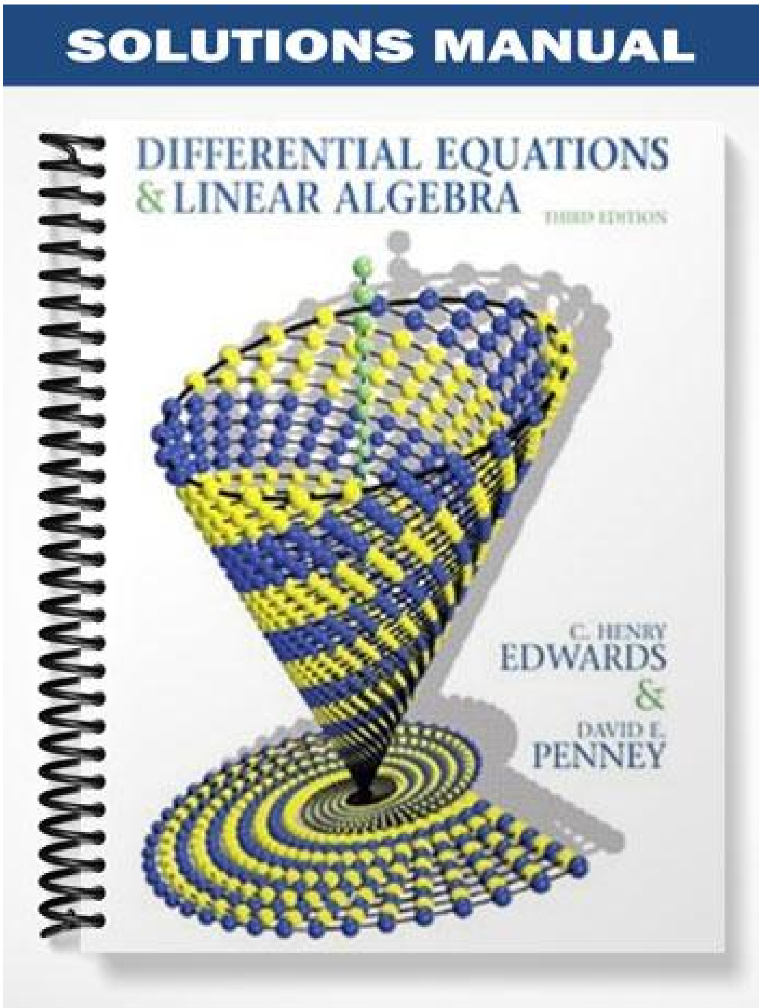
SOLUTIONS MANUAL
DIFFERENTIAL EQUATIONS
& LINEAR ALGEBRA
THIRD EDITION
C. HENRY
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CHAPTER 2

MATHEMATICAL MODELS AND NUMERICAL METHODS

SECTION 2.1

POPULATION MODELS

Section 2.1 introduces the first of the two major classes of mathematical models studied in the textbook, and is a prerequisite to the discussion of equilibrium solutions and stability in Section 2.2.

In Problems 1–8 we outline the derivation of the desired particular solution, and then sketch some typical solution curves.

1. Noting that $x > 1$ because $x(0) = 2$, we write

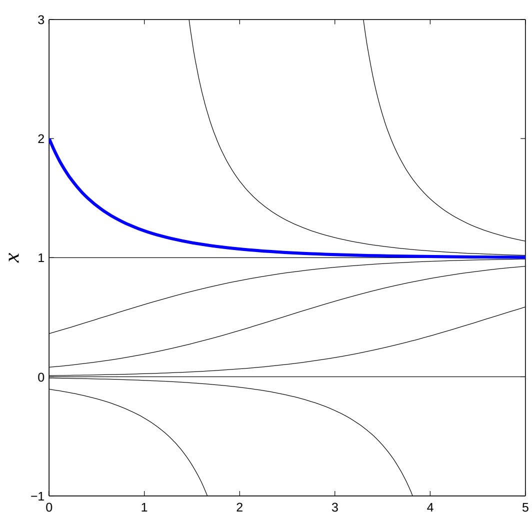
$$\int \frac{dx}{x(1-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx = \int 1\,dt$$

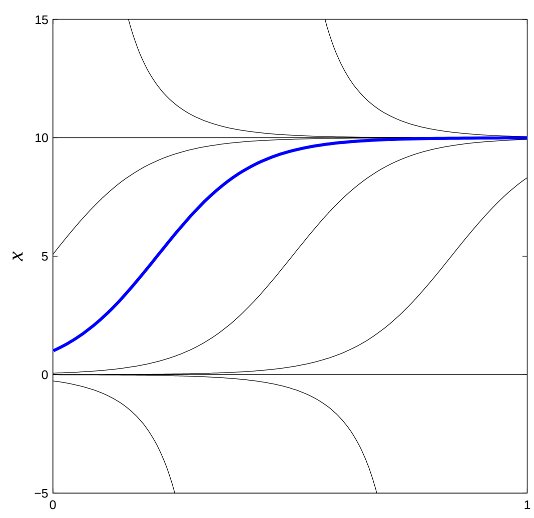
$$\ln x - \ln(x-1) = t + \ln C; \qquad \frac{x}{x-1} = C e^t$$

$$x(0) = 2 \quad \text{implies} \quad C = 2; \qquad x = 2(x-1)e^t$$

$$x(t) = \frac{2e^t}{2e^t - 1} = \frac{2}{2 - e^{-t}}.$$

Typical solution curves are shown in the figure on the left below.





2. Noting that $x < 10$ because $x(0) = 1$, we write

$$\int \frac{dx}{x(10-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{10-x}\right) dx = \int 10\,dt$$

$$\ln x - \ln(10-x) = 10t + \ln C; \qquad \frac{x}{10-x} = C\,e^{10t}$$

$$x(0) = 1 \text{ implies } C = \frac{1}{9}; \qquad 9x = (10-x)\,e^{10t}$$

$$x(t) = \frac{10e^{10t}}{9+e^{10t}} = \frac{10}{1+9e^{-10t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

3. Noting that $x > 1$ because $x(0) = 3$, we write

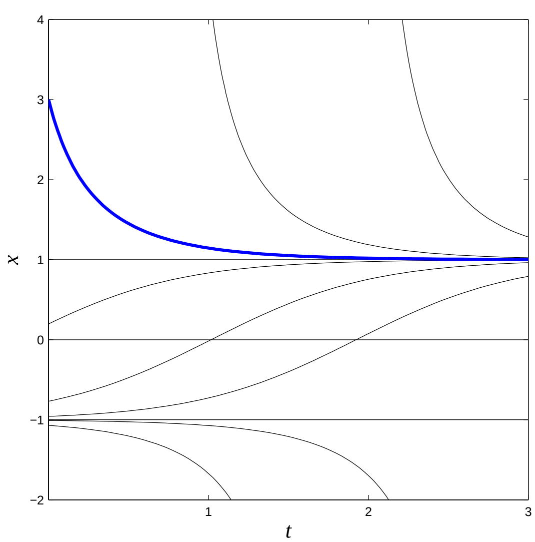
$$\int \frac{dx}{(1+x)(1-x)} = \int 1\,dt; \qquad \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \int (-2)\,dt$$

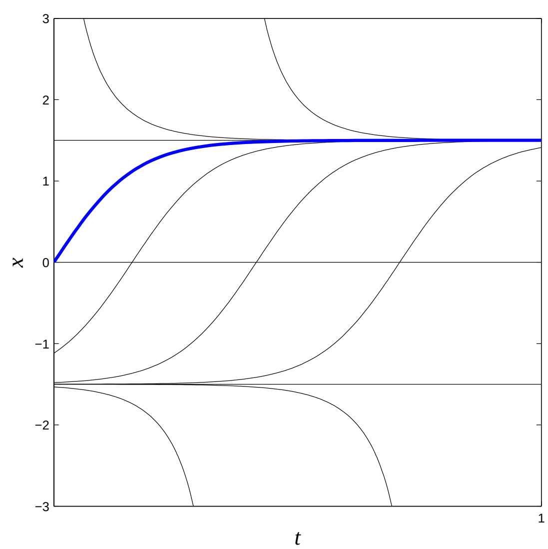
$$\ln(x-1) - \ln(x+1) = -2t + \ln C; \qquad \frac{x-1}{x+1} = C\,e^{-2t}$$

$$x(0) = 3 \text{ implies } C = \frac{1}{2}; \qquad 2(x-1) = (x+1)\,e^{-2t}$$

$$x(t) = \frac{2+e^{-2t}}{2-e^{-2t}} = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

Typical solution curves are shown in the figure on the left below.





4. Noting that $|x| < \frac{3}{2}$ because $x(0) = 0$, we write

$$\int \frac{dx}{(3+2x)(3-2x)} = \int 1\,dt; \qquad \int \left(\frac{1}{3+2x} + \frac{1}{3-2x}\right) dx = \int 6\,dt$$

$$\frac{1}{2}\ln(3+2x) - \frac{1}{2}\ln(3-2x) = 6t + \frac{1}{2}\ln C; \qquad \frac{3+2x}{3-2x} = C e^{12t}$$

$$x(0) = 0 \quad \text{implies} \quad C = 1; \qquad 3 + 2x = (3-2x)e^{12t}$$

$$x(t) = \frac{3e^{12t} - 3}{2e^{12t} + 2} = \frac{3\left(e^{12t} - 1\right)}{2\left(e^{12t} + 1\right)}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

5. Noting that $x > 5$ because $x(0) = 8$, we write

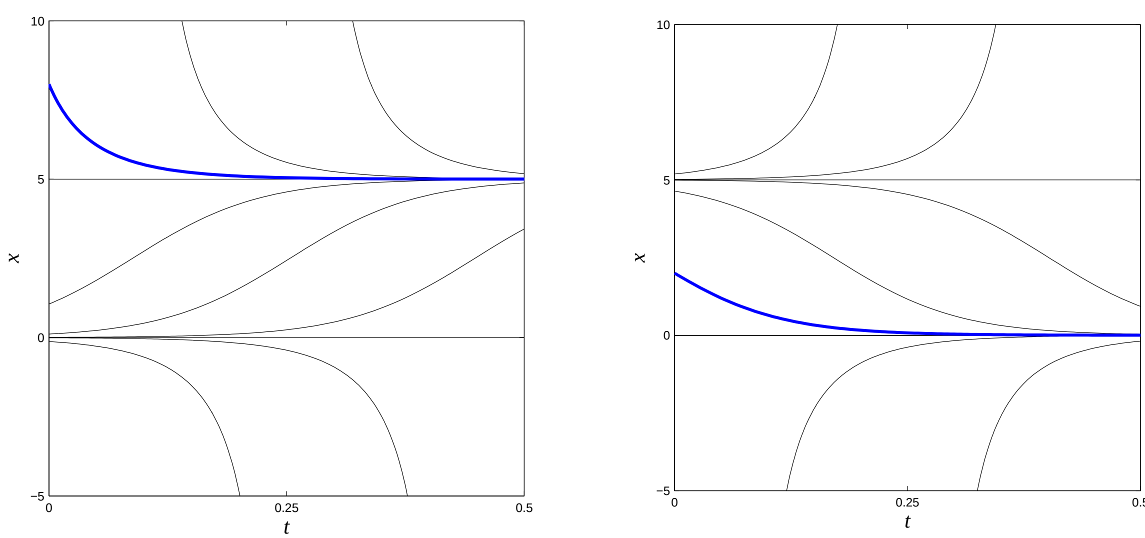
$$\int \frac{dx}{x(x-5)} = \int (-3)\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-5}\right) dx = \int 15\,dt$$

$$\ln x - \ln(x-5) = 15t + \ln C; \qquad \frac{x}{x-5} = C e^{15t}$$

$$x(0) = 8 \quad \text{implies} \quad C = 8/3; \qquad 3x = 8(x-5)e^{15t}$$

$$x(t) = \frac{-40e^{15t}}{3 - 8e^{15t}} = \frac{40}{8 - 3e^{-15t}}.$$

Typical solution curves are shown in the figure on the left below.



6. Noting that $x < 5$ because $x(0) = 2$, we write

$$\int \frac{dx}{x(5-x)} = \int (-3)\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{5-x}\right) dx = \int (-15)\,dt$$

$$\ln x - \ln(5-x) = -15t + \ln C; \qquad \frac{x}{5-x} = C\,e^{-15t}$$

$$x(0) = 2 \quad \text{implies} \quad C = 2/3; \qquad 3x = 2(5-x)\,e^{-15t}$$

$$x(t) = \frac{10\,e^{-15t}}{3 + 2\,e^{-15t}} = \frac{10}{2 + 3\,e^{15t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

7. Noting that $x > 7$ because $x(0) = 11$, we write

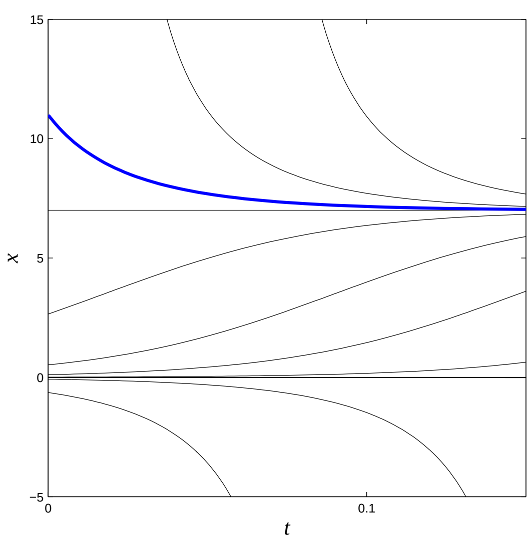
$$\int \frac{dx}{x(x-7)} = \int (-4)\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-7}\right) dx = \int 28\,dt$$

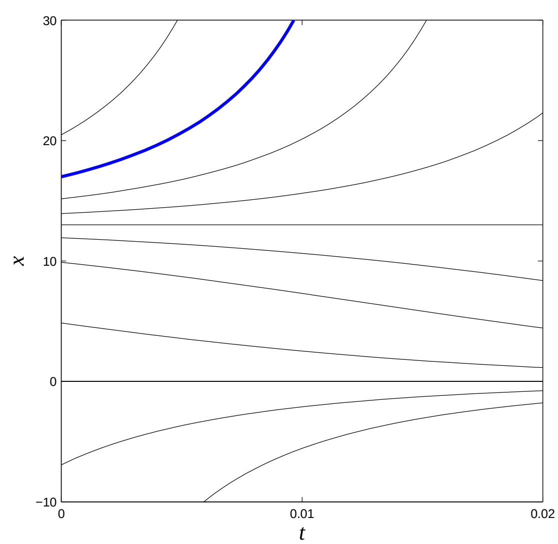
$$\ln x - \ln(x-7) = 28t + \ln C; \qquad \frac{x}{x-7} = C\,e^{28t}$$

$$x(0) = 11 \quad \text{implies} \quad C = 11/4; \qquad 4x = 11(x-17)\,e^{28t}$$

$$x(t) = \frac{-77\,e^{28t}}{4 - 11e^{28t}} = \frac{77}{11 - 4\,e^{-28t}}.$$

Typical solution curves are shown in the figure on the left below.





8. Noting that $x > 13$ because $x(0) = 17$, we write

$$\int \frac{dx}{x(x-13)} = \int 7\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-13}\right) dx = \int (-91)\,dt$$

$$\ln x - \ln(x-13) = -91t + \ln C; \qquad \frac{x}{x-13} = C e^{-91t}$$

$$x(0) = 17 \quad \text{implies} \quad C = 17/4; \qquad 4x = 17(x-13)e^{-91t}$$

$$x(t) = \frac{-221 e^{-91t}}{4 - 17 e^{-91t}} = \frac{221}{17 - 4e^{91t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

9. Substitution of $P(0) = 100$ and $P'(0) = 20$ into $P' = k\sqrt{P}$ yields $k = 2$, so the differential equation is $P' = 2\sqrt{P}$. Separation of variables and integration, $\int dP/2\sqrt{P} = \int dt$, gives $\sqrt{P} = t + C$. Then $P(0) = 100$ implies $C = 10$, so $P(t) = (t + 10)^2$. Hence the number of rabbits after one year is $P(12) = 484$.

