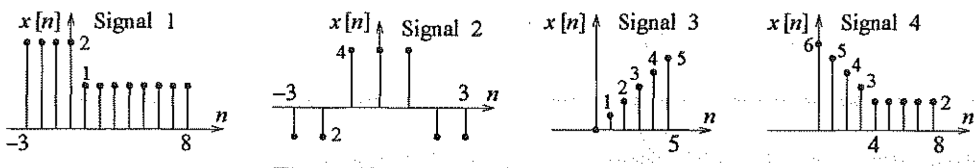


Figure P2.13 Signals for Problem 2.13

[**Hints and Suggestions:** In part (c), all signals must be turned off (by step functions) and any ramps must be first flattened out (by other ramps) . For example, signal 3 $= r[n] - r[n-5] - 5u[n-6]$. The second term flattens out the first ramp and last term turns the signal off after $n = 5$.]

(Solution) Refer to the sketches

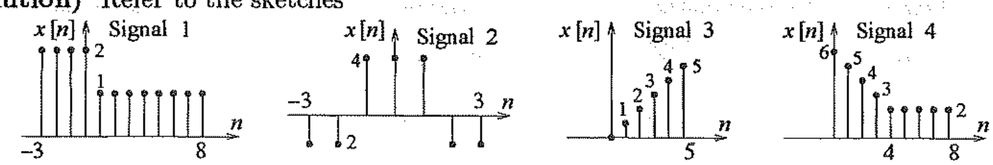


Figure P2.13. Signals for Problem 2.13.

(a) Signals as a numeric sequence:

- **(Signal 1:)** $x[n] = \{2, 2, 2, \overset{\Downarrow}{2}, 1, 1, 1, 1, 1, 1, 1, 1\}$
- **(Signal 2:)** $x[n] = \{-2, -2, 4, \overset{\Downarrow}{4}, 4, -2, -2\}$

- **(Signal 3:)** $x[n] = \{\overset{\Downarrow}{0}, 1, 2, 3, 4, 5\}$
- **(Signal 4:)** $x[n] = \{\overset{\Downarrow}{6}, 5, 4, 3, 2, 2, 2, 2, 2\}$

(b) Signal representation by impulses

- **(Signal 1:)** $x[n] = 2(\delta[n+3] + \delta[n+2] + \delta[n+1] + \delta[n]) + \delta[n-1] + \delta[n-2] + \ldots + \delta[n-8]$
- **(Signal 2:)** $x[n] = -2\delta[n+3] - 2\delta[n+2] + 4\delta[n+1] + 4\delta[n] + 4\delta[n-1] - 2\delta[n-2] - 2\delta[n-3]$
- **(Signal 3:)** $x[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + 5\delta[n-5]$
- **(Signal 4:)** $x[n] = 6\delta[n] + 5\delta[n-1] + 4\delta[n-2] + 3\delta[n-3]) + 2(\delta[n-4] + \delta[n-5] + \ldots + \delta[n-8]$

(c) Signal representation by steps and ramps:

- **(Signal 1:)** $x[n] = 2u[n+3] - u[n-1] - u[n-9]$
- **(Signal 2:)** $x[n] = -2u[n+3] + 6u[n+1] - 6u[n-2] + 2u[n-4]$
- **(Signal 3:)** $x[n] = r[n] - r[n-5] - 5u[n-6]$
- **(Signal 4:)** $x[n] = 6u[n] - r[n] + r[n-4] - 2u[n-9]$

(d) and (e) Signal energy and signal power (if periodic)

Signal 1: $E = \sum x^2[n] = 24 \quad N = 12 \quad P = \frac{1}{N}\sum x^2[n] = \frac{24}{12} = 2$

Signal 2: $E = \sum x^2[n] = 64 \quad N = 7 \quad P = \frac{1}{N}\sum x^2[n] = \frac{64}{7} = 9.1429$

Signal 3: $E = \sum x^2[n] = 55 \quad N = 6 \quad P = \frac{1}{N}\sum x^2[n] = \frac{55}{6} = 9.1667$

Signal 4: $E = \sum x^2[n] = 106 \quad N = 9 \quad P = \frac{1}{N}\sum x^2[n] = \frac{106}{9} = 11.7778$

2.14 (Discrete Exponentials) A causal discrete exponential has the form $x[n] = \alpha^n u[n]$.

(a) Assume that α is real and positive. Pick convenient values for $\alpha > 1$, $\alpha = 1$, and $\alpha < 1$; sketch $x[n]$; and describe the nature of the sketch for each choice of α.

(b) Assume that α is real and negative. Pick convenient values for $\alpha < -1$, $\alpha = -1$, and $\alpha > -1$; sketch $x[n]$; and describe the nature of the sketch for each choice of α.

(c) Assume that α is complex and of the form $\alpha = Ae^{j\theta}$, where A is a positive constant. Pick convenient values for θ and for $A < 1$, $A = 1$, and $A > 1$; sketch the real part and imaginary part of $x[n]$ for each choice of A; and describe the nature of each sketch.

(d) Assume that α is complex and of the form $\alpha = Ae^{j\theta}$, where A is a positive constant. Pick convenient values for θ and for $A < 1$, $A = 1$, and $A > 1$; sketch the magnitude and imaginary phase of $x[n]$ for each choice of A; and describe the nature of each sketch.

(Solution)

(a) $\alpha = 0.5$ $\quad (0.5)^n u[n] = \{1, 0.5, 0.25, 0.125, \ldots\}$. This is a decaying exponential.

$\alpha = 1$ $\quad (1)^n u[n] = \{1, 1, 1, 1, \ldots\}$. This is a unit step (constant).

$\alpha = 2$ $\quad (2)^n u[n] = \{1, 2, 4, 8, \ldots\}$. This is a growing exponential.

(b) $\alpha = -0.5$ $\quad (-0.5)^n u[n] = \{1, -0.5, 0.25, -0.125, \ldots\}$. This is a decaying exponential.

$\alpha = 1$ $\quad (-1)^n u[n] = \{1, -1, 1, -1, \ldots\}$. This is an alternating step.

$\alpha = 2$ $\quad (-2)^n u[n] = \{1, -2, 4, -8, \ldots\}$. This is a growing exponential.

(c) $\alpha = 0.5e^{j\theta}$ $\quad (0.5e^{j\theta})^n = (0.5)^n e^{jn\theta}$. This is an exponentially damped sinusoid.

$\alpha = e^{j\theta}$ $\quad (e^{j\theta})^n = e^{jn\theta}$. This is a sinusoid.

$\alpha = 2e^{j\theta}$ $\quad (2e^{j\theta})^n = (2)^n e^{jn\theta}$. This is an exponentially growing sinusoid.

(d) $\alpha = -0.5e^{j\theta}$ $\quad (-0.5e^{j\theta})^n = (-0.5)^n e^{jn\theta}$. Damped sinusoid with samples alternating in sign.

$\alpha = e^{j\theta}$ $\quad (e^{j\theta})^n = (-1)^n e^{jn\theta}$. Sinusoid with samples alternating in sign.

$\alpha = -2e^{j\theta}$ $\quad (-2e^{j\theta})^n = (-2)^n e^{jn\theta}$. Growing sinusoid with samples alternating in sign.

2.15 (Signal Representation) The two signals shown in Figure P2.15 may be expressed as

(a) $x[n] = A\alpha^n(u[n] - u[n-N])$ $\qquad$ (b) $y[n] = A\cos(2\pi Fn + \theta)$

Find the constants in each expression and then find the signal energy or power as appropriate.

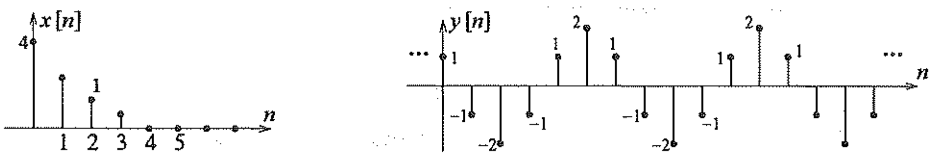


Figure P2.15 Signals for Problem 2.15

[**Hints and Suggestions:** For $y[n]$, first find the period to compute F. Then, evaluate $y[n]$ at two values of n to get two equations for, say $y[0]$ and $y[1]$. These will yield θ (from their ratio) and A.]

(Solution) Refer to the sketch.

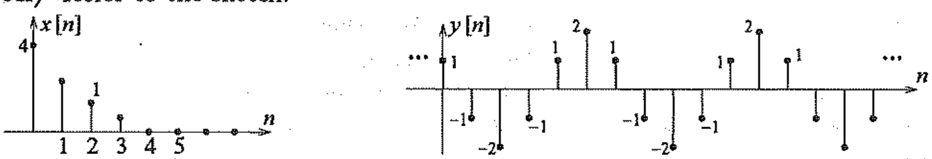


Figure P2.15. Signals for Problem 2.15.

(a) $x[n] = A\alpha^n(u[n] - u[n-N])$. From the figure, $N = 4$. Also $x[0] = 4 = A$

Also, $x[2] = 1 = 4\alpha^2$, so $\alpha = 0.5$.

(b) $y[n] = A\cos(2\pi Fn + \theta)$. From the figure, $N = 6$ or $F = 1/6$. At $n = 0$ and $n = 1$, we have

$y[0] = 1 = A\cos\theta \qquad y[1] = -1 = A\cos(\frac{\pi}{3} + \theta) = \frac{A}{2}\cos\theta - \frac{A\sqrt{3}}{2}\sin\theta$

Thus, $-1 = 0.5 - \frac{A\sqrt{3}}{2}\sin\theta$ or $A\sin\theta = \sqrt{3}$. So, $\dfrac{A\sin\theta}{A\cos\theta} = \tan\theta = \sqrt{3}$.

