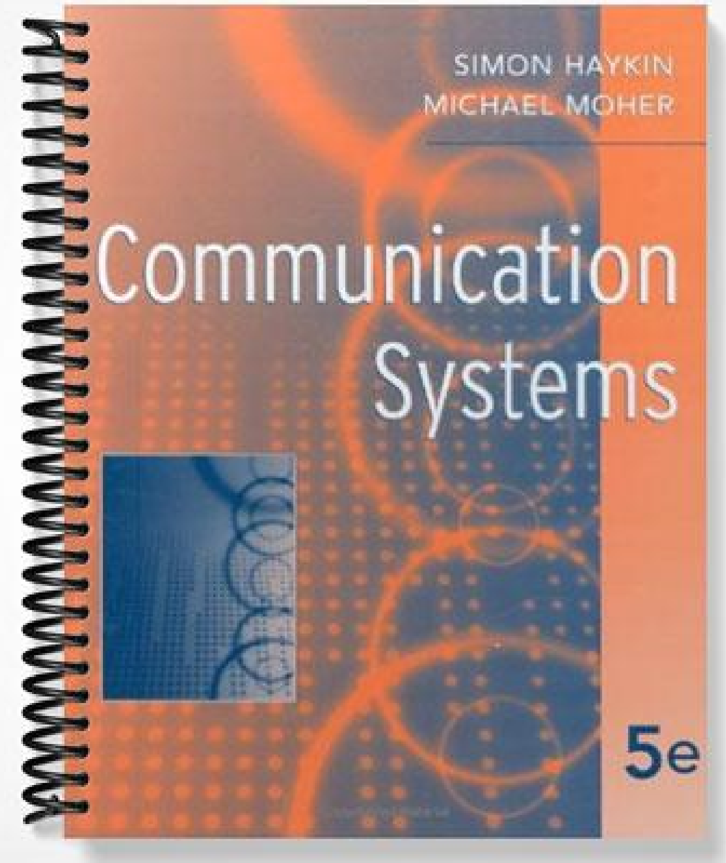
SOLUTIONS MANUAL



DRILL PROBLEMS

Problem 2.1

Evaluate the Fourier transform of the damped sinusoidal wave $g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$ where $u(t)$ is the unit step function

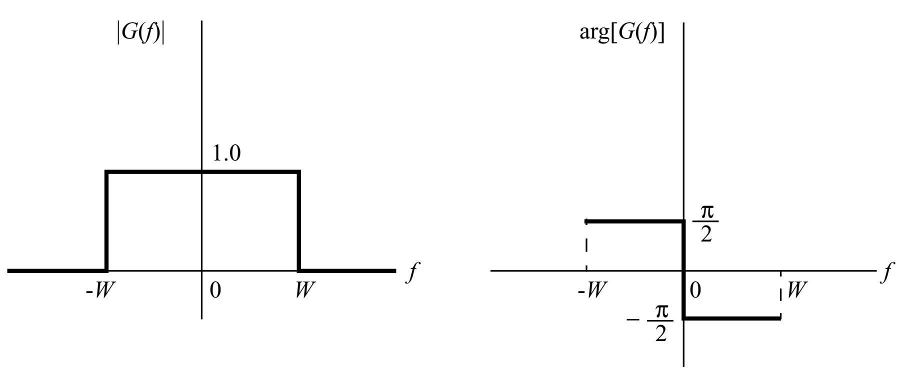
Solution

The Fourier transform of $g(t)$ is

$$
\begin{aligned}
G(f) &= \int_0^{\infty} \exp(-t)\sin(2\pi f_c t)\sin(-j2\pi f_c t)dt \\
&= \frac{1}{2j}\int_0^{\infty} \exp(-t)[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]\exp(-j2\pi f t)dt \\
&= \frac{1}{2j}\int_0^{\infty} [\exp(j2\pi(f_c - f)t - t)]dt \\
&= \frac{1}{2j}\left[\frac{1}{j2\pi(f_c - f) - 1}\exp(j2\pi(f_c\text{-}f)t\text{-}t) + \frac{1}{j2\pi(f_c - f) + 1}\exp((-j2\pi(f_c+f)t)\text{-}t)\right]_{t=0}^{\infty} \\
&= \frac{1}{2j}\left(\frac{1}{j2\pi(f_c - f) - 1} + \frac{1}{j2\pi(f_c - f) + 1}\right) \\
&= \frac{1}{2j}\left(\frac{(j2\pi(f_c - f) + 1) + (j2\pi(f_c - f) - 1)}{1 + 4\pi^2(f_c - f)^2}\right) \\
&= \frac{2\pi f_c}{1 + 4\pi^2(f - f_c)^2}
\end{aligned}
$$

Problem 2.2

Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. 2.5.



Solution

$$g(t) = \int_{-W}^{0} e^{j\pi/2} \cdot e^{j2\pi ft} df + \int_{0}^{W} e^{-j\pi/2} e^{j2\pi ft} df$$

$$= \left[\frac{1}{j2\pi t} e^{j\left(\frac{\pi}{2} + 2\pi ft\right)}\right]_{f=-W}^{0} + \left[\frac{1}{j2\pi t} e^{j\left(-\frac{\pi}{2} + 2\pi ft\right)}\right]_{f=0}^{W}$$

$$= \frac{1}{j2\pi t}\left(e^{j\left(\frac{\pi}{2}-2\pi Wt\right)} - e^{j\pi/2}\right) + \frac{1}{j2\pi t}\left(e^{-j\pi/2} - e^{j\left(-\frac{\pi}{2}-j2\pi Wt\right)}\right)$$

$$= \frac{1}{j2\pi t}(e^{-j\pi/2} - e^{j\pi/2}) + \frac{1}{j2\pi t} e^{-j2\pi Wt}(e^{j\pi/2} - e^{-j\pi/2})$$

$$= -\frac{1}{\pi t} + \frac{1}{\pi t} e^{-j2\pi Wt} = \frac{1}{\pi t}(e^{-j2\pi Wt} - 1)$$

Note: If we let $W \to \infty$, $G(f) \to j\,\mathrm{sgn}(t)$, the inverse of which $-\frac{1}{\pi t}$. This result agrees with the limiting value of the solution for $W = \infty$.

Problem 2.3

Suppose $g(t)$ is real valued with a complex-valued Fourier transform $G(f)$. Explain how the rule of Eq. (2.31) can be satisfied by such a signal.

Solution

With $G(f)$ being complex valued, we may express it as

$$G(f) = G_r(f) + jG_i(f)$$

where $G_r(f)$ is the real part of $G(f)$ and $G_i(f)$ is its imaginary part. Hence,

$$G(0) = G_r(0) + jG_i(0).$$

According to Eq. (2.31) in the text,

$$\int_{-\infty}^{\infty} g(t)dt = G_r(0) + jG_i(0)$$

With $g(t)$ being real valued, this condition can only be satisfied if the imaginary part $G_i(0)$ is zero.

Problem 2.4

Continuing with Problem 2.3, explain how the rule of Eq. (2.32) can be satisfied by the signal $g(t)$ described therein.

Solution

Since $g(t)$ is real valued, it follows that the integral $\int_{-\infty}^{\infty} G(f)df$ must likewise be real valued. For this condition to be satisfied, the imaginary part of $G(f)$ must be an odd function of f.

Problem 2.5

Develop the detailed steps that show that the modulation and convolution theorems are indeed the dual of each other.

Solution

The modulation theorem states that

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda \tag{1}$$

To apply the duality theorem, we say that if

$g_1(t)g_2(t) \rightleftharpoons X(f)$, then

$X(f) \rightleftharpoons g_1(-f)g_2(-f)$

For the problem at hand, we may therefore write

$$\int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda \rightleftharpoons g_1(-f)g_2(-f) \tag{2}$$

Next, we apply Eq. (2.21), which states that if $g(t) \rightleftharpoons G(f)$ then $g(-t) \rightleftharpoons G(-f)$. Hence, applying this rule to Eq. (2), we may write

$$\int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda-t)d\lambda \rightleftharpoons g_1(f)g_2(f)$$

which is a statement of the convolution theorem, with $G_1(t) \rightleftharpoons g_1(f)$ and $G_2(t) \rightleftharpoons g_2(f)$.

Problem 2.6

Develop the detailed steps involved in deriving Eq. (2.53), starting from Eq. (2.51).

Solution

According to Eq. (2.51),

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$$

According to Eq. (2.21), if $g(t) \rightleftharpoons G(f)$, then $g(-t) \rightleftharpoons G(-f)$. Hence, applying this rule to the problem at hand, we may write

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(\tau-t)d\tau \rightleftharpoons G_1(f)G_2(-f)$$

Next, we note that if we complex conjugate the term $g_2(\tau - t)$, then the conjugation theorem of Eq. (2.22) teaches us that

$$\int_{-\infty}^{\infty} g_1(\tau)g_2^*(\tau-t)d\tau \rightleftharpoons G_1(f)G_2^*(-f)$$

which is the desired result, except for the fact that we have interchanged the roles of variables t and τ.

Problem 2.7

Prove the following properties of the convolution process:

(a) The commutative property:

$$g_1(t)\bigstar g_2(t) = g_2(t)\bigstar g_1(t)$$

Proof:

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} g_2(t-\tau)g_1(\tau)d\tau$$

Replace t - τ with λ. That is, τ = t - λ. Hence

$$g_1(t)\bigstar g_2(t) = -\int_{+\infty}^{-\infty} g_2(\lambda)g_1(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} g_2(\lambda)g_1(t-\lambda)d\lambda$$

$$= g_2(t)\bigstar g_1(t)$$

(b) The associative property:

$$g_1(t)\bigstar[g_2(t)\bigstar g_3(t)] = [g_1(t)\bigstar g_2(t)]\bigstar g_3(t)$$

Proof:

.