10. Given $P' = -\delta P = -(k/\sqrt{P})P = -k\sqrt{P}$, separation of variables and integration as in Problem 9 yields $2\sqrt{P} = -kt + C$. The initial condition $P(0) = 900$ gives $C = 60$, and then the condition $P(6) = 441$ implies that $k = 3$. Therefore $2\sqrt{P} = -3t + 60$, so $P = 0$ after $t = 20$ weeks.

11. **(a)** Starting with $dP/dt = k\sqrt{P}$, $dP/dt = k\sqrt{P}$, we separate the variables and integrate to get $P(t) = (kt/2 + C)^2$. Clearly $P(0) = P_0$ implies $C = \sqrt{P_0}$.

(b) If $P(t) = (kt/2 + 10)^2$, then $P(6) = 169$ implies that $k = 1$. Hence $P(t) = (t/2 + 10)^2$, so there are 256 fish after 12 months.

12. Solution of the equation $P' = kP^2$ by separation of variables and integration,

$$\int \frac{dP}{P^2} = \int k\,dt; \qquad -\frac{1}{P} = kt - C,$$

gives $P(t) = 1/(C - kt)$. Now $P(0) = 12$ implies that $C = 1/12$, so now $P(t) = 12/(1 - 12kt)$. Then $P(10) = 24$ implies that $k = 1/240$, so finally $P(t) = 240/(20 - t)$. Hence $P = 48$ when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$.

13. **(a)** If the birth and death rates both are proportional to P^2 and $\beta > \delta$, then Eq. (1) in this section gives $P' = kP^2$ with k positive. Separating variables and integrating as in Problem 12, we find that $P(t) = 1/(C - kt)$. The initial condition $P(0) = P_0$ then gives $C = 1/P_0$, so $P(t) = 1/(1/P_0 - kt) = P_0/(1 - kP_0t)$.

(b) If $P_0 = 6$ then $P(t) = 6/(1 - 6kt)$. Now the fact that $P(10) = 9$ implies that $k = 180$, so $P(t) = 6/(1 - t/30) = 180/(30 - t)$. Hence it is clear that $P \to \infty$ as $t \to 30$ (doomsday).

14. Now $dP/dt = -kP^2$ with $k > 0$, and separation of variables yields $P(t) = 1/(kt + C)$. Clearly $C = 1/P_0$ as in Problem 13, so $P(t) = P_0/(1 + kP_0t)$. Therefore it is clear that $P(t) \to 0$ as $t \to \infty$, so the population dies out in the long run.

15. If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 16 and 17) that $a = B_0/P_0$ and $b = D_0/P_0^2 = k$.

16. The relations in Problem 15 give $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120 + 40e^{-t/15})$. We find that $P = 0.95M$ after about 27.69 months.

17. The relations in Problem 15 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240 - 60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months.

18. If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 19 and 20) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$.

19. The relations in Problem 18 give $k = 1/1000$ and $M = 90$. The solution is $P(t) = 9000/(100 - 10e^{9t/100})$. We find that $P = 10M$ after about 24.41 months.

20. The relations in Problem 18 give $k = 1/1100$ and $M = 120$. The solution is $P(t) = 13200/(110 + 10e^{6t/55})$. We find that $P = 0.1M$ after about 42.12 months.

21. Starting with the differential equation $dP/dt = kP(200 - P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P}+\frac{1}{200-P}\right) dP = \int 200k\,dt;$$

$$\ln\frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0)=100$ gives $C=1$, and $P'(0)=1$ implies that $1 = k\cdot 100(200-100)$, so we find that $k = 1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t) = 200/\left(1+e^{-t/50}\right)$. Finally, $P(60) = 200/\left(1+e^{-6/5}\right) \approx 153.7$ million.

22. We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100-50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50\times 100}{50+(100-50)e^{-0.04t}},$$

and we solve this equation for $t \approx 34.66$. Thus the rumor will have spread to 80% of the population in a little less than 35 days.

23. **(a)** $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50+150\,e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

24. The differential equation for $N(t)$ is $N'(t) = kN(15-N)$. When we substitute $N(0) = 5$ (thousands) and $N'(0) = 0.5$ (thousands/day) we find that $k = 0.01$. With N in place of P, this is the logistic equation in Eq. (3) of the text, so its solution is given by Equation (7):

$$N(t) = \frac{15 \times 5}{5 + 10\exp[-(0.01)(15)t]} = \frac{15}{1 + 2\,e^{-0.15\,t}}.$$

Upon substituting $N = 10$ on the left, we solve for $t = (\ln 4)/(0.15) \approx 9.24$ days.

25. Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M - 25.00) = 3/8, \qquad 47.54k(M - 47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Equation (4) gives the population function

$$P(t) = \frac{2500}{25 + 75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A. D.

26. The differential equation for $P(t)$ is

$$P'(t) = 0.001P^2 - \delta P.$$

When we substitute $P(0) = 100$ and $P'(0) = 8$ we find that $\delta = 0.02$, so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P - 20).$$

We separate variables and integrate, noting that $P > 20$ because $P_0 = 100$:

$$\int \frac{dP}{P(P-20)} = \int 0.001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-20} - \frac{1}{P} \right) dP = \int 0.02\,dt;$$

$$\ln \frac{P-20}{P} = \frac{1}{50}t + \ln C \quad \Rightarrow \quad \frac{P-20}{P} = Ce^{t/50}.$$

Now $P(0) = 100$ gives $C = 4/5$, hence

$$5(P - 20) = 4P\,e^{t/50} \quad \Rightarrow \quad P(t) = \frac{100}{5 - 4e^{t/50}}.$$

It follows readily that $P = 200$ when $t = 50 \ln(9/8) \approx 5.89$ months.

27. We are given that

$$P' = kP^2 - 0.01P,$$

When we substitute $P(0) = 200$ and $P'(0) = 2$ we find that $k = 0.0001$, so

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P-100).$$

We separate variables and integrate, noting that $P > 100$ because $P_0 = 200$:

$$\int \frac{dP}{P(P-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-100} - \frac{1}{P} \right) dP = \int 0.01\,dt;$$

$$\ln \frac{P-100}{P} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{P-100}{P} = Ce^{t/100}.$$

Now $P(0) = 100$ gives $C = 1/2$, hence

$$2(P-100) = P\,e^{t/100} \quad \Rightarrow \quad P(t) = \frac{200}{2 - e^{t/100}}.$$

(a) $P = 1000$ when $t = 100 \ln(9/5) \approx 58.78$.

(b) $P \to \infty$ as $t \to 100 \ln 2 \approx 69.31$.

28. Our alligator population satisfies the equation

$$\frac{dP}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x-100).$$

With x in place of P, this is the same differential equation as in Problem 27, but now we use absolute values to allow both possibilities $x < 100$ and $x > 100$:

$$\int \frac{dx}{x(x-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{x-100} - \frac{1}{x} \right) dP = \int 0.01\,dt;$$

$$\ln \frac{|x-100|}{x} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{|x-100|}{x} = Ce^{t/100}. \qquad (*)$$

(a) If $x(0) = 25$ then $x < 100$ and $|x-100| = 100 - x$, so (*) gives $C = 3$ and hence

$$100 - x = 3x\,e^{t/100} \quad \Rightarrow \quad x(t) = \frac{100}{1 + 3e^{t/100}}.$$

We therefore see that $x(t) \to 0$ as $t \to \infty$.

(b) But if $x(0)=150$ then $x>100$ and $|x-100|=x-100$, so (*) gives $C=1/3$ and hence

$$3(x-100) = x e^{t/100} \quad \Rightarrow \quad x(t) = \frac{300}{3-e^{t/100}}.$$

Now $x(t)\to+\infty$ as $t\to(100\ln 3)^{-}$, so doomsday occurs after about 109.86 months.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 = 0.0001489P(210.544-P)$$

where $k=0.0001489$ and $P=210.544$. With $P_0=3.9$ also, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{(3.9)+(210.544-3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9+206.644e^{-0.03135t}}.$$

(a) This solution gives $P(140)\approx 127.008$, fairly close to the actual 1930 U.S. census population of 123.2 million.

(b) The limiting population as $t\to\infty$ is $821.122/3.9 = 210.544$ million.

(c) Since the actual U.S. population in 200 was about 281 million — already exceeding the maximum population predicted by the logistic equation — we see that that this model did *not* continue to hold throughout the 20th century.

30. The equation is separable, so we have

$$\int \frac{dP}{P} = \int \beta_0 e^{-\alpha t}dt, \qquad \text{so} \qquad \ln P = -\frac{\beta_0}{\alpha}e^{-\alpha t}+C.$$

The initial condition $P(0)=P_0$ gives $C=\ln P_0+\beta_0/\alpha$, so

$$P(t) = P_0 \exp\left[\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)\right].$$

31. If we substitute $P(0) = 10^6$ and $P'(0) = 3\times10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t}P,$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 30 is

$$P(t) = P_0\exp[(0.3/\alpha)(1-e^{-\alpha t})].$$

The fact that $P(6) = 2P_0$ now yields the equation

$$f(\alpha) = (0.3)(1 - e^{-6\alpha}) - \alpha \ln 2 = 0$$

for α. We apply Newton's iterative formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

with $f'(\alpha) = 1.8e^{-6\alpha} - \ln 2$ and initial guess $\alpha_0 = 1$, and find that $\alpha \approx 0.3915$. Therefore the limiting cell population as $t \to \infty$ is

$$P_0 \exp(\beta_0/\alpha) = 10^6 \exp(0.3/0.3915) \approx 2.15 \times 10^6.$$

Thus the tumor does not grow much further after 6 months.

32. We separate the variables in the logistic equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(M-P)} = \int k\, dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int kM\, dt;$$

$$\ln \frac{P}{|M-P|} = kMt + \ln C \quad \Rightarrow \quad \frac{P}{|M-P|} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|M - P| = M - P$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = P_0/(M - P_0)$. It follows that

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|M - P| = P - M$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = P_0/(P_0 - M)$, and it follows that

$$\frac{P}{P-M} = \frac{P_0}{P_0-M} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(M - P_0)P = (M - P)P_0 e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0 e^{kMt}}{(M - P_0) + P_0 e^{kMt}},$$

which gives the desired result upon division of numerator and denominator by e^{kMt}.

33. (a) We separate the variables in the extinction-explosion equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(P-M)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-M} - \frac{1}{P} \right) dP = \int kM\,dt;$$

$$\ln \frac{|P-M|}{P} = kMt + \ln C \quad \Rightarrow \quad \frac{|P-M|}{P} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|P-M| = M - P$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = (M - P_0)/P_0$. It follows that

$$\frac{M-P}{P} = \frac{M-P_0}{P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|P-M| = P - M$, so substitution of $t = 0$, $P = P_0$ in (*) gives $C = (P_0 - M)/P_0$, and it follows that

$$\frac{P-M}{P} = \frac{P_0-M}{P_0} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(P-M)P_0 = (P_0-M)P\,e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0}{P_0 + (M-P_0)\,e^{kMt}}.$$

(b) If $P_0 < M$ then the coefficient $M - P_0$ is positive and the denominator increases without bound, so $P(t) \to 0$ as $t \to \infty$. But if $P_0 > M$, then the denominator $P_0 - (P_0 - M)\,e^{kMt}$ approaches zero — so $P(t) \to +\infty$ — as t approaches the value $(1/kM)\ln[P_0/(P_0 - M)] > 0$ from the left.

34. Differentiation of both sides of the logistic equation $P' = kP \cdot (M - P)$ yields

$$\begin{aligned} P'' &= \frac{dP'}{dP} \cdot \frac{dP}{dt} \\ &= [k \cdot (M-P) + kP \cdot (-1)] \cdot kP(M-P) \\ &= k[M-2P] \cdot kP(M-P) = 2k^2 P(M - \tfrac{1}{2}P)(M-P) \end{aligned}$$

as desired. The conclusions that $P'' > 0$ if $0 < P < \frac{1}{2}M$, that $P'' = 0$ if $P = \frac{1}{2}M$, and that $P'' < 0$ if $\frac{1}{2}M < P < M$ are then immediate. Thus it follows that each of the curves for which $P_0 < M$ has an inflection point where it crosses the horizontal line $P = \frac{1}{2}M$.

35. Any way you look at it, you should see that, the larger the parameter $k > 0$ is, the faster the logistic population $P(t)$ approaches its limiting population M.

36. With $x = e^{-50kM}$, $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$, Eqs. (7) in the text take the form

$$\frac{P_0 M}{P_0 + (M - P_0)\,x} = P_1, \qquad \frac{P_0 M}{P_0 + (M - P_0)\,x^2} = P_2$$

from which we get

$$P_0 + (M - P_0)\,x = P_0 M / P_1, \qquad P_0 + (M - P_0)\,x^2 = P_0 M / P_2$$

$$x = \frac{P_0(M - P_1)}{P_1(M - P_0)}, \qquad x^2 = \frac{P_0(M - P_2)}{P_2(M - P_0)} \qquad \text{(i)}$$

$$\frac{P_0^2(M - P_1)^2}{P_1^2(M - P_0)^2} = \frac{P_0(M - P_2)}{P_2(M - P_0)}$$

$$P_0 P_2 (M - P_1)^2 = P_1^2 (M - P_0)(M - P_2)$$

$$P_0 P_2 M^2 - 2P_0 P_1 P_2 M + P_0 P_1^2 P_2 = P_1^2 M^2 - P_1^2 (P_0 + P_2) M + P_0 P_1^2 P_2$$

We cancel the final terms on the two sides of this last equation and solve for

$$M = \frac{P_1(2P_0 P_2 - P_0 P_1 - P_1 P_2)}{P_0 P_2 - P_1^2}. \qquad \text{(ii)}$$

Substitution of the given values $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$ now gives $M = 188.121$. The first equation in (i) and $x = \exp(-kMt_1)$ yield

$$k = -\frac{1}{Mt_1} \ln \frac{P_0(M - P_1)}{P_1(M - P_0)}. \qquad \text{(iii)}$$

Now substitution of $t_1 = 50$ and our numerical values of M, P_0, P_1, P_2 gives $k = 0.000167716$. Finally, substitution of these values of k and M (and P_0) in the logistic solution (4) gives the logistic model of Eq. (8) in the text.

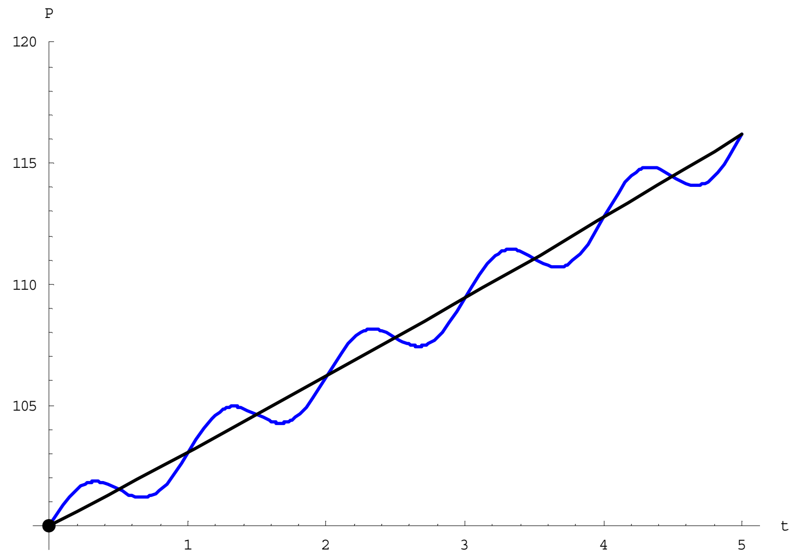
In Problems 37 and 38 we give just the values of k and M calculated using Eqs. (ii) and (iii) in Problem 36 above, the resulting logistic solution, and the predicted year 2000 population.

37. $k = 0.000146679$ and $M = 208.250$, so $P(t) = \dfrac{4829.73}{23.192 + 185.058\,e^{-0.0305458\,t}}$,

predicting $P = 248.856$ in the year 2000.

38. $k = 0.0000668717$ and $M = 338.027$, so $P(t) = \dfrac{25761.7}{76.212 + 261.815\,e^{-0.0226045\,t}}$,

predicting $P = 192.525$ in the year 2000.



39. We readily separate the variables and integrate:

$$\int \frac{dP}{P} = \int (k + b\cos 2\pi t)\,dt \quad \Rightarrow \quad \ln P = kt + \frac{b}{2\pi}\sin 2\pi t + \ln C.$$

Clearly $C = P_0$, so we find that $P(t) = P_0 \exp\left(kt + \dfrac{b}{2\pi}\sin 2\pi t\right)$. The colored curve in the figure above shows the graph that results with the typical numerical values $P_0 = 100$, $k = 0.03$, and $b = 0.06$. It oscillates about the black curve which represents natural growth with P_0 and $k = 0.03$. We see that the two agree at the end of each full year.