This gives $\theta = \frac{\pi}{3}$ and $A = \frac{1}{\cos\theta} = 2$

2.16 (Discrete-Time Harmonics) Check for the periodicity of the following signals, and compute the common period N if periodic.

(a) $x[n] = \cos(\frac{n\pi}{2})$ (b) $y[n] = \cos(\frac{n}{2})$
(c) $f[n] = \sin(\frac{n\pi}{4}) - 2\cos(\frac{n\pi}{6})$ (d) $g[n] = 2\cos(\frac{n\pi}{4}) + \cos^2(\frac{n\pi}{4})$
(e) $p[n] = 4 - 3\sin(\frac{7n\pi}{4})$ (f) $q[n] = \cos(\frac{5n\pi}{12}) + \cos(\frac{4n\pi}{9})$
(g) $r[n] = \cos(\frac{8}{3}n\pi) + \cos(\frac{8}{3}n)$ (h) $s[n] = \cos(\frac{8n\pi}{3})\cos(\frac{n\pi}{2})$
(i) $d[n] = e^{j0.3n\pi}$ (j) $e[n] = 2e^{j0.3n\pi} + 3e^{j0.4n\pi}$
(k) $v[n] = e^{j0.3n}$ (l) $w[n] = (j)^{n/2}$

[**Hints and Suggestions:** There is no periodicity if F is not a rational fraction for any component. Otherwise, work with the periods and find their LCM. For $w[n]$, note that $j = e^{j\pi/2}$.]

(Solution)

(a) $x[n] = \cos(0.5n\pi)$. So, $F = 0.25 = \frac{1}{4}$, so periodic with $N = 4$.

(b) $y[n] = \cos(0.5n)$. So, $F = \frac{1}{4\pi}$, so not periodic (F is not a rational fraction)

(c) $f[n] = \sin(\frac{n\pi}{4}) - 2\cos(\frac{n\pi}{6})$. So, $F_1 = \frac{1}{8}$, $F_2 = \frac{1}{12}$.
So, $N_1 = 8$, $N_2 = 12$. So, periodic with $N = \text{LCM}(8,12) = 24$

(d) $g[n] = 2\cos(\frac{n\pi}{4}) + \cos^2(\frac{n\pi}{4}) = 2\cos(\frac{n\pi}{4}) + 0.5 + 0.5\cos(\frac{n\pi}{2})$.
So, $F_1 = \frac{1}{8}$, $F_2 = \frac{1}{4}$, so $N_1 = 8$, $N_2 = 4$, so periodic with $N = \text{LCM}(8,4) = 8$

(e) $p[n] = 4 - 3\sin(\frac{7n\pi}{4})$ Periodic with $F = \frac{7}{8}$ and $N = 8$.

(f) $q[n] = \cos(\frac{5n\pi}{12}) + \cos(\frac{4n\pi}{9})$. So, $F_1 = \frac{5}{24}$, $F_2 = \frac{4}{18} = \frac{2}{9}$.
So, $N_1 = 24$, $N_2 = 9$. So, periodic with $N = \text{LCM}(24,9) = 72$

(g) $r[n] = \cos(\frac{8n\pi}{3}) + \cos(\frac{8n}{3})$. Not periodic because $F_2 = \frac{8}{6\pi}$ is not rational.

(h) $s[n] = \cos(\frac{8n\pi}{3})\cos(\frac{n\pi}{2}) = 0.5\cos(\frac{19n\pi}{6}) + 0.5\cos(\frac{13n\pi}{6})$. So, $F_1 = \frac{19}{12}$, $F_2 = \frac{13}{12}$.
So, $N_1 = 12$, $N_2 = 12$. So, periodic with $N = 12$.

(i) $d[n] = e^{j0.3n\pi}$. So, $F = 0.15 = \frac{15}{100} = \frac{3}{20}$. So, periodic with $N = 20$.

(j) $e[n] = 2e^{j0.3n\pi} + 3e^{j0.4n\pi}$. So, $F_1 = 0.15 = \frac{15}{100} = \frac{3}{20}$, $F_2 = 0.2 = \frac{1}{5}$.
So, $N_1 = 20$, $N_2 = 5$ and $N = \text{LCM}(20,5) = 20$

(k) $v[n] = e^{j0.3n}$. So, $F = \frac{0.15}{\pi}$. Not periodic because F is not rational.

(l) $w[n] = (j)^{n/2} = (e^{j\pi/2})^{n/2} = e^{jn\pi/4}$. So, $F = \frac{1}{8}$. So, periodic with $N = 8$.

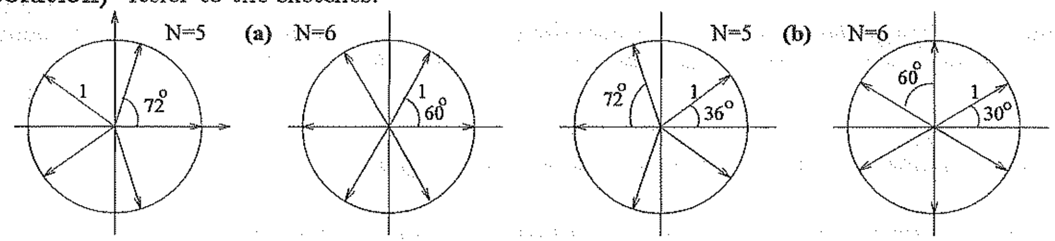
2.17 (The Roots of Unity) The N roots of the equation $z^N = 1$ can be found by writing it as $z^N = e^{j2k\pi}$ to give $z = e^{j2k\pi/N}$, $k = 0, 1, \ldots, N-1$. What is the magnitude and angle of each root? The roots can be displayed as vectors directed from the origin whose tips lie on a circle.

(a) What is the length of each vector and the angular spacing between adjacent vectors? Sketch for $N = 5$ and $N = 6$.

(b) Extend this concept to find the roots of $z^N = -1$ and sketch for $N = 5$ and $N = 6$.

[**Hints and Suggestions:** In part (b), note that $z^N = -1 = e^{j\pi}e^{j2k\pi} = e^{j(2k+1)\pi}$.]

(Solution) Refer to the sketches.



(a) $z^N = 1$ $\quad z = e^{j2k\pi/N}$, $k = 0, 1, \ldots, N-1$ with $|z| = 1$.

$N = 5$: Angular spacing $= \frac{2\pi}{N} = \frac{2\pi}{5} = 72°$ with $k = 0$ at $0°$).

$N = 6$: Angular spacing $= \frac{2\pi}{6} = \frac{2\pi}{6} = 60°$ (with $k = 0$ at $0°$).

(b) $z^N = -1 = e^{j\pi}e^{j2k\pi} = e^{j(2k+1)\pi}$ $\quad z = e^{j(2k+1)\pi/N}$, $k = 0, 1, \ldots, N-1$ with $|z| = 1$.

$N = 5$: Angular spacing $= \frac{2\pi}{N} = \frac{2\pi}{5} = 72°$ (with $k = 0$ at $36°$).

$N = 6$: Angular spacing $= \frac{2\pi}{6} = \frac{2\pi}{6} = 60°$ (with $k = 0$ at $30°$).

2.18 (Digital Frequency) Set up an expression for each signal, using a digital frequency $|F| < 0.5$, and another expression using a digital frequency in the range $4 < F < 5$.

(a) $x[n] = \cos(\frac{4n\pi}{3})$ $\quad$ (b) $x[n] = \sin(\frac{4n\pi}{3}) + 3\sin(\frac{8n\pi}{3})$

[**Hints and Suggestions:** First find the digital frequency of each component in the principal range ($-0.5 < F \le 0.5$). Then, add 4 or 5 as appropriate to bring each frequency into the required range.]

(Solution)

(a) $x[n] = \cos(4n\pi/3)$. So, $F = 4/6 = 2/3$. So, $F = 2/3 - 1 = -1/3$.

So, $x[n] = \cos(-2n\pi/3) = \cos(2n\pi/3)$.

For $4 < F < 5$, we have $F = -1/3 + 5 = 14/3$ and $x[n] = \cos(28n\pi/3)$.

(b) $x[n] = \sin(4n\pi/3) + 3\sin(8n\pi/3)$. So, $F_1 = 2/3 = -1/3$, $F_2 = 4/3 = 1/3$

So, $x[n] = \sin(-2n\pi/3) + 3\sin(2n\pi/3)$

[Note: This can be simplified to $x[n] = -\sin(2n\pi/3) + 3\sin(2n\pi/3) = 2\sin(2n\pi/3)$]

For $4 < F < 5$, we have $F_1 = -1/3 + 5 = 14/3$, $F_2 = 1/3 + 4 = 13/3$, and $x_2[n] = \sin(28n\pi/3) + 3\sin(26n\pi/3)$

2.19 (Digital Sinusoids) Find the period N of each signal if periodic. Express each signal using a digital frequency in the principal range ($|F| < 0.5$) and in the range $3 \le F \le 4$.

(a) $x[n] = \cos(\frac{7n\pi}{3})$ (b) $x[n] = \cos(\frac{7n\pi}{3}) + \sin(0.5n\pi)$ (c) $x[n] = \cos(n)$

(Solution)

(a) $x[n] = \cos(\frac{7n\pi}{3})$. So, $F = \frac{7}{6} \Rightarrow F = \frac{1}{6}$. For $3 < F < 4$, $F = 3 + \frac{1}{6} = \frac{19}{6}$
So, $x[n] = \cos(\frac{n\pi}{3})$ or $x[n] = \cos(\frac{19n\pi}{3})$, $3 < F < 4$.