Let

$$x(t) = g_2(t)\bigstar g_3(t)$$

$$= \int_{-\infty}^{\infty} g_2(\tau)g_3(t-\tau)d\tau$$

Hence

$$I(t) = g_1(t)\bigstar \underbrace{[g_2(t)\bigstar g_3(t)]} = \int_{-\infty}^{\infty} g_1(\lambda)x(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\tau)g_3(t-\tau-\lambda)d\tau d\lambda \qquad (1)$$

Replace τ + λ with μ; that is, τ = μ - λ. Hence, keeping λ fixed, we may write

$$I(t) = \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\mu-\lambda)g_3(t-\mu)d\mu d\lambda \qquad (2)$$

With μ fixed, the integral $\int_{-\infty}^{\infty} g_1(\lambda)g_2(\mu-\lambda)d\lambda$ is recognized as the convolution of $g_1(\mu)$ and $g_2(\mu)$, as shown by

$$g_{12}(\mu) = \int_{-\infty}^{\infty} g_1(\lambda)g_2(\mu-\lambda)d\lambda = g_1(\mu)\bigstar g_2(\mu)$$

We may therefore rewrite Eq. (1) as

$$I(t) = \int_{-\infty}^{\infty} g_{12}(\mu) g_3(t-\mu) d\mu = g_{12}(t) \star g_3(t) = [g_1(t) \star g_2(t)] \star g_3(t)$$

(c) The distributive property:

$$g_1(t) \star [g_2(t) + g_3(t)] = g_1(t) \star g_2(t) + g_1(t) \star g_3(t)$$

Proof:

$$\begin{aligned} g_1(t) \star [g_2(t) + g_3(t)] &= \int_{-\infty}^{\infty} g_1(\tau)[g_2(t-\tau) + g_3(t-\tau)] d\tau \\ &= \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau + \int_{-\infty}^{\infty} g_1(\tau) g_3(t-\tau) d\tau \\ &= g_1(t) \star g_2(t) + g_1(t) \star g_3(t) \end{aligned}$$

Problem 2.8

Considering the sinc pulse sinc(t), show that

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2(t) dt = 1$$

Solution

This integral may be viewed as

$$I = \int_{-\infty}^{\infty} \mathrm{sinc}(t) \cdot \mathrm{sinc}(t) dt$$

which, in light of Rayleigh's energy theorem, may also be expressed as

$$I = \int_{-\infty}^{\infty} |\mathbf{F}[\mathrm{sinc}(t)]|^2 df$$

From Eq. (2.25) in the text, we have

$\mathbf{F}[\mathrm{sinc}t] = \mathrm{rect}(f)$

Hence,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{rect}^2(f) df \\ &= \int_{-1/2}^{1/2} 1^2 df \\ &= 1 \end{aligned}$$

Problem 2.9

Determine the Fourier transform of the squared sinusoidal signals:

(i) $g(t) = \cos^2(2\pi f_c t)$

(ii) $g(t) = \sin^2(2\pi f_c t)$

Solution

(i) Using the trigonometric identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

we may express $g(t)$ as

$$g(t) = \frac{1}{2}(1 + \cos 4\pi f_c t)$$

Hence,

$$G(f) = \frac{1}{2}\delta(f) + \frac{1}{4}\delta(f - 2f_c) + \frac{1}{4}\delta(f + 2f_c)$$

(ii) Next, using the trigonometric identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

we may write

$$\sin^2(2\pi f_c t) = \frac{1}{2}(1 - \cos 4\pi f_c t)$$

Hence,

$$G(f) = \frac{1}{2}\delta(f) - \frac{1}{4}\delta(f - f_c) - \frac{1}{4}\delta(f + f_c)$$

Problem 2.10

Consider the function

$$g(t) = \delta\left(t + \frac{1}{2}\right) - \delta\left(t - \frac{1}{2}\right)$$

which consists of two delta functions at $t = \pm\frac{1}{2}$. The integration of $g(t)$ with respect to time t yields the unit rectangular function rect(t). Using Eq. (2.79), show that

$$\text{rect}(t) \rightleftharpoons \text{sinc}(f)$$

Solution

To begin, consider the transform pair

$$\delta(t) \rightleftharpoons 1$$

Hence, the Fourier transform of $g(t)$ is

$$G(f) = \exp(j\pi f t) - \exp(-j\pi f t)$$

from which we readily deduce that $G(0)$. Hence, applying Eq. (2.79) in the text yields

$$\mathbf{F}[\text{rect}(t)] = \frac{1}{j2\pi f}[\exp(j\pi f) - \exp(-j\pi f)]$$

$$= \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$$

where we have used the identity

$$\sin(\pi f) = \frac{1}{2j}(e^{j\pi f} - e^{-j\pi f})$$

Problem 2.11

Using the Euler formula

$$\cos x = \frac{1}{2}\exp[(jx) + \exp(-jx)]$$

reformulate Eqs. (2.91) and (2.92) in terms of cosinusoidal functions.

Solution

$$\sum_{n=-\infty}^{\infty} \exp(j2\pi n f_0 t) = \sum_{n=1}^{\infty} \exp(j2\pi n f_0 t) + 1 + \sum_{n=-\infty}^{-1} \exp(j2\pi n f_0 t)$$

$$= 1 + \sum_{n=1}^{\infty} [\exp(j2\pi n f_0 t) + \exp(-j2\pi n f_0 t)]$$

$$= 1 + 2\sum_{n=1}^{\infty} \cos(2\pi n f_0 t)$$

We may therefore reformulate Eq. (2.91) as

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = f_0 + 2f_0 \sum_{n=1}^{\infty} \cos(2\pi m f_0 t)$$

where $f_0 = 1/T$.

Similarly, we may write

$$\sum_{n=-\infty}^{\infty} \cos(j2\pi m f_0 t) = 1 + 2\sum_{m=1}^{\infty} \cos(2\pi m f_0 t)$$

Hence, we may reformulate Eq. (2.92) as

$$1 + 2\sum_{m=1}^{\infty} \cos(2\pi m f_0 t) = T_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$

Problem 2.12

Discuss the following two issues, citing examples for your answers:

(a) Is it possible for a linear time-invariant system to be causal but unstable?

(b) Is it possible for such a system to be noncausal but stable?

Solution

(a) It is possible for a system to be causal but unstable. Causality means that the impulse response of the system $h(t)$ must be zero for negative t. Instability means that the BIBO criterion

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

is violated. Such a system could be represented by the impulse response

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \exp(t) & \text{for } t > 0 \end{cases}$$

(b) By the same token, it is possible for the system to be stable but noncausal. In this second case, we may cite the impulse response

$$h(t) = \begin{cases} \exp(t) & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases}$$

What does Problem 2.12 teach us?

The problem teaches us that the properties of stability and causality are independent.

Problem 2.13

The impulse response of a linear system is defined by the Gaussian function

$$h(t) = \exp\left(-\frac{t^2}{2\tau^2}\right)$$

where the parameter τ is used to adjust duration of the impulse response. Determine the frequency response of the system.

Solution

From Eq. (2.40) in the text, recall that

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2)$$

Next, from the dilation property of the Fourier transform described in Eq. (2.20), recall that if $h(t) \rightleftharpoons H(f)$, then

$$h(at) \rightleftharpoons \frac{1}{|a|}H\left(\frac{f}{a}\right)$$

where a is the dilation parameter. For the problem at hand, we have

$$a = \sqrt{\frac{1}{2\pi}}\ \frac{1}{\tau}$$

Accordingly, the frequency response of the system is

$$H(f) = \sqrt{2\pi}\ \tau\exp(-2\pi^2\tau^2 f^2)$$

Problem 2.14

A tapped-delay-line filter consists of N weights, where N is odd. It is symmetric with respect to the center tap, that is, the weights satisfy the condition

$$w_n = w_{N-1-n}, \qquad 0 \le n \le N-1$$

(a) Find the amplitude response of the filter.

(b) Show that this filter has a linear phase response. What is the implication of this property?

Solution

The impulse response of the filter is

$$h(t) = \sum_{n=0}^{N-1} w_n \delta(t - n\Delta\tau)$$

Hence, the frequency response of the filter is

$$H(f) = \sum_{n=0}^{N-1} w_n \exp(-j2\pi n f \Delta\tau)$$

To illustrate, consider the example of $N = 5$. Then

$$\begin{aligned} H(f) &= w_0 + w_1 \exp(-j2\pi f\Delta\tau) + w_2 \exp(-j4\pi f\Delta\tau) + w_3 \exp(-j6\pi f\Delta\tau) + w_4 \exp(-j8\pi f\Delta\tau) \\ &= \exp(-j4\pi f\Delta\tau)[w_0 \exp(j4\pi f\Delta\tau) + w_1 \exp(j2\pi f\Delta\tau) + w_2 + w_3 \exp(-j2\pi f\Delta\tau) \\ &\qquad + w_4 \exp(-j4\pi f\Delta\tau)] \end{aligned} \tag{1}$$

For this example, the symmetry condition

$w_n = w_{N\text{-}1\text{-}n}$ for $0 \le n \le N\text{-}1$

reads as

$w_n = w_{4\text{-}n}$ for $0 \le n \le 4$

Hence, $w_0 = w_4$ and $w_1 = w_3$. Accordingly, we may rewrite Eq. (1) as

$$\begin{aligned} H(f) &= \exp(-j4\pi f\Delta\tau)[w_0 \exp(j4\pi f\Delta\tau) + \exp(-j4\pi f\Delta\tau) \\ &\qquad + w_1(\exp(j2\pi f\Delta\tau)) + \exp(-j2\pi f\Delta\tau) \\ &\qquad + w_2] \\ &= \exp(-j4\pi f\Delta\tau)[2w_0 \cos(4\pi f\Delta\tau) + 2w_1(2\pi f\Delta\tau) + w_2] \end{aligned}$$

We may therefore generalize this result as

$$H(f) = \exp\left(-j2\pi\left(\frac{N\text{-}1}{2}\right)f\Delta\tau\right)\left[w_{\frac{N\text{-}1}{2}} + 2\sum_{n=0}^{\frac{N\text{-}1}{2}-1} w_n \cos(2\pi n f\Delta\tau)\right]$$

(a) The amplitude response of the filter is therefore

$$|H(f)| = w_{\frac{N\text{-}1}{2}} + 2\sum_{n=0}^{\frac{N\text{-}1}{2}-1} w_n \cos(2\pi n f\Delta\tau)$$

(b) The phase response of the filter is therefore

$$\arg(H(f)) = \exp\left(-j2\pi\left(\frac{N\text{-}1}{2}\right)f\Delta\tau\right)$$

which is linear with respect to the frequency f. The implication of this condition is that except for a delay, there is no phase distortion produced by the filter.

Problem 2.15

Derive the relationship of Eq. (2.142) between the two cross-correlation factors $R_{xy}(\tau)$ and $R_{yx}(\tau)$.

Solution

By definition

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)x^*(t-\tau)dt$$

Complex conjugate both sides of the equation:

$$R^*_{yx}(\tau) = \int_{-\infty}^{\infty} x(t-\tau)y^*(t)dt$$

Next, replace τ with $-\tau$:

$$R^*_{yx}(-\tau) = \int_{-\infty}^{\infty} x(t+\tau)y^*(t)dt$$

Finally, replace $t + \tau$ with t, which is equivalent to replacing t with $t - \tau$; we therefore (since dt remains unchanged)

$$R^*_{yx}(-\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)dt = R_{xy}(\tau)$$

Problem 2.16

Consider the decaying exponential pulse

$$g(t) = \begin{cases} \exp(-at) & t > 0 \\ 1, & t = 0 \\ 0, & t < 0 \end{cases}$$

Determine the energy spectral density of the pulse $g(t)$.