SECTION 2.2

EQUILIBRIUM SOLUTIONS AND STABILITY

In Problems 1–12 we identify the stable and unstable critical points as well as the funnels and spouts along the equilibrium solutions. In each problem the indicated solution satisfying $x(0) = x_0$ is derived fairly routinely by separation of variables. In some cases, various signs in the solution depend on the initial value, and we give a typical solution. For each problem we show typical solution curves corresponding to different values of x_0.

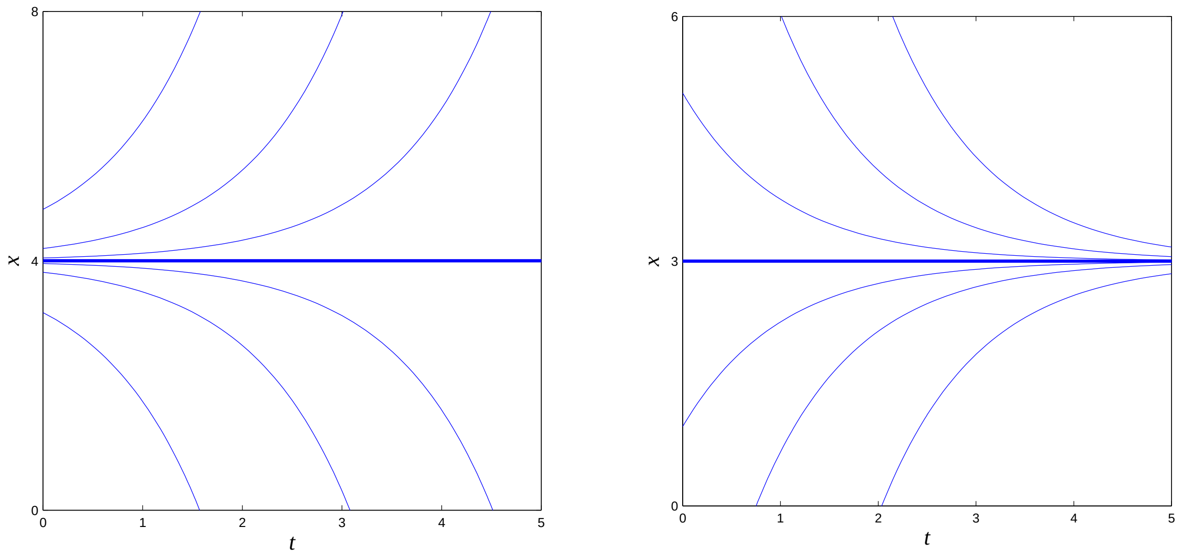
1. Unstable critical point: $x = 4$
Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

$$\int \frac{dx}{x-4} = \int dt; \quad \ln(x-4) = t + C; \quad C = \ln(x_0 - 4)$$

$$x - 4 = (x_0 - 4)e^t; \quad x(t) = 4 + (x_0 - 4)e^t.$$

Typical solution curves are shown in the figure on the left below.



2. Stable critical point: $x = 3$
Funnel: Along the equilibrium solution $x(t) = 3$

Solution: If $x_0 > 3$ then

$$\int \frac{dx}{x-3} = \int (-1)\,dt; \quad \ln(x-3) = -t + C; \quad C = \ln(x_0 - 3)$$

$$x - 3 = (x_0 - 3)e^{-t}; \quad x(t) = 3 + (x_0 - 3)e^{-t}.$$

Typical solution curves are shown in the figure on the right above.

3. Stable critical point: $x = 0$
Unstable critical point: $x = 4$
Funnel: Along the equilibrium solution $x(t) = 0$
Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

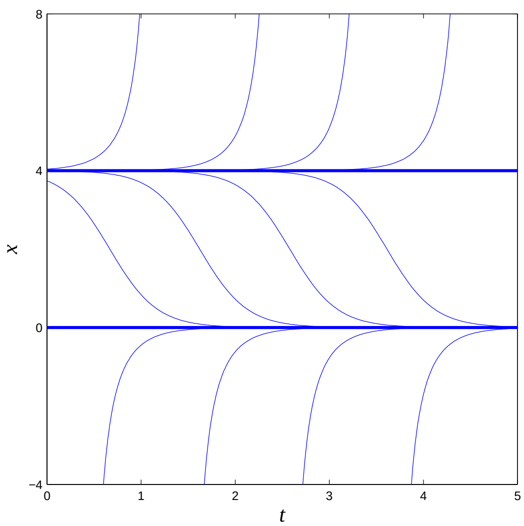
$$\int 4\,dt = \int \frac{4\,dx}{x(x-4)} = \int\left(\frac{1}{x-4}-\frac{1}{x}\right)dx$$

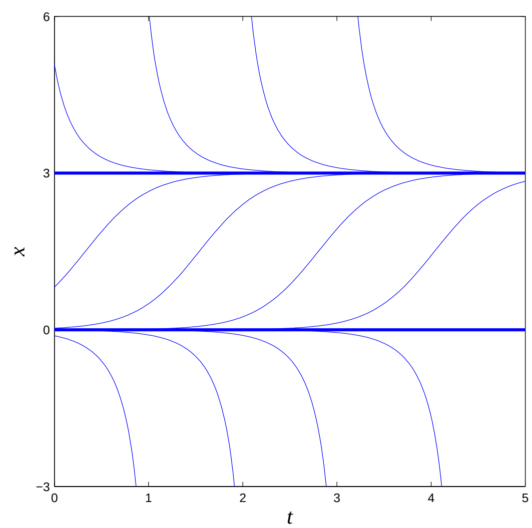
$$4t + C = \ln\frac{x-4}{x}; \quad C = \ln\frac{x_0-4}{x_0}$$

$$4t = \ln\frac{x_0(x-4)}{x(x_0-4)}; \quad e^{4t} = \frac{x_0(x-4)}{x(x_0-4)}$$

$$x(t) = \frac{4x_0}{x_0+(4-x_0)e^{4t}}.$$

Typical solution curves are shown in the figure on the left below.





4. Stable critical point: $x = 3$
Unstable critical point: $x = 0$
Funnel: Along the equilibrium solution $x(t) = 3$
Spout: Along the equilibrium solution $x(t) = 0$

Solution: If $x_0 > 3$ then

$$\int(-3)\,dt = \int \frac{3\,dx}{x(x-3)} = \int\left(\frac{1}{x-3}-\frac{1}{x}\right)dx$$

$$-3t + C = \ln\frac{x-3}{x}; \quad C = \ln\frac{x_0-3}{x_0}$$

$$-3t = \ln\frac{x_0(x-3)}{x(x_0-3)}; \quad e^{-3t} = \frac{x_0(x-3)}{x(x_0-3)}$$

$$x(t) = \frac{3x_0}{x_0 + (3 - x_0)e^{-3t}}.$$

Typical solution curves are shown in the figure on the right above.

5. Stable critical point: $x = -2$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = -2$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 2$ then

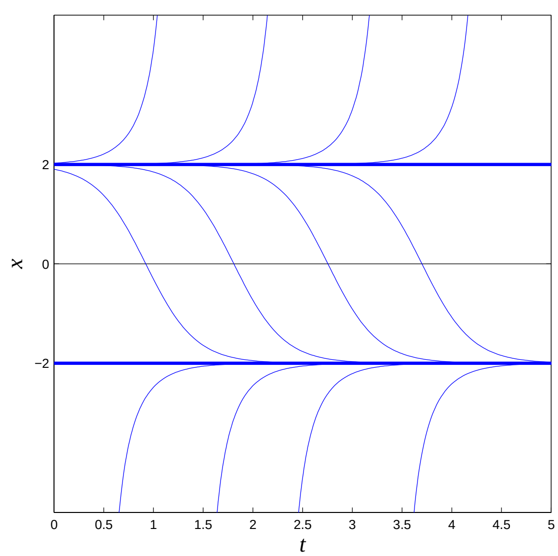
$$\int 4\,dt = \int \frac{4\,dx}{x^2 - 4} = \int \left(\frac{1}{x-2} - \frac{1}{x+2} \right) dx$$

$$4t + C = \ln \frac{x-2}{x+2}; \qquad C = \ln \frac{x_0 - 2}{x_0 + 2}$$

$$4t = \ln \frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}; \qquad e^{4t} = \frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}$$

$$x(t) = \frac{2\left[(x_0+2) + (x_0-2)e^{4t} \right]}{(x_0+2) - (x_0-2)e^{4t}}.$$

Typical solution curves are shown in the figure below.



6. Stable critical point: $x = 3$
Unstable critical point: $x = -3$
Funnel: Along the equilibrium solution $x(t) = 3$

Spout: Along the equilibrium solution $x(t) = -3$

Solution: If $x_0 > 3$ then

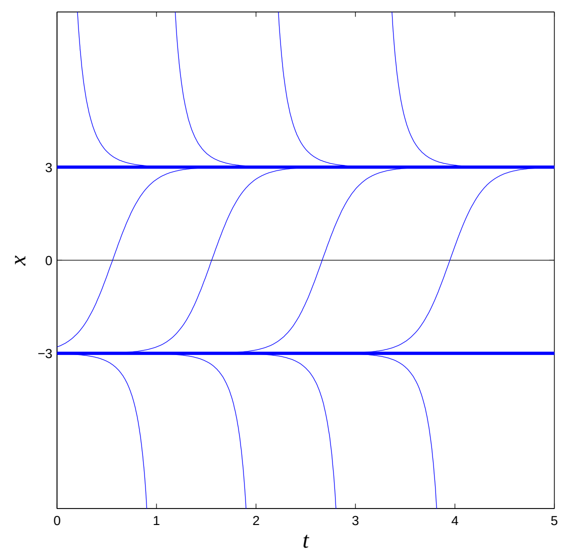
$$\int 6\,dt = \int \frac{6\,dx}{9-x^2} = \int \left(\frac{1}{3+x} + \frac{1}{3-x} \right) dx$$

$$6t + C = \ln\frac{x+3}{x-3}; \qquad C = \ln\frac{x_0+3}{x_0-3}$$

$$6t = \ln\frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}; \qquad e^{6t} = \frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}$$

$$x(t) = \frac{3\left[(x_0-3)+(x_0+3)e^{6t}\right]}{(3-x_0)+(x_0+3)e^{6t}}.$$

Typical solution curves are shown in the figure below.



7. Critical point: $x = 2$

This single critical point is *semi-stable*, meaning that solutions with $x_0 > 2$ go to infinity as t increases, while solutions with $x_0 < 2$ approach 2.

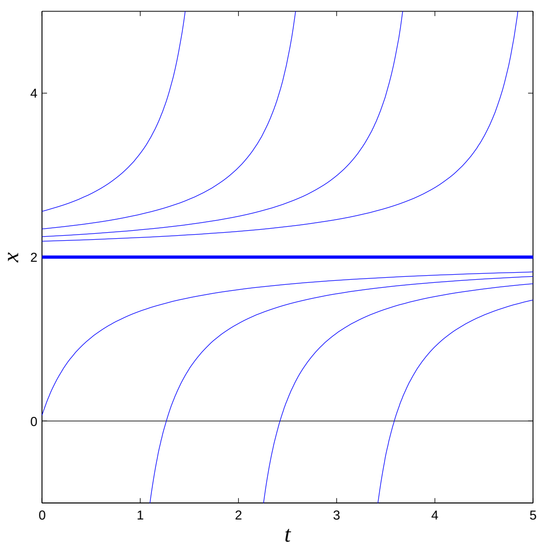
Solution: If $x_0 > 2$ then

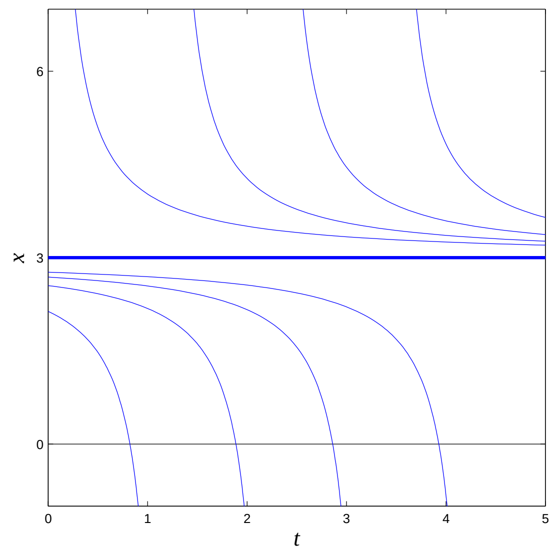
$$\int \frac{-dx}{(x-2)^2} = \int (-1)\,dt; \qquad \frac{1}{x-2} = -t + C; \qquad C = \frac{1}{x_0-2}$$

$$\frac{1}{x-2} = -t + \frac{1}{x_0 - 2} = \frac{1 - t(x_0 - 2)}{x_0 - 2}$$

$$x(t) = 2 + \frac{x_0 - 2}{1 - t(x_0 - 2)} = \frac{x_0(2t-1) - 4t}{t\,x_0 - 2t - 1}.$$

Typical solution curves are shown in the figure on the left below.





8. Critical point: $x = 3$

This single critical point is *semi-stable*, meaning that solutions with $x_0 < 3$ go to minus infinity as t increases, while solutions with $x_0 > 3$ approach 3.

Solution: If $x_0 > 3$ then

$$\int \frac{-dx}{(x-3)^2} = \int dt; \quad \frac{1}{x-3} = t + C; \quad C = \frac{1}{x_0 - 3}$$

$$\frac{1}{x-3} = t + \frac{1}{x_0 - 3} = \frac{1 + t(x_0 - 3)}{x_0 - 3}$$

$$x(t) = 3 + \frac{x_0 - 3}{1 + t(x_0 - 3)} = \frac{x_0(3t+1) - 9t}{t\,x_0 - 3t + 1}.$$

Typical solution curves are shown in the figure on the right above.

9. Stable critical point: $x = 1$
Unstable critical point: $x = 4$
Funnel: Along the equilibrium solution $x(t) = 1$

Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

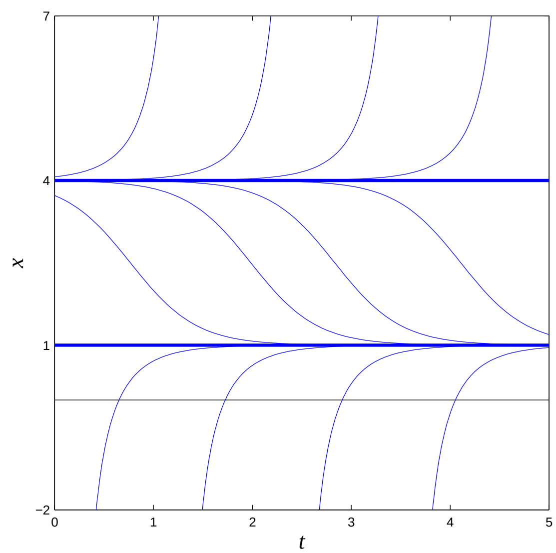
$$\int 3\,dt = \int \frac{3\,dx}{(x-4)(x-1)} = \int \left(\frac{1}{x-4} - \frac{1}{x-1}\right) dx$$

$$3t + C = \ln\frac{x-4}{x-1}; \qquad C = \ln\frac{x_0-4}{x_0-1}$$

$$3t = \ln\frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}; \qquad e^{3t} = \frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}$$

$$x(t) = \frac{4(1-x_0)+(x_0-4)e^{3t}}{(1-x_0)+(x_0-4)e^{3t}}.$$

Typical solution curves are shown in the figure below.