(b) $x[n] = \cos(\frac{7n\pi}{3}) + \sin(0.5n\pi)$.
So, $F_1 = \frac{7}{6} \Rightarrow F_1 = \frac{1}{6}$ or $F_1 = \frac{19}{6}$, $3 < F < 4$ and $F_2 = \frac{1}{4}$ or $F_2 = 3 + \frac{1}{4} = \frac{13}{4}$, $3 < F < 4$
So, $x[n] = \cos(\frac{n\pi}{3}) + \sin(\frac{n\pi}{2})$ or $x[n] = \cos(\frac{19n\pi}{3}) + \sin(\frac{13n\pi}{2})$, $3 < F < 4$

(c) $x[n] = \cos(n)$. So, $F = \frac{1}{2\pi}$ or $F = 3 + \frac{1}{2\pi}$, $3 < F < 4$.
So, $x[n] = \cos(n)$ or $x[n] = \cos[2n\pi(3 + \frac{1}{2\pi})]$, $3 < F < 4$

2.20 (Sampling and Aliasing) Each of the following sinusoids is sampled at $S = 100$ Hz. Determine if aliasing has occurred and set up an expression for each sampled signal using a digital frequency in the principal range ($|F| < 0.5$).

(a) $x(t) = \cos(320\pi t + \frac{\pi}{4})$ (b) $x(t) = \cos(140\pi t - \frac{\pi}{4})$ (c) $x(t) = \sin(60\pi t)$

[**Hints and Suggestions:** Find the frequency f_0. If $S > 2f_0$ there is no aliasing and $F < 0.5$. Otherwise, bring F into the principal range to write the expression for the sampled signal.]

(Solution)

(a) $x(t) = \cos(320\pi t + 0.25\pi)$. So, $f_0 = 160$ Hz, $S = 100$ Hz, and $F = f_0/S = 1.6$.
There is aliasing because $F > 0.5$ (or $S < 2f_0$).
Now, $F = 1.6 = -0.4$, so $x[n] = \cos(-0.8n\pi + 0.25\pi) = \cos(0.8n\pi - 0.25\pi)$.

(b) $x(t) = \cos(140\pi t - 0.25\pi)$. So, $f_0 = 70$ Hz, $S = 100$ Hz, and $F = f_0/S = 0.7$
There is aliasing because $F > 0.5$ (or $S < 2f_0$).
Now, $F = 0.7 = -0.3$, so $x[n] = \cos(-0.6n\pi - 0.25\pi) = \cos(0.6n\pi + 0.25\pi)$.

(c) $x(t) = \sin(60\pi t)$. So, $f_0 = 30$ Hz, $S = 100$ Hz, and $F = f_0/S = 0.3$
There is no aliasing because $F < 0.5$ (or $S > 2f_0$). So, $x[n] = \sin(0.6n\pi)$.

2.21 (Aliasing and Signal Reconstruction) The signal $x(t) = \cos(320\pi t + \frac{\pi}{4})$ is sampled at 100 Hz, and the sampled signal $x[n]$ is reconstructed at 200 Hz to recover the analog signal $x_r(t)$.

(a) Has aliasing occurred? What is the period N and the digital frequency F of $x[n]$?
(b) How many full periods of $x(t)$ are required to generate one period of $x[n]$?
(c) What is the analog frequency of the recovered signal $x_r(t)$?
(d) Write expressions for $x[n]$ (using $|F| < 0.5$) and for $x_r(t)$.

[**Hints and Suggestions:** For part (b), if the digital frequency is expressed as $F = k/N$ where N is the period and k is an integer, it takes k full cycles of the analog sinusoid to get N samples of the sampled signal. In part (c), the frequency of the reconstructed signal is found from the aliased frequency in the principal range.]

(Solution) $x(t) = \cos(320\pi t + \frac{\pi}{4})$ with $S = 100$ Hz and $S_R = 200$ Hz.

(a) Aliasing has occurred. $f_0 = 160$ Hz. So, $F_0 = \frac{160}{100} = \frac{8}{5}$. So, $N = 5$.

(b) 8 full periods of $x(t)$ generate one period (5 samples) of $x[n]$.

(c) The digital frequency in the principal range is $F_0 = 1.6 \Rightarrow F_0 = -0.4$.
The analog frequency of the recovered signal $x_r(t)$ is $f_a = SF_0 = 200(-0.4) = -80$ Hz

(d) $x[n] = \cos(-0.8n\pi + \frac{\pi}{4}) = \cos(0.8n\pi - \frac{\pi}{4})$ $\quad x_r(t) = \cos(-160\pi t + \frac{\pi}{4}) = \cos(160\pi t - \frac{\pi}{4})$.

2.22 (Digital Pitch Shifting) One way to accomplish *pitch shifting* is to play back (or reconstruct) a sampled signal at a *different* sampling rate. Let the analog signal $x(t) = \sin(15800\pi t + 0.25\pi)$ be sampled at a sampling rate of 8 kHz.

(a) Find its sampled representation with digital frequency $|F| < 0.5$.
(b) What frequencies are heard if the signal is reconstructed at a rate of 4 kHz?
(c) What frequencies are heard if the signal is reconstructed at a rate of 8 kHz?
(d) What frequencies are heard if the signal is reconstructed at a rate of 20 kHz?

[**Hints and Suggestions:** The frequency of the reconstructed signal is found from the aliased digital frequency in the principal range and the appropriate reconstruction rate.]

(Solution)

(a) $x(t) = \sin(15800\pi t + 0.25\pi)$ and $S = 8000$ Hz. So, $f_0 = 7900$ Hz and $F_0 = \frac{7900}{8000} = \frac{79}{80} \Rightarrow F_0 = \frac{-1}{80}$.
So, $x[n] = \sin(-\frac{2n\pi}{80} + 0.25\pi) = -\sin(\frac{2n\pi}{80} - 0.25\pi)$.

(b) If $S_R = 4$ kHz, the reconstructed frequency is $F_0 S_R = -\frac{4000}{80} = 50$ Hz (i.e., 50 Hz).

(c) If $S_R = 8$ kHz, the reconstructed frequency is $F_0 S_R = -\frac{8000}{80} = -100$ Hz (i.e., 100 Hz).

(d) If $S_R = 20$ kHz, the reconstructed frequency is $F_0 S_R = -\frac{20000}{80} = -250$ Hz (i.e., 250 Hz).

2.23 (Discrete-Time Chirp Signals) Consider the signal $x(t) = \cos[\phi(t)]$, where $\phi(t) = \alpha t^2$. Show that its instantaneous frequency $f_i(t) = \frac{1}{2\pi}\phi'(t)$ varies linearly with time.

(a) Choose α such that the frequency varies from 0 Hz to 2 Hz in 10 seconds, and generate the sampled signal $x[n]$ from $x(t)$, using a sampling rate of $S = 4$ Hz.
(b) It is claimed that, unlike $x(t)$, the signal $x[n]$ is periodic. Verify this claim, using the condition for periodicity ($x[n] = x[n + N]$), and determine the period N of $x[n]$.
(c) The signal $y[n] = \cos(\pi F_0 n^2/M)$, $n = 0, 1, \ldots, M-1$, describes an M-sample chirp whose digital frequency varies linearly from 0 to F_0. What is the period of $y[n]$ if $F_0 = 0.25$ and $M = 8$?

[**Hints and Suggestions:** In part (b), if $x[n] = \cos(\beta n^2)$, periodicity requires $x[n] = x[n + N]$ or $\cos(\beta n^2) = \cos[\beta(n^2 + 2nN + N^2)]$. Thus $2nN\beta = 2m\pi$ and $N^2\beta = 2k\pi$ where m and k are integers. Satisfy these conditions for the smallest integer N.]

(Solution) $x(t) = \cos[\phi(t)] = \cos(\alpha t^2)$. So, $f_i(t) = \frac{1}{2\pi}\phi'(t) = \frac{\alpha}{\pi}t$. This varies linearly with t.

(a) If the frequency varies from 0 to 2 Hz in 10 seconds, $\frac{\alpha}{\pi} = \frac{2}{10}$. So, $\alpha = 0.2\pi$.

Now, $S = 4$ Hz, so $t = nt_s = n/S$ and $x[n] = \cos(\alpha n^2/S^2) = \cos(\frac{\pi n^2}{80}$.

(b) $x[n+N] = \cos[\frac{\pi}{80}(n+N)^2] = \cos[\frac{\pi}{80}(n^2 + 2nN + N^2)]$.

So, for $x[n] = x[n+N]$, we require $\frac{N}{80} = m$ and $\frac{N^2}{80} = 2k$ (where N, m and k are integers that make the last two terms integer multiples of 2π). The smallest N that satisfies these results is $N = 80$. So, $x[n]$ is periodic with period $N = 80$.

(c) $y[n] = \cos(\pi F_0 n^2/M)$. With $F_0 = 0.25$ and $M = 8$, $y[n] = \cos(\frac{\pi}{32}n^2)$. Following part (b), $y[n]$ is periodic with period $N = 32$.

2.24 (Time Constant) For exponentially decaying discrete signals, the **time constant** is a measure of how fast a signal decays. The 60-dB time constant describes the (integer) number of samples it takes for the signal level to decay by a factor of 1000 (or $20\log 1000 = 60$ dB).

