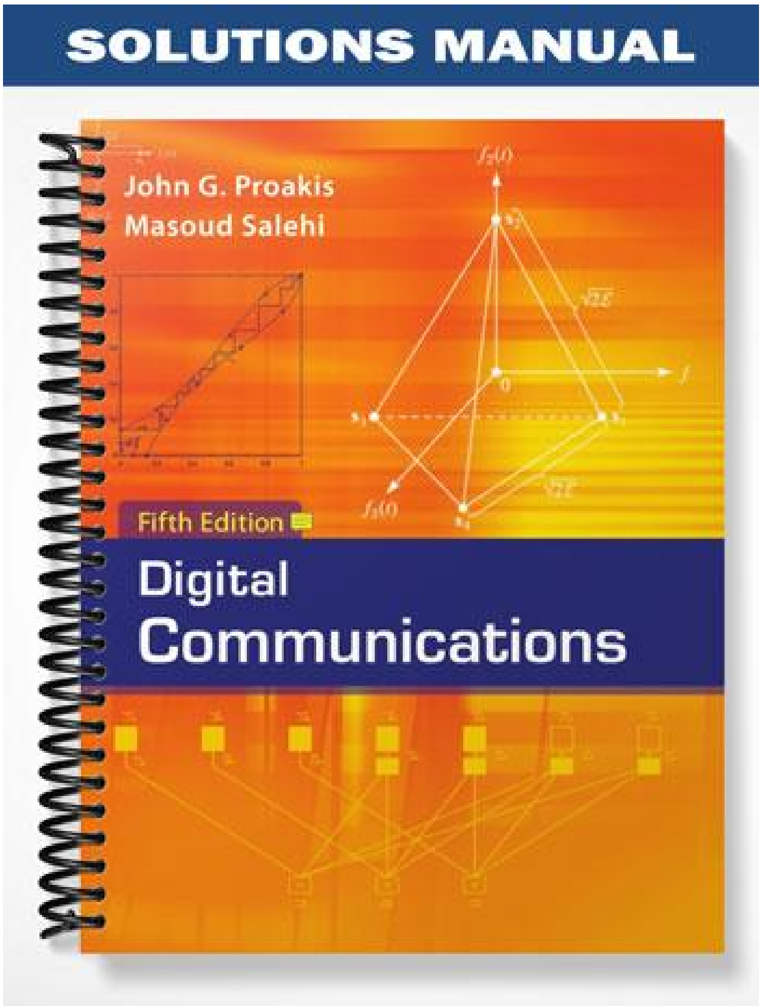
SOLUTIONS MANUAL
John G. Proakis
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Fifth Edition
Digital
Communications

Solutions Manual
for
Digital Communications, 5th Edition
(Chapter 2) [1]

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Problem 2.1

a.

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(a)}{t-a} da$$

Hence :

$$\begin{aligned} -\hat{x}(-t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(a)}{-t-a} da \\ &= -\frac{1}{\pi}\int_{\infty}^{-\infty} \frac{x(-b)}{-t+b}(-db) \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(b)}{-t+b} db \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(b)}{t-b} db = \hat{x}(t) \end{aligned}$$

where we have made the change of variables : $b = -a$ and used the relationship : $x(b) = x(-b)$.

b. In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

c. $x(t) = \cos\omega_0 t$, so its Fourier transform is : $X(f) = \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right],\ f_0 = 2\pi\omega_0$. Exploiting the phase-shifting property (2-1-4) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2}\left[-j\delta(f-f_0) + j\delta(f+f_0)\right] = \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] = F^{-1}\left\{\sin 2\pi f_0 t\right\}$$

Hence, $\hat{x}(t) = \sin\omega_0 t$.

d. In a similar way to part (c) :

$$x(t) = \sin\omega_0 t \Rightarrow X(f) = \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] \Rightarrow \hat{X}(f) = \frac{1}{2}\left[-\delta(f-f_0) - \delta(f+f_0)\right]$$

$$\Rightarrow \hat{X}(f) = -\frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] = -F^{-1}\left\{\cos 2\pi\omega_0 t\right\} \Rightarrow \hat{x}(t) = -\cos\omega_0 t$$

e. The positive frequency content of the new signal will be : $(-j)(-j)X(f) = -X(f),\ f > 0$, while the negative frequency content will be : $j \cdot jX(f) = -X(f),\ f < 0$. Hence, since $\hat{\hat{X}}(f) = -X(f)$, we have : $\hat{\hat{x}}(t) = -x(t)$.

f. Since the magnitude response of the Hilbert transformer is characterized by : $|H(f)| = 1$, we have that : $\left|\hat{X}(f)\right| = |H(f)|\,|X(f)| = |X(f)|$. Hence :

$$\int_{-\infty}^{\infty} \left|\hat{X}(f)\right|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

g. From parts (a) and (b) above, we note that if $x(t)$ is even, $\hat{x}(t)$ is odd and vice-versa. Therefore, $x(t)\hat{x}(t)$ is always odd and hence : $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.

Problem 2.2

1. Using relations

$$X(f) = \frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0)$$
$$Y(f) = \frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0)$$

and Parseval's relation, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)y(t)\,dt &= \int_{-\infty}^{\infty} X(f)Y^*(f)\,df \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0)\right]\left[\frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0)\right]^* df \\
&= \frac{1}{4}\int_{-\infty}^{\infty} X_l(f - f_0)Y_l^*(f - f_0)\,df + \frac{1}{4}\int_{-\infty}^{\infty} X_l(-f - f_0)Y_l(-f - f_0)\,df \\
&= \frac{1}{4}\int_{-\infty}^{\infty} X_l(u)Y_l^*(u)\,du + \frac{1}{4}X_l^*(v)Y(v)\,dv \\
&= \frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^{\infty} X_l(f)Y_l^*(f)\,df\right] \\
&= \frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^{\infty} x_l(t)y_l^*(t)\,dt\right]
\end{aligned}$$

where we have used the fact that since $X_l(f - f_0)$ and $Y_l(-f - f_0)$ do not overlap, $X_l(f - f_0)Y_l(-f - f_0) = 0$ and similarly $X_l(-f - f_0)Y_l(f - f_0) = 0$.

2. Putting $y(t) = x(t)$ we get the desired result from the result of part 1.

Problem 2.3

A well-known result in estimation theory based on the minimum mean-squared-error criterion states that the minimum of $\mathcal{E}_e$ is obtained when the error is orthogonal to each of the functions in the series expansion. Hence :

$$\int_{-\infty}^{\infty}\left[s(t)-\sum_{k=1}^{K}s_kf_k(t)\right]f_n^*(t)dt=0, \qquad n=1,2,...,K \tag{1}$$

since the functions $\{f_n(t)\}$ are orthonormal, only the term with $k=n$ will remain in the sum, so :

$$\int_{-\infty}^{\infty}s(t)f_n^*(t)dt-s_n=0, \qquad n=1,2,...,K$$

or:

$$s_n=\int_{-\infty}^{\infty}s(t)f_n^*(t)dt \qquad n=1,2,...,K$$

The corresponding residual error $\mathcal{E}_e$ is :

$$\begin{aligned}
\mathcal{E}_{\min} &= \int_{-\infty}^{\infty}\left[s(t)-\sum_{k=1}^{K}s_kf_k(t)\right]\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]^*dt \\
&= \int_{-\infty}^{\infty}|s(t)|^2dt-\int_{-\infty}^{\infty}\sum_{k=1}^{K}s_kf_k(t)s^*(t)dt-\sum_{n=1}^{K}s_n^*\int_{-\infty}^{\infty}\left[s(t)-\sum_{k=1}^{K}s_kf_k(t)\right]f_n^*(t)dt \\
&= \int_{-\infty}^{\infty}|s(t)|^2dt-\int_{-\infty}^{\infty}\sum_{k=1}^{K}s_kf_k(t)s^*(t)dt \\
&= \mathcal{E}_s-\sum_{k=1}^{K}|s_k|^2
\end{aligned}$$

where we have exploited relationship (1) to go from the second to the third step in the above calculation.

Note : Relationship (1) can also be obtained by simple differentiation of the residual error with respect to the coefficients $\{s_n\}$. Since s_n is, in general, complex-valued $s_n=a_n+jb_n$ we have to differentiate with respect to both real and imaginary parts :

$$\frac{d}{da_n}\mathcal{E}_e=\frac{d}{da_n}\int_{-\infty}^{\infty}\left[s(t)-\sum_{k=1}^{K}s_kf_k(t)\right]\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]^*dt=0$$

$$\Rightarrow -\int_{-\infty}^{\infty}a_nf_n(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]^*+a_n^*f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]dt=0$$

$$\Rightarrow -2a_n\int_{-\infty}^{\infty}Re\left\{f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]\right\}dt=0$$

$$\Rightarrow \int_{-\infty}^{\infty}Re\left\{f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]\right\}dt=0, \qquad n=1,2,...,K$$

where we have exploited the identity : $(x+x^*) = 2Re\{x\}$. Differentiation of $\mathcal{E}_e$ with respect to b_n will give the corresponding relationship for the imaginary part; combining the two we get (1).

Problem 2.4

The procedure is very similar to the one for the real-valued signals described in the book (pages 33-37). The only difference is that the projections should conform to the complex-valued vector space :

$$c_{12}=\int_{-\infty}^{\infty} s_2(t) f_1^*(t)dt$$

and, in general for the k-th function :

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i^*(t)dt, \quad i = 1, 2, ..., k-1$$

Problem 2.5

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \left\{ \begin{array}{cc} -1/\sqrt{3}, & 0 \le t \le 3 \\ 0, & \text{o.w.} \end{array} \right\}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t) g_4(t) dt = -1/\sqrt{3} \Rightarrow g_3'(t) = s_3(t) - c_{43} g_4(t) = \left\{ \begin{array}{cc} 2/3, & 0 \le t \le 2 \\ -4/3, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{array} \right\}$$

Hence :

$$g_3(t) = \frac{g_3'(t)}{\sqrt{E_3}} = \left\{ \begin{array}{cc} 1/\sqrt{6}, & 0 \le t \le 2 \\ -2/\sqrt{6}, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{array} \right\}$$

where E_3 denotes the energy of $g_3'(t)$: $E_3 = \int_0^3 (g_3'(t))^2 dt = 8/3$.

For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t) g_4(t) dt = 0 \quad \text{and} \quad c_{32} = \int_{-\infty}^{\infty} s_2(t) g_3(t) dt = 0$$

Hence :

$$g_2'(t) = s_2(t) - c_{42}g_4(t) - c_{32}g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g_2'(t)}{\sqrt{\mathcal{E}_2}} = \left\{ \begin{array}{cc} 1/\sqrt{2}, & 0 \le t \le 1 \\ -1/\sqrt{2}, & 1 \le t \le 2 \\ 0, & \text{o.w} \end{array} \right\}$$

where : $\mathcal{E}_2 = \int_0^2 (s_2(t))^2\, dt = 2$.

Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t)g_4(t)dt = -2/\sqrt{3},\ c_{31} = \int_{-\infty}^{\infty} s_1(t)g_3(t)dt = 2/\sqrt{6},\ c_{21} = 0$$

Hence :

$$g_1'(t) = s_1(t) - c_{41}g_4(t) - c_{31}g_3(t) - c_{21}g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions $(g_2(t), g_3(t), g_4(t))$ the basis representation of the signals is :

$$\begin{aligned}
\mathbf{s}_4 &= \left(0, 0, \sqrt{3}\right) \Rightarrow \mathcal{E}_4 = 3 \\
\mathbf{s}_3 &= \left(0, \sqrt{8/3}, -1/\sqrt{3}\right) \Rightarrow \mathcal{E}_3 = 3 \\
\mathbf{s}_2 &= \left(\sqrt{2}, 0, 0\right) \Rightarrow \mathcal{E}_2 = 2 \\
\mathbf{s}_1 &= \left(2/\sqrt{6}, -2/\sqrt{3}, 0\right) \Rightarrow \mathcal{E}_1 = 2
\end{aligned}$$

Problem 2.6

Consider the set of signals $\widetilde{\phi}_{nl}(t) = j\phi_{nl}(t),\ \ 1 \le n \le N$, then by definition of lowpass equivalent signals and by Equations 2.2-49 and 2.2-54, we see that $\phi_n(t)$'s are $\sqrt{2}$ times the lowpass equivalents of $\phi_{nl}(t)$'s and $\widetilde{\phi}_n(t)$'s are $\sqrt{2}$ times the lowpass equivalents of $\widetilde{\phi}_{nl}(t)$'s. We also note that since $\phi_n(t)$'s have unit energy, $\langle \phi_{nl}(t), \widetilde{\phi}_{nl}(t) \rangle = \langle \phi_{nl}(t), j\phi_{nl}(t) \rangle = -j$ and since the inner product is pure imaginary, we conclude that $\phi_n(t)$ and $\widetilde{\phi}_n(t)$ are orthogonal. Using the orthonormality of the set $\phi_{nl}(t)$, we have

$$\langle \phi_{nl}(t), -j\phi_{ml}(t) \rangle = j\delta_{mn}$$

and using the result of problem 2.2 we have

$$\langle \phi_n(t), \widetilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n, m$$

We also have

$$\langle \phi_n(t), \phi_m(t) \rangle = 0 \quad \text{for all } n \ne m$$

and

$$\langle \widetilde{\phi}_n(t), \widetilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n \neq m$$

Using the fact that the energy in lowpass equivalent signal is twice the energy in the bandpass signal we conclude that the energy in $\phi_n(t)$'s and $\widetilde{\phi}_n(t)$'s is unity and hence the set of $2N$ signals $\{\phi_n(t), \widetilde{\phi}_n(t)\}$ constitute an orthonormal set. The fact that this orthonormal set is sufficient for expansion of bandpass signals follows from Equation 2.2-57.

Problem 2.7

Let $x(t) = m(t)\cos 2\pi f_0 t$ where $m(t)$ is real and lowpass with bandwidth less than f_0. Then $\mathcal{F}[\hat{x}(t)] = -j\,\mathrm{sgn}(f)\left[\frac{1}{2}M(f-f_0) + \frac{1}{2}M(f+f_0)\right]$ and hence $\mathcal{F}[\hat{x}(t)] = -\frac{j}{2}M(f-f_0) + \frac{j}{2}M(f+f_0)$ where we have used that fact that $M(f-f_0) = 0$ for $f < 0$ and $M(f+f_0) = 0$ for $f > 0$. This shows that $\hat{x}(t) = m(t)\sin 2\pi f_0 t$. Similarly we can show that Hilbert transform of $m(t)\sin 2\pi f_0 t$ is $-m(t)\cos 2\pi f_0 t$. From above and Equation 2.2-54 we have

$$\mathcal{H}[\phi_n(t)] = \sqrt{2}\phi_{ni}(t)\sin 2\pi f_0 t + \sqrt{2}\phi_{nq}(t)\cos 2\pi f_0 t = -\widetilde{\phi}_n(t)$$

Problem 2.8

For real-valued signals the correlation coefficients are given by : $\rho_{km} = \frac{1}{\sqrt{\mathcal{E}_k\mathcal{E}_m}}\int_{-\infty}^{\infty} s_k(t)s_m(t)dt$ and the Euclidean distances by : $d_{km}^{(e)} = \left\{\mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k\mathcal{E}_m}\rho_{km}\right\}^{1/2}$. For the signals in this problem :

$$\mathcal{E}_1 = 2,\ \mathcal{E}_2 = 2,\ \mathcal{E}_3 = 3,\ \mathcal{E}_4 = 3$$

$$\begin{array}{lll} \rho_{12} = 0 & \rho_{13} = \frac{2}{\sqrt{6}} & \rho_{14} = -\frac{2}{\sqrt{6}} \\ \rho_{23} = 0 & \rho_{24} = 0 & \\ \rho_{34} = -\frac{1}{3} & & \end{array}$$

and:

$$\begin{array}{lll} d_{12}^{(e)} = 2 & d_{13}^{(e)} = \sqrt{2+3-2\sqrt{6}\frac{2}{\sqrt{6}}} = 1 & d_{14}^{(e)} = \sqrt{2+3+2\sqrt{6}\frac{2}{\sqrt{6}}} = 3 \\ d_{23}^{(e)} = \sqrt{2+3=}\sqrt{5} & d_{24}^{(e)} = \sqrt{5} & \\ d_{34}^{(e)} = \sqrt{3+3+2*3\frac{1}{3}} = 2\sqrt{2} & & \end{array}$$

Problem 2.9

We know from Fourier transform properties that if a signal $x(t)$ is real-valued then its Fourier transform satisfies : $X(-f) = X^*(f)$ (Hermitian property). Hence the condition under which $s_l(t)$ is real-valued is : $S_l(-f) = S_l^*(f)$ or going back to the bandpass signal $s(t)$ (using 2-1-5):

$$S_+(f_c - f) = S_+^*(f_c + f)$$

The last condition shows that in order to have a real-valued lowpass signal $s_l(t)$, the positive frequency content of the corresponding bandpass signal must exhibit hermitian symmetry around the center frequency f_c. In general, bandpass signals do not satisfy this property (they have Hermitian symmetry around $f = 0$), hence, the lowpass equivalent is generally complex-valued.

Problem 2.10

a. To show that the waveforms $f_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t) f_n(t) dt = 0, \qquad m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \int_0^4 f_1(t) f_2(t) dt \\ &= \int_0^2 f_1(t) f_2(t) dt + \int_2^4 f_1(t) f_2(t) dt \\ &= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\ &= 0 \end{aligned}$$

Similarly:

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t) dt = \int_0^4 f_1(t) f_3(t) dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

and :

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t) dt = \int_0^4 f_2(t) f_3(t) dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1,2,3$$

Hence, they are orthonormal.

b. We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t) dt, \qquad n = 1,2,3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t) f_1(t) dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t) f_2(t) dt = \frac{1}{2}\int_0^4 x(t) dt = 0 \\
x_3 &= \int_0^4 x(t) f_3(t) dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1,2,3$ and thus it can not represented as a linear combination of these functions.

Problem 2.11

a. As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

b. The representation vectors are

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix}\\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix}\\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}\\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix}\end{aligned}$$

c. The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned}d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5}\\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12}\\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14}\\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31}\\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19}\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 2.12

As a set of orthonormal functions we consider the waveforms

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}\\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}\\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix}\\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 2.13

1. $P(E_2) = P(R2, R3, R4) = 3/7$.
2. $P(E_3|E_2) = \frac{P(E_3 E_2)}{P(E_2)} = \frac{P(R2)}{3/7} = \frac{1}{3}$.
3. Here $E_4 = \{R2, R4, B2, R1, B1\}$ and $P(E_2|E_4 E_3) = \frac{P(E_2 E_3 E_4)}{P(E_3 E_4)} = \frac{P(R2)}{P(R2,B2,R1,B1)} = \frac{1}{4}$.
4. $E_5 = \{R_2, R_4, B_2\}$. We have $P(E_3 E_5) = P(R_2, B_2) = \frac{2}{7}$ and $P(E_3) = P(R1, R2, B1, B2) = \frac{4}{7}$ and $P(E_5) = \frac{3}{7}$. Obviously $P(E_3 E_5) \neq P(E_3)P(E_5)$ and the events are not independent.

Problem 2.14

1. $P(R) = P(A)P(R|A) + P(B)P(R|B) + P(C)P(R|C) = 0.2 \times 0.05 + 0.3 \times 0.1 + 0.5 \times 0.15 = 0.01 + 0.03 + 0.075 = 0.115$.
2. $P(A|R) = \frac{P(A)P(R|A)}{P(R)} = \frac{0.01}{0.115} \approx 0.087$.

Problem 2.15

The relationship holds for $n = 2$ (2-1-34) : $p(x_1, x_2) = p(x_2|x_1)p(x_1)$
Suppose it holds for $n = k$, i.e : $p(x_1, x_2, ..., x_k) = p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \, ...p(x_1)$
Then for $n = k + 1$:

$$\begin{aligned} p(x_1, x_2, ..., x_k, x_{k+1}) &= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k, x_{k-1} ..., x_1) \\ &= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \, ...p(x_1) \end{aligned}$$

Hence the relationship holds for $n = k + 1$, and by induction it holds for any n.

Problem 2.16

1. Let T and R denote channel input and outputs respectively. Using Bayes rule we have

$$\begin{aligned} p(T=0|R=A) &= \frac{p(T=0)p(R=A|T=0)}{p(T=0)p(R=A|T=0)+p(T=1)p(R=A|T=1)} \\ &= \frac{0.4\times\frac{1}{6}}{0.4\times\frac{1}{6}+0.6\times\frac{1}{3}} \\ &= \frac{1}{4} \end{aligned}$$

   and therefore $p(T=1|R=A)=\frac{3}{4}$, obviously if $R=A$ is observed, the best decision would be to declare that a 1 was sent, i.e., $T=1$, because $T=1$ is more probable that $T=0$. Similarly it can be verified that $p(T=0|R=B)=\frac{4}{7}$ and $p(T=0|R=C)=\frac{1}{4}$. Therefore, when the output is B, the best decision is 0 and when the output is C, the best decision is $T=1$. Therefore the decision function d can be defined as

$$d(R)=\begin{cases}1, & R=A \text{ or } C\\ 0, & R=B\end{cases}$$

   This is the optimal decision scheme.

2. Here we know that a 0 is transmitted, therefore we are looking for $p(\text{error}|T=0)$, this is the probability that the receiver declares a 1 was sent when actually a 0 was transmitted. Since by the decision method described in part 1 the receiver declares that a 1 was sent when $R=A$ or $R=C$, therefore, $p(\text{error}|T=0)=p(R=A|T=0)+p(R=C|T=0)=\frac{1}{3}$.