Solution

The Fourier transform of $g(t)$ is (see Eq. (2.12) in the text

$$G(f) = \frac{1}{a + j2\pi f}$$

The energy spectral density of the pulse is therefore

$$\begin{aligned} E_g(f) &= |G(f)|^2 \\ &= \frac{1}{a^2 + 4\pi^2 f^2}, \qquad -\infty < f < \infty \end{aligned}$$

Problem 2.17

Repeat Problem 2.16 for the double exponential pulse

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$

Solution

The Fourier transform of $g(t)$ is (see Eq. (2.16))

$$G(f) = \frac{2a}{a^2 + 4\pi^2 f^2}$$

The energy spectral density of the double exponential pulse is

$$E_g(f) = \frac{4a^2}{(a^2 + 4\pi^2 f^2)^2}, \qquad -\infty < f < \infty$$

Problem 2.18

In an implicit sense, Eq. (2.153) embodies *Parseval's power theorem*, which states that for a *periodic signal* $x(t)$ we have

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X(nf_0)|^2$$

where T is the period of the signal, f_0 is the fundamental frequency, and $X(nf_0)$ is the Fourier transform of $x(t)$ evaluated at the frequency nf_0. Prove this theorem.

Solution

Adapting Eq. (2.86) to the problem at hand, we may write

$$x_T(t) = f_0 \sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t) \tag{1}$$

where

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f_0 = \frac{1}{T}$$

and $X(nf_0)$ is the Fourier transform of $g(t)$, evaluated at the frequency $f = nf_0$. Using Eq. (1) to evaluate the integral

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

we write

$$I = f_0\int_{-T/2}^{T/2}\left(\sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t)\right)\left(\sum_{m=-\infty}^{\infty} X^*(mf_0)\exp(-j2\pi m f_0 t)\right)dt$$

$$= f_0 \sum_{n=-\infty}^{\infty} X(nf_0)X^*(mf_0)\int_{-T/2}^{T/2} \exp(j2\pi(n-m)f_0 t)dt \tag{2}$$

To evaluate the integral on the right-hand side of Eq. (2), we write

$$\int_{-T/2}^{T/2} \exp(j2\pi(n-m)f_0 t)dt = \frac{1}{j2\pi(n-m)f_0}\exp(j2\pi(n-m)f_0 t)\Big|_{t=-T/2}^{T/2}$$

$$= \frac{1}{j2\pi(n-m)f_0}[\exp((j\pi(n-m)) - \exp - j\pi(n-m))]$$

$$= \frac{1}{\pi(n-m)f_0}\sin(\pi(n-m)) \tag{3}$$

Whenever the indices *n* and *m* are assigned different integer values, Eq. (3) assumes the value zero. On the other hand, whenever the indices are assigned the same integer value, the integral in Eq. (3) assumes the limiting value

$$\frac{1}{f_0}\lim_{n=m}\frac{\sin(\pi(n-m))}{\pi(n-m)} = \frac{1}{f_0}$$

Accordingly, we may simplify Eq. (3) as

$$\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0t)dt = \begin{cases} \frac{1}{f_0}, & n = m \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Hence, substituting Eq. (4) into (2), we get

$$I = \sum_{n=-\infty}^{\infty}|X(nf_0)|^2 \tag{5}$$

We finally write

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2dt = \sum_{n=-\infty}^{\infty}|X(nf_0)|^2$$

which is the desired result.

ADDITIONAL PROBLEMS

Problem 2.19

(a) The half-cosine pulse $g(t)$ of Fig. 2.40(a) may be considered as the product of the rectangular function rect(t/T) and the sinusoidal wave Acos($\pi t/T$). Since

$$\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\text{sinc}(fT)$$

$$A\cos\left(\frac{\pi t}{T}\right) \rightleftharpoons \frac{A}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]$$

and multiplication in the time domain is transformed into convolution in the frequency domain, it follows that

$$G(f) = [T\,\text{sinc}(fT)]\star\left\{\frac{A}{2}\left[\delta\left(f-\frac{1}{2T}+f+\frac{1}{2T}\right)\right]\right\}$$

where ★ denotes convolution. Therefore, noting that

$$\text{sinc}(fT)\star\delta\left(f-\frac{1}{2T}\right) = \text{sinc}\left[T\left(f-\frac{1}{2T}\right)\right]$$

$$\text{sinc}(fT)\star\delta\left(f+\frac{1}{2T}\right) = \text{sinc}\left[T\left(f+\frac{1}{2T}\right)\right]$$

we obtain the desired result

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]$$

(b) The half-sine pulse of Fig. 2.40(b) may be obtained by shifting the half-cosine pulse to the right by $T/2$ seconds. Since a time shift of $T/2$ seconds is equivalent to multiplication by exp($-j\pi fT$) in the frequency domain, it follows that the Fourier transform of the half-sine pulse is

$$G(f) = \frac{AT}{2}\left[\text{sinc}\left(fT-\frac{1}{2}\right)+\text{sinc}\left(fT+\frac{1}{2}\right)\right]\exp(-j\pi fT)$$

(c) The Fourier transform of a half-sine pulse of duration aT is equal to

$$\frac{|a|AT}{2}\left[\text{sinc}\left(afT-\frac{1}{2}\right)+\text{sinc}\left(afT+\frac{1}{2}\right)\right]\exp(-j\pi faT)$$

(d) The Fourier transform of the negative half-sine pulse shown in Fig. 2.40(c) is obtained from the result by putting a = -1, and multiplying the result by -1, and so we find that its Fourier transform is equal to

$$-\frac{AT}{2}\left[\text{sinc}\left(fT+\frac{1}{2}\right)+\text{sinc}\left(fT-\frac{1}{2}\right)\right]\exp(j\pi fT)$$

(e) The full-sine pulse of Fig. 2.40(d) may be considered as the superposition of the half-sine pulses shown in parts (b) and (c) of the figure. The Fourier transform of this pulse is therefore

$$G(f) = \frac{AT}{2}\left[\mathrm{sinc}\left(fT - \frac{1}{2}\right) + \mathrm{sinc}\left(fT + \frac{1}{2}\right)\right][\exp(-j\pi fT) - \exp(j\pi fT)]$$

$$= -jAT\left[\mathrm{sinc}\left(fT - \frac{1}{2}\right) + \mathrm{sinc}\left(fT + \frac{1}{2}\right)\right]\sin(\pi fT)$$

$$= -jAT\left[\frac{\sin\left(\pi fT - \frac{\pi}{2}\right)}{\pi fT - \frac{\pi}{2}} + \frac{\sin\left(\pi fT + \frac{\pi}{2}\right)}{\pi fT + \frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= -jAT\left[-\frac{\cos(\pi fT)}{\pi fT - \frac{\pi}{2}} + \frac{\cos(\pi fT)}{\pi fT + \frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= jAT\left[\frac{\sin(2\pi fT)}{2\pi fT - \pi} - \frac{\sin(2\pi fT)}{2\pi fT + \pi}\right]$$

$$= jAT\left[-\frac{\sin(2\pi fT - \pi)}{2\pi fT - \pi} + \frac{\sin(2\pi fT + \pi)}{2\pi fT + \pi}\right]$$

$$= jAT[\mathrm{sinc}(2fT + 1) - \mathrm{sinc}(2fT - 1)]$$

Problem 2.20

(a) The even part $g_e(t)$ of a pulse $g(t)$ is given by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

Therefore, for $g(t) = A\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$ we obtain

$$g_e(t) = \frac{A}{2}\left[\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right) + \mathrm{rect}\left(-\frac{t}{T} - \frac{1}{2}\right)\right]$$

$$= \frac{A}{2}\left[\mathrm{rect}\left(\frac{t}{2T}\right)\right]$$

which is shown illustrated in Fig. 1:

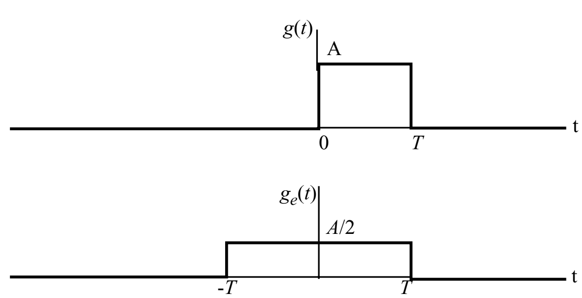


Figure 1

The odd part of $g(t)$ is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

$$= \frac{A}{2}\left[\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right) - \mathrm{rect}\left(-\frac{t}{T} - \frac{1}{2}\right)\right]$$

which is illustrated in Fig. 2:

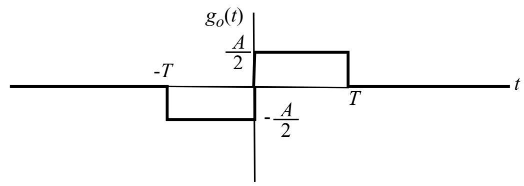


Figure 2

(b) The Fourier transform of the even part is

$$G_e(f) = AT\,\mathrm{sinc}(2fT)$$

The Fourier transform of the odd part is

$$G_o(f) = \frac{AT}{2}\mathrm{sinc}(fT)\exp(-j\pi fT) - \frac{AT}{2}\mathrm{sinc}(fT)\exp(j\pi fT)$$

$$= \frac{AT}{j}\mathrm{sinc}(fT)\sin(\pi fT)$$

Problem 2.21

Express $g(t)$ as

$$g(t) = g_1(t) + g_2(t)$$

where

$$g_1(t) = \frac{1}{\tau}\int_{t-T}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$g_2(t) = \frac{1}{\tau}\int_{0}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

Therefore,

$$g_1(t+T) = \frac{1}{\tau}\int_{t}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{0}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du + \frac{1}{2}$$

where we have made use of the fact that

$$\frac{1}{\tau}\int_{-\infty}^{t}\exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

Similarly

$$g_2(t\text{-}T) = \int_0^t \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= \frac{1}{\tau}\int_{-\infty}^{t}\exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

Next, noting the following four relationships

$$\mathbf{F}[g_1(t+T)] = G_1(f)\exp(j2\pi fT)$$

$$\mathbf{F}[g_2(t\text{-}T)] = G_2(f)\exp(-j2\pi fT)$$

$$\exp\left(-\frac{\pi t^2}{\tau^2}\right) \rightleftharpoons \tau\exp(-\pi\tau^2 f^2)$$

$$\int_{-\infty}^{t} g(u)du = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

we find that taking the Fourier transforms of $g_1(t+T)$ and $g_2(t\text{-}T)$ respectively yields

$$G_1(f)\exp(j2\pi fT) = -\frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

$$G_2(f)\exp(-j2\pi fT) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

Therefore,

$$G_1(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(-j2\pi fT)$$

$$G_2(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(j2\pi fT)$$

Thus the Fourier transform of $g(t)$ is

$$G(f) = G_1(f) + G_2(f)$$

$$= \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)[\exp(-j2\pi fT) + \exp(j2\pi fT)]$$

$$= \frac{1}{\pi f}\exp(-\pi\tau^2 f^2)\sin(2\pi fT)$$

$$= 2T\exp(-\pi\tau^2 f^2)\operatorname{sinc}(2fT)$$

When τ approaches zero, $G(f)$ approaches the limiting value $2T\operatorname{sinc}(2fT)$, which corresponds to the Fourier transform of a rectangular pulse of unit amplitude and duration $2T$, which is correct.