(a) Let $x[n] = (0.5)^n u[n]$. Compute its 60-dB time constant and 40-dB time constant.
(b) Compute the time constant in seconds if the discrete-time signal is derived from an analog signal sampled at 1 kHz.

(Solution)

(a) $x[n] = (0.5)^n u[n]$. So, $x[0] = 1$. The 60-dB time constant is found from $(0.5)^n = 0.001$ and gives $n\log(0.5) = \log(0.001)$ or $n = 9.9658 \approx 10$. The 40-dB time constant is found from $(0.5)^n = 0.01$ and gives $n\log(0.5) = \log(0.01)$ or $n = 6.6439 \approx 7$.

(b) With $S = 1$ kHz, the 60-dB and 40-dB time constants are 10 ms and 7 ms, respectively.

2.25 (Signal Delay) The delay D of a discrete-time energy signal $x[n]$ is defined by

$$D = \frac{\displaystyle\sum_{k=-\infty}^{\infty} kx^2[k]}{\displaystyle\sum_{k=-\infty}^{\infty} x^2[k]}$$

(a) Verify that the delay of the symmetric sequence $x[n] = \{4, 3, 2, 1, \overset{\Downarrow}{0}, 1, 2, 3, 4\}$ is zero.
(b) Compute the delay of the signals $g[n] = x[n-1]$ and $h[n] = x[n-2]$.
(c) What is the delay of the signal $y[n] = 1.5(0.5)^n u[n] - 2\delta[n]$?

[**Hints and Suggestions:** For part (c), compute the summations required in the expression for the delay by using tables and the fact that $y[n] = -0.5$ for $n = 0$ and $y[n] = 1.5(0.5)^n$ for $n \geq 1$.]

(Solution)

(a) $x[n] = \{4, 3, 2, 1, \overset{\Downarrow}{0}, 1, 2, 3, 4\}, \quad -4 \leq k \leq 4$

We compute $D = \dfrac{\displaystyle\sum_{k=-4}^{4} kx^2[k]}{\sum x^2[k]} = \frac{0}{60} = 0.$

(b) $g[n] = \{4, 3, 2, \overset{\Downarrow}{1}, 0, 1, 2, 3, 4\}, \quad -3 \le k \le 5$

We compute $D = \dfrac{\sum_{k=-3}^{5} kg^2[k]}{\sum g^2[k]} = \frac{60}{60} = 1.$

$h[n] = \{4, 3, \overset{\Downarrow}{2}, 1, 0, 1, 2, 3, 4\}, \quad -2 \le k \le 6$

We compute $D = \dfrac{\sum_{k=-2}^{6} kh^2[k]}{\sum h^2[k]} = \frac{120}{60} = 2.$

(c) $y[n] = 1.5(0.5)^n u[n] - 2\delta[n]$. Since $y[0] = -0.5$ and $y[n] = 1.5(0.5)^n$, $n \ge 1$, we compute

$P = \sum h^2[k] = (-0.5)^2 + (1.5)^2 \sum_{k=1}^{\infty}(0.25)^k = 0.25 + (1.5)^2 \dfrac{0.25}{1-0.25} = 1$

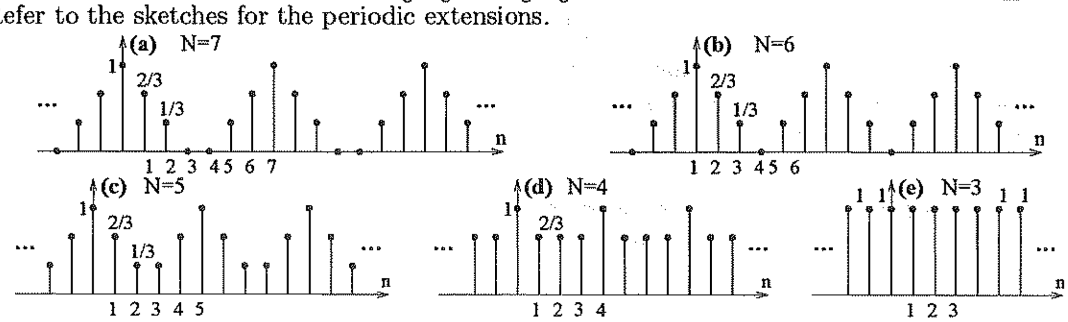
$Q = \sum kh^2[k] = 0(-0.5)^2 + (1.5)^2 \sum_{k=1}^{\infty} k(0.25)^k = (1.5)^2 \dfrac{(0.25)}{(1-0.25)^2} = 1$

So, the delay is $D = \frac{P}{Q} = 1$.

2.26 (Periodicity) It is claimed that the sum of an absolutely summable signal $x[n]$ and its shifted (by multiples of N) replicas is a periodic signal $x_p[n]$ with period N. Verify this claim by sketching the following and, for each case, compute the power in the resulting periodic signal $x_p[n]$ and compare the sum and energy of one period of $x_p[n]$ with the sum and energy of $x[n]$.

(a) The sum of $x[n] = \text{tri}(n/3)$ and its replicas shifted by $N = 7$
(b) The sum of $x[n] = \text{tri}(n/3)$ and its replicas shifted by $N = 6$
(c) The sum of $x[n] = \text{tri}(n/3)$ and its replicas shifted by $N = 5$
(d) The sum of $x[n] = \text{tri}(n/3)$ and its replicas shifted by $N = 4$
(e) The sum of $x[n] = \text{tri}(n/3)$ and its replicas shifted by $N = 3$

(Solution) $x[n] = \text{tri}(n/3) = \{0, \frac{1}{3}, \frac{2}{3}, \overset{\Downarrow}{1}, \frac{2}{3}, \frac{1}{3}, 0\}$. We have $\sum x[n] = 3$ and $E = \sum x^2[n] = \frac{19}{9}$. Refer to the sketches for the periodic extensions.



(a) For $N = 7$, we have $\sum x_p[n] = 3$, $E_p = \frac{19}{9}$ and $P = \frac{19}{63}$

(b) For $N = 6$, we have $\sum x_p[n] = 3$, $E_p = \frac{19}{9}$ and $P = \frac{19}{54}$

(c) For $N = 5$, we have $\sum x_p[n] = 3$, $E_p = \frac{19}{9}$ and $P = \frac{19}{45}$

(d) For $N = 4$, we have $\sum x_p[n] = 3$, $E_p = \frac{7}{3}$ and $P = \frac{7}{12}$

(e) For $N = 3$, we have $\sum x_p[n] = 3$, $E_p = 3$ and $P = 1$

2.27 (Periodic Extension) The sum of an absolutely summable signal $x[n]$ and its shifted (by multiples of N) replicas is called the *periodic extension* of $x[n]$ with period N.

(a) Show that one period of the periodic extension of the signal $x[n] = \alpha^n u[n]$ with period N is

$$y[n] = \frac{\alpha^n}{1-\alpha^N}, \quad 0 \le n \le N-1$$

(b) How does the sum of one period of the periodic extension $y[n]$ compare with the sum of $x[n]$?
(c) With $\alpha = 0.5$ and $N = 3$, compute the signal energy in $x[n]$ and the signal power in $y[n]$.

[**Hints and Suggestions:** For one period ($n = 0$ to $n = N-1$), only $x[n]$ and the tails of the replicas to its left contribute. So, find the sum of $x[n+kN] = \alpha^{n+kN}$ only from $k = 0$ to $k = \infty$.]

(Solution)

(a) $x[n] = \alpha^n u[n]$. So, its periodic extension with period N is

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^{n-kN} u[n-kN] = \sum_{k=0}^{\infty} \alpha^{n+kN} = \alpha^n \sum_{k=0}^{\infty} (\alpha^N)^k = \frac{\alpha^n}{1-\alpha^N}$$

Thus, $y[n] = \dfrac{x[n]}{1-\alpha^N}$, $0 \le n \le N-1$.

(b) The sum of $x[n]$ equals the one-period sum of $y[n]$:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad \sum_{n=0}^{N-1} \frac{\alpha^n}{1-\alpha^N} = \left(\frac{1}{1-\alpha^N}\right)\left(\frac{1-\alpha^N}{1-\alpha}\right) = \frac{1}{1-\alpha}$$

(c) For $\alpha = 0.5$ and $N = 3$, $x[n] = (0.5)^n u[n]$ and $y[n] = \dfrac{(0.5)^n}{1-(0.5)^3}$.

The signal energy in $x[n]$ is $E = \sum_{n=0}^{\infty} (0.25)^n = \dfrac{4}{3} = 1.3333$

The signal power in $y[n]$ is

$$P = \tfrac{1}{3} \sum_{n=0}^{2} y^2[n] = \tfrac{4}{7} = 0.5714$$

2.28 (Signal Norms) Norms provide a measure of the *size* of a signal. The ***p*-norm**, or **Hölder norm**, $\|x\|_p$ for discrete signals is defined by $\|x\|_p = (\sum |x|^p)^{1/p}$, where $0 < p < \infty$ is a positive integer. For $p = \infty$, we also define $\|x\|_\infty$ as the peak absolute value $|x|_{\max}$.

(a) Let $x[n] = \{3, \ -j4, \ 3+j4\}$. Find $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$.
(b) What is the significance of each of these norms?

(Solution)

(a) $x[n] = \{3, -j4, 3+j4\}$. So $|x[n]| = \{3, 4, 5\}$. Thus,

$\|x\|_1 = \sum |x[n]| = 3+4+5 = 12$

$\|x\|_2 = \left(\sum |x[n]|^2\right)^{1/2} = (9+16+25)^{1/2} = \sqrt{50} = 7.0711$

$\|x\|_\infty = \max(|x[n]|) = 5.$

(b) The norm $\|x\|_1$ signifies the absolute area.