3. We have $p(\text{error}|T=0)=\frac{1}{3}$, and $p(\text{error}|T=1)=p(R=B|T=1)=\frac{1}{3}$. Therefore, by the total probability theorem

$$\begin{aligned} p(\text{error}) &= p(T=0)p(\text{error}|T=0)+p(T=1)p(\text{error}|T=1) \\ &= 0.4\times\frac{1}{3}+0.6\times\frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

Problem 2.17

Following the same procedure as in example 2-1-1, we prove :

$$p_Y(y)=\frac{1}{|a|}p_X\left(\frac{y-b}{a}\right)$$

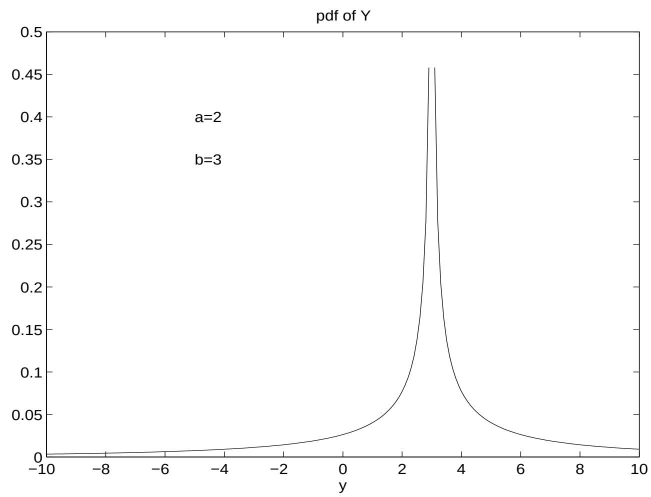
Problem 2.18

Relationship (2-1-44) gives :

$$p_Y(y) = \frac{1}{3a\left[(y-b)/a\right]^{2/3}} p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

X is a gaussian r.v. with zero mean and unit variance : $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
Hence :

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}\left[(y-b)/a\right]^{2/3}} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$



Problem 2.19

1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $p(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned} p(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\ p(X > 4 \times 10^{-4}) &= Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5} \\ p(-2 \times 10^{-4} < X \leq 10^{-4}) &= 1 - Q(1) - Q(2) = .8182 \end{aligned}$$

2)

$$p(X > 10^{-4} | X > 0) = \frac{p(X > 10^{-4}, \ X > 0)}{p(X > 0)} = \frac{p(X > 10^{-4})}{p(X > 0)} = \frac{.159}{.5} = .318$$

Problem 2.20

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\
&= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\
&= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}}
\end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \leq y \leq b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \leq b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned}
f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\
&= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}[u_{-1}(y+b) - u_{-1}(y-b)]
\end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned}
p(Y = b) &= p(X < 0) = F_X(0) = \frac{1}{2} \\
p(Y = a) &= p(X > 0) = 1 - F_X(0) = \frac{1}{2}
\end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} [F_X(a_{i+1}) - F_X(a_i)]\, \delta(y - x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right) \right] \delta(y - x_i) \end{aligned}$$

Problem 2.21

For n odd, x^n is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval $[-\infty, \infty]$ is zero, we obtain $E[X^n] = 0$ for n odd. Let $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) dx$. Obviously I_n is a constant and its derivative with respect to x is zero, i.e.,

$$\frac{d}{dx} I_n = \int_{-\infty}^{\infty} \left[n x^{n-1} e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2} x^{n+1} e^{-\frac{x^2}{2\sigma^2}} \right] dx = 0$$

which results in the recursion

$$I_{n+1} = n\sigma^2 I_{n-1}$$

This is true for all n. Now let $n = 2k - 1$, we will have $I_{2k} = (2k-1)\sigma^2 I_{2k-2}$, with the initial condition $I_0 = \sqrt{2\pi\sigma^2}$. Substituting we have

$$\begin{aligned} I_2 &= \sigma^2 \sqrt{2\pi\sigma^2} \\ I_4 &= 3\sigma^2 I_2 = 3\sigma^4 \sqrt{2\pi\sigma^2} \\ I_6 &= 5 \times 3\sigma^2 I_4 = 5 \times 3\sigma^6 \sqrt{2\pi\sigma^2} \\ I_8 &= 7 \times \sigma^2 I_6 = 7 \times 5 \times 3\sigma^8 \sqrt{2\pi\sigma^2} \\ \vdots &= \vdots \end{aligned}$$

and in general if $I_{2k} = (2k-1)(2k-3)(2k-5) \times \cdots \times 3 \times 1\sigma^{2k}\sqrt{2\pi\sigma^2}$, then $I_{2k+2} = (2k+1)\sigma^2 I_{2k} = (2k+1)(2k-1)(2k-3)(2k-5) \times \cdots \times 3 \times 1\sigma^{2k+2}\sqrt{2\pi\sigma^2}$. Using the fact that $E[X^{2k}] = I_{2k}/\sqrt{2\pi\sigma^2}$, we obtain

$$I_n = 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^n$$

for n even.

Problem 2.22

a. Since (X_r, X_i) are statistically independent :

$$p_{\mathbf{X}}(x_r, x_i) = p_X(x_r)p_X(x_i) = \frac{1}{2\pi\sigma^2}e^{-\left(x_r^2+x_i^2\right)/2\sigma^2}$$

Also :

$$Y_r + jY_i = (X_r + X_i)e^{j\phi} \Rightarrow$$

$$X_r + X_i = (Y_r + jY_i)\, e^{-j\phi} = Y_r \cos\phi + Y_i \sin\phi + j(-Y_r \sin\phi + Y_i \cos\phi) \Rightarrow$$

$$\left\{ \begin{array}{rl} X_r = & Y_r \cos\phi + Y_i \sin\phi \\ X_i = & -Y_r \sin\phi + Y_i \cos\phi \end{array} \right\}$$

The Jacobian of the above transformation is :

$$J = \left| \begin{array}{cc} \frac{\partial X_r}{\partial Y_r} & \frac{\partial X_i}{\partial Y_r} \\ \frac{\partial Xr}{\partial Y_i} & \frac{\partial X_i}{\partial Y_i} \end{array} \right| = \left| \begin{array}{cc} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array} \right| = 1$$

Hence, by (2-1-55) :

$$\begin{array}{rcl} p_{\mathbf{Y}}(y_r, y_i) & = & p_{\mathbf{X}}((Y_r \cos\phi + Y_i \sin\phi), (-Y_r \sin\phi + Y_i \cos\phi)) \\ & = & \frac{1}{2\pi\sigma^2}e^{-\left(y_r^2+y_i^2\right)/2\sigma^2} \end{array}$$

b. $\mathbf{Y} = \mathbf{AX}$ and $\mathbf{X} = \mathbf{A^{-1}Y}$

Now, $p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}}e^{-\mathbf{x}'\mathbf{x}/2\sigma^2}$ (the covariance matrix $\mathbf{M}$ of the random variables $x_1, ..., x_n$ is $\mathbf{M} = \sigma^2\mathbf{I}$, since they are i.i.d) and $J = 1/|\det(\mathbf{A})|$. Hence :

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi\sigma^2)^{n/2}}\frac{1}{|\det(\mathbf{A})|}e^{-\mathbf{y}'(\mathbf{A^{-1}})'\mathbf{A^{-1}y}/2\sigma^2}$$

For the pdf's of X and Y to be identical we require that :

$$|\det(\mathbf{A})| = 1 \; and \; (\mathbf{A}^{-1})'\mathbf{A}^{-1} = \mathbf{I} \implies \mathbf{A}^{-1} = \mathbf{A}'$$

Hence, $\mathbf{A}$ must be a unitary (orthogonal) matrix .

Problem 2.23

Since we are dealing with linear combinations of jointly Gaussian random variables, it is clear that $\boldsymbol{Y}$ is jointly Gaussian. We clearly have $\boldsymbol{m}_Y = E[\boldsymbol{AX}] = \boldsymbol{Am}_X$. This means that $\boldsymbol{Y} - \boldsymbol{m}_Y = \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{m}_X)$. Also note that

$$\boldsymbol{C}_Y = E\left[(\boldsymbol{Y} - \boldsymbol{m}_Y)(\boldsymbol{Y} - \boldsymbol{m}_Y)'\right] = E\left[\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{m}_X)(\boldsymbol{X} - \boldsymbol{m}_X)\boldsymbol{A}'\right]$$

resulting in $\boldsymbol{C}_Y = \boldsymbol{A}\boldsymbol{C}_X\boldsymbol{A}'$.

Problem 2.24

a.

$$\psi_Y(jv) = E\left[e^{jvY}\right] = E\left[e^{jv\sum_{i=1}^{n} x_i}\right] = E\left[\prod_{i=1}^{n} e^{jvx_i}\right] = \prod_{i=1}^{n} E\left[e^{jvX}\right] = \left(\psi_X(e^{jv})\right)^n$$

But,

$$p_X(x) = p\delta(x-1) + (1-p)\delta(x) \Rightarrow \psi_X(e^{jv}) = 1 + p + pe^{jv}$$

$$\Rightarrow \psi_Y(jv) = \left(1 + p + pe^{jv}\right)^n$$

b.

$$E(Y) = -j\frac{d\psi_Y(jv)}{dv}|_{v=0} = -jn(1-p+pe^{jv})^{n-1}jpe^{jv}|_{v=0} = np$$