Problem 2.22

(a) $$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$
$$= \int_{-\infty}^{0} g(t)\exp(-j2\pi ft)dt + \int_{0}^{\infty} g(t)\exp(-j2\pi ft)dt$$
$$= \int_{-\infty}^{0} g(t)\cos(2\pi ft)dt - \int_{-\infty}^{0} jg(t)\sin(2\pi ft)dt$$
$$+ \int_{0}^{\infty} g(t)\cos(2\pi ft)dt - \int_{0}^{\infty} jg(t)\sin(2\pi ft)dt$$

If $g(t)$ is even, then $g(t) = g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = \int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$
$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = -\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$

and so

$G(f) = 2\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$. which is purely real.

If, on the other hand, $g(t)$ is odd, $g(t) = -g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = \int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$
$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = -\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$

and thus

$G(f) = -2j\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$ which is purely imaginary.

(b) The Fourier transform of $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$

Differentiating both sides of this relation n times with respect to f:

$$\frac{d^n G(f)}{df^n} = (-j2\pi)^n \int_{-\infty}^{\infty} t^n \exp(-j2\pi ft)dt \tag{1}$$

That is,

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n \frac{d^n G(f)}{df^n}$$

(c) Putting $f = 0$ in Eq. (1), we get

$$\int_{-\infty}^{\infty} t^n g(t)dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$$

where $G^{(n)}(f) = \dfrac{d^n G(f)}{d fn}$

(d) Since

$g_2^*(t) \rightleftharpoons G_2^*(-f)$

it follows that

$g_1(t) g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2^*(\lambda - f) d\lambda$

From this result we deduce the Fourier transform

$$\mathbf{F}[g_1(t) g_2^*(t)] = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) \exp(-j2\pi ft) dt$$

$$= \int_{-\infty}^{\infty} G_1(\lambda) G_2^*(\lambda - f) d\lambda \qquad (2)$$

Setting $f = 0$ in Eq. (2), we get the desired relation

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) = \int_{-\infty}^{\infty} G_1(\lambda) G_2^*(\lambda) d\lambda$$

Problem 2.23

We are given the following inequalities:

$$|G(f)| \le \int_{-\infty}^{\infty} |g_1(t)| \ dt$$

$$|j2\pi f G(f)| \le \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

$$\left|j2\pi f^2 G(f)\right| \le \int_{-\infty}^{\infty} \left|\frac{d^2 g(t)}{dt}\right| dt$$

Considering the triangular pulse $g(t)$ of Fig. 2.41 in the text, its first and second derivatives with respect to time t are illustrated in Fig. 1:

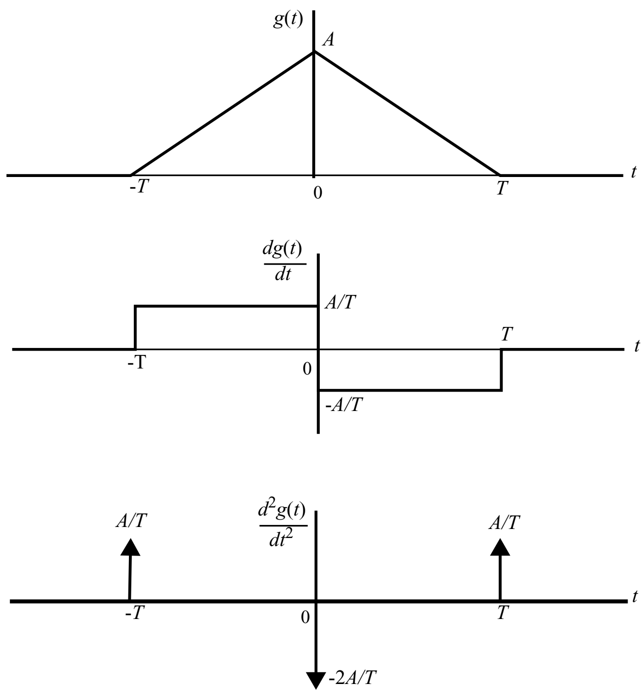


Figure 1

We thus have

$$\int_{-\infty}^{\infty} |g(t)| dt \ = \ AT$$

$$\int_{-\infty}^{\infty} \frac{dg(t)}{dt} dt \ = \ 2A$$

$$\int_{-\infty}^{\infty} \frac{d^2 g(t)}{dt} dt \ = \ \int_{-\infty}^{\infty} \frac{A}{T} |\delta(t+T) - 2\delta(t) + \delta(t-T)| dt$$

$$= \ \frac{4A}{T}$$

The bounds on the amplitude spectrum $|G(f)|$ are therefore as follows:

$$|G(f)| \le AT$$

$$|G(f)| \le \frac{A}{\pi|f|}$$

$$|G(f)| \le \frac{A}{\pi^2 f^2 T}$$

which are shown plotted in Fig. 2.

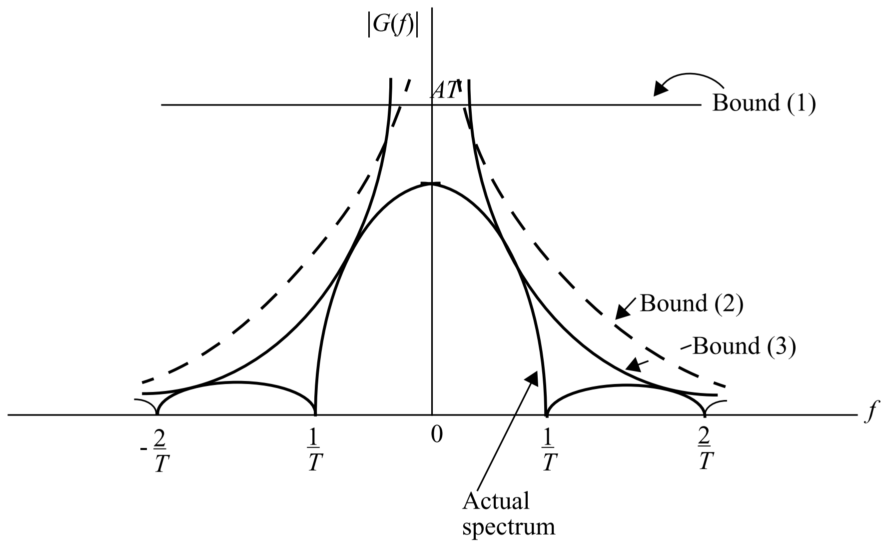


Figure 2

The actual amplitude spectrum of the triangular pulse is given by

$$|G(f)| = AT\,\text{sinc}^2(fT)$$

which is also plotted in Fig. 1. From this figure we see that bounds (1) and (3) define boundaries on the actual spectrum $|G(f)|$.

Problem 2.24

(a) The convolution of $g_1(t)$ and $g_2(t)$ is defined by

$$g_1(t) \star g_2(t) = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \tag{1}$$

Differentiating both sides of Eq. (1) with respect to time t:

$$\begin{aligned}
\frac{d}{dt}[g_1(t) \star g_2(t)] &= \int_{-\infty}^{\infty} g(\tau) \frac{d}{dt} g_2(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} g_1(\tau) \frac{d}{d(t-\tau)} g_2(t-\tau) d\tau \\
&= g_1(t) \star \left[\frac{d}{dt} g_2(t)\right]
\end{aligned}$$

Since convolution is commutative, we may also write

$$\frac{d}{dt}[g_1(t) \star g_2(t)] = \left[\frac{d}{dt} g_1(t)\right] \star g_2(t)$$

In other words, the derivative of a convolution produce of two signals is equivalent to the convolution of one of the signals and the derivative of the other.

(b) Changing variables in Eq. (1), we may write

$$g_1(t) \bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\lambda) g_2(t-\lambda) d\lambda \qquad (2)$$

Integrating both sides of Eq. (2) with respect to t:

$$\int_{-\infty}^{t} [g_1(\tau) \bigstar g_2(\tau)] d\tau = \int_{-\infty}^{t} \int_{-\infty}^{\infty} g_1(\lambda) g_2(\tau-\lambda) \lambda d\tau$$

Interchanging the order of integration and rearranging terms:

$$\int_{-\infty}^{t} [g_1(\tau) \bigstar g_2(\tau)] d\tau = \int_{-\infty}^{\infty} g_1(\lambda) \int_{-\infty}^{\infty} g_2(\tau-\lambda) \tau d\lambda d \qquad (3)$$

Recognizing that

$$\int_{-\infty}^{t} g_2(\tau-\lambda) d\lambda = \int_{-\infty}^{t-\lambda} g_2(\tau) d\tau$$

we may rewrite Eq. (3) as

$$\int_{-\infty}^{t} [g_1(\tau) \bigstar g_2(\tau)] d\tau = \int_{-\infty}^{\infty} g_1(\lambda) \int_{-\infty}^{t-\lambda} g_2(\tau) d\tau d\lambda$$

$$= g_1(t) \bigstar \left[\int_{-\infty}^{t} g_2(\tau) d\tau \right]$$

In other words, the integral of a convolution product of two signals is equivalent to the convolution of one of the signals and the integral of the other.

Problem 2.25

Express $y(t)$ as

$$y(t) = x^2(t)$$
$$= x(t)x(t)$$

Since multiplication in the time domain corresponds to convolution in the frequency domain, we may express the Fourier transform of $y(t)$ as

$$Y(f) = \int_{-\infty}^{\infty} X(\lambda) X(f-\lambda) d\lambda$$

where $X(f)$ is the Fourier transform of $x(t)$. However, $X(f)$ is zero for $|f| > W$. Therefore,

$$Y(f) = \int_{-W}^{W} X(\lambda) X(f-\lambda) d\lambda$$

In this integral we note that $X(f - \lambda)$ is limited to $-W \le f - \lambda \le W$. When $\lambda = -W$, we find that $-2W \le f \le 0$. When $\lambda = W$, we find that $0 \le f \le 2W$. Accordingly, the Fourier transform $Y(f)$ is limited to the frequency interval $-2W \le f \le 2W$.

Problem 2.26

(a) Consider a rectangular pulse $g(t)$ of duration T and amplitude $1/T$, centered at $t = 0$, as shown in Fig. 1:

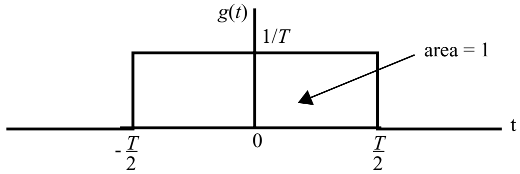


Figure 1

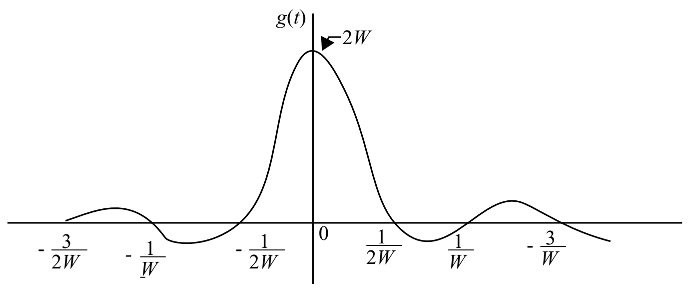
The Fourier transform of $g(t)$ is

$$G(f) = \frac{\sin(\pi f T)}{\pi f T}$$

As the duration T approaches zero, $g(t)$ approaches a delta function, and so we find that in the limit:

$$\lim_{T \to 0} G(f) = \lim_{T \to 0} \frac{\sin(\pi f T)}{\pi f T} = 1$$

(b) Consider next the sinc pulse $2W \,\mathrm{sinc}(2Wt)$ of unit area, as shown in Fig. 2:



The Fourier transform of $g(t)$ is

$$G(f) = \mathrm{rect}\left(\frac{f}{2W}\right)$$

which has unit amplitude and width $2W$, centered at $f = 0$. As W approaches infinity, $g(t)$ approaches a delta funct6ion, and the corresponding Fourier transform becomes equal to unity for all f.