The norm $\|x\|_2$ is similar to the rms value (and related to the signal power).

The norm $\|x\|_\infty$ is the peak absolute deviation from zero.

COMPUTATION AND DESIGN

2.29 (Discrete Signals) For each part, plot the signals $x[n]$ and $y[n]$ over $-10 \le n \le 10$ and compare.

(a) $x[n] = u[n+4] - u[n-4] + 2\delta[n+6] - \delta[n-3]$ $\quad y[n] = x[-n-4]$
(b) $x[n] = r[n+6] - r[n+3] - r[n-3] + r[n-6]$ $\quad y[n] = x[n-4]$
(c) $x[n] = \text{rect}(\frac{n}{10}) - \text{rect}(\frac{n-3}{6})$ $\quad y[n] = x[n+4]$
(d) $x[n] = 6\,\text{tri}(\frac{n}{6}) - 3\,\text{tri}(\frac{n}{3})$ $\quad y[n] = x[-n+4]$

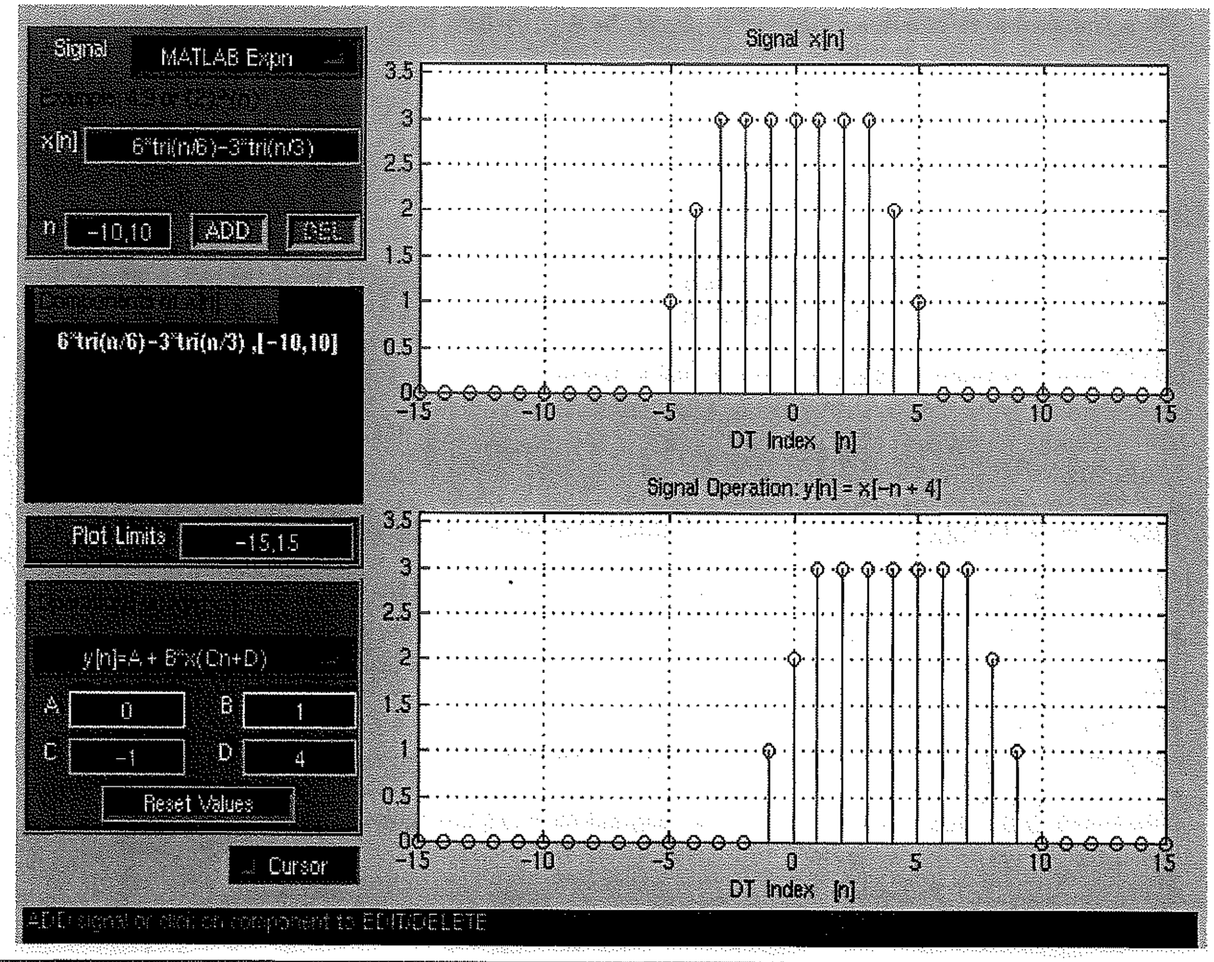
(Solution) Uses the author's routines `ustep, uramp, urect, udelta, tri, operate, dtplot`

```
%PROBLEM 3.29
%PART (a)
n=-10:10;x=ustep(n+4)-ustep(n-4)+2*udelta(n+6)-udelta(n-3);ax=[-15 15 0 3];
[ny,y]=operate(n,x,-1,-4);subplot(2,1,1),dtplot(n,x,'o'),axis(ax),
subplot(2,1,2),dtplot(ny,y,'o'),axis(ax),pause
%PART (b)
x=uramp(n+6)-uramp(n+3)-uramp(n-3)+uramp(n-6);ax=[-15 15 0 4];
[ny,y]=operate(n,x,1,-4);subplot(2,1,1),dtplot(n,x,'o'),axis(ax),
subplot(2,1,2),dtplot(ny,y,'o'),axis(ax),pause

%PART (c)
x=urect(n/10)-urect((n-3)/6);ax=[-15 15 0 2];
[ny,y]=operate(n,x,1,4);subplot(2,1,1),dtplot(n,x,'o'),axis(ax),
subplot(2,1,2),dtplot(ny,y,'o'),axis(ax),pause

%PART (d)
x=6*tri(n/6)-3*tri(n/3);ax=[-15 15 0 4];
[ny,y]=operate(n,x,-1,4);subplot(2,1,1),dtplot(n,x,'o'),axis(ax),
subplot(2,1,2),dtplot(ny,y,'o'),axis(ax)
```

You could also use the routine dtsiggui. The plot shown is for part (d)



2.30 (Signal Interpolation) Let $h[n] = \sin(n\pi/3)$, $0 \leq n \leq 10$. Plot the signal $h[n]$. Use this to generate and plot the *zero-interpolated*, *step-interpolated*, and *linearly interpolated* signals assuming interpolation by 3.

(Solution) Uses the author's routines `dtplot, interpol`

```
%PROBLEM 3.30
n=0:10;h=sin(n*pi/3);dtplot(n,h,'o'),pause
hz=interpol(h,'z',3);L=length(hz)-1;dtplot(0:L,hz,'o'),pause %Zero interp
hz=interpol(h,'c',3);dtplot(0:L,hz,'o'),pause                %Step interp
hz=interpol(h,'l',3);dtplot(0:L,hz,'o')                      %Linear interp
```

2.31 (Discrete Exponentials) A causal discrete exponential may be expressed as $x[n] = \alpha^n u[n]$, where the nature of α dictates the form of $x[n]$. Plot the following over $0 \leq n \leq 40$ and comment on the nature of each plot.

(a) The signal $x[n]$ for $\alpha = 1.2$, $\alpha = 1$, and $\alpha = 0.8$.

(b) The signal $x[n]$ for $\alpha = -1.2$, $\alpha = -1$, and $\alpha = -0.8$.

(c) The real part and imaginary parts of $x[n]$ for $\alpha = Ae^{j\pi/4}$, with $A = 1.2$, $A = 1$, and $A = 0.8$.

(d) The magnitude and phase of $x[n]$ for $\alpha = Ae^{j\pi/4}$, with $A = 1.2$, $A = 1$, and $A = 0.8$.

(Solution) Uses the author's routine dtplot

```
%PROBLEM 3.31
n=0:40;a=1.2;x=a.^n;subplot(3,1,1),dtplot(n,x,'o')
a=1;x=a.^n;subplot(3,1,2),dtplot(n,x,'o')
a=0.8;x=a.^n;subplot(3,1,3),dtplot(n,x,'o'),pause
%PART (b)
a=-1.2;x=a.^n;subplot(3,1,1),dtplot(n,x,'o')
a=-1;x=a.^n;subplot(3,1,2),dtplot(n,x,'o')
a=-0.8;x=a.^n;subplot(3,1,3),dtplot(n,x,'o'),pause
%PART (c)
a=1.2*exp(j*pi/4);x=a.^n;
subplot(3,2,1),dtplot(n,real(x),'.'),subplot(3,2,2),dtplot(n,imag(x),'.')
a=1.0*exp(j*pi/4);x=a.^n;
subplot(3,2,3),dtplot(n,real(x),'.'),subplot(3,2,4),dtplot(n,imag(x),'.')
a=0.8*exp(j*pi/4);x=a.^n;
subplot(3,2,5),dtplot(n,real(x),'.'),subplot(3,2,6),dtplot(n,imag(x),'.'),pause
%PART (d)
a=1.2*exp(j*pi/4);x=a.^n;
subplot(3,2,1),dtplot(n,abs(x),'.'),subplot(3,2,2),dtplot(n,angle(x),'.')
a=1.0*exp(j*pi/4);x=a.^n;
subplot(3,2,3),dtplot(n,abs(x),'.'),subplot(3,2,4),dtplot(n,angle(x),'.')
a=0.8*exp(j*pi/4);x=a.^n;
subplot(3,2,5),dtplot(n,abs(x),'.'),subplot(3,2,6),dtplot(n,angle(x),'.')
```

2.32 (Discrete-Time Sinusoids) Which of the following signals are periodic and with what period? Plot each signal over $-10 \le n \le 30$. Do the plots confirm your expectations?