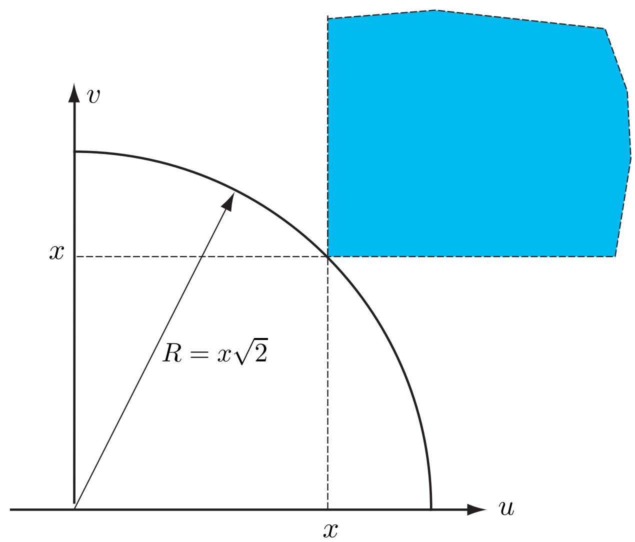
and

$$E(Y^2) = -\frac{d^2\psi_Y(jv)}{d^2v}|_{v=0} = -\frac{d}{dv}\left[jn(1-p+pe^{jv})^{n-1}pe^{jv}\right]_{v=0} = np + np(n-1)p$$

$$\Rightarrow E(Y^2) = n^2p^2 + np(1-p)$$

Problem 2.25

1. In the figure shown below



let us consider the region $u > x, v > x$ shown as the colored region extending to infinity, call this region $\mathcal{R}$, and let us integrate $e^{-\frac{u^2+v^2}{2}}$ over this region. We have

$$\begin{aligned}\iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}}\, du\, dv &= \iint_{\mathcal{R}} e^{-\frac{r^2}{2}} r\, dr\, d\theta \\ &\le \int_{x\sqrt{2}}^{\infty} r e^{-\frac{r^2}{2}}\, dr \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{\pi}{2}\left[-e^{-\frac{r^2}{2}}\right]_{x\sqrt{2}}^{\infty} \\ &= \frac{\pi}{2} e^{-x^2}\end{aligned}$$

where we have used the fact that region $\mathcal{R}$ is included in the region outside the quarter circle as shown in the figure. On the other hand we have

$$\begin{aligned}\iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}}\, du\, dv &= \int_x^{\infty} e^{-\frac{u^2}{2}}\, du \int_x^{\infty} e^{-\frac{v^2}{2}}\, dv \\ &= \left(\int_x^{\infty} e^{-\frac{u^2}{2}}\, du\right)^2 \\ &= \left(\sqrt{2\pi} Q(x)\right)^2 \\ &= 2\pi \left(Q(x)\right)^2\end{aligned}$$

From the above relations we conclude that

$$2\pi \left(Q(x)\right)^2 \le \frac{\pi}{2} e^{-x^2}$$

and therefore, $Q(x) \le \frac{1}{2} e^{-\frac{x^2}{2}}$.

2. In $\int_x^\infty e^{-\frac{y^2}{2}}\frac{dy}{y^2}$ define $u = e^{-\frac{y^2}{2}}$ and $dv = \frac{dy}{y^2}$ and use the integration by parts relation $\int u\,dv = uv - \int v\,du$. We have $v = -\frac{1}{y}$ and $du = -ye^{-\frac{y^2}{2}}\,dy$. Therefore

$$\int_x^\infty e^{-\frac{y^2}{2}}\frac{dy}{y^2} = \left[-\frac{e^{-\frac{y^2}{2}}}{y}\right]_x^\infty - \int_x^\infty e^{-\frac{y^2}{2}}\,dy = \frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x)$$

Now note that $\int_x^\infty e^{-\frac{y^2}{2}}\frac{dy}{y^2} > 0$ which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) > 0 \Rightarrow Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

On the other hand, note that

$$\int_x^\infty e^{-\frac{y^2}{2}}\frac{dy}{y^2} < \frac{1}{x^2}\int_x^\infty e^{-\frac{y^2}{2}}\,dy = \frac{\sqrt{2\pi}}{x^2}Q(x)$$

which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) < \frac{\sqrt{2\pi}}{x^2}Q(x)$$

or, $\sqrt{2\pi}\frac{1+x^2}{x^2}Q(x) > \frac{e^{-\frac{x^2}{2}}}{x}$ which results in

$$Q(x) > \frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}}$$

3. From

$$\frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

we have

$$\frac{1}{\sqrt{2\pi}(\frac{1}{x}+x)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

As x becomes large $\frac{1}{x}$ in the denominator of the left hand side becomes small and the two bounds become equal, therefore for large x we have

$$Q(x) \approx \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

Problem 2.26

1. $F_{Y_n}(y) = P[Y_n \leq y] = 1 - P[Y_n > y] = 1 - P[x_1 > y, X_2 > y, \ldots, X_n > y] = 1 - (P[X > y])^n$ where we have used the independence of X_i's in the last step. But $P[X > y] = \int_y^A \frac{1}{A}\, dy = \frac{A-y}{A}$. Therefore, $F_{Y_n}(y) = 1 - \frac{(A-y)^n}{A^n}$, and $f_{Y_n}(y) = \frac{d}{dy} F_{Y_n}(y) = n\frac{(A-y)^{n-1}}{A^n}$, $\;0 < y < A$.

2.

$$\begin{aligned} f(y) &= \frac{n}{A}\left(1 - \frac{y}{A}\right)^{n-1} \\ &= \frac{\lambda}{1-\frac{y}{A}}\left(1 - \frac{ny}{nA}\right)^n \\ &= \frac{\lambda}{1-\frac{y}{A}}\left(1 - \frac{\lambda y}{n}\right)^n \to \lambda e^{-\lambda y} \quad y > 0 \end{aligned}$$

Problem 2.27

$$\psi(jv_1, jv_2, jv_3, jv_4) = E\left[e^{j(v_1x_1+v_2x_2+v_3x_3+v_4x_4)}\right]$$

$$E\left(X_1X_2X_3X_4\right) = (-j)^4 \frac{\partial^4\psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given rv's :

$$\psi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mathbf{v} = [v_1, v_2, v_3, v_4]'\,, \mathbf{M} = [\mu_{ij}]$.

We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that :

$$\frac{\partial\psi(j\mathbf{v})}{\partial v_i} = -\mu_{\mathbf{i}}'\mathbf{v}e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mu_{\mathbf{i}}' = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. Also note that :

$$\frac{\partial \mu_{\mathbf{j}}'\mathbf{v}}{\partial v_i} = \mu_{ij} = \mu_{ji}$$

Hence :

$$\frac{\partial^4\psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}|_{\mathbf{V}=\mathbf{0}} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Problem 2.28

1) By Chernov bound, for $t > 0$,

$$P[X \geq \alpha] \leq e^{-t\alpha} E[e^{tX}] = e^{-t\alpha} \Theta_X(t)$$

This is true for all $t > 0$, hence

$$\ln P[X \geq \alpha] \leq \min_{t \geq 0} [-t\alpha + \ln \Theta_X(t)] = -\max_{t \geq 0} [t\alpha - \ln \Theta_X(t)]$$

2) Here

$$\ln P[S_n \geq \alpha] = \ln P[Y \geq n\alpha] \leq -\max_{t \geq 0} [tn\alpha - \ln \Theta_Y(t)]$$

where $Y = X_1 + X_2 + \cdots + X_n$, and $\Theta_Y(t) = E[e^{X_1 + X_2 + \cdots + X_n}] = [\Theta_X(t)]^n$. Hence,

$$\ln P[S_n \geq \alpha] = -\max_{t \geq 0} n\, [t\alpha - \ln \Theta_X(t)] = -nI(\alpha) \Rightarrow \frac{1}{n} P[S_n \geq \alpha] \leq e^{-nI(\alpha)}$$

$\Theta_X(t) = \int_0^\infty e^{tx} e^{-x}\, dx = \frac{1}{1-t}$ as long as $t < 1$. $I(\alpha) = \max_{t \geq 0}(t\alpha + \ln(1-t))$, hence $\frac{d}{dt}(t\alpha + \ln(1-t)) = 0$ and $t^* = \frac{\alpha - 1}{\alpha}$. Since $\alpha \geq 0$, $t^* \geq 0$ and also obviously $t^* < 1$. $I(\alpha) = \alpha - 1 + \ln\left(1 - \frac{\alpha-1}{\alpha}\right) = \alpha - 1 - \ln \alpha$, using the large deviation theorem

$$\ln P[S_n \geq \alpha] = e^{-n(\alpha - 1 - \ln \alpha) + o(n)} = \alpha^n e^{-n(\alpha-1) + o(n)}$$

Problem 2.29

For the central chi-square with n degress of freedom :

$$\psi(jv) = \frac{1}{\left(1 - j2v\sigma^2\right)^{n/2}}$$

Now :

$$\frac{d\psi(jv)}{dv} = \frac{jn\sigma^2}{\left(1 - j2v\sigma^2\right)^{n/2+1}} \Rightarrow E\left(Y\right) = -j \frac{d\psi(jv)}{dv}|_{v=0} = n\sigma^2$$

$$\frac{d^2\psi(jv)}{dv^2} = \frac{-2n\sigma^4\left(n/2+1\right)}{\left(1 - j2v\sigma^2\right)^{n/2+2}} \Rightarrow E\left(Y^2\right) = -\frac{d^2\psi(jv)}{dv^2}|_{v=0} = n(n+2)\sigma^2$$

The variance is $\sigma_Y^2 = E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 = 2n\sigma^4$

For the non-central chi-square with n degrees of freedom :

$$\psi(jv) = \frac{1}{\left(1 - j2v\sigma^2\right)^{n/2}} e^{jvs^2/\left(1 - j2v\sigma^2\right)}$$

where by definition : $s^2 = \sum_{i=1}^{n} m_i^2$.

$$\frac{d\psi(jv)}{dv} = \left[\frac{jn\sigma^2}{(1-j2v\sigma^2)^{n/2+1}} + \frac{js^2}{(1-j2v\sigma^2)^{n/2+2}}\right] e^{jvs^2/\left(1-j2v\sigma^2\right)}$$

Hence, $E\left(Y\right) = -j\frac{d\psi(jv)}{dv}|_{v=0} = n\sigma^2 + s^2$

$$\frac{d^2\psi(jv)}{dv^2} = \left[\frac{-n\sigma^4\left(n+2\right)}{(1-j2v\sigma^2)^{n/2+2}} + \frac{-s^2(n+4)\sigma^2 - ns^2\sigma^2}{(1-j2v\sigma^2)^{n/2+3}} + \frac{-s^4}{(1-j2v\sigma^2)^{n/2+4}}\right] e^{jvs^2/\left(1-j2v\sigma^2\right)}$$

Hence,

$$E\left(Y^2\right) = -\frac{d^2\psi(jv)}{dv^2}|_{v=0} = 2n\sigma^4 + 4s^2\sigma^2 + \left(n\sigma^2 + s^2\right)$$

and

$$\sigma_Y^2 = E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

Problem 2.30

The Cauchy r.v. has : $p(x) = \frac{a/\pi}{x^2+a^2}, -\infty < x < \infty$

a.

$$E\left(X\right) = \int_{-\infty}^{\infty} xp(x)dx = 0$$

since $p(x)$ is an even function.