Problem 2.27

The $G(f)$ is in the form of a unit step function defined in the frequency domain, as shown in Fig. 1

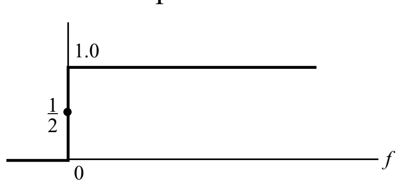


Figure 1

Now, for a unit step function defined in the time domain, we have

$$u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Applying the duality property of the Fourier transform to this relation, we get

$$-\frac{1}{j2\pi t} + \frac{1}{2}\delta(f) \rightleftharpoons u(f)$$

where we have used the fact that $\delta(-t) = \delta(t)$. Therefore, the time function $g(t)$ whose Fourier transform is depicted in Fig. 1, is given by

$$g(t) = -\frac{1}{j2\pi t} + \frac{1}{2}\delta(t)$$

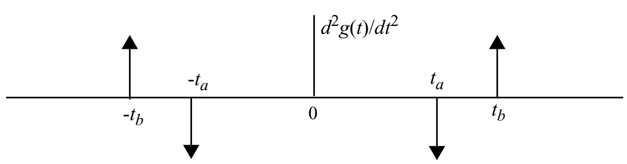
Problem 2.28

(a) Taking the Fourier transform of both sides of $\frac{d^2 g(t)}{dt^2} = \sum_i k_i \delta(t - t_i)$, we get

$$(j2\pi f)^2 G(f) = \sum_i k_i \exp(-j2\pi f t_i)$$

Therefore, $G(f) = \frac{-1}{4\pi^2 f^2} \sum_i k_i \exp(-j2\pi f t_i)$

(b) Differentiating the trapezoidal pulse of Fig. 2.42 twice, we get:



Hence,

$$G(f) = \frac{-A}{4\pi^2 f^2 (t_b - t_a)} [\exp(j2\pi f t_b) - \exp(j2\pi f t_a) - \exp(j2\pi f t_a) - \quad + \exp(j2\pi f t_b)]$$

$$= \frac{-A}{2\pi^2 f^2 (t_b - t_a)} [\cos(j2\pi f t_b) - \cos(j2\pi f t_a)]$$

$$= \frac{-A}{2\pi^2 f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a) \sin \pi f (t_b + t_a)]$$

Problem 2.29

(a) From part (b) of Problem 2.28, we have

$$G(f) = \frac{A}{\pi^2 f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a)] \sin[\pi f (t_b + t_a)] \qquad (1)$$

As t_b approaches t_a, we get the following result:

$$\lim_{t_b \to t_a} \frac{1}{\pi f^2 (t_b - t_a)} \sin[\pi f (t_b - t_a)] = 1$$

and

$$\lim_{t_b \to t_a} \sin[\pi f (t_b + t_a)] = \sin(\pi f t_a)$$

Accordingly, the Fourier transform of Eq. (1) approaches the limiting value

$$\begin{aligned}\lim_{t_b \to t_a} G(f) &= \frac{A}{\pi f} \sin(2\pi f t_a) \\ &= 2t_a A \frac{\sin(2\pi f t_a)}{2\pi f t_a} \\ &= 2t_a A \operatorname{sinc}(2\pi f t_a) \qquad (2)\end{aligned}$$

which is the desired result.

(b) The limiting Fourier transform of Eq. (2) is recognized as the Fourier transform of a rectangular pulse of amplitude A and duration $T = 2t_a$.

Problem 2.30

The transfer function $H(f)$ and impulse response $h(t)$ of a linear time-invariant filter are related by

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi f t)\, dt$$

Applying a special form of Schwarz's inequality (see Appendix 5), we may write

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t) \exp(-j2\pi f t)|\, dt$$

Since $|\exp(-j2\pi f t)| = 1$, we may simplify this relation as

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t)|\, dt$$

If the filter is stable, the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Therefore, the amplitude response of a stable filter is bounded for every value of the frequency f, as shown by

$$|H(f)| = \infty$$

According to Rayleigh's energy theorem, the energy of the input signal $x(t)$ is given by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and the energy of the output signal $y(t)$ is

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

The Fourier transforms $Y(f)$ and $X(f)$ are related by

$Y(f) = H(f)X(f)$

Therefore,

$$E_y = \int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \tag{1}$$

For a stable filter, we may express $|H(f)|$ in the form $K|H_n(f)|$ where K is a scaling factor equal to the maximum value of $|H(f)|$ and $|H_n(f)| \le 1$ for all f. Thus, we may rewrite Eq. (1) in the form:

$$E_y = K^2 \int_{-\infty}^{\infty} |H_n(f)|^2 |X(f)|^2 df$$

Since $|H_n(f)| \le 1$ for all f, it follows that

$$\int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \le \int_{-\infty}^{\infty} |X(f)|^2 df$$

or equivalently

$$E_y \le K^2 \int_{-\infty}^{\infty} |X(f)|^2 df$$

If the input signal has finite energy, we then have

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \infty$$

Accordingly, we find that $E_y < \infty$, which means that the output signal $y(t)$ also has finite energy.

Problem 2.31

(a) The transfer function of the ith stage of the system of Fig. 2.43 is

$$H_i(f) = \frac{1}{1 + j2\pi fRC}$$

$$= \frac{1}{1 + j2\pi f\tau_0}, \qquad T_0 = RC$$

where it is assumed that the buffer amplifier has a constant gain of unity. The overall transfer function of the system is therefore

$$H(f) = \prod_{i=1}^{N} H_i(f)$$

$$= \frac{1}{(1 + j2\pi f\tau_0)^N}$$

The corresponding amplitude response is

$$|H(f)| = \frac{1}{[1 + (2\pi f\tau_0)^2]^{N/2}} \tag{1}$$

(b) Let

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

Then, we may rewrite Eq. (1) for the amplitude response as

$$|H(f)| = \left[1 + \frac{1}{N}(fT)^2\right]^{-N/2}$$

In the limit, as N approaches infinity we have

$$\begin{aligned}
|H(f)| &= \lim_{N \to \infty}\left[1 + \frac{1}{N}(fT)^2\right]^{-N/2} \\
&= \exp\left[\frac{N}{2} \cdot \frac{1}{N}(fT)^2\right] \\
&= \exp\left(-\frac{f^2T^2}{2}\right)
\end{aligned}$$

Problem 2.32

(a) The integrator output is

$$y(t) = \int_{t-T}^{t} x(\tau)d\tau$$

Let $x(t) \rightleftharpoons X(f)$; then

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Therefore,

$$y(t) = \int_{t-T}^{t}\left[\int_{-\infty}^{\infty} X(f)\exp(j2\pi f\tau)df\right]d\tau$$

Interchanging the order of integration:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} X(f)\left[\int_{t-T}^{t} \exp(j2\pi f\tau)d\tau\right]df \\
&= \int_{-\infty}^{\infty} [TX(f)\operatorname{sinc}(fT)\exp(-j\pi fT)]\exp(j2\pi ft)df
\end{aligned}$$

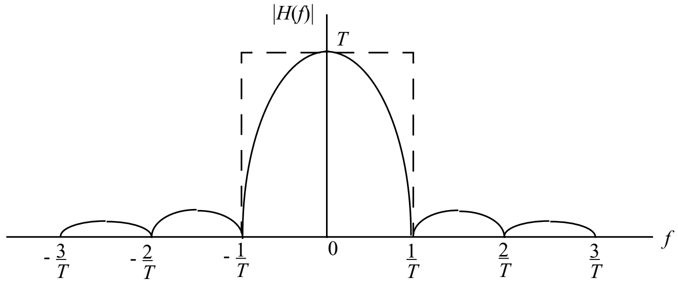
The Fourier transform of the integrator output is therefore

$$Y(f) = TX(f)\operatorname{sinc}(fT)\exp(-j\pi fT) \tag{1}$$

Equation (1) shows that $y(t)$ can be obtained by passing the input signal $x(t)$ through a linear filter whose transfer function is equal to $T\operatorname{sinc}(fT)\exp(-j\pi fT)$.

(b) The amplitude response of this filter is shown in Fig. 1:

Figure 1



The approximation with an ideal low-pass filter of bandwidth $1/T$, gain T, and delay $T/2$, is shown dashed in Fig. 1. The response of this ideal filter to a unit step function applied at $t = 0$ is given by

$$y_{\text{ideal}}(t) = \frac{T}{\pi}\int_{-\infty}^{\frac{2\pi}{T}\left(t-\frac{T}{2}\right)} \frac{\sin\lambda}{\lambda}d\lambda$$

At time $t = T$. we therefore have

$$\begin{aligned} y_{\text{ideal}}(t) &= \frac{T}{\pi}\int_{-\infty}^{\pi} \frac{\sin\lambda}{\lambda}d\lambda \\ &= \frac{T}{\pi}\left[\int_{-\infty}^{0} \frac{\sin\lambda}{\lambda}d\lambda + \int_{0}^{\pi}\frac{\sin\lambda}{\lambda}d\lambda\right] \\ &= \frac{T}{\pi}[\mathrm{Si}(\infty) + \mathrm{Si}(\pi)] \\ &= \frac{T}{\pi}\left(\frac{\pi}{2} + 1.85\right) \\ &= 1.09T \qquad (2) \end{aligned}$$

On the other hand, the output of the ideal integrator to a unit step function, evaluated at time $t = T$, is given by

$$\begin{aligned} y(T) &= \int_0^T u(\tau)d\tau \\ &= T \qquad (3) \end{aligned}$$

Thus, comparing Eqs. (2) and (3) we see that the ideal low-pass filter output exceeds the ideal integrator output by only nine percent for $T = 1$.