(a) $x[n] = 2\cos(\frac{n\pi}{2}) + 5\sin(\frac{n\pi}{5})$ (b) $x[n] = 2\cos(\frac{n\pi}{2})\sin(\frac{n\pi}{3})$

(c) $x[n] = \cos(0.5n)$ (d) $x[n] = 5\sin(\frac{n\pi}{8} + \frac{\pi}{4}) - 5\cos(\frac{n\pi}{8} - \frac{\pi}{4})$

(Solution) Uses the author's routine dtplot

```
%PROBLEM 3.32
n=-10:30;
x1=2*cos(n*pi/2)+5*sin(n*pi/5);subplot(2,2,1),dtplot(n,x1,'.')  %Period N=20
x2=2*cos(n*pi/2).*sin(n*pi/3);subplot(2,2,2),dtplot(n,x2,'.')   %Period N=12
x3=2*cos(n/2);subplot(2,2,3),dtplot(n,x3,'.')                    %Not periodic
x4=5*sin(n*pi/8+0.25*pi)-5*cos(n*pi/8-0.25*pi);
subplot(2,2,4),dtplot(n,x4,'.') %Zero (Note the y-scale)
```

2.33 (Complex-Valued Signals) A complex-valued signal $x[n]$ requires *two* plots for a complete description in one of two forms—the *magnitude* and *phase* vs. n or the *real part* vs. n and *imaginary part* vs. n.

(a) Let $x[n] = \{\overset{\Downarrow}{2},\ 1+j,\ -j2,\ 2-j2,\ -4\}$. Sketch each form for $x[n]$ by hand.

(b) Let $x[n] = e^{-j0.3n\pi}$. Use MATLAB to plot each form over $-30 \le n \le 30$. Is $x[n]$ periodic? If so, can you identify its period from the MATLAB plots? From which form, and how?

(Solution) $x[n] = \{2,\ 1+j,\ -j2,\ 2-j2,\ -4\} = \{2,\ \sqrt{2}\angle 45^\circ,\ 2\angle -90^\circ,\ 2\sqrt{2}\angle -45^\circ,\ 4\angle 180^\circ\}$

```
%PROBLEM 3.33 Uses the author's routine dtplot
n=-30:30;x=exp(-j*0.3*n*pi);axis([-30 30 -1 1])
subplot(2,2,1),dtplot(n,real(x)),axis([-30 30 -1 1]),
subplot(2,2,2),dtplot(n,imag(x)),axis([-30 30 -1 1])
subplot(2,2,3),dtplot(n,abs(x)),axis([-30 30 0 1])
subplot(2,2,4),dtplot(n,angle(x)),axis([-30 30 -pi pi])

%All (except magnitude) allow the period to be determined. N=20
```

2.34 (Complex Exponentials) Let $x[n] = 5\sqrt{2}e^{j(\frac{n\pi}{9}-\frac{\pi}{4})}$. Plot the following signals and, for each case, derive analytic expressions for the signals plotted and compare with your plots. Is the signal $x[n]$ periodic? What is the period N? Which plots allow you determine the period of $x[n]$?

(a) The real part and imaginary part of $x[n]$ over $-20 \le n \le 20$

(b) The magnitude and phase of $x[n]$ over $-20 \le n \le 20$

(c) The sum of the real and imaginary parts over $-20 \le n \le 20$

(d) The difference of the real and imaginary parts over $-20 \le n \le 20$

(Solution) $x[n] = 5\sqrt{2}e^{j(\frac{\pi}{9}n-\frac{\pi}{4})}$.

Real part: $x_r = 5\sqrt{2}\cos(\frac{n\pi}{9} - 0.25\pi)$, Imaginary part: $x_i = 5\sqrt{2}\sin(\frac{n\pi}{9} - 0.25\pi)$

Magnitude: $x_m = 5\sqrt{2}$, Phase: $x_p = \frac{n\pi}{9} - 0.25\pi$

$x_r + x_i = 10\sin(\frac{n\pi}{9})$, $x_r - x_i = 10\cos(\frac{n\pi}{9})$

```
%PROBLEM 3.34 Uses the author's routine dtplot
n=-20:20;x=5*sqrt(2)*exp(j*(n*(pi/9)-pi/4));
%PART (a)
subplot(2,2,1),dtplot(n,real(x),'o')
subplot(2,2,2),dtplot(n,imag(x),'o')
%PART (b)
subplot(2,2,3),dtplot(n,abs(x),'o')
subplot(2,2,4),dtplot(n,angle(x),'o'),pause
%PART (c)
subplot(2,1,1),dtplot(n,real(x)+imag(x),'o')
subplot(2,1,2),dtplot(n,real(x)-imag(x),'o')

%All plots (except magnitude) allow the period to be determined. N=18
```

2.35 (Complex Exponentials) Let $x[n] = (\sqrt{j})^n + (\sqrt{j})^{-n}$. Plot the following signals and, for each case, derive analytic expressions for the sequences plotted and compare with your plots. Is the signal $x[n]$ periodic? What is the period N? Which plots allow you determine the period of $x[n]$?

(a) The real part and imaginary part of $x[n]$ over $-20 \le n \le 20$

(b) The magnitude and phase of $x[n]$ over $-20 \le n \le 20$

(Solution) $x[n] = (\sqrt{j})^n + (\sqrt{j})^{-n} = e^{jn\pi/4} + e^{-jn\pi/4} = 2\cos(n\pi/4)$.

```
%PROBLEM 3.34  Uses the author's routine dtplot
n=-20:20;j=sqrt(-1);x=sqrt(j).^n+sqrt(j).^(-n);
%PART (a)
subplot(2,2,1),dtplot(n,real(x),'o')
subplot(2,2,2),dtplot(n,imag(x),'o')
%PART (b)
subplot(2,2,3),dtplot(n,abs(x),'o')
subplot(2,2,4),dtplot(n,angle(x),'o')

%The real part and angle allow the period to be determined. N=5
```

2.36 (Discrete-Time Chirp Signals) An N-sample *chirp* signal $x[n]$ whose digital frequency *varies linearly* from F_0 to F_1 is described by

$$x[n] = \cos\left[2\pi\left(F_0 n + \frac{F_1 - F_0}{2N}n^2\right)\right], \qquad n = 0, 1, \ldots, N-1$$

(a) Generate and plot 800 samples of a chirp signal x whose digital frequency varies from $F = 0$ to $F = 0.5$. Using the MATLAB based routine `timefreq` (from the author's website), observe how the frequency of x varies linearly with time.

(b) Generate and plot 800 samples of a chirp signal x whose digital frequency varies from $F = 0$ to $F = 1$. Is the frequency always increasing? If not, what is the likely explanation?

(Solution) $x[n] = \cos\left[2\pi\left(F_0 n + \dfrac{F_1 - F_0}{2N}n^2\right)\right] \qquad n = 0, 1, \ldots N-1$

```
%PROBLEM 3.36  Uses the author's routine timefreq
N=800;n=0:N-1;F0=0;F1=0.5;
x=cos(2*pi*(n*F0+0.5*n.*n*(F1-F0)/N));
timefreq(x);pause  %Frequency increases linearly from F=0 to F=0.5
F0=0;F1=1;
x=cos(2*pi*(n*F0+0.5*n.*n*(F1-F0)/N));
timefreq(x);  %Frequency increases up to F=0.5, then decreases (aliasing)
```

2.37 (Chirp Signals) It is claimed that the chirp signal $x[n] = \cos(\pi n^2/6)$ is periodic (unlike the analog chirp signal $x(t) = \cos(\pi t^2/6)$). Plot $x[n]$ over $0 \le n \le 20$. Does $x[n]$ appear periodic? If so, can you identify the period N? Justify your results by trying to find an integer N such that $x[n] = x[n+N]$ (the basis for periodicity).

(Solution) Uses the author's routine `dtplot`

For periodicity, $x[n] = \cos(\pi n^2/6) = x[n+N] = \cos[\pi(n+N)^2/6] = \cos[\pi(n^2 + 2nN + N^2)/6]$.

So, $\dfrac{\pi(2nN + N^2)}{6} = 2k\pi$ or $2nN + N^2 = 12k$. This is satisfied for $N = 6$.

```
%PROBLEM 3.37
n=0:20;x=cos(pi*n.*n/6);dtplot(n,x,'o')   %Period N=6
```

2.38 (Signal Averaging) Extraction of signals from noise is an important signal-processing application. Signal averaging relies on averaging the results of many runs. The noise tends to average out to zero, and the signal quality or **signal-to-noise ratio** (SNR) improves.

(a) Generate samples of the sinusoid $x(t) = \sin(800\pi t)$ sampled at $S = 8192$ Hz for 2 seconds. The sampling rate is chosen so that you may also listen to the signal if your machine allows.

(b) Create a noisy signal $s[n]$ by adding $x[n]$ to samples of uniformly distributed noise such that $s[n]$ has an SNR of 10 dB. Compare the noisy signal with the original and compute the actual SNR of the noisy signal.