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{x^2+a^2}dx$$

Note that for large x, $\frac{x^2}{x^2+a^2} \to 1$ (i.e non-zero value). Hence,

$$E\left(X^2\right) = \infty, \sigma^2 = \infty$$

b.

$$\psi(jv) = E\left(^{jvX}\right) = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2+a^2}e^{jvx}dx = \int_{-\infty}^{\infty} \frac{a/\pi}{\left(x+ja\right)\left(x-ja\right)}e^{jvx}dx$$

This integral can be evaluated by using the residue theorem in complex variable theory. Then, for $v \geq 0$:

$$\psi(jv) = 2\pi j\left(\frac{a/\pi}{x+ja}e^{jvx}\right)_{x=ja} = e^{-av}$$

For $v < 0$:

$$\psi(jv) = -2\pi j\left(\frac{a/\pi}{x-ja}e^{jvx}\right)_{x=-ja} = e^{av}v$$

Therefore :

$$\psi(jv) = e^{-a|v|}$$

Note: an alternative way to find the characteristic function is to use the Fourier transform relationship between $p(x), \psi(jv)$ and the Fourier pair :

$$e^{-b|t|} \leftrightarrow \frac{1}{\pi}\frac{c}{c^2+f^2}, \; c = b/2\pi, \; f = 2\pi v$$

Problem 2.31

Since R_0 and R_1 are independent $f_{R_0,R_1}(r_0,r_1) = f_{R_0}(r_0)f_{R_1}(r_1)$ and

$$f_{R_0,R_1}(r_0,r_1) = \begin{cases} \frac{r_0 r_1}{\sigma^4} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{r_1^2+r_0^2}{2\sigma^2}}, & r_0, r_1 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned}
P(R_0 > R_1) &= \iint\limits_{r_0 > r_1} f(r_0,r_1)\, dr_1 dr_0 \\
&= \int_0^\infty dr_1 \int_{r_1}^\infty f(r_0,r_1)\, dr_0 \\
&= \int_0^\infty f_{R_1}(r_1) \left(\int_{r_1}^\infty f_{R_0}(r_0)\, dr_0\right) dr_1 \\
&= \int_0^\infty f_{R_1}(r_1) \left(\int_{r_1}^\infty \frac{r_0}{\sigma^2} e^{-\frac{r_0^2}{2\sigma^2}}\, dr_0\right) dr_1 \\
&= \int_0^\infty f_{R_1}(r_1) \left[-e^{-\frac{r_0^2}{2\sigma^2}}\right]_{r_1}^\infty dr_1 \\
&= \int_0^\infty e^{-\frac{r_1^2}{2\sigma^2}} f_{R_1}(r_1)\, dr_1 \\
&= \int_0^\infty \frac{r_1}{\sigma^2} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2+2r_1^2}{2\sigma^2}}\, dr_1
\end{aligned}$$

Now using the change of variable $y = \sqrt{2}r_1$ and letting $s = \frac{\mu}{\sqrt{2}}$ we obtain

$$\begin{aligned}
P(R_0 > R_1) &= \int_0^\infty \frac{y}{\sqrt{2}\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{2s^2+y^2}{2\sigma^2}} \frac{dy}{\sqrt{2}} \\
&= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2+y^2}{2\sigma^2}}\, dy \\
&= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \\
&= \frac{1}{2} e^{-\frac{\mu^2}{4\sigma^2}}
\end{aligned}$$

where we have used the fact that $\int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2+y^2}{2\sigma^2}}\, dy = 1$ because it is the integral of a Rician pdf.

Problem 2.32

1. The joint pdf of a, b is :

$$p_{ab}(a,b) = p_{xy}(a-m_r, b-m_i) = p_x(a-m_r)p_y(b-m_i) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(a-m_r)^2+(b-m_i)^2\right]}$$

2. $u = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1} b/a \Rightarrow a = u\cos\phi,\ b = u\sin\phi$ The Jacobian of the transformation is : $J(a,b) = \begin{vmatrix} \partial a/\partial u & \partial a/\partial\phi \\ \partial b/\partial u & \partial b/\partial\phi \end{vmatrix} = u$, hence :

$$\begin{aligned} p_{u\phi}(u,\phi) &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(u\cos\phi-m_r)^2+(u\sin\phi-m_i)^2\right]} \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[u^2+M^2-2uM\cos(\phi-\theta)\right]} \end{aligned}$$

where we have used the transformation :

$$\left\{ \begin{array}{l} M = \sqrt{m_r^2+m_i^2} \\ \theta = \tan^{-1} m_i/m_r \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} m_r = M\cos\theta \\ m_i = M\sin\theta \end{array} \right\}$$

3.

$$\begin{aligned} p_u(u) &= \int_0^{2\pi} p_{u\phi}(u,\phi)d\phi \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[-2uM\cos(\phi-\theta)]} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{uM\cos(\phi-\theta)/\sigma^2} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} I_o\left(uM/\sigma^2\right) \end{aligned}$$

Problem 2.33

a. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i,\ \psi_{X_i}(jv) = e^{-a|v|}$

$$\psi_Y(jv) = E\left[e^{jv\frac{1}{n}\sum_{i=1}^{n} X_i}\right] = \prod_{i=1}^{n} E\left[e^{j\frac{v}{n}X_i}\right] = \prod_{i=1}^{n} \psi_{X_i}(jv/n) = \left[e^{-a|v|/n}\right]^n = e^{-a|v|}$$

b. Since $\psi_Y(jv) = \psi_{X_i}(jv) \Rightarrow p_Y(y) = p_{X_i}(x_i) \Rightarrow p_Y(y) = \frac{a/\pi}{y^2+a^2}$.

c. As $n \to \infty$, $p_Y(y) = \frac{a/\pi}{y^2+a^2}$, which is not Gaussian ; hence, the central limit theorem does not hold. The reason is that the Cauchy distribution does not have a finite variance.

Problem 2.34

Since $\boldsymbol{Z}$ and $\boldsymbol{Z}e^{j\theta}$ have the same pdf, we have $E[\boldsymbol{Z}] = E\left[\boldsymbol{Z}e^{j\theta}\right] = e^{j\theta}E[\boldsymbol{Z}]$ for all θ. Putting $\theta = \pi$ gives $E[\boldsymbol{Z}] = \boldsymbol{0}$. We also have $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = E\left[\boldsymbol{Z}e^{j\theta}\left(\boldsymbol{Z}e^{j\theta}\right)^t\right]$ or $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = e^{2j\theta}E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right]$, for all θ. Putting $\theta = \frac{\pi}{2}$ gives $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \boldsymbol{0}$. Since $\boldsymbol{Z}$ is zero-mean and $E\left[\boldsymbol{Z}\boldsymbol{Z}^t\right] = \boldsymbol{0}$, we conclude that it is proper.

Problem 2.35

Using Equation 2.6-29 we note that for the zero-mean proper case if $\boldsymbol{W} = e^{j\theta}\boldsymbol{Z}$, it is sufficient to show that $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$ and $\boldsymbol{w}^H\boldsymbol{C}_{\boldsymbol{W}}^{-1}\boldsymbol{w} = \boldsymbol{z}^H\boldsymbol{C}_{\boldsymbol{Z}}^{-1}\boldsymbol{z}$. But $\boldsymbol{C_W} = [\boldsymbol{W}\boldsymbol{W}^H] = E[e^{j\theta}\boldsymbol{Z}e^{-j\theta}\boldsymbol{Z}^H] = E[\boldsymbol{Z}\boldsymbol{Z}^H] = \boldsymbol{C_Z}$, hence $\det(\boldsymbol{C_W}) = \det(\boldsymbol{C_Z})$. Similarly, $\boldsymbol{w}^H\boldsymbol{C}_{\boldsymbol{W}}^{-1}\boldsymbol{w} = e^{-j\theta}\boldsymbol{z}^H\boldsymbol{C}_{\boldsymbol{Z}}^{-1}\boldsymbol{z}e^{j\theta} = \boldsymbol{z}^H\boldsymbol{C}_{\boldsymbol{Z}}^{-1}\boldsymbol{z}$. Substituting into Equation 2.6-29, we conclude that $p(\boldsymbol{w}) = p(\boldsymbol{z})$.

Problem 2.36

Since $\boldsymbol{Z}$ is proper, we have $E[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t] = \boldsymbol{0}$. Let $\boldsymbol{W} = \boldsymbol{A}\boldsymbol{Z} + \boldsymbol{b}$, then

$$E[(\boldsymbol{W} - E(\boldsymbol{W}))(\boldsymbol{W} - E(\boldsymbol{W}))^t] = \boldsymbol{A}E[(\boldsymbol{Z} - E(\boldsymbol{Z}))(\boldsymbol{Z} - E(\boldsymbol{Z}))^t]\boldsymbol{A}^t = \boldsymbol{0}$$

hence $\boldsymbol{W}$ is proper.

Problem 2.37

We assume that $x(t), y(t), z(t)$ are real-valued stochastic processes. The treatment of complex-valued processes is similar.

a.

$$R_{zz}(\tau) = E\left\{[x(t+\tau)+y(t+\tau)]\,[x(t)+y(t)]\right\} = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

b. When $x(t), y(t)$ are uncorrelated :

$$R_{xy}(\tau) = E\left[x(t+\tau)y(t)\right] = E\left[x(t+\tau)\right]E\left[y(t)\right] = m_x m_y$$

Similarly :

$$R_{yx}(\tau) = m_x m_y$$

Hence :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + 2m_x m_y$$

c. When $x(t), y(t)$ are uncorrelated and have zero means :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau)$$

Problem 2.38

The power spectral density of the random process $x(t)$ is :

$$\mathcal{S}_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j2\pi f\tau}d\tau = N_0/2.$$

The power spectral density at the output of the filter will be :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Hence, the total power at the output of the filter will be :

$$R_{yy}(\tau=0)=\int_{-\infty}^{\infty}\mathcal{S}_{yy}(f)df=\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2df=\frac{N_0}{2}(2B)=N_0B$$

Problem 2.39

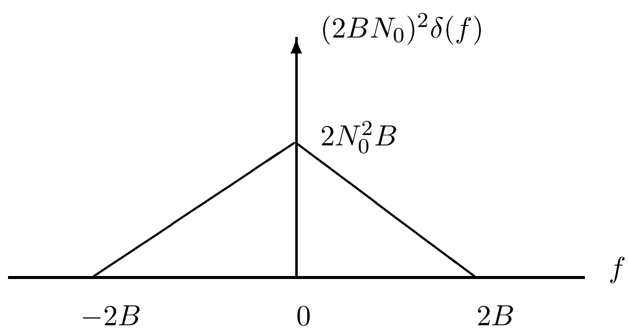
The power spectral density of $X(t)$ corresponds to : $R_{xx}(t)=2BN_0\frac{\sin 2\pi Bt}{2\pi Bt}$. From the result of Problem 2.14 :

$$R_{yy}(\tau)=R_{xx}^2(0)+2R_{xx}^2(\tau)=(2BN_0)^2+8B^2N_0^2\left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2$$

Also :

$$\mathcal{S}_{yy}(f)=R_{xx}^2(0)\delta(f)+2\mathcal{S}_{xx}(f)*\mathcal{S}_{xx}(f)$$

The following figure shows the power spectral density of $Y(t)$:



Problem 2.40

$$\mathbf{M}_X=E\left[(\mathbf{X}-\mathbf{m}_x)(\mathbf{X}-\mathbf{m}_x)'\right],\quad \mathbf{X}=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix},\quad \mathbf{m}_x \text{ is the corresponding vector of mean values.}$$