10. Stable critical point: $x = 5$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = 5$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 5$ then

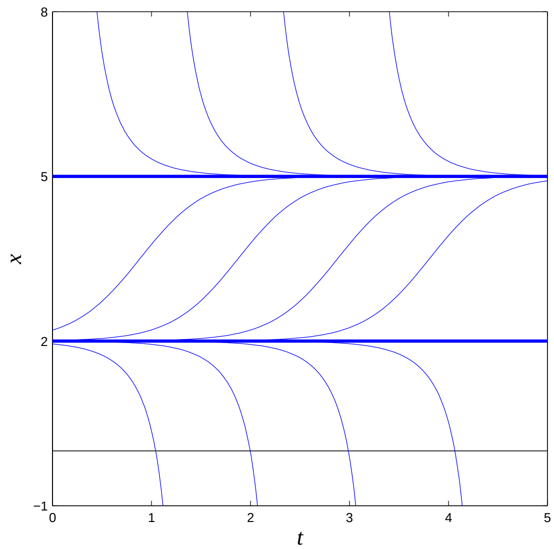
$$\int 3\,dt = \int \frac{(-3)\,dx}{(x-5)(x-2)} = \int \left(\frac{1}{x-2} - \frac{1}{x-5}\right) dx$$

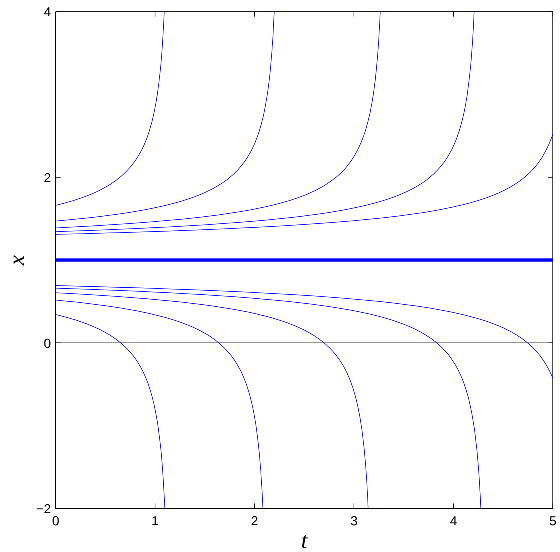
$$3t + C = \ln\frac{x-2}{x-5}; \qquad C = \ln\frac{x_0-2}{x_0-5}$$

$$3t = \ln\frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}; \quad e^{3t} = \frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}$$

$$x(t) = \frac{2(5-x_0)+5(x_0-2)e^{3t}}{(5-x_0)+(x_0-2)e^{3t}}.$$

Typical solution curves are shown in the figure on the left below.



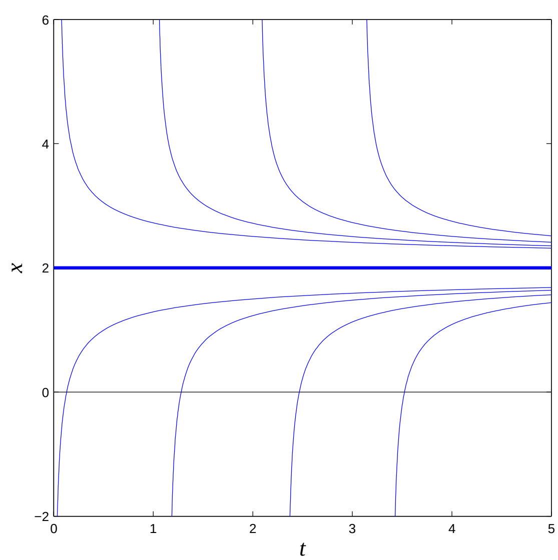


11. Unstable critical point: $x = 1$

Spout: Along the equilibrium solution $x(t) = 1$

Solution: $\displaystyle\int \frac{-2\,dx}{(x-1)^3} = \int(-2)\,dt; \quad \frac{1}{(x-1)^2} = -2t + \frac{1}{(x_0-1)^2}.$

Typical solution curves are shown in the figure on the right above.



12. Stable critical point: $x = 2$

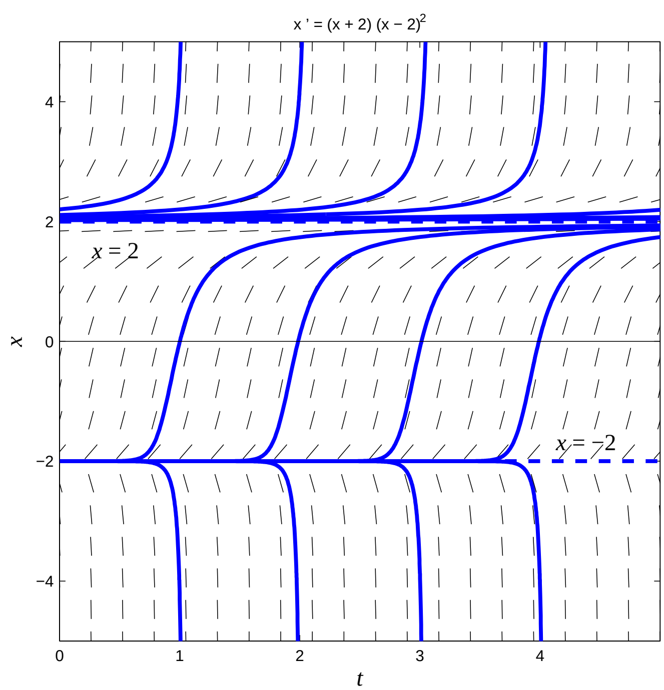
Funnel: Along the equilibrium solution $x(t) = 2$

Solution: $\displaystyle\int \frac{2\,dx}{(2-x)^3} = \int 2\,dt; \qquad \frac{1}{(2-x)^2} = 2t + \frac{1}{(2-x_0)^2}.$

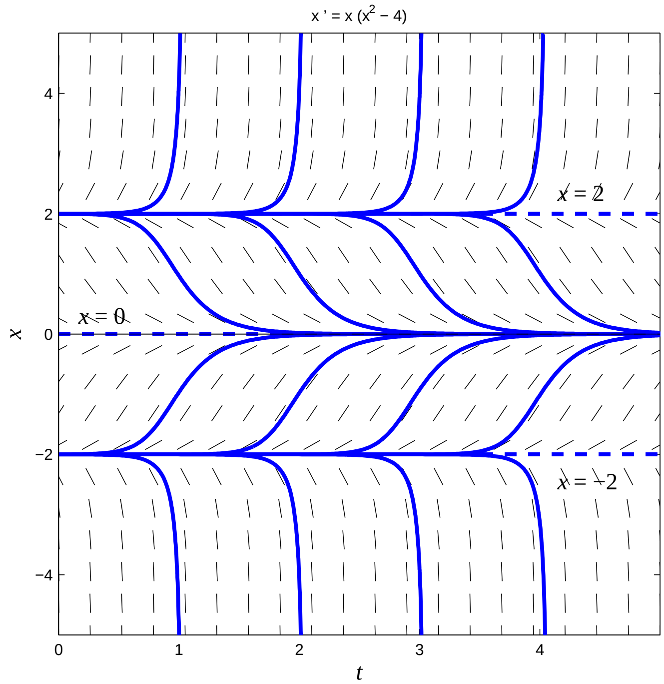
Typical solution curves are shown in the figure at the bottom of the preceding page.

In each of Problems 13 through 18 we present the figure showing slope field and typical solution curves, and then record the visually apparent classification of critical points for the given differential equation.

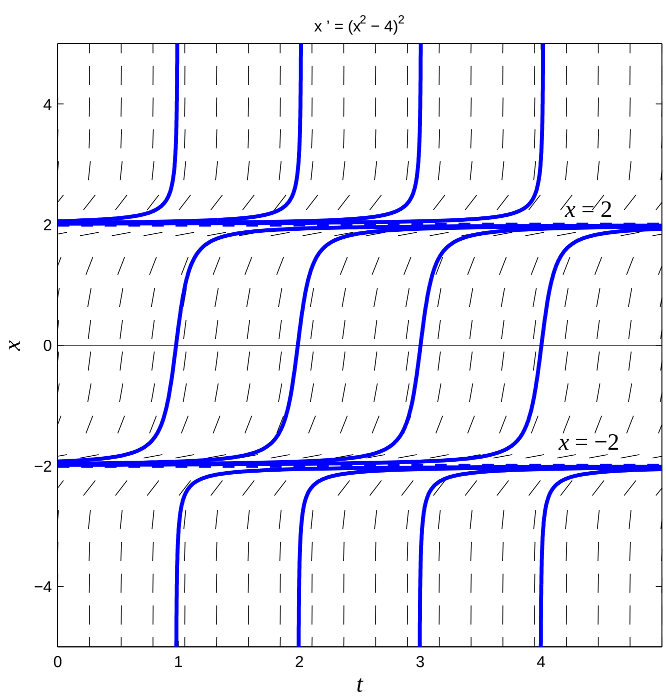
13. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



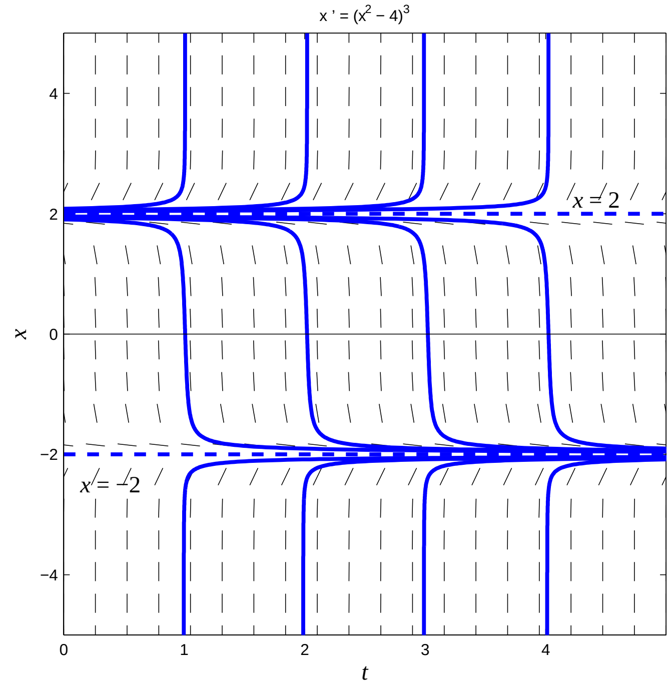
14. The critical points $x = \pm 2$ are both unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



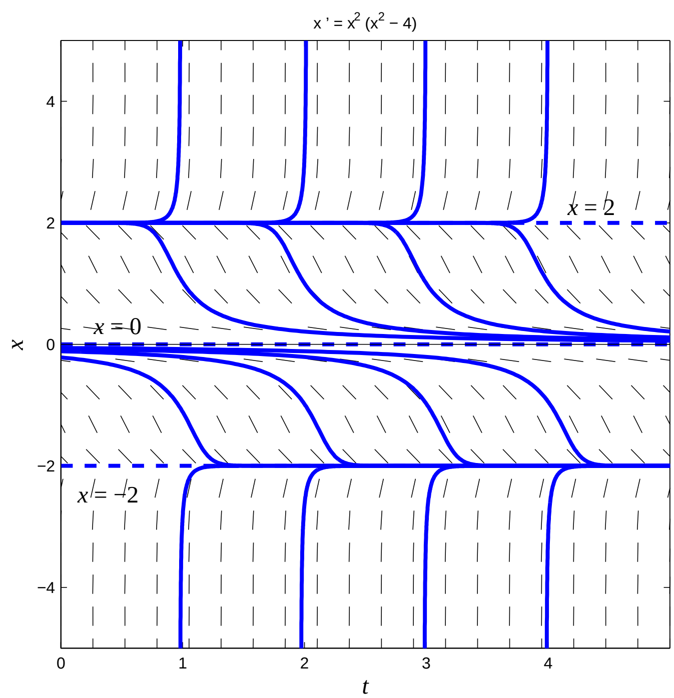
15. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



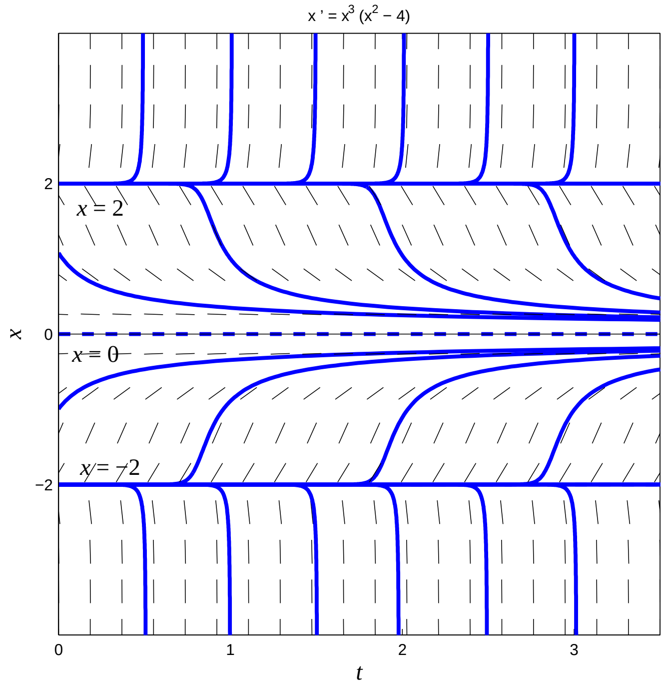
16. The critical point $x = 2$ is unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



17. The critical points $x = 2$ and $x = 0$ are unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



18. The critical points $x = 2$ and $x = -2$ are unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



19. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{10}x(10-x)-h = 0, \qquad \text{that is,} \qquad x^2-10x+10h = 0.$$

Thus a critical point c is given in terms of h by

$$c = \frac{10 \pm \sqrt{100-40h}}{2} = 5 \pm \sqrt{25-10h}.$$

It follows that there is no critical point if $h > 2\tfrac{1}{2}$, only the single critical point $c = 0$ if $h = 2\tfrac{1}{2}$, and two distinct critical points if $h < 2\tfrac{1}{2}$ (so $10-25h > 0$). Hence the bifurcation diagram in the *hc*-plane is the parabola with the $(c-5)^2 = 25-10h$ that is obtained upon squaring to eliminate the square root above.

20. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{100}x(x-5)+s = 0, \qquad \text{that is,} \qquad x^2-5x+100s = 0.$$

Thus a critical point c is given in terms of s by

$$c = \frac{5 \pm \sqrt{25-400s}}{2} = \frac{5}{2} \pm \frac{5}{2}\sqrt{1-16s}.$$

It follows that there is no critical point if $s > \frac{1}{16}$, only the single critical point $c = 0$ if $s = \frac{1}{16}$, and two distinct critical points if $s < \frac{1}{16}$ (so $1-16s > 0$). Hence the bifurcation diagram in the *sc*-plane is the parabola $(2c-5)^2 = 25(1-16s)$ that is obtained upon elimination of the radical above.