Problem 2.33

The half cosine pulse in Fig. 2.33(a) is

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)\cos\left(\frac{\pi t}{T}\right)$$

Fourier transforming both sides gives

$$G(f) = AT\frac{\sin[\pi f T]}{\pi f T}\star\left\{\frac{1}{2}\left[\delta\left(f - \frac{1}{2T}\right) + \delta\left(f + \frac{1}{2T}\right)\right]\right\}$$

$$= \frac{AT\sin\left[\pi fT - \frac{\pi}{2}\right]}{2\left(\pi fT - \frac{\pi}{2}\right)} + \frac{AT\sin\left[\pi fT + \frac{\pi}{2}\right]}{2\left(\pi fT + \frac{\pi}{2}\right)}$$

$$= \frac{2AT\cos(\pi fT)}{\pi(1-2fT)(1+2fT)}$$

Therefore, the energy density of $g(t)$ is

$$\Psi(f) = |G(f)|^2 = \frac{4A^2T^2\cos^2(\pi fT)}{\pi^2(1-4f^2T^2)^2} = \frac{4A^2T^2\cos^2(\pi fT)}{\pi^2(4T^2f^2-1)^2} \qquad (1)$$

Consider next the half-sine pulse in Fig. 2.33(b), which is the same as that of Fig. 2.33(b) shifted to the right by $T/2$. This time-shift corresponds to multiplication by $\exp(-j2\pi fT)$, which has unit amplitude for all f. Therefore, both pulses have exactly the same energy density defined in Eq. (1).

Problem 2.34

The autocorrelation function of a deterministic signal $g(t)$ is defined by

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt \qquad (1)$$

This formula applies to a real-valued signal, which is satisfied by all three signals specified under parts (a) through (c).

(a) $g(t) = \exp(-at)u(t)$, $\quad u(t)$: unit step function

Applying Eq. (1) yields

$$R_g(\tau) = \int_{\tau}^{\infty} \exp(-at)\exp(-a(t-\tau))dt$$

$$= \exp(a\tau)\int_{\tau}^{\infty} \exp(-2at)dt$$

$$= \exp(a\tau)\left[-\frac{1}{2a}\exp(-2at)\right]_{t=\tau}^{\infty}$$

$$= \frac{1}{2a}\exp(-a\tau)$$

which is depicted in Fig. 1

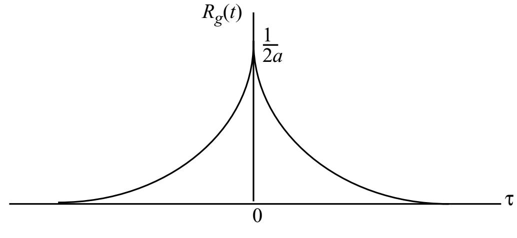


Figure 1

(b) $g(t) = \exp(-a|t|)$

which is sketched in Fig. 2(a). Part (b) of the figure sketches $g(t - \tau) = \exp(-a|t - \tau|)$

(a)

(b)

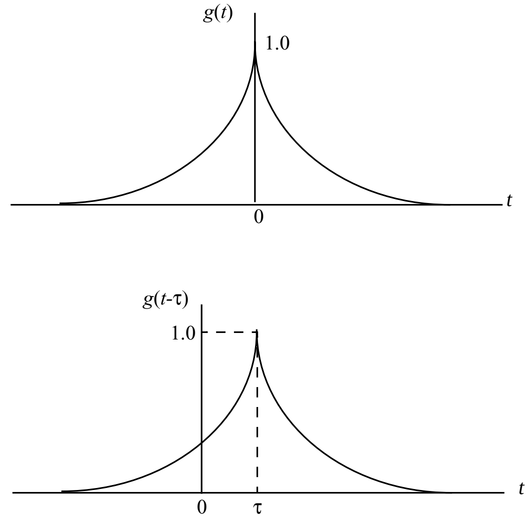


Figure 2

In light of Fig. 1, applying Eq. (1):

$$R_g(\tau) = \int_{\tau}^{\infty} \exp(-at)\exp(-a(t-\tau))dt \quad + \int_{0}^{\tau} \exp(-at)\exp(-a(t-\tau))dt$$

$$+ \int_{0}^{\tau} \exp(at)\exp(a(t-\tau))dt$$

$$= \frac{1}{2a}\exp(-at) + \tau\exp(-at) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} + \tau\right)\exp(-a\tau)$$

which is sketched in Fig. 3.

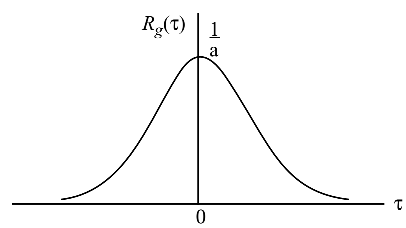


Figure 3

(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

which is sketched in Fig. 4(a). Part (b) of the figure sketches $g(t - \tau)$

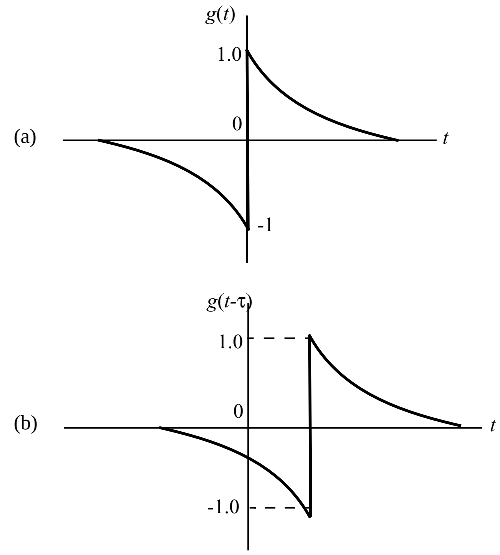


Figure 4

In light of Fig. 4, applying Eq. (1) for $\tau \geq 0$:

$$R_g(\tau) = \int_\tau^\infty \exp(-at)\exp(-a(t-\tau))dt$$

$$+\int_0^\tau \exp(-at)[-\exp(a(t-\tau))]dt$$

$$+\int_{-\infty}^0 [-\exp(-at)][-\exp(a(t-\tau))]dt$$

$$= \frac{1}{2a}\exp(-a\tau) - \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} - \tau\right)\exp(-a\tau), \qquad \tau \geq 0$$

Similarly, for $\tau \leq 0$ we have

$$R_g(\tau) = \left(\frac{1}{a} + \tau\right)\exp(a\tau)$$

Summing up these two results:

$$R_g(\tau) = \begin{cases} \left(\frac{1}{a} - \tau\right)\exp(-a\tau), & \tau \geq 0 \\ \left(\frac{1}{a} + \tau\right)\exp(a\tau), & \tau \leq 0 \end{cases}$$

which is sketched in Fig. 5.

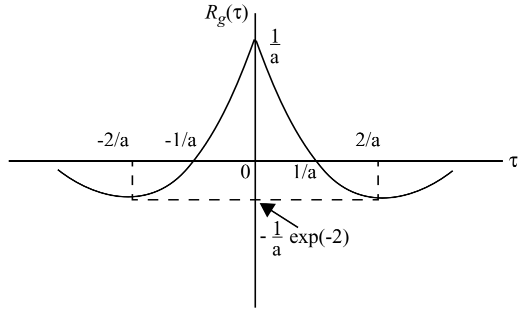


Figure 5

Problem 2.35

Applying the formula for the autocorrelation function

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t-\tau)dt$$

to the specified signal

$$g(t) = \frac{1}{t_0}\exp\left(-\frac{\pi t^2}{t_0^2}\right), \qquad -\infty < t < \infty$$

we get

$$R_g(\tau) = \int_{-\infty}^{\infty} \frac{1}{t_0^2}\exp\left[\frac{\pi}{t_0^2}(t^2 + (t-\tau)^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)(2t^2 - 2t\tau + \tau^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right) - \frac{\pi}{t_0^2}\frac{\tau^2}{2}\right]dt$$

$$= \frac{1}{t_0^2}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)^2\right]dt \qquad (1)$$

Let $x = \frac{1}{t_0}\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)$, and therefore (for fixed τ)

$$dt = \frac{t_0}{\sqrt{2}}dx$$

We may then rewrite Eq. (1) as

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp(-\pi x^2)dx \qquad (2)$$

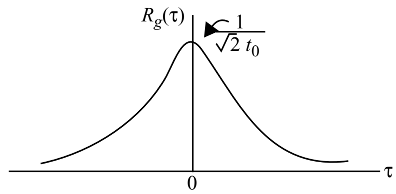
Recognizing that

$$\int_{-\infty}^{\infty} \exp(-\pi x^2)dx = 1$$

we find that Eq. (2) simplifies to

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)$$

which has the same form as the bell-shaped Gaussian curve:



Problem 2.36

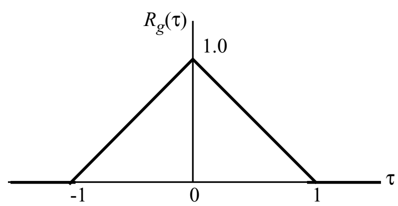
We are given the Fourier transform

$$G(f) = \text{sinc}(f)$$

Using the transform pair

$$R_g(\tau) \rightleftharpoons |G(f)|^2$$

we may therefore express the autocorrelation function $R_g(\tau)$ as the inverse Fourier transform of $\text{sinc}^2(f)$. From Eq. (2.43) in the text, we readily deduce that $R_g(\tau)$ has the triangular form



Problem 2.37

Recognizing two facts:

1. $R_g(\tau) = |G(f)|^2$, and
2. the spectrum $G(f)$ is invariant to a time shift,

we infer that a signal $g(t)$ and the time-shifted version $g(t - t_0)$ for any t_0 will have exactly the same autocorrelation function.

Problem 2.38

(a) We are given the power signal

$$g(t) = A_0 + A_1\cos(2\pi f_1 t + \theta_1) + A_2\cos(2\pi f_2 t)$$

The three components of $g(t)$ are uncorrelated with each other. Therefore, the power spectral density of $g(t)$ is the sum of the power spectral densities of the three constituent components, as shown by

$$S_g(f) = \frac{A^2}{2}\delta(f) + \frac{A_1^2}{4}[\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4}[\delta(f - f_2) + \delta(f + f_2)]$$

Correspondingly, the autocorrelation function $R_g(\tau)$ is given by

$$R_g(\tau) = \frac{A^2}{2} + \frac{A_1^2}{2}\cos(2\pi f_1\tau) + \frac{A_2^2}{2}\cos(2\pi f_2\tau)$$

(Here we are postulating a fundamental result that, as with energy signals, the autocorrelation function and power spectral density of a power signal constitute a Fourier-transform pair).

(b) $R_g(0) = \frac{A^2}{2}$.

(c) In calculating the autocorrelation function, information about the phase shifts θ_1 and θ_2 is completely lost.