(c) Sum the signal $s[n]$ 64 times and average the result to obtain the signal $s_a[n]$. Compare the averaged signal $s_a[n]$, the noisy signal $s[n]$, and the original signal $x[n]$. Compute the SNR of the averaged signal $x_a[n]$. Is there an improvement in the SNR? Do you notice any (visual and audible) improvement? Should you?

(d) Create the averaged result $x_b[n]$ of 64 *different* noisy signals and compare the averaged signal $x_b[n]$ with the original signal $x[n]$. Compute the SNR of the averaged signal $x_b[n]$. Is there an improvement in the SNR? Do you notice any (visual and/or audible) improvement? Explain how the signal $x_b[n]$ differs from $x_a[n]$.

(e) The reduction in SNR is a function of the noise distribution. Generate averaged signals, using different noise distributions (such as Gaussian noise) and comment on the results.

(Solution) Uses the author's routine `randist`

```
%PROBLEM 3.38
S=8192;ts=1/S;t=0:ts:2;              % Time array
ax=[0 0.01 -1.5 1.5];                % Plot for 0.01 s for all signals
x=sin(800*pi*t);                     % Pure sinusoid
%PART (b)
xn=randist(x,'uni',0);               % uniform noise (mean=0)
snr=10;                              % Desired SNR
A=std(x)/std(xn)/(10^(snr/20));      % Compute A
y=A*xn;zn=x+y;                        % Generate noisy signal
subplot(3,1,1),plot(t,x),axis(ax)
subplot(3,1,2),plot(t,y),axis(ax)
subplot(3,1,3),plot(t,zn),axis(ax),pause
SNR=10*log10(sum(x.*x)/sum(y.*y))    % Compute actual SNR
```

```
%PART (c)
z=0;for n=1:64;                       % Initialize z and start loop
z=z+x+y;                              % Sum
if n==16, a16=z/16; end               % Save average of 16 runs
end                                   % End of summing 64-runs
a64=z/64;                             % Average of 64 runs
subplot(3,1,1),plot(t,zn),axis(ax)    % Plot noisy signal
subplot(3,1,2),plot(t,a16),axis(ax)   % and 16-run average
subplot(3,1,3),plot(t,a64),axis(ax),pause     % and 64-run (not any better)

%PART (d)
z=0;for n=1:64;                       % Initialize z and start loop
z=z+x+A*randist(x,'uni',0);           % Keep summing each run
if n==16, a16=z/16; end               % Save average of 16 runs
end                                   % End of summing 64-runs
a64=z/64;                             % Average of 64 runs
subplot(3,1,1),plot(t,zn),axis(ax)    % Plot noisy signal
subplot(3,1,2),plot(t,a16),axis(ax)   % and 16-run average
subplot(3,1,3),plot(t,a64),axis(ax),pause  % and 64-run (much better)

%PART (e)
xn=randist(x,'nor',0);                % Gaussian noise (mean=0)
snr=10;                               % Desired SNR
A=std(x)/std(xn)/(10^(snr/20));       % Compute A
y=A*xn;
SNR2=10*log10(sum(x.*x)/sum(y.*y))    % Compute actual SNR
z=0;for n=1:64;                       % Initialize z and start loop
z=z+x+A*randist(x,'nor',0);           % Keep summing each run
if n==16, a16=z/16; end               % Save average of 16 runs
end                                   % End of summing 64-runs
a64=z/64;                             % Average of 64 runs
subplot(3,1,1),plot(t,zn),axis(ax)    % Plot noisy signal
subplot(3,1,2),plot(t,a16),axis(ax)   % and 16-run average
subplot(3,1,3),plot(t,a64),axis(ax)   % and 64-run (much better)
```

2.39 (The Central Limit Theorem) The central limit theorem asserts that the sum of independent noise distributions tends to a Gaussian distribution as the number N of distributions in the sum increases. In fact, one way to generate a random signal with a Gaussian distribution is to add many (typically 6 to 12) uniformly distributed signals.

(a) Generate the sum of uniformly distributed random signals using $N = 2$, $N = 6$, and $N = 12$ and plot the histograms of each sum. Does the histogram begin to take on a Gaussian shape as N increases? Comment on the shape of the histogram for $N = 2$.

(b) Generate the sum of random signals with different distributions using $N = 6$ and $N = 12$. Does the central limit theorem appear to hold even when the distributions are not identical (as long as you select a large enough N)? Comment on the physical significance of this result.

(Solution) Uses the author's routine `randist`

```
%PROBLEM 3.39
S=0;N=2;x=randist(500,'uni',0);for k=1:2,S=S+randist(500,'uni',0);end
%For N=2, should ideally be triangular (convolution of uniform distributions)
subplot(2,2,1),hist(x,20),subplot(2,2,2),hist(S,20),pause
S=0;for k=1:6,S=S+randist(500,'uni',0);end
subplot(2,2,1),hist(x,20),subplot(2,2,2),hist(S,20)
S=0;for k=1:12,S=S+randist(500,'uni',0);end
subplot(2,2,3),hist(x,20),subplot(2,2,4),hist(S,20),pause
%PART (b)
S=0;for k=1:6,S=S+randist(500,'exp');end     % Exponential distribution
subplot(2,2,1),hist(x,20),subplot(2,2,2),hist(S,20)
S=0;for k=1:12,S=S+randist(500,'exp');end
subplot(2,2,3),hist(x,20),subplot(2,2,4),hist(S,20)
```

2.40 (Music Synthesis I) A musical composition is a combination of *notes*, or signals, at various frequencies. An *octave* covers a range of frequencies from f_0 to $2f_0$. In the western musical scale, there are 12 notes per octave, *logarithmically equispaced*. The frequencies of the notes from f_0 to $2f_0$ correspond to

$$f = 2^{k/12} f_0 \qquad k = 0, 1, 2, \ldots, 11$$

The 12 notes are as follows (the $\sharp$ and $\flat$ stand for *sharp* and *flat*, and each pair of notes in parentheses has the same frequency):

A (A$^\sharp$ or B$^\flat$) B C (C$^\sharp$ or D$^\flat$) D (D$^\sharp$ or E$^\flat$) E F (F$^\sharp$ or G$^\flat$) G (G$^\sharp$ or A$^\flat$)

An Example: Raga Malkauns: In Indian classical music, a *raga* is a musical composition based on an ascending and descending scale. The notes and their order form the musical alphabet and grammar from which the performer constructs musical passages, using only the notes allowed. The performance of a raga can last from a few minutes to an hour or more! Raga *malkauns* is a pentatonic raga (with five notes) and the following scales:

Ascending: D F G B$^\flat$ C D   Descending: C B$^\flat$ G F D

The final note in each scale is held twice as long as the rest. To synthesize this scale in MATLAB, we start with a frequency f_0 corresponding to the first note D and go up in frequency to get the notes in the ascending scale; when we reach the note D, which is an octave higher, we go down in frequency to get the notes in the descending scale. Here is a MATLAB code fragment.

```
f0=340; d=f0;                        % Pick a frequency and the note D
f=f0*(2^(3/12)); g=f0*(2^(5/12));    % The notes F and G
bf=f0*(2^(8/12)); c=f0*(2^(10/12));  % The notes B(flat) and C
d2=2*d;                              % The note D (an octave higher)
```

Generate sampled sinusoids at these frequencies, using an appropriate sampling rate (say, 8192 Hz); concatenate them, assuming silent passages between each note; and play the resulting signal, using the MATLAB command `sound`. Use the following MATLAB code fragment as a guide:

```
ts=1/8192;                                    % Sampling interval
t=0:ts:0.4;                                   % Time for each note (0.4 s)
s1=0*(0:ts:0.1);                              % Silent period (0.1 s)
s2=0*(0:ts:0.05);                             % Shorter silent period (0.05 s)
tl=0:ts:1;                                    % Time for last note of each scale
d1=sin(2*pi*d*t);                             % Start generating the notes
f1=sin(2*pi*f*t); g1=sin(2*pi*g*t);
bf1=sin(2*pi*bf*t); c1=sin(2*pi*c*t);
dl1=sin(2*pi*d2*tl); dl2=sin(2*pi*d*tl);
asc=[d1 s1 f1 s1 g1 s1 bf1 s1 c1 s2 dl1];   % Create ascending scale
dsc=[c1 s1 bf1 s1 g1 s1 f1 s1 dl2];          % Create descending scale
y=[asc s1 dsc s1]; sound(y)                   % Malkauns scale (y)
```

(Solution)

```
%PROBLEM 3.40
f0=340;
d=f0;
f=f0*(2^(3/12));
g=f0*(2^(5/12));
bf=f0*(2^(8/12));
c=f0*(2^(10/12));
d2=2*d;
ts=1/8192; %Sampling interval
t=0:ts:0.4; %time for each note
s1=0*(0:ts:0.1); %silent period
s2=0*(0:ts:0.05); %shorter silent period
tl=0:ts:1; %time for last note of asc and desc scale
d1=sin(2*pi*d*t);f1=sin(2*pi*f*t);g1=sin(2*pi*g*t);bf1=sin(2*pi*bf*t);
c1=sin(2*pi*c*t);dl1=sin(2*pi*d2*tl);dl2=sin(2*pi*d*tl);
asc=[d1 s1 f1 s1 g1 s1 bf1 s1 c1 s2 dl1]; %Create asc scale
dsc=[c1 s1 bf1 s1 g1 s1 f1 s1 dl2]; %Create desc scale
y=[asc s1 dsc s1]; %Malkauns scale
sound(y)
```

2.41 (Music Synthesis II) The raw scale of raga *malkauns* will sound pretty dry! The reason for this is the manner in which the sound from a musical instrument is generated. Musical instruments produce sounds by the vibrations of a string (in string instruments) or a column of air (in woodwind instruments). Each instrument has its characteristic sound. In a guitar, for example, the strings are plucked, held, and then released to sound the notes. Once plucked, the sound dies out and decays. Furthermore, the notes are never pure but contain overtones (harmonics). For a realistic sound, we must include the overtones and the attack, sustain, and release (decay) characteristics. The sound signal may be considered to have the form $x(t) = \alpha(t)\cos(2\pi f_0 t + \theta)$, where f_0 is the pitch and $\alpha(t)$ is the envelope that describes the attack-sustain-release characteristics of the instrument played. A crude representation of some envelopes is shown in Figure P2.41 (the piecewise linear approximations will work just as well for our purposes). Woodwind instruments have a much longer sustain time and a much shorter release time than do plucked string and keyboard instruments.