Then :

$$\begin{aligned}\mathbf{M}_Y &= E\left[(\mathbf{Y}-\mathbf{m}_y)(\mathbf{Y}-\mathbf{m}_y)'\right]\\ &= E\left[\mathbf{A}(\mathbf{X}-\mathbf{m}_x)(\mathbf{A}(\mathbf{X}-\mathbf{m}_x))'\right]\\ &= E\left[\mathbf{A}(\mathbf{X}-\mathbf{m}_x)(\mathbf{X}-\mathbf{m}_x)'\mathbf{A}'\right]\\ &= \mathbf{AE}\left[(\mathbf{X}-\mathbf{m_x})(\mathbf{X}-\mathbf{m_x})'\right]\mathbf{A}'\\ &= \mathbf{AM_xA}'\end{aligned}$$

Hence :

$$\mathbf{M}_Y = \begin{bmatrix} \mu_{11} & 0 & \mu_{11}+\mu_{13} \\ 0 & 4\mu_{22} & 0 \\ \mu_{11}+\mu_{31} & 0 & \mu_{11}+\mu_{13}+\mu_{31}+\mu_{33} \end{bmatrix}$$

Problem 2.41

$Y(t) = X^2(t),\ R_{xx}(\tau) = E\left[x(t+\tau)x(t)\right]$

$$R_{yy}(\tau) = E\left[y(t+\tau)y(t)\right] = E\left[x^2(t+\tau)x^2(t)\right]$$

Let $X_1 = X_2 = x(t),\ \ X_3 = X_4 = x(t+\tau)$. Then, from problem 2.7 :

$$E\left(X_1X_2X_3X_4\right) = E\left(X_1X_2\right)E\left(X_3X_4\right) + E\left(X_1X_3\right)E\left(X_2X_4\right) + E\left(X_1X_4\right)E\left(X_2X_3\right)$$

Hence :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

Problem 2.42

$p_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1}e^{-mr^2/\Omega},\ \ X = \frac{1}{\sqrt{\Omega}}R$

We know that : $p_X(x) = \frac{1}{1/\sqrt{\Omega}}p_R\left(\frac{x}{1/\sqrt{\Omega}}\right)$.

Hence :

$$p_X(x) = \frac{1}{1/\sqrt{\Omega}}\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\left(x\sqrt{\Omega}\right)^{2m-1}e^{-m(x\sqrt{\Omega})^2/\Omega} = \frac{2}{\Gamma(m)}m^m x^{2m-1}e^{-mx^2}$$

Problem 2.43

The transfer function of the filter is :

$$H(f) = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1} = \frac{1}{j2\pi fRC+1}$$

a.

$$\mathcal{S}_{xx}(f) = \sigma^2 \Rightarrow \mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)\,|H(f)|^2 = \frac{\sigma^2}{(2\pi RC)^2 f^2 + 1}$$

b.

$$R_{yy}(\tau) = F^{-1}\{\mathcal{S}_{xx}(f)\} = \frac{\sigma^2}{RC}\int_{-\infty}^{\infty} \frac{\frac{1}{RC}}{(\frac{1}{RC})^2 + (2\pi f)^2} e^{j2\pi f\tau} df$$

Let : $a = RC,\ \ v = 2\pi f$. Then :

$$R_{yy}(\tau) = \frac{\sigma^2}{2RC}\int_{-\infty}^{\infty} \frac{a/\pi}{a^2+v^2} e^{jv\tau} dv = \frac{\sigma^2}{2RC} e^{-a|\tau|} = \frac{\sigma^2}{2RC} e^{-|\tau|/RC}$$

where the last integral is evaluated in the same way as in problem P-2.9 . Finally :

$$E\left[Y^2(t)\right] = R_{yy}(0) = \frac{\sigma^2}{2RC}$$

Problem 2.44

If $\mathcal{S}_X(f) = 0$ for $|f| > W$, then $\mathcal{S}_X(f)e^{-j2\pi fa}$ is also bandlimited. The corresponding autocorrelation function can be represented as (remember that $\mathcal{S}_X(f)$ is deterministic) :

$$R_X(\tau - a) = \sum_{n=-\infty}^{\infty} R_X(\frac{n}{2W} - a)\frac{\sin 2\pi W\left(\tau - \frac{n}{2W}\right)}{2\pi W\left(\tau - \frac{n}{2W}\right)} \qquad (1)$$

Let us define :

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X(\frac{n}{2W})\frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{2\pi W\left(t - \frac{n}{2W}\right)}$$

We must show that :

$$E\left[|X(t) - \hat{X}(t)|^2\right] = 0$$

or

$$E\left[\left(X(t) - \hat{X}(t)\right)\left(X(t) - \sum_{m=-\infty}^{\infty} X(\frac{m}{2W})\frac{\sin 2\pi W\left(t - \frac{m}{2W}\right)}{2\pi W\left(t - \frac{m}{2W}\right)}\right)\right] = 0 \qquad (2)$$

First we have :

$$E\left[\left(X(t) - \hat{X}(t)\right) X(\frac{m}{2W})\right] = R_X(t - \frac{m}{2W}) - \sum_{n=-\infty}^{\infty} R_X(\frac{n-m}{2W})\frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{2\pi W\left(t - \frac{n}{2W}\right)}$$

But the right-hand-side of this equation is equal to zero by application of (1) with $a = m/2W$. Since this is true for any m, it follows that $E\left[\left(X(t) - \hat{X}(t)\right)\hat{X}(t)\right] = 0$. Also

$$E\left[\left(X(t) - \hat{X}(t)\right)X(t)\right] = R_X(0) - \sum_{n=-\infty}^{\infty} R_X(\frac{n}{2W} - t)\frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{2\pi W\left(t - \frac{n}{2W}\right)}$$

Again, by applying (1) with $a = t$ anf $\tau = t$, we observe that the right-hand-side of the equation is also zero. Hence (2) holds.

Problem 2.45

$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt = P\left[N \geq x\right]$, where N is a Gaussian r.v with zero mean and unit variance. From the Chernoff bound :

$$P\left[N \geq x\right] \leq e^{-\hat{v}x}E\left(e^{\hat{v}N}\right) \qquad (1)$$

where $\hat{v}$ is the solution to :

$$E\left(Ne^{vN}\right) - xE\left(e^{vN}\right) = 0 \qquad (2)$$

Now :

$$\begin{aligned} E\left(e^{vN}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{vt}e^{-t^2/2}dt \\ &= e^{v^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t-v)^2/2}dt \\ &= e^{v^2/2} \end{aligned}$$

and

$$E\left(Ne^{vN}\right) = \frac{d}{dv}E\left(e^{vN}\right) = ve^{v^2/2}$$

Hence (2) gives :

$$\hat{v} = x$$

and then :

$$(1) \Rightarrow Q(x) \leq e^{-x^2}e^{x^2/2} \Rightarrow Q(x) \leq e^{-x^2/2}$$

Problem 2.46

Since $H(0) = \sum_{-\infty}^{\infty} h(n) = 0 \Rightarrow m_y = m_x H(0) = 0$

The autocorrelation of the output sequence is

$$R_{yy}(k) = \sum_i \sum_j h(i)h(j)R_{xx}(k-j+i) = \sigma_x^2 \sum_{i=-\infty}^{\infty} h(i)h(k+i)$$

where the last equality stems from the autocorrelation function of $X(n)$:

$$R_{xx}(k-j+i) = \sigma_x^2 \delta(k-j+i) = \left\{ \begin{array}{cc} \sigma_x^2, & j=k+i \\ 0, & o.w. \end{array} \right\}$$

Hence, $R_{yy}(0) = 6\sigma_x^2,\ R_{yy}(1) = R_{yy}(-1) = -4\sigma_x^2,\ R_{yy}(2) = R_{yy}(-2) = \sigma_x^2,\ R_{yy}(k) = 0$ otherwise.

Finally, the frequency response of the discrete-time system is :

$$\begin{array}{rcl} H(f) & = & \sum_{-\infty}^{\infty} h(n)e^{-j2\pi fn} \\ & = & 1 - 2e^{-j2\pi f} + e^{-j4\pi f} \\ & = & \left(1 - e^{-j2\pi f}\right)^2 \\ & = & e^{-j2\pi f}\left(e^{j\pi f} - e^{-j\pi f}\right)^2 \\ & = & -4e^{-j\pi f}\sin^2 \pi f \end{array}$$

which gives the power density spectrum of the output :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)|H(f)|^2 = \sigma_x^2\left[16\sin^4 \pi f\right] = 16\sigma_x^2 \sin^4 \pi f$$

Problem 2.47

$$R(k) = \left(\frac{1}{2}\right)^{|k|}$$

The power density spectrum is

$$\begin{aligned}
\mathcal{S}(f) &= \sum_{k=-\infty}^{\infty} R(k) e^{-j2\pi fk} \\
&= \sum_{k=-\infty}^{-1} \left(\tfrac{1}{2}\right)^{-k} e^{-j2\pi fk} + \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k} e^{-j2\pi fk} \\
&= \sum_{k=0}^{\infty} (\tfrac{1}{2} e^{j2\pi fk})^k + \sum_{k=0}^{\infty} (\tfrac{1}{2} e^{-j2\pi f})^k - 1 \\
&= \tfrac{1}{1-e^{j2\pi f}/2} + \tfrac{1}{1-e^{-j2\pi f}/2} - 1 \\
&= \tfrac{2-\cos 2\pi f}{5/4-\cos 2\pi f} - 1 \\
&= \tfrac{3}{5-4\cos 2\pi f}
\end{aligned}$$

Problem 2.48

We will denote the discrete-time process by the subscript d and the continuous-time (analog) process by the subscript a. Also, f will denote the analog frequency and f_d the discrete-time frequency.

a.

$$\begin{aligned}
R_d(k) &= E\left[X^*(n)X(n+k)\right] \\
&= E\left[X^*(nT)X(nT+kT)\right] \\
&= R_a(kT)
\end{aligned}$$

Hence, the autocorrelation function of the sampled signal is equal to the sampled autocorrelation function of $X(t)$.

b.

$$\begin{aligned}
R_d(k) &= R_a(kT) = \int_{-\infty}^{\infty} \mathcal{S}_a(F) e^{j2\pi fkT} df \\
&= \sum_{l=-\infty}^{\infty} \int_{(2l-1)/2T}^{(2l+1)/2T} \mathcal{S}_a(F) e^{j2\pi fkT} df \\
&= \sum_{l=-\infty}^{\infty} \int_{-1/2T}^{1/2T} \mathcal{S}_a(f + \tfrac{l}{T}) e^{j2\pi FkT} df \\
&= \int_{-1/2T}^{1/2T} \left[\sum_{l=-\infty}^{\infty} \mathcal{S}_a(f + \tfrac{l}{T})\right] e^{j2\pi FkT} df
\end{aligned}$$

Let $f_d = fT$. Then :

$$R_d(k) = \int_{-1/2}^{1/2} \left[\frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a((f_d + l)/T) \right] e^{j2\pi f_d k} df_d \qquad (1)$$

We know that the autocorrelation function of a discrete-time process is the inverse Fourier transform of its power spectral density

$$R_d(k) = \int_{-1/2}^{1/2} \mathcal{S}_d(f_d) e^{j2\pi f_d k} df_d \qquad (2)$$

Comparing (1),(2) :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a(\frac{f_d + l}{T}) \qquad (3)$$

c. From (3) we conclude that :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \mathcal{S}_a(\frac{f_d}{T})$$

iff :

$$\mathcal{S}_a(f) = 0, \quad \forall\, f : |f| > 1/2T$$

Otherwise, the sum of the shifted copies of $\mathcal{S}_a$ (in (3)) will overlap and aliasing will occur.