21. **(a)** If $k = -a^2$ where $a \geq 0$, then $kx - x^3 = -a^2x - x^3 = -x(a^2+x^2)$ is 0 only if $x = 0$, so the only critical point is $c = 0$. If $a > 0$ then we can solve the differential equation by writing

$$\int \frac{a^2 dx}{x(a^2+x^2)} = \int \left(\frac{1}{x} - \frac{x}{a^2+x^2} \right) dx = -\int a^2\, dt,$$

$$\ln x - \frac{1}{2}\ln(a^2+x^2) = -a^2 t + \frac{1}{2}\ln C,$$

$$\frac{x^2}{a^2+x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2Ce^{-2a^2t}}{1-Ce^{-2a^2t}}.$$

It follows that $x \to 0$ as $t \to 0$, so the critical point $c = 0$ is *stable.*

(b) If $k = +a^2$ where $a > 0$ then $kx - x^3 = +a^2x - x^3 = -x(x+a)(x-a)$ is 0 if either $x = 0$ or $x = \pm a = \pm\sqrt{k}$. Thus we have the three critical points $c = 0, \pm\sqrt{k}$, and this observation together with part (a) yields the pitchfork bifurcation diagram shown in Fig. 2.2.13 of the textbook. If $x \neq 0$ then we can solve the differential equation by writing

$$\int \frac{2a^2 dx}{x(x-a)(x+a)} = \int \left(-\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a} \right) dx = -\int 2a^2\, dt,$$

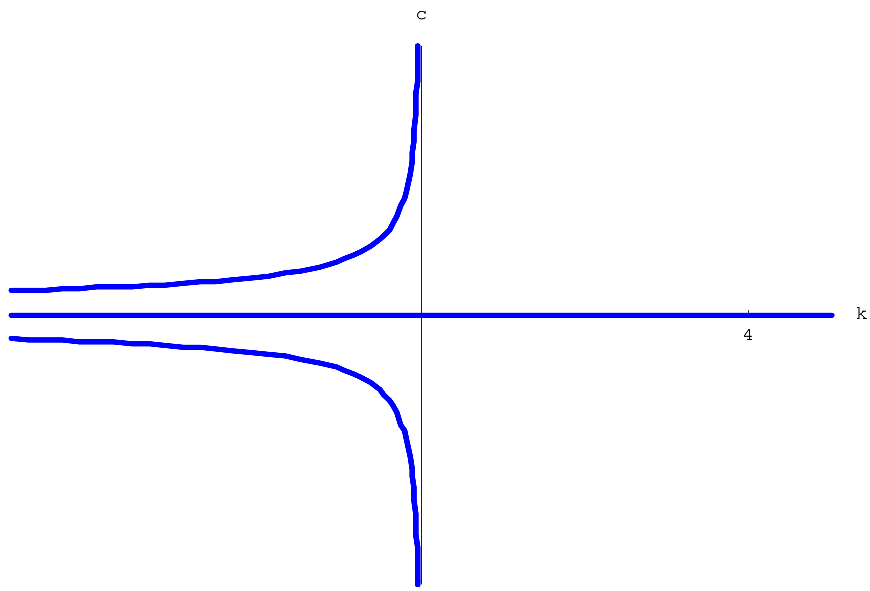
$$-2\ln x + \ln(x-a) + \ln(x-a) = -2a^2t + \ln C,$$

$$\frac{x^2-a^2}{x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2}{1-Ce^{-2a^2t}} \quad \Rightarrow \quad x = \frac{\pm\sqrt{k}}{\sqrt{1-Ce^{-2a^2t}}}.$$

It follows that if $x(0) \neq 0$ then $x \to \sqrt{k}$ if $x > 0$, $x \to -\sqrt{k}$ if $x < 0$. This implies that the critical point $c = 0$ is *unstable*, while the critical points $c = \pm\sqrt{k}$ are *stable.*

22. If $k = 0$ then the only critical point $c = 0$ of the equation $x' = x$ is unstable, because the solutions $x(t) = x_0 e^t$ diverge to infinity if $x_0 \neq 0$. If $k = +a^2 > 0$, then $x + a^2x^3 = x(1+a^2x^2) = 0$ only if $x = 0$, so again $c = 0$ is the only critical point. If

$k = -a^2 < 0$, then $x - a^2x^3 = x(1 - a^2x^2) = x(1 - ax)(1 + ax) = 0$ if either $x = 0$ or $x = \pm 1/a = \pm\sqrt{-1/k}$. Hence the bifurcation diagram of the differential equation $x' = x + kx^3$ looks as pictured below:



23. **(a)** If $h < kM$ then the differential equation is $x' = k\,x\big((M - h/k) - x\big)$, which is a logistic equation with the *reduced* limiting population $M - h/k$.

(b) If $h > kM$ then the differential equation can be rewritten in the form $x' = -ax - bx^2$ with a and b both positive. The solution of this equation is

$$x(t) = \frac{a\,x_0}{(a + b\,x_0)e^{at} - bx_0}$$

so it is obvious that $x(t) \to 0$ as $t \to \infty$.

24. If $x_0 > N$ then

$$\int -k(N - H)\,dt = \int \frac{(N - H)\,dx}{(x - N)(x - H)} = \int \left(\frac{1}{x - N} - \frac{1}{x - H}\right)dx$$

$$-k(N - H)t + C = \ln\frac{x - N}{x - H}; \qquad C = \ln\frac{x_0 - N}{x_0 - H}$$

$$-k(N - H)t = \ln\frac{(x - N)(x_0 - H)}{(x - H)(x_0 - M)}; \qquad e^{-k(N-H)t} = \frac{(x - N)(x_0 - H)}{(x - H)(x_0 - M)}$$

$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}$$

25. **(i)** In the first alternative form that is given, all of the coefficients within parentheses are positive if $H < x_0 < N$. Hence it is obvious that $x(t) \to N$ as $t \to \infty$.

(ii) In the second alternative form that is given, all of the coefficients within parentheses are positive if $x_0 < H$. Hence the denominator is initially equal to $N - H > 0$, but decreases as t increases, and reaches the value 0 when

$$t = \frac{1}{N-H} \ln \frac{N - x_0}{H - x_0} > 0.$$

26. If $4h = kM^2$ then Eqs. (13) and (14) in the text show that the differential equation takes the form $x' = -k(M/2 - x)^2$ with the single critical point $x = M/2$. This equation is readily solved by separation of variables, but clearly x' is negative whether x is less than or greater than $M/2$.

27. Separation of variables in the differential equation $x' = -k\left((x-a)^2 + b^2\right)$ yields

$$x(t) = a - b \tan\left(bk\,t + \tan^{-1}\frac{a - x_0}{b}\right).$$

It therefore follows that $x(t)$ goes to minus infinity in a finite period of time.

28. Aside from a change in sign, this calculation is the same as that indicated in Eqs. (13) and (14) in the text.

29. This is simply a matter of analyzing the signs of x' in the cases $x < a$, $a < x < b$, $b < x < c$, and $c > x$. Alternatively, plot slope fields and typical solution curves for the two differential equations using typical numerical values such as $a = -1, b = 1, c = 2$.

SECTION 2.3

ACCELERATION-VELOCITY MODELS

This section consists of three essentially independent subsections that can be studied separately: resistance proportional to velocity, resistance proportional to velocity-squared, and inverse-square gravitational acceleration.

1. Equation: $v' = k(250 - v), \quad v(0) = 0, \quad v(10) = 100$

Solution: $\int \frac{(-1)\,dv}{250 - v} = -\int k\,dt; \quad \ln(250 - v) = -kt + \ln C,$

$v(0) = 0$ implies $C = 250; \quad v(t) = 250(1 - e^{-kt})$

$v(10) = 100$ implies $k = \frac{1}{10}\ln(250/150) \approx 0.0511;$

Answer: $v = 200$ when $t = -(\ln 50/250)/k \approx 31.5$ sec

2. Equation: $v' = -kv, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $x'(t) = v(t) = v_0 e^{-kt}; \quad x(t) = -(v_0/k)e^{-kt} + C$

$C = x_0 + (v_0/k)e^{-kt}; \quad x(t) = x_0 + (v_0/k)\left(1 - e^{-kt}\right)$

Answer: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x_0 + (v_0/k)\left(1 - e^{-kt}\right)\right] = x_0 + (v_0/k)$

3. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $v(t) = 40\,e^{-kt}$ with $k = (1/10)\ln 2$

$x(t) = (40/k)(1 - e^{-kt})$

Answer: $x(\infty) = \lim_{t\to\infty}(40/k)(1 - e^{-kt}) = 40/k = 400/\ln 2 \approx 577$ ft

4. Equation: $v' = -kv^2, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{v^2} = \int k\,dt; \quad \frac{1}{v} = kt + C; \quad C = \frac{1}{v_0}$

$x'(t) = v(t) = \frac{v_0}{1 + v_0 kt}; \quad x(t) = \frac{1}{k}\ln\left(1 + v_0 kt\right) + x_0$

$x(t) \to \infty$ as $x(t) \to \infty$

5. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $v = \frac{40}{1 + 40kt}$ (as in Problem 3)

$v(10) = 20$ implies $40k = 1/10$, so $v(t) = \frac{400}{10 + t}$

$x(t) = 400\ln[(10 + t)/10]$

Answer: $x(60) = 400\ln 7 \approx 778$ ft

6. Equation: $v' = -kv^{3/2}, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{2v^{3/2}} = \int \frac{k\,dt}{2}; \quad \frac{1}{\sqrt{v}} = \frac{kt}{2} + C; \quad C = \frac{1}{\sqrt{v_0}}$

$$x'(t) = v(t) = \frac{4v_0}{\left(2 + kt\sqrt{v_0}\right)^2}; \quad x(t) = -\frac{4\sqrt{v_0}}{k\left(2 + kt\sqrt{v_0}\right)} + C$$

$$C = x_0 + \frac{2\sqrt{v_0}}{k}; \quad x(t) = x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{2 + kt\sqrt{v_0}}\right)$$

$$x(\infty) = x_0 + 2\sqrt{v_0}/k$$

7. Equation: $v' = 10 - 0.1v, \quad x(0) = v(0) = 0$

(a) $\int \frac{-0.1\,dv}{10 - 0.1v} = \int (-0.1)\,dt; \quad \ln(10 - 0.1v) = -t/10 + \ln C$

$v(0) = 0$ implies $C = 10; \quad \ln\left[(10 - 0.1v)/10\right] = -t/10$

$v(t) = 100(1 - e^{-t/10}); \quad v(\infty) = 100$ ft/sec (limiting velocity)

(b) $x(t) = 100t - 1000(1 - e^{-t/10})$

$v = 90$ ft/sec when $t = 23.0259$ sec and $x = 1402.59$ ft

8. Equation: $v' = 10 - 0.001v^2, \quad x(0) = v(0) = 0$

(a) $\int \frac{0.01\,dv}{1 - 0.0001v^2} = \int \frac{dt}{10}; \quad \tanh^{-1}\frac{v}{100} = \frac{t}{10} + C$

$v(0) = 0$ implies $C = 0$ so $v(t) = 100\tanh(t/10)$

$$v(\infty) = \lim_{t\to\infty} 100\tanh(t/10) = 100 \lim_{t\to\infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100 \text{ ft/sec}$$

(b) $x(t) = 1000\ln(\cosh t/10)$

$v = 90$ ft/sec when $t = 14.7222$ sec and $x = 830.366$ ft

9. The solution of the initial value problem

$$1000\,v' = 5000 - 100\,v, \qquad v(0) = 0$$

is

$$v(t) = 50(1 - e^{-t/10}).$$

Hence, as $t \to \infty$, we see that $v(t)$ approaches $v_{max} = 50$ ft/sec $\approx$ 34 mph.

10. Before opening parachute:

$$v' = -32 - 0.15v, \quad v(0) = 0, \quad y(0) = 10000$$
$$v(t) = 213.333(e^{-0.15t} - 1), \qquad v(20) = -202.712 \text{ ft/sec}$$
$$y(t) = 11422.2 - 1422.22e^{-0.15ty} - 213.333t, \quad y(20) = 7084.75 \text{ ft}$$

After opening parachute:

$$v' = -32 - 1.5v, \quad v(0) = -202.712, \quad y(0) = 7084.75$$
$$v(t) = -21.3333 - 181.379e^{-1.5t}$$
$$y(t) = 6964.83 + 120.919e^{-1.5t} - 21.3333t,$$
$$y = 0 \text{ when } t = 326.476$$

Thus she opens her parachute after 20 sec at a height of 7085 feet, and the total time of descent is 20 + 326.476 = 346.476 sec, about 5 minutes and 46.5 seconds. Her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper′s terminal velocity was 100 mph = 440/3 ft/sec, then Equation (7) in the text yields $\rho = 12/55$. Then we find by solving Equation (9) numerically with $y_0 = 1200$ and $v_0 = 0$ that $y = 0$ when $t \approx 12.5$ sec. Thus the newspaper account is inaccurate.

12. With $m = 640/32 = 20$ slugs, $W = 640$ lb, $B = (8)(62.5) = 500$ lb, and $F_R = -v$ lb (F_R is upward when $v < 0$), the differential equation is

$$20\,v'(t) = -640 + 500 - v = -140 - v\,.$$

Its solution with $v(0) = 0$ is

$$v(t) = 140\left(e^{-0.05t} - 1\right),$$

and integration with $y(0) = 0$ yields

$$y(t) = 2800\left(e^{-0.05t} - 1\right) - 140t\,.$$

Using these equations we find that $t = 20\ln(28/13) \approx 15.35$ sec when $v = -75$ ft/sec, and that $y(15.35) \approx -648.31$ ft. Thus the maximum safe depth is just under 650 ft.

Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. To solve the initial value problem $v' = -9.8 - 0.0011v^2, \quad v(0) = 49$ we write

$$\int \frac{dv}{9.8 + 0.0011v^2} = -\int dt; \qquad \int \frac{0.010595\,dv}{1 + (0.010595\,v)^2} = -\int 0.103827\,dt$$

$$\tan^{-1}(0.010595\,v) = -0.103827\,t + C; \qquad v(0) = 49 \text{ implies } C = 0.478854$$

$$v(t) = 94.3841\tan(0.478854 - 0.103827\,t)$$

Integration with $y(0) = 0$ gives

$$y(t) = 108.468 + 909.052\ \ln(\cos(0.478854 - 0.103827\,t)).$$

We solve $v(0) = 0$ for $t = 4.612$, and then calculate $y(4.612) = 108.468$.

18. We solve the initial value problem $v' = -9.8 + 0.0011v^2, \quad v(0) = 0$ much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

$$v(t) = -94.3841\tanh(0.103827\,t),$$

and then integration with $y(0) = 108.47$ gives

$$y(t) = 108.47 - 909.052\ \ln(\cosh(0.103827\,t)).$$

We solve $y(0) = 0$ for $t = \cosh^{-1}(\exp(108.47/909.052))/0.103.827 \approx 4.7992$, and then calculate $v(4.7992) = -43.489$.

19. Equation: $v' = 4 - (1/400)v^2, \quad v(0) = 0$

Solution: $$\int \frac{dv}{4 - (1/400)v^2} = \int dt; \qquad \int \frac{(1/40)\,dv}{1 - (v/40)^2} = \int \frac{1}{10}\,dt$$

$$\tanh^{-1}(v/40) = t/10 + C; \qquad C = 0; \qquad v(t) = 40\tanh(t/10)$$

Answer: $v(10) \approx 30.46$ ft/sec, $\quad v(\infty) = 40$ ft/sec

20. Equation: $v' = -32 - (1/800)v^2, \quad v(0) = 160, \quad y(0) = 0$

Solution: $$\int \frac{dv}{32 + (1/800)v^2} = -\int dt; \qquad \int \frac{(1/160)\,dv}{1 + (v/160)^2} = -\int \frac{1}{5}\,dt;$$

$$\tan^{-1}(v/160) = -t/5 + C; \quad v(0) = 160 \text{ implies } C = \pi/4$$

$$v(t) = 160\tan\left(\frac{\pi}{4} - \frac{t}{5}\right)$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4} - \frac{t}{5}\right)\right) + 400\ln 2$$

We solve $v(t) = 0$ for $t = 3.92699$ and then calculate $y(3.92699) = 277.26$ ft.

21. Equation: $v' = -g - \rho v^2, \quad v(0) = v_0, \quad y(0) = 0$

Solution: $$\int \frac{dv}{g + \rho v^2} = -\int dt; \quad \int \frac{\sqrt{\rho/g}\,dv}{1 + \left(\sqrt{\rho/g}\,v\right)^2} = -\int \sqrt{g\rho}\,dt;$$

$$\tan^{-1}\left(\sqrt{\rho/g}\,v\right) = -\sqrt{g\rho}\,t + C; \qquad v(0) = v_0 \text{ implies } C = \tan^{-1}\left(\sqrt{\rho/g}\,v_0\right)$$

$$v(t) = -\sqrt{\frac{g}{\rho}}\tan\left(t\sqrt{g\rho} - \tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)$$

We solve $v(t) = 0$ for $t = \frac{1}{\sqrt{g\rho}}\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)$ and substitute in Eq. (17) for $y(t)$:

$$y_{\max} = \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1} v_0\sqrt{\rho/g} - \tan^{-1} v_0\sqrt{\rho/g}\right)}{\cos\left(\tan^{-1} v_0\sqrt{\rho/g}\right)}\right|$$

$$= \frac{1}{\rho}\ln\left(\sec\left(\tan^{-1} v_0\sqrt{\rho/g}\right)\right) = \frac{1}{\rho}\ln\sqrt{1 + \frac{\rho v_0^2}{g}}$$

$$y_{\max} = \frac{1}{2\rho}\ln\left(1 + \frac{\rho v_0^2}{g}\right)$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem $v' = -32 + 0.075v^2, \quad v(0) = 0$ is

$$v(t) = -20.666\tanh(1.54919\,t),$$

so the terminal speed is 20.666 ft/sec. Then a further integration with $y(0) = 0$ gives

$$y(t) = 10000 - 13.333\ln(\cosh(1.54919\,t)).$$

We solve $y(0) = 0$ for $t = 484.57$. Thus the descent takes about 8 min 5 sec.