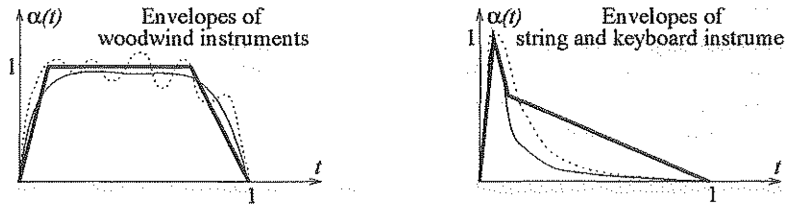


Figure P2.41 Envelopes and their piecewise linear approximations (dark) for Problem 2.41

Experiment with the scale of raga *malkauns* and try to produce a guitar-like sound, using the appropriate envelope form. You should be able to discern an audible improvement.

(Solution)

```
%PROBLEM 3.41
%FROM Prob 3.40
f0=340;d=f0;f=f0*(2^(3/12));g=f0*(2^(5/12));bf=f0*(2^(8/12));
c=f0*(2^(10/12));d2=2*d;ts=1/8192;t=0:ts:0.4;s1=0*(0:ts:0.1);
s2=0*(0:ts:0.05);tl=0:ts:1;d1=sin(2*pi*d*t);f1=sin(2*pi*f*t);
g1=sin(2*pi*g*t);bf1=sin(2*pi*bf*t);
c1=sin(2*pi*c*t);dl1=sin(2*pi*d2*tl);dl2=sin(2*pi*d*tl);
asc=[d1 s1 f1 s1 g1 s1 bf1 s1 c1 s2 dl1];
dsc=[c1 s1 bf1 s1 g1 s1 f1 s1 dl2];y=[asc s1 dsc s1];sound(y)


%We now let each note decay by adding an exponential decay
ex=exp(-5*t);ex1=exp(-4*tl);
asc=[d1.*ex s1 f1.*ex s1 g1.*ex s1 bf1.*ex s1 c1.*ex s2 dl1.*ex1];
dsc=[c1.*ex s1 bf1.*ex s1 g1.*ex s1 f1.*ex s1 dl2.*ex1];
yhd=[asc s1 dsc s1];
sound(yhd)
```

2.42 (Music Synthesis III) Synthesize the following notes, using a woodwind envelope, and synthesize the same notes using a plucked string envelope.

$$F^\sharp(0.3) \quad D(0.4) \quad E(0.4) \quad A(1) \quad A(0.4) \quad E(0.4) \quad F^\sharp(0.3) \quad D(1)$$

All the notes cover one octave, and the numbers in parentheses give a rough indication of their relative duration. Can you identify the music? (It is *Big Ben*.)

(Solution)

```
%PROBLEM 3.42
f0=260;a=f0;fs=f0*(2^(9/12));d=f0*(2^(5/12));e=f0*(2^(7/12));
ts=1/8192;t1=0:ts:0.7;t2=0:ts:1;s1=0*(0:ts:0.05);tl=0:ts:1;
f1=sin(2*pi*fs*t1);d1=sin(2*pi*d*t1);e1=sin(2*pi*e*t1);d2=sin(2*pi*d*t2);
a1=sin(2*pi*a*t1);a2=sin(2*pi*a*t2);
bb=[f1 s1 d1 s1 e1 s1 a2 s1 s1 s1 a1 s1 e1 s1 f1 s1 d2]; %Big Ben Notes
sound(bb)
```

```
%Then add decay for string-like sound
ex1=exp(-4*t1);ex2=exp(-3*t2);
bb2=[f1.*ex1, s1, d1.*ex1, s1, e1.*ex1, s1, a2.*ex2];
bb2=[bb2, s1, s1, s1, s1, s1, a1.*ex1, s1, e1.*ex1, s1, f1.*ex1];
bb2=[bb2, s1, d2.*ex2];sound(bb2)
```

2.43 (Music Synthesis IV) Synthesize the first bar of *Pictures at an Exhibition* by Mussorgsky, which has the following notes:

A(3) G(3) C(3) D(2) G*(1) E(3) D(2) G*(1) E(3) C(3) D(3) A(3) G(3)

All the notes cover one octave except the note G*, which is an octave above G. The numbers in parentheses give a rough indication of the relative duration of the notes (for more details, you may want to listen to an actual recording). Assume that a keyboard instrument (such as a piano) is played.

(Solution)

```
%PROBLEM 3.43
f0=440;
a=f0;g=f0*(2^(-2/12));c=f0*(2^(3/12));d=f0*(2^(5/12));
gg=f0*(2^(10/12));e=f0*(2^(7/12));
ts=1/8192;t1=0:ts:0.6;t2=0:ts:0.3;s1=0*(0:ts:0.06);tl=0:ts:1;
a1=sin(2*pi*a*t1);g1=sin(2*pi*g*t1);c1=sin(2*pi*c*t1);
d1=sin(2*pi*d*t1);d2=sin(2*pi*d*t2);
g2=sin(2*pi*gg*t2);
e1=sin(2*pi*e*t1);
p=[a1 s1 g1 s1 c1 s1 d2 s1 g2 s1 e1 s1 d2 s1 g2 s1 e1 s1];
p=[p c1 s1 d1 s1 a1 s1 g1 s1]; %First bar of Pictures
sound(p)

%Add decay for string-like sound
ex1=exp(-4*t1);ex2=exp(-3*t2);
pd=[a1.*ex1 s1 g1.*ex1 s1 c1.*ex1 s1 d2.*ex2 s1 g2.*ex2 s1 e1.*ex1 s1 d2.*ex2];
pd=[pd  s1 g2.*ex2 s1 e1.*ex1 s1 c1.*ex1 s1 d1.*ex1 s1 a1.*ex1 s1 g1.*ex1 s1];
sound(pd)
```

2.44 (DTMF Tones) In dual-tone multi-frequency (DTMF) or touch-tone telephone dialing, each number is represented by a dual-frequency tone. The frequencies for each digit are listed in Chapter 18.

(a) Generate DTMF tones corresponding to the telephone number 487-2550, by sampling the sum of two sinusoids at the required frequencies at $S = 8192$ Hz for each digit. Concatenate the signals by putting 50 zeros between each signal (to represent silence) and listen to the signal using the MATLAB command sound.

(b) Write a MATLAB program that generates DTMF signals corresponding to a vector input representing the digits in a phone number. Use a sampling frequency of $S = 8192$ Hz.

(Solution)

(a)
```
%PROBLEM 3.44
%PART (a)
%1=[697,1209]    2=[697,1336]    3=[697,1477]    4=[770,1209]    5=[770,1336]
%6=[770,1477]    7=[852,1209]    8=[852,1336]    9=[852,1477]    0=[941,1336]

S=8192; wh1=2*pi*1209/S;wh2=2*pi*1336/S;wh3=2*pi*1477/S;
wl1=2*pi*697/S;wl2=2*pi*770/S;wl3=2*pi*852/S;wl4=2*pi*941/S;
%Generate 1000 samples each
z=zeros(1,1000);      %50 zeros is too short!!
N=1000;n=0:N-1;
four=[sin(n*wl1)+sin(n*wh1),z];
eight=[sin(n*wl3)+sin(n*wh2),z];
seven=[sin(n*wl3)+sin(n*wh1),z,z,z];
two=[sin(n*wl1)+sin(n*wh2),z];
five=[sin(n*wl2)+sin(n*wh2),z];
oh=[sin(n*wl4)+sin(n*wh2),z];
ph=[four,eight,seven,two,five,five,oh];sound(ph)
```

(b) Here is one of the many possible solutions.

```
function [sigtt,snd]=dialtt(n,S)
%dial(n,S) Sound of tones in a phone number at sampling freq S
%S=sampling freq (Defaults to 8192 Hz)
%n=vector or array of digits in the phone number
%NOTE: Use 10 for * and 12 for # on the dial
% Use sound(sigtt) to listen to the tones
%See getnum.m (in Chapter 18) to decode the signal

if nargin<2,S=8192;end
low=[697 770 852 941];
hi=[1209 1336 1477];
n=n(:);                         %make column
i=find(n==0);n(i)=11+0*i;       %Change 0 to 11
M=1000;z=zeros(1,M);            %silent passage
N=0:M-1;
l=length(n);
sigtt=[];snd=[];for k=1:l
n(k);ih=rem(n(k)-1,3)+1;il=fix((n(k)-1)/3)+1;
tone=[low(il) hi(ih)];
snd=[snd;tone];
sigtt=[sigtt, cos(2*pi*tone(1)*N/S)+cos(2*pi*tone(2)*N/S)];
if k<l,sigtt=[sigtt, z];end     %add zeros to all except last digit
end
```