Problem 2.49

$$u(t) = X \cos 2\pi f t - Y \sin 2\pi f t$$
$$E[u(t)] = E(X) \cos 2\pi f t - E(Y) \sin 2\pi f t$$

and :

$$\begin{aligned} R_{uu}(t, t+\tau) &= E\{[X \cos 2\pi f t - Y \sin 2\pi f t][X \cos 2\pi f(t+\tau) - Y \sin 2\pi f(t+\tau)]\} \\ &= E\left(X^2\right)[\cos 2\pi f(2t+\tau) + \cos 2\pi f \tau] + E\left(Y^2\right)[-\cos 2\pi f(2t+\tau) + \cos 2\pi f \tau] \\ &\quad - E(XY) \sin 2\pi f(2t+\tau) \end{aligned}$$

For $u(t)$ to be wide-sense stationary, we must have : $E[u(t)]$ =constant and $R_{uu}(t, t+\tau) = R_{uu}(\tau)$. We note that if $E(X) = E(Y) = 0$, and $E(XY) = 0$ and $E(X^2) = E(Y^2)$, then the above requirements for WSS hold; hence these conditions are necessary. Conversely, if any of the above conditions does not hold, then either $E[u(t)] \neq$constant, or $R_{uu}(t, t+\tau) \neq R_{uu}(\tau)$. Hence, the conditions are also necessary.

Problem 2.50

a.

$$\begin{aligned} R_a(\tau) &= \int_{-\infty}^{\infty} \mathcal{S}_a(f) e^{j2\pi f\tau} df \\ &= \int_{-W}^{W} e^{j2\pi f\tau} df \\ &= \frac{\sin 2\pi W\tau}{\pi\tau} \end{aligned}$$

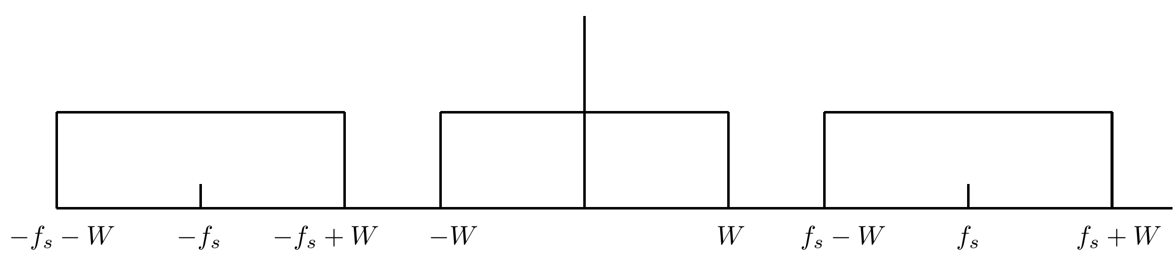
By applying the result in problem 2.21, we have

$$R_d(k) = f_a(kT) = \frac{\sin 2\pi WkT}{\pi kT}$$

b. If $T = \frac{1}{2W}$, then :

$$R_d(k) = \left\{ \begin{array}{cc} 2W = 1/T, & \text{k} = 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

Thus, the sequence $X(n)$ is a white-noise sequence. The fact that this is the minimum value of T can be shown from the following figure of the power spectral density of the sampled process:



We see that the maximum sampling rate f_s that gives a spectrally flat sequence is obtained when :

$$W = f_s - W \Rightarrow f_s = 2W \Rightarrow T = \frac{1}{2W}$$

c. The triangular-shaped spectrum $\mathcal{S}(f) = 1 - \frac{|f|}{W}, \ |f| \leq W$ may be obtained by convolving the rectangular-shaped spectrum $\mathcal{S}_1(f) = 1/\sqrt{W}, \ |f| \leq W/2$. Hence, $R(\tau) = R_1^2(\tau) = \frac{1}{W}\left(\frac{\sin \pi W \tau}{\pi \tau}\right)^2$.Therefore, sampling X(t) at a rate $\frac{1}{T} = W$ samples/sec produces a white sequence with autocorrelation function :

$$R_d(k) = \frac{1}{W}\left(\frac{\sin \pi W k T}{\pi k T}\right)^2 = W\left(\frac{\sin \pi k}{\pi k}\right)^2 = \left\{ \begin{array}{cc} W, & \text{k} = 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

Problem 2.51

Let's denote : $y(t) = f_k(t) f_j(t)$.Then :

$$\int_{-\infty}^{\infty} f_k(t) f_j(t) dt = \int_{-\infty}^{\infty} y(t) dt = Y(f)|_{f=0}$$

where $Y(f)$ is the Fourier transform of $y(t)$. Since : $y(t) = f_k(t) f_j(t) \longleftrightarrow Y(f) = F_k(f) * F_j(f)$. But :

$$F_k(f) = \int_{-\infty}^{\infty} f_k(t) e^{-j2\pi f t} dt = \frac{1}{2W} e^{-j2\pi f k/2W}$$

Then :

$$Y(f) = F_k(f) * F_j(f) = \int_{-\infty}^{\infty} F_k(a) * F_j(f-a) da$$

and at $f = 0$:

$$\begin{aligned} Y(f)|_{f=0} &= \int_{-\infty}^{\infty} F_k(a) * F_j(-a) da \\ &= \left(\frac{1}{2W}\right)^2 \int_{-\infty}^{\infty} e^{-j2\pi a(k-j)/2W} da \\ &= \left\{ \begin{array}{cc} 1/2W, & \text{k} = \text{j} \\ 0, & \text{k} \neq \text{j} \end{array} \right\} \end{aligned}$$

Problem 2.52

$$B_{eq} = \frac{1}{G} \int_0^{\infty} |H(f)|^2 df$$

For the filter shown in Fig. P2-12 we have $G = 1$ and

$$B_{eq} = \int_0^{\infty} |H(f)|^2 df = B$$

For the lowpass filter shown in Fig. P2-16 we have

$$H(f) = \frac{1}{1+j2\pi fRC} \Rightarrow |H(f)|^2 = \frac{1}{1+(2\pi fRC)^2}$$

So $G=1$ and

$$\begin{aligned} B_{eq} &= \int_0^\infty |H(f)|^2 df \\ &= \tfrac{1}{2}\int_{-\infty}^\infty |H(f)|^2 df \\ &= \tfrac{1}{4RC} \end{aligned}$$

where the last integral is evaluated in the same way as in problem P-2.9 .

Problem 2.53

a.

$$\begin{aligned} E\left[z(t)z(t+\tau)\right] &= E\left[\{x(t+\tau)+jy(t+t)\}\{x(t)+jy(t)\}\right] \\ &= E\left[x(t)x(t+\tau)\right] - E\left[y(t)y(t+\tau)\right] + jE\left[x(t)y(t+\tau)\right] \\ &\quad +E\left[y(t)x(t+\tau)\right] \\ &= R_{xx}(\tau) - R_{yy}(\tau) + j\left[R_{yx}(\tau)+R_{xy}(\tau)\right] \end{aligned}$$

But $R_{xx}(\tau) = R_{yy}(\tau)$and $R_{yx}(\tau) = -R_{xy}(\tau)$. Therefore :

$$E\left[z(t)z(t+\tau)\right] = 0$$

b.

$$V = \int_0^T z(t)dt$$

$$E\left(V^2\right) = \int_0^T \int_0^T E\left[z(a)z(b)\right] dadb = 0$$

from the result in (a) above. Also :

$$\begin{aligned} E\left(VV^*\right) &= \int_0^T \int_0^T E\left[z(a)z^*(b)\right] dadb \\ &= \int_0^T \int_0^T N_0\delta(a-b)dadb \\ &= \int_0^T N_0 da = N_0 T \end{aligned}$$

Problem 2.54

$$\begin{aligned} E\left[x(t+\tau)x(t)\right] &= A^2 E\left[\sin\left(2\pi f_c(t+\tau)+\theta\right)\sin\left(2\pi f_c t+\theta\right)\right] \\ &= \tfrac{A^2}{2}\cos 2\pi f_c\tau - \tfrac{A^2}{2}E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] \end{aligned}$$

where the last equality follows from the trigonometric identity :
$\sin A \sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$. But :

$$\begin{aligned} E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] &= \int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)p(\theta)d\theta \\ &= \tfrac{1}{2\pi}\int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)d\theta = 0 \end{aligned}$$

Hence :

$$E\left[x(t+\tau)x(t)\right] = \frac{A^2}{2}\cos 2\pi f_c\tau$$

Problem 2.55

1) We have $E[Z(t)] = E[X(t)] + jE[Y(t)] = 0 + j0 = 0$ and

$$\begin{aligned} R_Z(t+\tau,t) &= E\left[\left(X(t+\tau)+jY(t+\tau)\right)\left(X(t)-jY(t)\right)\right] \\ &= R_X(\tau)+R_Y(\tau) \\ &= 2R_X(\tau) \end{aligned}$$

because $E\left[X(t+\tau)Y(t)\right] = E\left[Y(t+\tau)X(t)\right] = E[X(t+\tau)]E[Y(t)] = 0$ (by independence) and therefore $Z(t)$ is obviously stationary. We also note that $R_X(\tau) = R_Y(\tau) = \mathcal{F}^{-1}\left[N_0\Pi\left(\frac{f}{2W}\right)\right] = 2WN_0\,\mathrm{sinc}(2W\tau)$

2) To compute the power spectral density of $Z(t)$, we have $\mathcal{S}_Z(f) = \mathcal{F}\left[2R_X(\tau)\right] = 2\mathcal{S}_X(f) = 2N_0\Pi\left(\frac{f}{2W}\right)$. Note that $\Pi(t)$ is a rectangular pulse defined as

$$\Pi(t) = \begin{cases} 1, & |t|<1 \\ \frac{1}{2}, & |t|=1 \\ 0, & \text{otherwise.} \end{cases}$$