23. Before opening parachute:

$$v' = -32 + 0.00075v^2, \quad v(0) = 0, \quad y(0) = 10000$$
$$v(t) = -206.559\tanh(0.154919t) \qquad v(30) = -206.521 \text{ ft/sec}$$
$$y(t) = 10000 - 1333.33\ln(\cosh(0.154919t)), \quad y(30) = 4727.30 \text{ ft}$$

After opening parachute:

$$v' = -32 + 0.075v^2, \quad v(0) = -206.521, \quad y(0) = 4727.30$$
$$v(t) = -20.6559\tanh(1.54919t + 0.00519595)$$
$$y(t) = 4727.30 - 13.3333\ln(\cosh(1.54919t + 0.00519595))$$
$$y = 0 \text{ when } t = 229.304$$

Thus she opens her parachute after 30 sec at a height of 4727 feet, and the total time of descent is 30 + 229.304 = 259.304 sec, about 4 minutes and 19.3 seconds.

24. Let M denote the mass of the Earth. Then

(a) $\sqrt{2GM/R} = c$ implies $R = 0.884\times10^{-3}$ meters, about 0.88 cm;

(b) $\sqrt{2G(329320M)/R} = c$ implies $R = 2.91\times10^{3}$ meters, about 2.91 kilometers.

25. **(a)** The rocket's apex occurs when $v = 0$. We get the desired formula when we set $v = 0$ in Eq. (23),

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right),$$

and solve for r.

(b) We substitute $v = 0$, $r = R + 10^5$ (100 km = 10^5 m) and the mks values $G = 6.6726\times10^{-11}$, $M = 5.975\times10^{24}$, $R = 6.378\times10^{6}$ in Eq. (23) and solve for $v_0 = 1389.21$ m/s ≈ 1.389 km/s.

(c) When we substitute $v_0 = (9/10)\sqrt{2GM/R}$ in the formula derived in part (a), we find that $r_{\max} = 100R/19$.

26. By an elementary computation (as in Section 1.2) we find that an initial velocity of $v_0 = 16$ ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let r denote the ratio of the radius of the asteroid to the radius $R = 3960$ miles of the earth, so that

$$r = \frac{1.5}{3960} = \frac{1}{2640}.$$

Then the mass and radius of the asteroid are given by

$$M_a = r^3 M \quad \text{and} \quad R_a = rR$$

in terms of the mass M and radius R of the earth. Hence the escape velocity from the asteroid's surface is given by

$$v_a = \sqrt{\frac{2GM_a}{R_a}} = \sqrt{\frac{2G \cdot r^3 M}{rR_a}} = r\sqrt{\frac{2GM}{R}} = r v_0$$

in terms of the escape velocity v_0 from the earth's surface. Hence $v_a \approx 36680/2640$ ≈ 13.9 ft/sec. Since the escape velocity from this asteroid is thus less than the initial velocity of 16 ft/sec that your legs can provide, you can indeed jump right off this asteroid into space.

27. **(a)** Substitution of $v_0^2 = 2GM/R = k^2/R$ in Eq. (23) of the textbook gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r}} = \frac{k}{\sqrt{r}}.$$

We separate variables and proceed to integrate:

$$\int \sqrt{r}\, dr = \int k\, dt \quad \Rightarrow \quad \frac{2}{3} r^{3/2} = kt + \frac{2}{3} R^{3/2}$$

(using the fact that $r = R$ when $t = 0$). We solve for $r(t) = \left(\tfrac{3}{2}kt + R^{3/2}\right)^{2/3}$ and note that $r(t) \to \infty$ as $t \to \infty$.

(b) If $v_0 > 2GM/R$ then Eq. (23) gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r} + \left(v_0^2 - \frac{2GM}{R}\right)} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Therefore, at every instant in its ascent, the upward velocity of the projectile in this part is greater than the velocity at the same instant of the projectile of part (a). It's as though the projectile of part (a) is the fox, and the projectile of this part is a rabbit that runs faster. Since the fox goes to infinity, so does the faster rabbit.

28. **(a)** Integration of gives

$$\frac{1}{2} v^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

and we solve for

$$\frac{dr}{dt} = v = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

taking the negative square root because $v < 0$ in descent. Hence

$$\begin{aligned} t &= -\sqrt{\frac{r_0}{2GM}} \int \sqrt{\frac{r}{r_0 - r}}\, dr \qquad (r = r_0 \cos^2 \theta) \\ &= \sqrt{r_0 / 2GM} \int 2r_0 \cos^2 \theta \, d\theta \\ &= \frac{r_0^{3/2}}{\sqrt{2GM}} (\theta + \sin\theta \cos\theta) \\ t &= \sqrt{\frac{r_0}{2GM}} \left(\sqrt{rr_0 - r^2} + r_0 \cos^{-1} \sqrt{\frac{r}{r_0}} \right) \end{aligned}$$

(b) Substitution of $G = 6.6726 \times 10^{-11}$, $M = 5.975 \times 10^{24}$ kg, $r = R = 6.378 \times 10^6$ m, and $r_0 = R + 10^6$ yields $t = 510.504$, that is, about $8\frac{1}{2}$ minutes for the descent to the surface of the earth. (Recall that we are ignoring air resistance.)

(c) Substitution of the same numerical values along with $v_0 = 0$ in the original differential equation of part (a) yields $v = -4116.42$ m/s ≈ -4.116 km/s for the velocity at impact with the earth's surface where $r = R$.

29. Integration of $v\dfrac{dv}{dy} = -\dfrac{GM}{(y+R)^2}$, $y(0) = 0$, $v(0) = v_0$ gives

$$\frac{1}{2}v^2 = \frac{GM}{y+R} - \frac{GM}{R} + \frac{1}{2}v_0^2$$

which simplifies to the desired formula for v^2. Then substitution of $G = 6.6726 \times 10^{-11}$, $M = 5.975 \times 10^{24}$ kg, $R = 6.378 \times 10^6$ m $v = 0$, and $v_0 = 1$ yields an equation that we easily solve for $y = 51427.3$, that is, about 51.427 km.

30. When we integrate

$$v\frac{dv}{dr} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, \quad r(0) = R, \quad r'(0) = v_0$$

in the usual way and solve for v, we get

$$v = \sqrt{\frac{2GM_e}{r} - \frac{2GM_e}{R} - \frac{2GM_m}{r-S} + \frac{2GM_m}{R-S} + v_0^2}.$$

The earth and moon attractions balance at the point where the right-hand side in the acceleration equation vanishes, which is when

$$r = \frac{\sqrt{M_e}\, S}{\sqrt{M_e} - \sqrt{M_m}}.$$

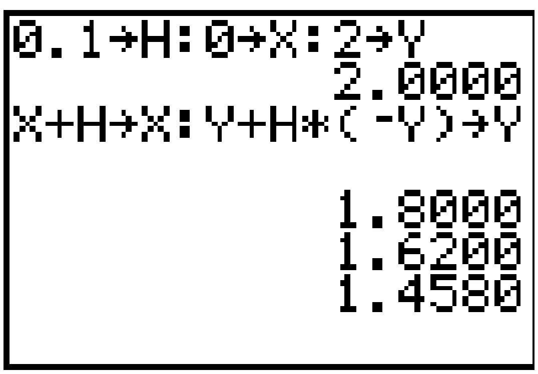
If we substitute this value of r, $M_m = 7.35\times10^{22}$ kg, $S = 384.4\times10^6$, and the usual values of the other constants involved, then set $v = 0$ (to just reach the balancing point), we can solve the resulting equation for $v_0 = 11{,}109$ m/s. Note that this is only 71 m/s less than the earth escape velocity of 11,180 m/s, so the moon really doesn't help much.

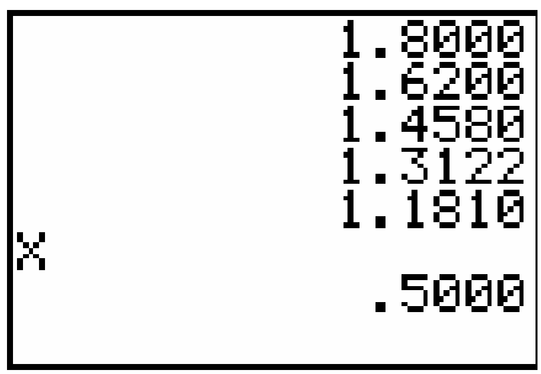
SECTION 2.4

NUMERICAL APPROXIMATION: EULER'S METHOD

In each of Problems 1–10 we also give first the explicit form of Euler's iterative formula for the given differential equation $y' = f(x, y)$. As we illustrate in Problem 1, the desired iterations are readily implemented, either manually or with a computer system or graphing calculator. Then we list the indicated values of $y(\frac{1}{2})$ rounded off accurate to 3 decimal places.

1. For the differential equation $y' = f(x, y)$ with $f(x, y) = -y$, the iterative formula of Euler's method is $y_{n+1} = y_n + h(-y_n)$. The TI-83 screen on the left shows a graphing calculator implementation of this iterative formula.





After the variables are initialized (in the first line), and the formula is entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.181$.

Approximate values 1.125 and 1.181; true value $y(\frac{1}{2}) \approx 1.213$

The following *Mathematica* instructions produce precisely this line of data.

```
f[x_,y_] = -y;
g[x_] = 2 Exp[-x];
```

```
h = 0.25;   x = 0;   y1 = y0;
Do[  k = f[x,y1];           (* the left-hand slope     *)
     y1 = y1 + h*k;         (* Euler step to update y  *)
     x = x + h,             (* update x                *)
     {i,1,2} ]

h = 0.1;   x = 0;   y2 = y0;
Do[  k = f[x,y2];           (* the left-hand slope     *)
     y2 = y2 + h*k;         (* Euler step to update y  *)
     x = x + h,             (* update x                *)
     {i,1,5} ]

Print[x,"     ",y1,"     ",y2,"     ",g[0.5]]

0.5     1.125     1.18098     1.21306
```

2. Iterative formula: $y_{n+1} = y_n + h(2y_n)$

Approximate values 1.125 and 1.244; true value $y(\frac{1}{2}) \approx 1.359$

3. Iterative formula: $y_{n+1} = y_n + h(y_n + 1)$

Approximate values 2.125 and 2.221; true value $y(\frac{1}{2}) \approx 2.297$

4. Iterative formula: $y_{n+1} = y_n + h(x_n - y_n)$

Approximate values 0.625 and 0.681; true value $y(\frac{1}{2}) \approx 0.713$

5. Iterative formula: $y_{n+1} = y_n + h(y_n - x_n - 1)$

Approximate values 0.938 and 0.889; true value $y(\frac{1}{2}) \approx 0.851$

6. Iterative formula: $y_{n+1} = y_n + h(-2x_n y_n)$

Approximate values 1.750 and 1.627; true value $y(\frac{1}{2}) \approx 1.558$

7. Iterative formula: $y_{n+1} = y_n + h(-3x_n^2 y_n)$

Approximate values 2.859 and 2.737; true value $y(\frac{1}{2}) \approx 2.647$

8. Iterative formula: $y_{n+1} = y_n + h\exp(-y_n)$

Approximate values 0.445 and 0.420; true value $y(\frac{1}{2}) \approx 0.405$

9. Iterative formula: $y_{n+1} = y_n + h(1 + y_n^2)/4$

Approximate values 1.267 and 1.278; true value $y(\frac{1}{2}) \approx 1.287$

10. Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$

Approximate values 1.125 and 1.231; true value $y(\frac{1}{2}) \approx 1.333$

The tables of approximate and actual values called for in Problems 11–16 were produced using the following MATLAB script (appropriately altered for each problem).

```
% Section 2.4, Problems 11-16
x0 = 0;    y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   y = y + h*(y-2);
   y1 = [y1,y];
   x = x + h;
   end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;
for  n = 1:200
   y = y + h*(y-2);
   y2 = [y2,y];
   x = x + h;
   end
% exact values
x = x0 : 0.2 : x0+1;
ye = 2 - exp(x);
% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
[x; y1(1:20:101); ya; ye; err]
```

11. The iterative formula of Euler's method is $y_{n+1} = y_n + h(y_n - 2)$, and the exact solution is $y(x) = 2 - e^x$. The resulting table of approximate and actual values is

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.0000	0.7798	0.5111	0.1833	–0.2167	–0.7048
y (h=0.005)	1.0000	0.7792	0.5097	0.1806	–0.2211	–0.7115
y actual	1.0000	0.7786	0.5082	0.1779	–0.2255	–0.7183
error	0%	–0.08%	–0.29%	–1.53%	1.97%	0.94%

12. Iterative formula: $y_{n+1} = y_n + h(y_n - 1)^2/2$

Exact solution: $y(x) = 1 + 2/(2 - x)$

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.0000	2.1105	2.2483	2.4250	2.6597	2.9864
y (h=0.005)	2.0000	2.1108	2.2491	2.4268	2.6597	2.9931
y actual	2.0000	2.1111	2.2500	2.4286	2.6597	3.0000
error	0%	0.02%	0.04%	0.07%	0.13%	0.23%

13. Iterative formula: $y_{n+1} = y_n + 2hx_n^3/y_n$

Exact solution: $y(x) = (8 + x^4)^{1/2}$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.0000	3.1718	3.4368	3.8084	4.2924	4.8890
y (h=0.005)	3.0000	3.1729	3.4390	3.8117	4.2967	4.8940
y actual	3.0000	3.1739	3.4412	3.8149	4.3009	4.8990
error	0%	0.03%	0.06%	0.09%	0.10%	0.10%

14. Iterative formula: $y_{n+1} = y_n + hy_n^2/x_{\text{n}}$

Exact solution: $y(x) = 1/(1 - \ln x)$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.0000	1.2215	1.5026	1.8761	2.4020	3.2031
y (h=0.005)	1.0000	1.2222	1.5048	1.8814	2.4138	3.2304
y actual	1.0000	1.2230	1.5071	1.8868	2.4259	3.2589
error	0%	0.06%	0.15%	0.29%	0.50%	0.87%

15. Iterative formula: $y_{n+1} = y_n + h(3 - 2y_n/x_n)$

Exact solution: $y(x) = x + 4/x^2$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	3.0253	3.0927	3.1897	3.3080	3.4422
y (h=0.005)	3.0000	3.0259	3.0936	3.1907	3.3091	3.4433
y actual	3.0000	3.0264	3.0944	3.1917	3.3102	3.4444
error	0%	0.019%	0.028%	0.032%	0.033%	0.032%

16. Iterative formula: $y_{n+1} = y_n + 2hx_n^5/y_n^2$

Exact solution: $y(x) = (x^6 - 37)^{1/3}$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	4.2476	5.3650	6.4805	7.6343	8.8440
y (h=0.005)	3.0000	4.2452	5.3631	6.4795	7.6341	8.8445
y actual	3.0000	4.2429	5.3613	6.4786	7.6340	8.8451
error	0%	–0.056%	–0.034%	–0.015%	0.002%	0.006%

The tables of approximate values called for in Problems 17–24 were produced using a MATLAB script similar to the one listed preceding the Problem 11 solution above.

17.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.1)	0.0000	0.0010	0.0140	0.0551	0.1413	0.2925
y (h=0.02)	0.0000	0.0023	0.0198	0.0688	0.1672	0.3379
y (h=0.004)	0.0000	0.0026	0.0210	0.0717	0.1727	0.3477
y (h=0.0008)	0.0000	0.0027	0.0213	0.0723	0.1738	0.3497

These data indicate that $y(1) \approx 0.35$, in contrast with Example 5 in the text, where the initial condition is $y(0) = 1$.