Problem 2.39

We will determine the autocorrelation function of the signal $g(t)$ depicted in Fig. 2.45 by proceeding on a segment-by-segment basis:

1. The maximum value of $R_g(\tau)$ occurs at $\tau = 0$, for then $g(t)$ and $g(t\text{-}\tau)$ overlap exactly, yielding

$$R_g(0) = 3(A^2)(T) = 3A^2T$$

2. For $0 < \tau < (T/2)$, we have the picture depicted in Fig. 1. From this figure, we obtain

$$\begin{aligned} R_g(\tau) &= \int_{-(3T/2)+\tau}^{-T/2}(+A)(+A)dt + \int_{-T/2}^{-(T/2)+\tau}(+A)(\text{-}A)dt \\ &\quad + \int_{-(T/2)+\tau}^{T/2}(+A)(+A)dt + \int_{T/2}^{(T/2)+\tau}(\text{-}A)(+A)dt \\ &\quad + \int_{(T/2)+\tau}^{3T/2}(\text{-}A)(\text{-}A)dt \\ &= A^2(T-\tau) - A^2\tau + A^2(T-\tau) - A^2\tau + A^2(T-\tau) \\ &= A^2(3T - 5\tau), \qquad 0 < |\tau| < (T/2) \end{aligned}$$

where the use of $|\tau|$ is invoked in light of the symmetric property of the autocorrelation function.

3. Next, for $(T/2) < \tau < T$, we have the picture depicted in Fig. 2, from which we obtain

$$R_g(\tau) = \int_{-(3T/2)+\tau}^{-T/2}(\text{-}A)(\text{-}A)dt + \int_{-T/2}^{-(T/2)+\tau}(+A)(\text{-}A)dt$$

$$+\int_{-(T/2)+\tau}^{T/2}(+A)(+A)dt+\int_{T/2}^{(T/2)+\tau}(\text{-}A)(+A)dt$$

$$+\int_{(T/2+\tau)}^{3T/2}(\text{-}A)(\text{-}A)dt$$

$$= A^2(T-\tau)-A^2\tau+A^2(T-\tau)-A^2\tau+A^2(T-\tau)$$

$$= A^2(3T-5\tau), \qquad \frac{T}{2}<|\tau|<T$$

4. Next, for $T < \tau < 3T/2$, we have the picture depicted in Fig. 3, from which we obtain

$$R_g(\tau) = \int_{-(3T/2)+\tau}^{T/2}(+A)(\text{-}A)dt+\int_{T/2}^{-(T/2)+\tau}(\text{-}A)(\text{-}A)dt$$

$$+\int_{-(T/2)+\tau}^{3T/2}(\text{-}A)(+A)dt$$

$$= -A^2(2T-\tau)+A^2(-T+\tau)-A^2(2T-\tau)$$

$$= A^2(-5T+3\tau) \qquad \text{for } T<|\tau|<3T/2$$

5. For $(3T/2) < \tau < 2T$ we have the picture depicted in Fig. 4, from which we obtain

$$R_g(\tau) = \int_{-(3T/2)+\tau}^{T/2}(+A)(\text{-}A)dt+\int_{T/2}^{-(T/2)+\tau}(\text{-}A)(\text{-}A)dt+\int_{-(T/2)+\tau}^{3T/2}(\text{-}A)(+A)dt$$

$$= -A^2(2T-\tau)+A^2(-T+\tau)-A^2(2T-\tau)$$

$$= A^2(-5T+3\tau) \qquad \text{for } (3T/2)+|\tau|<2T$$

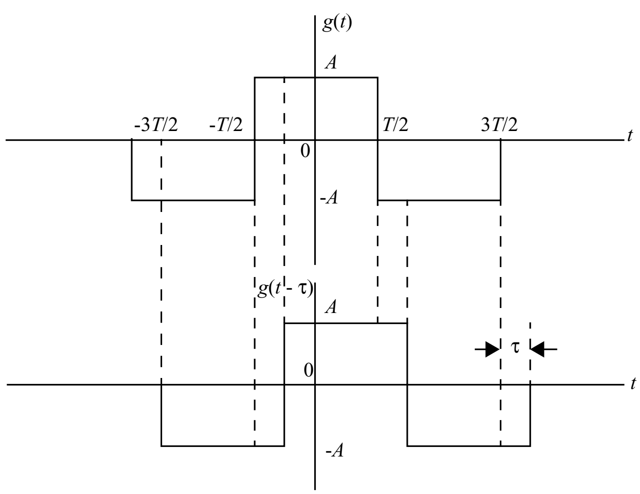


Figure 1: $-(T/2) < \tau < (T/2)$

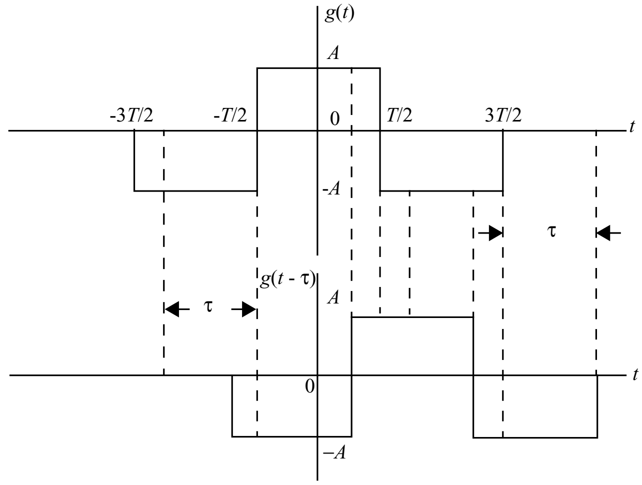


Figure 2: $(T/2) < \tau < (3T/2)$

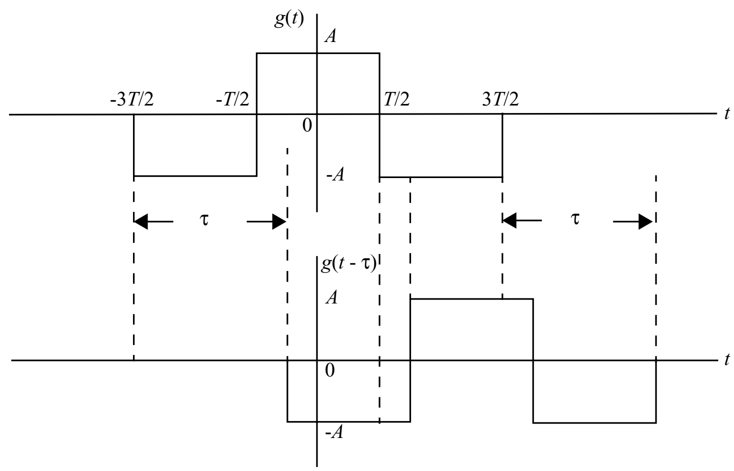


Figure 3: $T < \tau < (3T/2)$

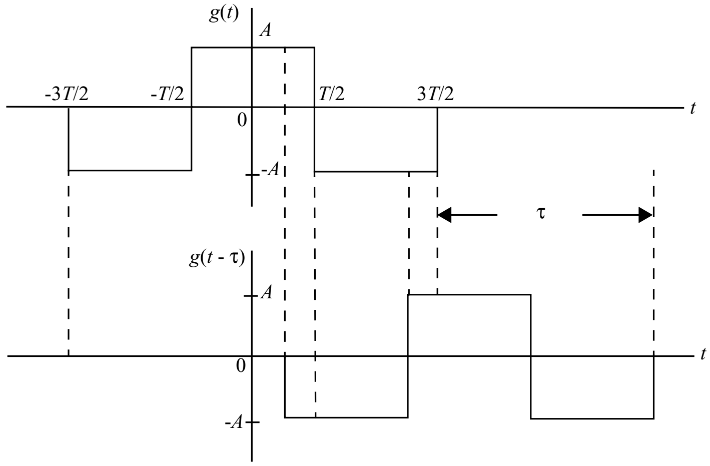


Figure 4: $(3T/2) < \tau < 2T$

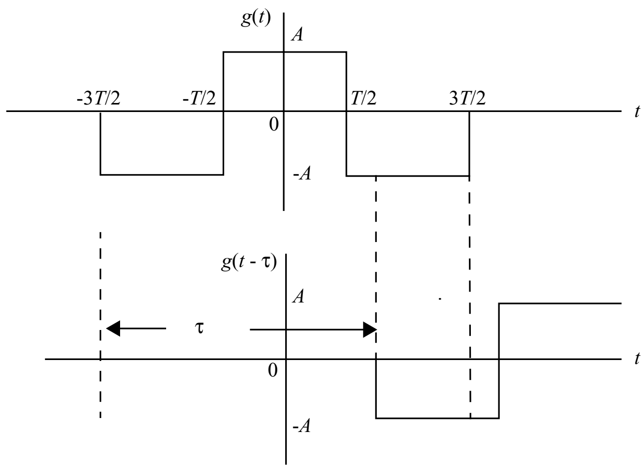


Figure 5: $2T < \tau < 3T$

6. Next, for $2T < \tau < 3T$ we have the picture depicted in Fig. 5, from which we obtain

$$R_g(\tau) = \int_{-3T/2+\tau}^{3T/2} (-A)(-A)dt$$

$$= A^2(3T - \tau) \qquad \text{for } 2T < |\tau| < 3T$$

7. Finally, for $|\tau| > 3T$, we find that $R_g(\tau) = 0$.

Putting all these pieces together, we get the autocorrelation function $R_g(\tau)$ plotted in Fig. 6, which is symmetric about the origin $\tau = 0$.

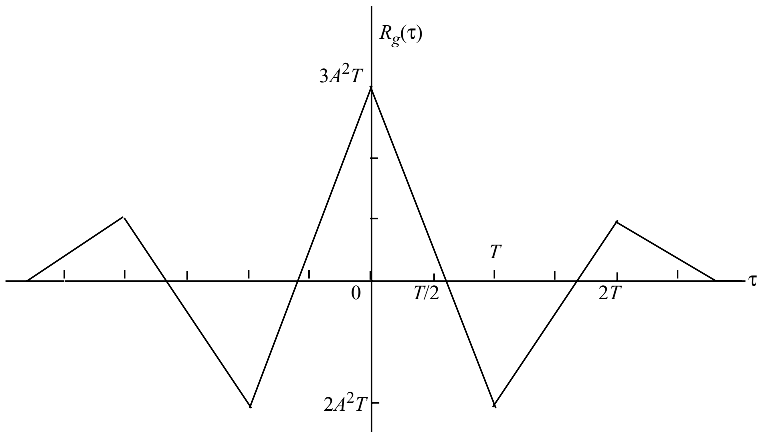


Figure 6: Plot of the autocorrelation function $R_g(t\tau)$

Problem 2.40

We start with the Fourier-transform pair

$$R_g(\tau) \rightleftharpoons \Psi_g(f) \tag{1}$$

where $R_g(\tau)$ denotes the autocorrelation function of energy signal $g(t)$ and $\Psi_g(f)$ denotes the corresponding energy spectral density. Differentiating $R_g(\tau)$ with respect to the lag τ changes Eq. (1) into the form

$$\frac{dR_g(\tau)}{d\tau} \rightleftharpoons j2\pi f \Psi_g(f) \tag{2}$$

Next, we apply Rayleigh's energy theorem, which, in the context of the problem at hand, yields

$$\int_{-\infty}^{\infty} \left|\frac{dR_g(\tau)}{d\tau}\right|^2 d\tau = \int_{-\infty}^{\infty} |j2\pi f \Psi_g(f)|^2 df$$

$$= 4\pi^2 \int_{-\infty}^{\infty} f^2 \Psi_g^2(f) df \tag{3}$$

Since $g(t)$ is real valued, $R_g(\tau)$ will likewise be real valued; hence we may write

$$\left|\frac{dR_g(\tau)}{d\tau}\right|^2 = \left[\frac{dR_g(\tau)}{d\tau}\right]^2$$

Moreover, by definition,

$$\Psi_g^2(f) = |G(f)|^2$$

where $G(f)$ is the Fourier transform of $g(t)$. Accordingly, we may rewrite Eq. (3) as follows:

$$\int_{-\infty}^{\infty} \left[\frac{dR_g(\tau)}{d\tau}\right]^2 d\tau = 4\pi^2 \int_{-\infty}^{\infty} |G(f)|^4 df \tag{4}$$

which is the desired result.