3) $E[Z_j] = E\left[\int_{-\infty}^{\infty} Z(t)R_j^*(t)\,dt\right] = \int_{-\infty}^{\infty} E[Z(t)]R_j^*(t)\,dt = 0$ since $Z(t)$ is zero-mean. For the

correlation we have

$$\begin{aligned}
E[Z_j Z_k^*] &= E\left[\int_{-\infty}^{\infty} Z(s)R_j^*(s)\,ds \int_{-\infty}^{\infty} Z^*(t)R_k(t)\,dt\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_Z(s-t)R_j^*(s)R_k(t)\,dsdt \\
&= \int_{-\infty}^{\infty} R_k(t)\left[\int_{-\infty}^{\infty} R_Z(s-t)R_j^*(s)\,ds\right]dt \quad (**)
\end{aligned}$$

Using Parseval's Theorem, $\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df$, we have ($\mathcal{S}_j(f)$ is the Fourier transform of $R_j(t)$).

$$\begin{aligned}
\int_{-\infty}^{\infty} R_Z(s-t)R_j^*(s)\,ds &= \int_{-\infty}^{\infty} e^{-j2\pi ft}2N_0\Pi\left(\frac{f}{2W}\right)\mathcal{S}_j^*(f)\,df \\
&\overset{a}{=} 2\int_{-W}^{W} N_0 e^{-j2\pi ft}\mathcal{S}_j^*(f)\,df \\
&\overset{b}{=} 2\int_{-\infty}^{\infty} N_0 e^{-j2\pi ft}\mathcal{S}_j^*(f)\,df
\end{aligned}$$

where (a) is due to the fact that $\Pi\left(\frac{f}{2W}\right)$ is zero outside the $[-W, W]$ interval and (b) follows from $R_j(t)$ being bandlimited to $[-W, W]$. From above we have

$$\begin{aligned}
\int_{-\infty}^{\infty} R_Z(s-t)R_j^*(s)\,ds &= 2N_0\left[\int_{-\infty}^{\infty} e^{j2\pi ft}\mathcal{S}_j(f)\,df\right]^* \\
&= 2N_0R_j^*(t)
\end{aligned}$$

Substituting this result in equation (**) we have

$$\begin{aligned}
E[Z_j Z_k^*] &= 2\int_{-\infty}^{\infty} N_0R_j^*(t)R_k(t)\,dt \\
&= \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}
\end{aligned}$$

This shows that Z_j's are Gaussian random variables (since they are the result of linear operation on a Gaussian process) with mean zero and variance $2N_0$, i.e., $Z_j \sim \mathcal{N}(0, 2N_0)$. Also note that for $j \neq k$, Z_j and Z_k are independent since they are Gaussian and uncorrelated.

4) This is done similar to part 3 (lengthy but straightforward) and the result is that for any k, Z_{kr} and Z_{ki} are zero-mean, independent Gaussian random variables with $E(Z_{kr}^2) = E(Z_{ki}^2) = N_0$ and therefore the random vector $(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \cdots, Z_{nr}, Z_{ni})$ is a $2n$-dimensional Gaussian vector with independent zero-mean components each having variance N_0. In standard notation

$$(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \cdots, Z_{nr}, Z_{ni}) \sim \mathcal{N}(\mathbf{0}, N_0\boldsymbol{I})$$

where $\mathbf{0}$ is a $2n$-dimensional zero vector and $\boldsymbol{I}$ is a $2n \times 2n$ identity matrix.

5) We have

$$E[\hat{Z}(t)Z_k^*] = E[(Z(t) - \sum_{j=1}^{N} Z_j R_j(t))Z_k^*]$$
$$= E[Z(t)Z_k^*] - 2N_0 R_k(t)$$

where we have used

$$E[Z_j Z_k^*] = \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}$$

Now we have

$$\begin{aligned} E[Z(t)Z_k^*] &= E\left[Z(t)\int_{-\infty}^{\infty} Z^*(s)R_k(s)\,ds\right] \\ &= \int_{-\infty}^{\infty} R_Z(t-s)R_k(s)\,ds \\ &= \int_{-\infty}^{\infty} R_k(s)R_Z^*(s-t)\,ds \\ &= 2\int_{-\infty}^{\infty} \mathcal{S}_k(f)e^{j2\pi ft}N_0\Pi\left(\frac{f}{2W}\right)\,df \\ &= \int_{-W}^{W} 2N_0\mathcal{S}_k(f)e^{j2\pi ft}\,dt \\ &\stackrel{a}{=} 2N_0\int_{-\infty}^{\infty} \mathcal{S}_k(f)e^{j2\pi ft}\,df \\ &= 2N_0 R_k(t) \end{aligned}$$

(a): because $R_k(t)$ is bandlimited to $[-W, W]$.
From above it follows that $E[\hat{Z}(t)Z_k^*] = 0$ for all $k = 1, 2, \cdots, N$. This means that the error term is independent of the projections.

Problem 2.56

1. $\mathcal{S}_{\hat{X}}(f) = |-j\,\mathrm{sgn}(f)|^2\mathcal{S}_X(f) = \mathcal{S}_X(f)$, hence $R_{\hat{X}}(\tau) = R_X(\tau)$.
2. $\mathcal{S}_{X\hat{X}}(f) = \mathcal{S}_X(f)(-j\,\mathrm{sgn}(f))^* = j\,\mathrm{sgn}(f)\mathcal{S}_X(f)$, therefore, $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$.
3. $R_Z(\tau) = E\left[\left(X(t+\tau) + j\hat{X}(t+\tau)\right)\left(X(t) - j\hat{X}(t)\right)\right]$, expanding we have

$$R_Z(\tau) = R_X(\tau) + R_{\hat{X}}(\tau) - j\left[R_{X\hat{X}}(\tau) - R_{\hat{X}X}(\tau)\right]$$

Using $R_{\hat{X}}(\tau) = R_X(\tau)$, and the fact that $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$ is an odd function (since it is the HT of an even signal) we have $R_{\hat{X}X}(\tau) = R_{X\hat{X}}(-\tau) = -R_{X\hat{X}}(\tau)$, we have

$$R_Z(\tau) = 2R_X(\tau) - j2R_{X\hat{X}}(\tau) = 2R_X(\tau) + j2\hat{R}_X(\tau)$$

Taking FT of both sides we have

$$\mathcal{S}_Z(f) = 2\mathcal{S}_X(f) + j2\left(-j\,\mathrm{sgn}(f)\mathcal{S}_X(f)\right) = 2\left(1 + \mathrm{sgn}(f)\right)\mathcal{S}_X(f) = 4\mathcal{S}_X(f)u_{-1}(f)$$

4. We have

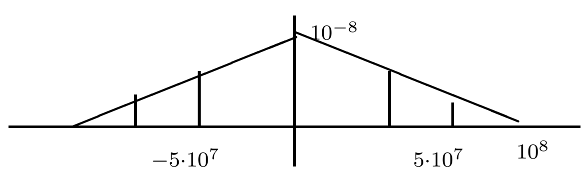
$$\begin{aligned} R_{X_l}(t+\tau, t) &= E\left[Z(t+\tau)e^{-j2\pi f_0(t+\tau)}Z^*(t)e^{j2\pi f_0 t}\right] \\ &= e^{-j2\pi f_0 \tau}R_Z(\tau) \end{aligned}$$

This shows that $X_l(t)$ is WSS (we already know it is zero-mean). Taking FT, we have $\mathcal{S}_{X_l}(f) = \mathcal{S}_Z(f - f_0) = 4\mathcal{S}_X(f - f_0)u_{-1}(f - f_0)$, this shows that $X_l(t)$ is lowpass. Also from above $R_X(\tau) = \frac{1}{2}\mathrm{Re}\left[R_Z(t)\right] = \frac{1}{2}\mathrm{Re}\left[R_{X_l}(\tau)e^{j2\pi f_0\tau}\right]$. This shows that $R_{X_l}(\tau)$ is twice the LP equivalent of $R_X(\tau)$.

Problem 2.57

1) The power spectral density $\mathcal{S}_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$\begin{aligned} P &= \int_{-51\times10^6}^{-49\times10^6} 10^{-8}\left(1 + \frac{f}{10^8}\right)df + \int_{49\times10^6}^{51\times10^6} 10^{-8}\left(1 - \frac{f}{10^8}\right)df \\ &= 10^{-8}x\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-16}\frac{1}{2}x^2\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-8}x\Big|_{49\times10^6}^{51\times10^6} - 10^{-16}\frac{1}{2}x^2\Big|_{49\times10^6}^{51\times10^6} \\ &= 2\times10^{-2} \end{aligned}$$



2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50 \times 10^6 t) - N_s(t)\sin(2\pi 50 \times 10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t)\cos(2\pi 50\times 10^6 t) + \hat{N}(t)\sin(2\pi 50\times 10^6 t) \\ N_s(t) &= \hat{N}(t)\cos(2\pi 50\times 10^6 t) - N(t)\sin(2\pi 50\times 10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi 50\times 10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi 50\times 10^6 t) \\ &= E[|N(t)|^2] = 2\times 10^{-2} \end{aligned}$$

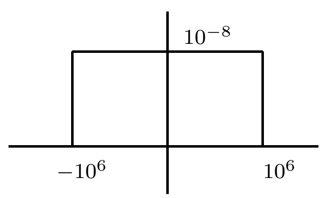
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2\times 10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$\mathcal{S}_{N_c}(f) = \mathcal{S}_{N_s}(f) = \begin{cases} \mathcal{S}_N(f - 50\times 10^6) + \mathcal{S}_N(f + 50\times 10^6) & |f| \leq 50\times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}_{N_c}(f)$ is depicted in the next figure. The power content of $\mathcal{S}_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2\times 10^{-2}$$



4) The power spectral density of the output is given by

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = 10^{-6}(|f| - 49\times 10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49\times 10^6 \leq |f| \leq 51\times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= 10^{-6}\left(\int_{-51\times 10^6}^{-49\times 10^6} (-f - 49\times 10^6)(10^{-8} + 10^{-16} f) df\right) \\ &\quad +10^{-6}\left(\int_{49\times 10^6}^{51\times 10^6} (f - 49\times 10^6)(10^{-8} - 10^{-16} f) df\right) \\ &= 10^{-6}(2\times 10^4 - \frac{4}{3}10^2) \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} \mathcal{S}_{Y_c}(f) = \mathcal{S}_{Y_s}(f) &= 10^{-6}\left(\left((f + 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} - 10^{-16}(f + 50\times 10^6)\right)\right) \\ &\quad +10^{-6}\left(\left(-(f - 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} + 10^{-16}(f - 50\times 10^6)\right)\right) \\ &= 10^{-6}(-2\times 10^{-16} f^2 + 10^{-2}) \end{aligned}$$

for $|f| \leq 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned} P_{Y_c} = P_{Y_s} &= 10^{-6}\int_{-10^6}^{10^6}(-2\times 10^{-16}f^2+10^{-2})df \\ &= 10^{-6}(-2\times 10^{-16}\frac{1}{3}f^3\Big|_{-10^6}^{10^6}+10^{-2}f\Big|_{-10^6}^{10^6}) \\ &= 10^{-6}(2\times 10^4-\frac{4}{3}10^2) = P_Y \end{aligned}$$