In Problems 18–24 we give only the final approximate values of y obtained using Euler's method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.6680	1.6771	1.6790	1.6794

19. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.1831	6.3653	6.4022	6.4096

20. With $x_0 = 0$ and $y_0 = -1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	−1.3792	−1.2843	−1.2649	−1.2610

21. With $x_0 = 1$ and $y_0 = 2$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	2.8508	2.8681	2.8716	2.8723

22. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.9879	7.2601	7.3154	7.3264

23. With $x_0 = 0$ and $y_0 = 0$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.2262	1.2300	1.2306	1.2307

24. With $x_0 = -1$ and $y_0 = 1$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	0.9585	0.9918	0.9984	0.9997

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0,\ v_0 = 0.$

With $h = 0.01,$ 100 iterations of $v_{n+1} = v_n + h\, f(t_n, v_n)$ yield $v(1) \approx 16.014,$ and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.998.$ Thus we observe an approximate velocity of 16.0 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01,$ 200 iterations yield $v(2) \approx 19.2056,$ and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1952.$ Thus we observe an approximate velocity of 19.2 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25.$

With $h = 1,$ 60 iterations of $P_{n+1} = P_n + h\, f(t_n, P_n)$ yield $P(60) \approx 49.3888,$ and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3903.$ Thus we observe a population of 49 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1,$ 120 iterations yield $P(120) \approx 66.1803,$ and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1469.$ Thus we observe a population of 66 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0,\ y_0 = 0.$ The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	20	200	2000	20000	200000
$y(2)$	0.7772	0.9777	1.0017	1.0042	1.0044

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2,\ y_0 = 0.$ The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	40	400	4000	40000	400000
$y(2)$	1.2900	1.4435	1.4613	1.4631	1.4633

29. With step sizes $h = 0.15$, $h = 0.03$, and $h = 0.006$ we get the following results:

x	y with h=0.15	y with h=0.03	y with h=0.006
−1.0	1.0000	1.0000	1.0000
−0.7	1.0472	1.0512	1.0521
−0.4	1.1213	1.1358	1.1390
−0.1	1.2826	1.3612	1.3835
+0.2	0.8900	1.4711	0.8210
+0.5	0.7460	1.2808	0.7192

While the values for $h = 0.15$ alone are not conclusive, a comparison of the values of y for all three step sizes with $x > 0$ suggests some anomaly in the transition from negative to positive values of x.

30. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	0.0000	0.0000
0.1	0.0000	0.0003
0.2	0.0010	0.0025
0.3	0.0050	0.0086
.	.	.
.	.	.
.	.	.
1.8	2.8200	4.3308
1.9	3.9393	7.9425
2.0	5.8521	28.3926

Clearly there is some difficulty near $x = 2$.

31. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	1.0000	1.0000
0.1	1.2000	1.2200
0.2	1.4428	1.4967
.	.	.
.	.	.
.	.	.
0.7	4.3460	6.4643

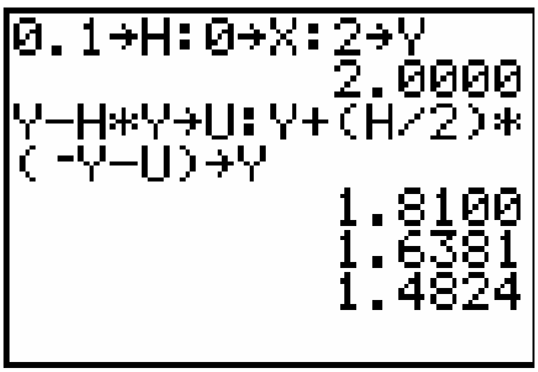
0.8	5.8670	11.8425
0.9	8.3349	39.5010

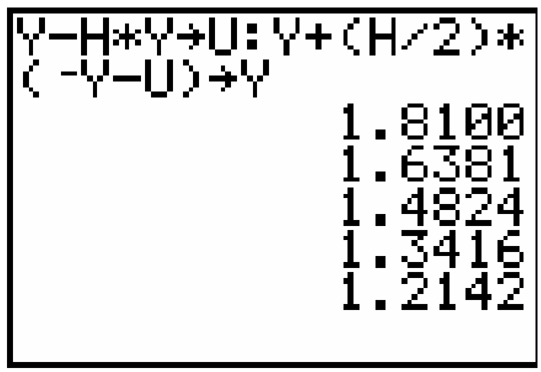
Clearly there is some difficulty near $x = 0.9$.

SECTION 2.5

A CLOSER LOOK AT THE EULER METHOD

In each of Problems 1–10 we give first the predictor formula for u_{n+1} and then the improved Euler corrector for y_{n+1}. These predictor-corrector iterations are readily implemented, either manually or with a computer system or graphing calculator (as we illustrate in Problem 1). We give in each problem a table showing the approximate values obtained, as well as the corresponding values of the exact solution.





1. $u_{n+1} = y_n + h(-y_n)$

$y_{n+1} = y_n + (h/2)[-y_n - u_{n+1}]$

The TI-83 screen on the left above shows a graphing calculator implementation of this iteration. After the variables are initialized (in the first line), and the formulas are entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.2142$ — and we see the approximate values shown in the second row of the table below.

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.8100	1.6381	1.4824	1.3416	1.2142
y actual	2.0000	1.8097	1.6375	1.4816	1.3406	1.2131

2. $u_{n+1} = y_n + 2hy_n$

$y_{n+1} = y_n + (h/2)[2y_n + 2u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.5000	0.6100	0.7422	0.9079	1.1077	1.3514
y actual	0.5000	0.6107	0.7459	0.9111	1.1128	1.3591

3. $u_{n+1} = y_n + h(y_n + 1)$

$y_{n+1} = y_n + (h/2)[(y_n + 1) + (u_{n+1} + 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.2100	1.4421	1.6985	1.9818	2.2949
y actual	1.0000	1.2103	1.4428	1.6997	1.9837	2.2974

4. $u_{n+1} = y_n + h(x_n - y_n)$

$y_{n+1} = y_n + (h/2)[(x_n - y_n) + (x_n + h - u_{n+1})]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9100	0.8381	0.7824	0.7416	0.7142
y actual	1.0000	0.9097	0.8375	0.7816	0.7406	0.7131

5. $u_{n+1} = y_n + h(y_n - x_n - 1)$

$y_{n+1} = y_n + (h/2)[(y_n - x_n - 1) + (u_{n+1} - x_n - h - 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9950	0.9790	0.9508	0.9091	0.8526
y actual	1.0000	0.9948	0.9786	0.9501	0.9082	0.8513

6. $u_{n+1} = y_n - 2x_n y_n h$

$y_{n+1} = y_n - (h/2)[2x_n y_n + 2(x_n + h)u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.9800	1.9214	1.8276	1.7041	1.5575
y actual	2.0000	1.9801	1.9216	1.8279	1.7043	1.5576

7. $u_{n+1} = y_n - 3x_n^2 y_n h$

$y_{n+1} = y_n - (h/2)[3x_n^2 y_n + 3(x_n + h)^2 u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	3.0000	2.9955	2.9731	2.9156	2.8082	2.6405
y actual	3.0000	2.9970	2.9761	2.9201	2.8140	2.6475

8. $u_{n+1} = y_n + h\exp(-y_n)$

$y_{n+1} = y_n + (h/2)[\exp(-y_n) + \exp(-u_{n+1})]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.0000	0.0952	0.1822	0.2622	0.3363	0.4053
y actual	0.0000	0.0953	0.1823	0.2624	0.3365	0.4055

9. $u_{n+1} = y_n + h(1 + y_n^2)/4$

$y_{n+1} = y_n + h[1 + y_n^2 + 1 + (u_{n+1})^2]/8$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0513	1.1053	1.1625	1.2230	1.2873
y actual	1.0000	1.0513	1.1054	1.1625	1.2231	1.2874

10. $u_{n+1} = y_n + 2x_n y_n^2 h$

$y_{n+1} = y_n + h[x_n y_n^2 + (x_n + h)(u_{n+1})^2]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0100	1.0414	1.0984	1.1895	1.3309
y actual	1.0000	1.0101	1.0417	1.0989	1.1905	1.3333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 2.5, Problems 11-16
x0 = 0;  y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   u = y + h*f(x,y);                  %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));   %corrector
   y1 = [y1,y];
   x = x + h;
   end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;
```

```
for  n = 1:200
   u = y + h*f(x,y);                        %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));    %corrector
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
x = sprintf('%10.5f',x), sprintf('\n');
y1 = sprintf('%10.5f',y1(1:20:101)), sprintf('\n');
ya = sprintf('%10.5f',ya), sprintf('\n');
ye = sprintf('%10.5f',ye), sprintf('\n');
err = sprintf('%10.5f',err), sprintf('\n');
table = [x; y1; ya; ye; err]
```

For each problem the differential equation $y' = f(x, y)$ and the known exact solution $y = g(x)$ are stored in the files **f.m** and **g.m** — for instance, the files

```
function  yp = f(x,y)
yp = y-2;

function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11. (The exact solutions for Problems 11–16 here are given in the solutions for Problems 11–16 in Section 2.4.)

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.00000	0.77860	0.50819	0.17790	–0.22551	–0.71824
y (h=0.005)	1.00000	0.77860	0.50818	0.17789	–0.22553	–0.71827
y actual	1.00000	0.77860	0.50818	0.17788	–0.22554	–0.71828
error	0.000%	–0.000%	–0.001%	–0.003%	0.003%	0.002%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.00000	2.11111	2.25000	2.42856	2.66664	2.99995
y (h=0.005)	2.00000	2.11111	2.25000	2.42857	2.66666	2.99999
y actual	2.00000	2.11111	2.25000	2.42857	2.66667	3.00000
error	0.0000%	0.0000%	0.0001%	0.0001%	0.0002%	0.0004%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.00000	3.17390	3.44118	3.81494	4.30091	4.89901
y (h=0.005)	3.00000	3.17390	3.44117	3.81492	4.30089	4.89899
y actual	3.00000	3.17389	3.44116	3.81492	4.30088	4.89898
error	0.0000%	–0.0001%	–0.0001%	0.0001%	–0.0002%	–0.0002%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.00000	1.22296	1.50707	1.88673	2.42576	3.25847
y (h=0.005)	1.00000	1.22297	1.50709	1.88679	2.42589	3.25878
y actual	1.00000	1.22297	1.50710	1.88681	2.42593	3.25889
error	0.0000%	0.0002%	0.0005%	0.0010%	0.0018%	0.0033%

15.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444448
y (h=0.005)	3.000000	3.026447	3.094445	3.191717	3.310205	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.00000%	–0.00002%	–0.00002%	–0.00002%	–0.00002%	–0.00002%

16.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	4.242859	5.361304	6.478567	7.633999	8.845112
y (h=0.005)	3.000000	4.242867	5.361303	6.478558	7.633984	8.845092
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.00000%	0.00006%	–0.00001%	–0.00005%	–0.00007%	–0.00007%

17. With $h = 0.1$: $y(1) \approx 0.35183$
With $h = 0.02$: $y(1) \approx 0.35030$
With $h = 0.004$: $y(1) \approx 0.35023$
With $h = 0.0008$: $y(1) \approx 0.35023$

The table of numerical results is

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.004$	y with $h = 0.0008$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.00300	0.00268	0.00267	0.00267
0.4	0.02202	0.02139	0.02136	0.02136
0.6	0.07344	0.07249	0.07245	0.07245
0.8	0.17540	0.17413	0.17408	0.17408
1.0	0.35183	0.35030	0.35023	0.35023

In Problems 18–24 we give only the final approximate values of y obtained using the improved Euler method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $h = 0.1$: $y(2) \approx 1.68043$
With $h = 0.02$: $y(2) \approx 1.67949$
With $h = 0.004$: $y(2) \approx 1.67946$
With $h = 0.0008$: $y(2) \approx 1.67946$

19. With $h = 0.1$: $y(2) \approx 6.40834$
With $h = 0.02$: $y(2) \approx 6.41134$
With $h = 0.004$: $y(2) \approx 6.41147$
With $h = 0.0008$: $y(2) \approx 6.41147$

20. With $h = 0.1$: $y(2) \approx -1.26092$
With $h = 0.02$: $y(2) \approx -1.26003$
With $h = 0.004$: $y(2) \approx -1.25999$
With $h = 0.0008$: $y(2) \approx -1.25999$

21. With $h = 0.1$: $y(2) \approx 2.87204$
With $h = 0.02$: $y(2) \approx 2.87245$
With $h = 0.004$: $y(2) \approx 2.87247$
With $h = 0.0008$: $y(2) \approx 2.87247$

22. With $h = 0.1$: $y(2) \approx 7.31578$
With $h = 0.02$: $y(2) \approx 7.32841$
With $h = 0.004$: $y(2) \approx 7.32916$
With $h = 0.0008$: $y(2) \approx 7.32920$

23. With $h = 0.1$: $y(1) \approx 1.22967$
With $h = 0.02$: $y(1) \approx 1.23069$
With $h = 0.004$: $y(1) \approx 1.23073$
With $h = 0.0008$: $y(1) \approx 1.23073$

24. With $h = 0.1$: $y(1) \approx 1.00006$
With $h = 0.02$: $y(1) \approx 1.00000$
With $h = 0.004$: $y(1) \approx 1.00000$
With $h = 0.0008$: $y(1) \approx 1.00000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0$, $v_0 = 0$.

With $h = 0.01$, 100 iterations of

$$k_1 = f(t, v_n), \quad k_2 = f(t+h, v_n + hk_1), \quad v_{n+1} = v_n + \frac{h}{2}(k_1 + k_2)$$

yield $v(1) \approx 15.9618$, and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.9620$. Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01$, 200 iterations yield $v(2) \approx 19.1846$, and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1847$. Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25$.

With $h = 1$, 60 iterations of

$$k_1 = f(t, P_n), \qquad k_2 = f(t+h, P_n + hk_1), \qquad P_{n+1} = P_n + \frac{h}{2}(k_1 + k_2)$$

yield $P(60) \approx 49.3909$, and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3913$. Thus we observe an approximate population of 49.391 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1$, 120 iterations yield $P(120) \approx 66.1129$, and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1134$. Thus we observe an approximate population of 66.113 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.0045$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	20	200	2000	20000
$y(2)$	1.01087	1.00452	1.00445	1.00445

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.4633$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	40	400	4000	40000
$y(2)$	1.46620	1.46335	1.46332	1.46331

In the solutions for Problems 29 and 30 we illustrate the following general MATLAB ode solver.

```
function  [t,y] = ode(method, yp, t0,b, y0, n)
%  [t,y] = ode(method, yp, t0,b, y0, n)
%  calls the method described by 'method' for the
```

```
%  ODE 'yp' with function header
%
%             y' = yp(t,y)
%
%  on the interval  [t0,b]  with initial (column)
%  vector  y0.  Choices for method are 'euler',
%  'impeuler', 'rk' (Runge-Kutta), 'ode23', 'ode45'.
%  Results are saved at the endPoints of n subintervals,
%  that is, in steps of length  h = (b - t0)/n.  The
%  result  t  is an (n+1)-column vector from b to t1,
%  while  y  is a matrix with  n+1  rows (one for each
%  t-value) and one column for each dependent variable.

h = (b - t0)/n;             % step size
t = t0 : h : b;
t = t';                          % col. vector of t-values
y = y0';                    % 1st row of result matrix
for  i = 2 : n+1            % for i=2 to i=n+1
   t0 = t(i-1);             % old t
   t1 = t(i);                    % new t
   y0 = y(i-1,:)';               % old y-row-vector
   [T,Y] = feval(method, yp, t0,t1, y0);
   y = [y;Y'];                   % adjoin new y-row-vector
end
```

To use the improved Euler method, we call as **'method'** the following function.

```
function [t,y] = impeuler(yp, t0,t1, y0)
%
%  [t,y] = impeuler(yp, t0,t1, y0)
%  Takes one improved Euler step for
%
%       y' = yprime( t,y ),
%
%  from t0  to  t1  with initial value  the
%  column vector  y0.

h = t1 - t0;
k1 = feval( yp, t0, y0        );
k2 = feval( yp, t1, y0 + h*k1 );
k  = (k1 + k2)/2;
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 50;
[t1,v1] = ode('impeuler','vpbolt1',0,10,49,n);
n = 100;
[t2,v2] = ode('impeuler','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:5:51), v2(1:10:101), ve]
```

generate the table

t	with $n = 50$	with $n = 100$	actual *v*
0	49.0000	49.0000	49.0000
1	37.4722	37.4721	37.4721
2	26.3964	26.3963	26.3962
3	15.7549	15.7547	15.7546
4	5.5307	5.5304	5.5303
5	-4.2926	-4.2930	-4.2932
6	-13.7308	-13.7313	-13.7314
7	-22.7989	-22.7994	-22.7996
8	-31.5115	-31.5120	-31.5122
9	-39.8824	-39.8830	-39.8832
10	-47.9251	-47.9257	-47.9259

We notice first that the final two columns agree to 3 decimal places (each difference being less than 0.0005). Scanning the $n = 100$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.5$ and $t = 4.6$ in the table

```
[t2(40:51),v2(40:51)]

    3.9000    6.5345
    4.0000    5.5304
    4.1000    4.5303
    4.2000    3.5341
    4.3000    2.5420
    4.4000    1.5538
    4.5000    0.5696
    4.6000   -0.4108
    4.7000   -1.3872
    4.8000   -2.3597
    4.9000   -3.3283
    5.0000   -4.2930
```

indicates that $t = 4.56$ at the bolt's apex. Finally, interpolation in

```
[t2(95:96),v2(95:96)]

    9.4000  -43.1387
    9.5000  -43.9445
```

gives the impact velocity $v(9.41) \approx -43.22$ m/s.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('impeuler','vpbolt2',0,10,49,n);
n = 200;
[t2,v2] = ode('impeuler','vpbolt2',0,10,49,n);
t = (0:10)';
[t, v1(1:10:101), v2(1:20:201)]
```

generate the table

t	with $n = 100$	with $n = 200$
0	49.0000	49.0000
1	37.1547	37.1547
2	26.2428	26.2429
3.	15.9453	15.9455
4	6.0041	6.0044
5	-3.8020	-3.8016
6	-13.5105	-13.5102
7	-22.9356	-22.9355
8	-31.8984	-31.8985
9	-40.2557	-40.2559
10	-47.9066	-47.9070

We notice first that the final two columns agree to 2 decimal places (each difference being less than 0.005). Scanning the $n = 200$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.60$ and $t = 4.65$ in the table

```
[t2(91:101),v2(91:101)]

    4.5000    1.0964
    4.5500    0.6063
    4.6000    0.1163
    4.6500   -0.3737
    4.7000   -0.8636
    4.7500   -1.3536
    4.8000   -1.8434
    4.8500   -2.3332
    4.9000   -2.8228
```

```
    4.9500   -3.3123
    5.0000   -3.8016
```

indicates that $t = 4.61$ at the bolt's apex. Finally, interpolation in

```
[t2(189:190),v2(189:190)]

    9.4000  -43.4052
    9.4500  -43.7907
```

gives the impact velocity $v(9.41) \approx -43.48$ m/s.

SECTION 2.6

THE RUNGE-KUTTA METHOD

Each problem can be solved with a "template" of computations like those listed in Problem 1. We include a table showing the slope values k_1, k_2, k_3, k_4 and the *xy*-values at the ends of two successive steps of size $h = 0.25$.

1. To make the first step of size $h = 0.25$ we start with the function defined by

```
f[x_, y_] := -y
```

and the initial values

```
x = 0;     y = 2;    h = 0.25;
```

and then perform the calculations

```
k1 = f[x, y]
k2 = f[x + h/2, y + h*k1/2]
k3 = f[x + h/2, y + h*k2/2]
k4 = f[x + h, y + h*k3]
y  = y + (h/6)*(k1 + 2*k2 + 2*k3 + k4)
x  = x + h
```

in turn. Here we are using Mathematica notation that translates transparently to standard mathematical notation describing the corresponding manual computations. A repetition of this same block of calculations carries out a second step of size $h = 0.25$. The following table lists the intermediate and final results obtained in these two steps.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–2	–1/75	–1.78125	–1.55469	0.25	1.55762	1.55760
–1.55762	–1.36292	–1.38725	–1.2108	0.5	1.21309	1.21306

2.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	1.25	1.3125	1.65625	0.25	0.82422	0.82436
1.64844	2.06055	2.16357	2.73022	0.5	1.35867	1.35914

3.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
2	2.25	2.28125	2.57031	0.25	1.56803	1.56805
2.56803	2.88904	2.92916	3.30032	0.5	2.29740	2.29744

4.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–1	–0.75	–0.78128	–55469	0.25	0.80762	0.80760
–0.55762	–0.36292	–0.38725	–0.21080	0.5	0.71309	0.71306

5.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.125	–0.14063	–0.28516	0.25	0.96598	0.96597
–28402	–0.44452	–0.46458	–0.65016	0.5	0.85130	0.85128

6.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.5	–0.48438	–0.93945	0.25	1.87882	1.87883
–0.93941	–1.32105	–1.28527	–1.55751	0.5	1.55759	1.55760

7.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.14063	–0.13980	–0.55595	0.25	2.95347	2.95349
–0.55378	–1.21679	–1.18183	–1.99351	0.5	2.6475	2.64749

8.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	0.88250	0.89556	0.79940	0.25	0.22315	0.22314
0.80000	0.72387	0.73079	0.66641	0.5	0.40547	0.40547

9.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0.5	0.53223	0.53437	0.57126	0.25	1.13352	1.13352
0.57122	0.61296	0.61611	0.66444	0.5	1.28743	1.28743

10.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	0.25	0.26587	0.56868	0.25	1.06668	1.06667
0.56891	0.97094	1.05860	1.77245	0.5	1.33337	1.33333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 2.6, Problems 11-16
x0 = 0;  y0 = 1;

% first run:
h = 0.2;
x = x0;  y = y0;  y1 = y0;
for  n = 1:5
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y1 = [y1,y];
   x = x + h;
   end

% second run:
h = 0.1;
x = x0;  y = y0;  y2 = y0;
for  n = 1:10
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
y2 = y2(1:2:11);
err = 100*(ye-y2)./ye;
x = sprintf('%10.6f',x), sprintf('\n');
y1 = sprintf('%10.6f',y1), sprintf('\n');
y2 = sprintf('%10.6f',y2), sprintf('\n');
ye = sprintf('%10.6f',ye), sprintf('\n');
err = sprintf('%10.6f',err), sprintf('\n');
table = [x;y1;y2;ye;err]
```

For each problem the differential equation $y' = f(x, y)$ and the known exact solution $y = g(x)$ are stored in the files **f.m** and **g.m** — for instance, the files

```
function  yp = f(x,y)
yp = y-2;
```

and

```
function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11.

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	1.000000	0.778600	0.508182	0.177894	–0.225521	–0.718251
y (h=0.1)	1.000000	0.778597	0.508176	0.177882	–0.225540	–0.718280
y actual	1.000000	0.778597	0.508175	0.177881	–0.225541	–0.718282
error	0.00000%	–0.00002%	–0.00009%	–0.00047%	–0.00061%	–0.00029%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	2.000000	2.111110	2.249998	2.428566	2.666653	2.999963
y (h=0.1)	2.000000	2.111111	2.250000	2.428571	2.666666	2.999998
y actual	2.000000	2.111111	2.250000	2.428571	2.666667	3.000000
error	0.000000%	0.000002%	0.000006%	0.000014%	0.000032%	0.000080%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	3.000000	3.173896	3.441170	3.814932	4.300904	4.899004
y (h=0.1)	3.000000	3.173894	3.441163	3.814919	4.300885	4.898981
y actual	3.000000	3.173894	3.441163	3.814918	4.300884	4.898979
error	0.00000%	–0.00001%	–0.00001%	–0.00002%	–0.00003%	–0.00003%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	1.000000	1.222957	1.507040	1.886667	2.425586	3.257946
y (h=0.1)	1.000000	1.222973	1.507092	1.886795	2.425903	3.258821
y actual	1.000000	1.222975	1.507096	1.886805	2.425928	3.258891
error	0.0000%	0.0001%	0.0003%	0.0005%	0.0010%	0.0021%

15.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444447
y (h=0.1)	3.000000	3.026446	3.094445	3.191716	3.310204	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.000000%	–0.000004%	–0.000005%	–0.000005%	–0.000005%	–0.000004%

16.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	4.243067	5.361409	6.478634	7.634049	8.845150
y (h=0.1)	3.000000	4.242879	5.361308	6.478559	7.633983	8.845089
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.000000%	–0.000221%	–0.000094%	–0.000061%	–0.000047%	–0.000039%

17. With $h = 0.2$: $y(1) \approx 0.350258$
With $h = 0.1$: $y(1) \approx 0.350234$
With $h = 0.05$: $y(1) \approx 0.350232$
With $h = 0.025$: $y(1) \approx 0.350232$

The table of numerical results is

x	y with $h = 0.2$	y with $h = 0.1$	y with $h = 0.05$	y with $h = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002667	0.002667	0.002667	0.002667
0.4	0.021360	0.021359	0.021359	0.021359
0.6	0.072451	0.072448	0.072448	0.072448
0.8	0.174090	0.174081	0.174080	0.174080
1.0	0.350258	0.350234	0.350232	0.350232

In Problems 18–24 we give only the final approximate values of y obtained using the Runge-Kutta method with step sizes $h = 0.2$, $h = 0.1$, $h = 0.05$, and $h = 0.025$.

18. With $h = 0.2$: $y(2) \approx 1.679513$
With $h = 0.1$: $y(2) \approx 1.679461$
With $h = 0.05$: $y(2) \approx 1.679459$
With $h = 0.025$: $y(2) \approx 1.679459$

19. With $h = 0.2$: $y(2) \approx 6.411464$
With $h = 0.1$: $y(2) \approx 6.411474$
With $h = 0.05$: $y(2) \approx 6.411474$
With $h = 0.025$: $y(2) \approx 6.411474$

20. With $h = 0.2$: $y(2) \approx -1.259990$
With $h = 0.1$: $y(2) \approx -1.259992$
With $h = 0.05$: $y(2) \approx -1.259993$
With $h = 0.025$: $y(2) \approx -1.259993$

21. With $h = 0.2$: $y(2) \approx 2.872467$
With $h = 0.1$: $y(2) \approx 2.872468$

With $h = 0.05$: $y(2) \approx 2.872468$
With $h = 0.025$: $y(2) \approx 2.872468$

22. With $h = 0.2$: $y(2) \approx 7.326761$
With $h = 0.1$: $y(2) \approx 7.328452$
With $h = 0.05$: $y(2) \approx 7.328971$
With $h = 0.025$: $y(2) \approx 7.329134$

23. With $h = 0.2$: $y(1) \approx 1.230725$
With $h = 0.1$: $y(1) \approx 1.230731$
With $h = 0.05$: $y(1) \approx 1.230731$
With $h = 0.025$: $y(1) \approx 1.230731$

24. With $h = 0.2$: $y(1) \approx 1.000000$
With $h = 0.1$: $y(1) \approx 1.000000$
With $h = 0.05$: $y(1) \approx 1.000000$
With $h = 0.025$: $y(1) \approx 1.000000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0,\ v_0 = 0.$

With $h = 0.1,$ 10 iterations of

$$k_1 = f(t_n, v_n), \qquad k_2 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_1),$$
$$k_3 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), \qquad k_4 = f(t_n + h, v_n + hk_3),$$
$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad v_{n+1} = v_n + hk$$

yield $v(1) \approx 15.9620,$ and 20 iterations with $h = 0.05$ yield $v(1) \approx 15.9621.$ Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.1,$ 20 iterations yield $v(2) \approx 19.1847,$ and 40 iterations with $h = 0.05$ yield $v(2) \approx 19.1848.$ Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25.$

With $h = 6,$ 10 iterations of

$$k_1 = f(t_n, P_n), \qquad k_2 = f(t_n + \tfrac{1}{2}h, P_n + \tfrac{1}{2}hk_1),$$
$$k_3 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), \qquad k_4 = f(t_n + h, P_n + hk_3),$$
$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad P_{n+1} = P_n + hk$$

yield $P(60) \approx 49.3915,$ as do 20 iterations with $h = 3.$ Thus we observe an approximate population of 49.3915 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 6$, 20 iterations yield $P(120) \approx 66.1136$, as do 40 iterations with $h = 3$. Thus we observe an approximate population of 66.1136 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x, y) = x^2 + y^2 - 1$ and $x_0 = 0$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00445$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	2	20	200	2000
$y(2)$	1.05722	1.00447	1.00445	1.00445

28. Here $f(x, y) = x + \frac{1}{2}y^2$ and $x_0 = -2$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46331$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	4	40	00	40000
$y(2)$	1.48990	1.46332	1.46331	1.46331

In the solutions for Problems 29 and 30 we use the general MATLAB solver **ode** that was listed prior to the Problem 29 solution in Section 2.5. To use the Runge-Kutta method, we call as **'method'** the following function.

```
function [t,y] = rk(yp, t0,t1, y0)

%  [t, y] = rk(yp, t0, t1, y0)
%  Takes one Runge-Kutta step for
%
%       y' = yp( t,y ),
%
%  from t0  to  t1  with initial value  the
%  column vector  y0.

h = t1 - t0;
k1 = feval(yp, t0      , y0             );
k2 = feval(yp, t0 + h/2, y0 + (h/2)*k1 );
k3 = feval(yp, t0 + h/2, y0 + (h/2)*k2 );
k4 = feval(yp, t0 + h   ,y0 +    h *k3 );
k  = (1/6)*(k1 + 2*k2 + 2*k3 + k4);
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('rk','vpbolt1',0,10,49,n);
n = 200;
[t2,v] = ode('rk','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 100$	with $n = 200$	actual v
0	49.0000	49.0000	49.0000
1	37.4721	37.4721	37.4721
2	26.3962	26.3962	26.3962
3	15.7546	15.7546	15.7546
4	5.5303	5.5303	5.5303
5	-4.2932	-4.2932	-4.2932
6	-13.7314	-13.7314	-13.7314
7	-22.7996	-22.7996	-22.7996
8	-31.5122	-31.5122	-31.5122
9	-39.8832	-39.8832	-39.8832
10	-47.9259	-47.9259	-47.9259

We notice first that the final three columns agree to the 4 displayed decimal places. Scanning the last column for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.55$ and $t = 4.60$ in the table

```
[t2(91:95),v(91:95)]

   4.5000    0.5694
   4.5500    0.0788
   4.6000   -0.4109
   4.6500   -0.8996
   4.7000   -1.3873
```

indicates that $t = 4.56$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
```

```
for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h +
                  0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = 7350*(1 - exp(-t/25)) - 245*t;
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0
1	43.1974	43.1976
2	75.0945	75.0949
3	96.1342	96.1348
4	106.7424	106.7432
5	107.3281	107.3290
6	98.2842	98.2852
7	79.9883	79.9895
8	52.8032	52.8046
9	17.0775	17.0790
10	-26.8540	-26.8523

We see at least 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.40$ and $t = 9.45$ in the table

```
[t2(187:191),y(187:191)]

   9.3000    4.7448
   9.3500    2.6182
   9.4000    0.4713
   9.4500   -1.6957
   9.5000   -3.8829
```

indicates that the bolt is aloft for about 9.41 seconds.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 200;
[t1,v1] = ode('rk','vpbolt2',0,10,49,n);
n = 2*n;
[t2,v] = ode('rk','vpbolt2',0,10,49,n);
```

```
t = (0:10)';
ve = zeros(size(t));
ve(1:5)= 94.388*tan(0.478837 - 0.103827*t(1:5));
ve(6:11)= -94.388*tanh(0.103827*(t(6:11)-4.6119));

[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 200$	with $n = 400$	actual v
0	49.0000	49.0000	49.0000
1	37.1548	37.1548	37.1547
2	26.2430	26.2430	26.2429
3	15.9456	15.9456	15.9455
4	6.0046	6.0046	6.0045
5	-3.8015	-3.8015	-3.8013
6	13.5101	-13.5101	-13.5100
7	-22.9354	-22.9354	-22.9353
8	-31.8985	-31.8985	-31.8984
9	-40.2559	-40.2559	-40.2559
10	-47.9071	-47.9071	-47.9071

We notice first that the final three columns almost agree to the 4 displayed decimal places. Scanning the last colmun for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.600$ and $t = 4.625$ in the table

```
[t2(185:189),v(185:189)]

   4.6000    0.1165
   4.6250   -0.1285
   4.6500   -0.3735
   4.6750   -0.6185
   4.7000   -0.8635
```

indicates that $t = 4.61$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h + 0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = zeros(size(t));
ye(1:5)= 108.465+909.091*log(cos(0.478837 -
0.103827*t(1:5)));
ye(6:11)= 108.465-909.091*log(cosh(0.103827
                      *(t(6:11)-4.6119)));
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0.0001
1	42.9881	42.9841
2	74.6217	74.6197
3	95.6719	95.6742
4	106.6232	106.6292
5	107.7206	107.7272
6	99.0526	99.0560
7	80.8027	80.8018
8	53.3439	53.3398
9	17.2113	17.2072
10	-26.9369	-26.9363

We see almost 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.400$ and $t = 9.425$ in the table

```
[t2(377:381),y(377:381)]

   9.4000    0.4740
   9.4250   -0.6137
   9.4500   -1.7062
   9.4750   -2.8035
   9.5000   -3.9055
```

indicates that the bolt is aloft for about 9.41 seconds.