Note: In the first publication of the book, the power 2 in $\left[\frac{dR_g(\tau)}{d\tau}\right]^2$ was missed out in error.

Problem 2.41

To determine the cross-correlation function $R_{12}(\tau)$ of the two pulses $g_1(t)$ and $g_2(t)$ in Fig. 2.41, we proceed on a stage-by-stage basis as follows:

1. For $\tau = 0$, we have

$$R_{12}(0) = \int_{-\infty}^{\infty} g_1(t) g_2(t) dt$$

$$= \int_{-3}^{-1} (1.0)(-1.0) dt + \int_{-1}^{+1} (1.0)(1.0) dt + \int_{-3}^{+3} (1.0)(-1.0) dt$$

$$= -2 + 2 - 2 = -2$$

2. For $0 < \tau < 2.0$, we have the picture depicted in Fig. 1, from which we obtain

$$
\begin{aligned}
R_{12}(\tau) &= \int_{1+\tau}^{3} (1)(-1)\,dt \\
&= \tau - 2, \qquad 0 < \tau < 2
\end{aligned}
$$

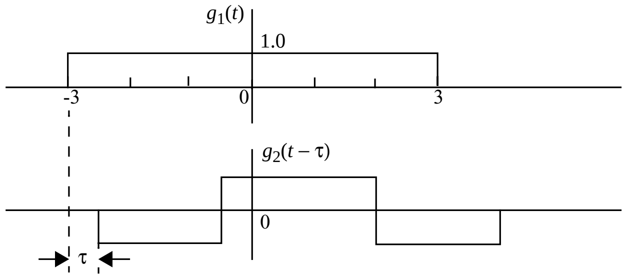


Figure 1: $0 < \tau < 2.0$

3. Next, for $2 < \tau < 4$ we have the picture depicted in Fig. 2, from which we obtain

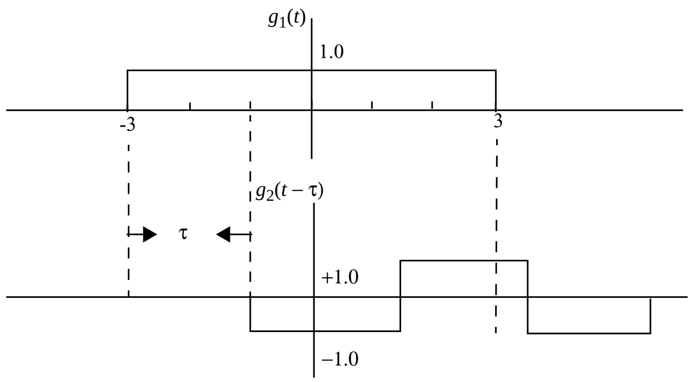


Figure 2

$$
\begin{aligned}
R_{12}(\tau) &= \int_{-3+\tau}^{-1+\tau} (1)(-1)\,dt + \int_{-1+\tau}^{3} (1)(1)\,dt \\
&= -2 + (4 - \tau) \\
&= 2 - \tau, \qquad 2 < \tau < 4
\end{aligned}
$$

4. For $4 < \tau < 6$ we have the picture depicted in Fig. 3, from which we obtain:

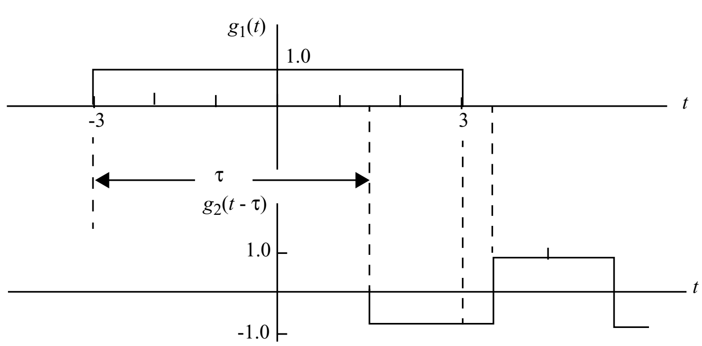


Figure 3

$$R_{12}(\tau) = \int_{-3+\tau}^{3} (1)(-1)dt$$

$$= \tau - 6, \qquad 4 < \tau < 6$$

5. Finally, for $\tau > 6$ we find that $R_{12}(\tau)$ is zero.

Putting all these results together, we get the cross-correlation function $R_{12}(\tau)$ plotted in Fig. 4.

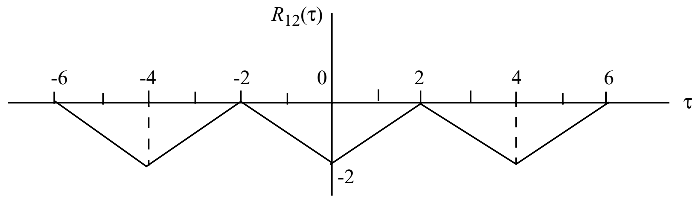


Figure 4

For the problem at hand, $R_{12}(\tau)$ is symmetric about the origin $\tau = 0$. Hence, the pulses defined in Fig. 2.46 satisfy the property

$$R_{12}(\tau) = R_{21}(\tau)$$

Moreover, the fact that $R_{12}(0)$ is nonzero implies that the pulses $g_1(t)$ and $g_2(t)$ are *not* orthogonal.

Problem 2.42

The convolution of energy signals $g_1(t)$ and $g_2(t)$ is defined by

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)dt$$

With $g_1(t)$ delayed by t_1 and $g_2(t)$ delayed by t_2, we have

$$g_1(t-t_1)\bigstar g_2(t-t_2) = \int_{-\infty}^{\infty} g_1(\tau - t_1)g_2(t-t_2-\tau)d\tau \qquad (1)$$

The cross-correlation of $g_1(t)$ and $g_2(t)$ is defined by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t)g_2(t-\tau)dt$$

When $g_1(t)$ is delayed by t_1 and $g_2(t)$ is delayed by t_2, we have

$$R_{12}(\tau) = \int_{-\infty}^{\infty} g_1(t-t_1)g_2(t-t_2-\tau)dt \qquad (2)$$

Comparing Eqs. (1) and (2), we make the following observations:

1. The convolution integral in Eq. (1) involves the dummy variable τ as the integration variable, whereas the cross-correlation function of Eq. (2) involves time t as the integration variable.
2. By virtue of the difference identified under point 1, the arguments of g_1 and g_2 in Eq. (1) are additive, in which case the time delays t_1 and t_2 are additive. On the other hand, the argument of g_2 is subtracted from the argument of g_2 in Eq. (2); hence, the time delays t_1 and t_2 are subtractive.

Problem 2.43

(a) The relationship between the Fourier transform $X(f)$ of an energy signal $g(t)$, its autocorrelation function $R_x(\tau)$, and energy spectral density $\Psi_x(f)$ is illustrated by the flow-graph in Fig. 1.

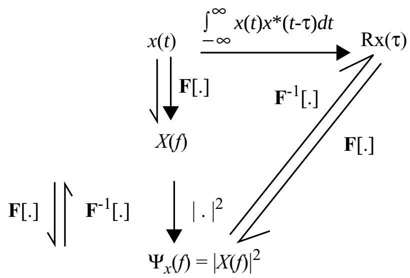


Figure 1

(b) One way of calculating the autocorrelation function $R_x(\tau)$ from the Fourier transform $X(f)$ is to proceed as follows (in accordance with Fig. 1):

- Take the squared magnitude of $X(f)$, obtaining the energy spectral density $\Psi_x(f) = |X(f)|^2$.
- Take the inverse Fourier transform of $\Psi_x(f)$ to obtain the autocorrelation function $R_x(\tau)$ as desired.

Another way of calculating $R_x(\tau)$ from $X(f)$ is to proceed as follows (again in accordance with Fig. 1:

- Use the inverse Fourier transform to calculate $x(t)$ from $X(f)$.
- Then use the formula

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

to calculate $R_x(\tau)$ for prescribed lag τ.

Problem 2.44

The autocorrelation function of a power signal is the inverse Fourier transform of the power spectral density. Given the power spectral density of Fig. 2.47, we write

$$\begin{aligned}
R_x(\tau) &= \int_{-\infty}^{\infty} S_x(f)\exp(j2\pi f\tau)d\tau \\
&= \int_{-2}^{-1} 1 \cdot \exp(j2\pi f\tau)d\tau + \int_{-1}^{1} 2 \cdot \exp(j2\pi f\tau)d\tau \\
&\quad + \int_{1}^{2} 1 \cdot \exp(j2\pi f\tau)d\tau \\
&= \frac{1}{j2\pi\tau}\{\exp(j2\pi f\tau)|_{f=-2}^{-1} + 2\exp(j2\pi f\tau)|_{f=-1}^{-1} + \exp(j2\pi f\tau)|_{f=1}^{2}\} \\
&= \frac{1}{j2\pi\tau}\{\exp(-j2\pi f\tau) - \exp(-j4\pi f\tau) + 2[\exp(j2\pi\tau) - \exp(-j2\pi\tau)]
\end{aligned}$$

$$+ \exp(j4\pi\tau) - \exp(j2\pi\tau)\}$$

$$= \frac{1}{\pi\tau}\{\sin(4\pi\tau) - \sin(2\pi\tau) + 2\sin(2\pi\tau)\}$$

$$= \frac{1}{\pi\tau}\{\sin(4\pi\tau) + \sin(2\pi\tau)\}$$

The value of the autocorrelation function $R_x(\tau)$ at $\tau = 0$ is given by

$$R_x(0) = \lim_{\tau \to 0}\left[4 \cdot \frac{\sin(4\pi\tau)}{4\pi\tau} + 2 \cdot \frac{\sin(2\pi\tau)}{2\pi\tau}\right]$$

$$= 4 + 2 = 6 \tag{1}$$

As a check, $R_x(0)$ equals the total area under the power spectral density $S_x(f)$. For the example of Fig. 2.47, we write

$$R_x(0) = |\mathrm{x}| + 2 \times 2 + |\mathrm{x}|$$

$$= 6$$

which checks the result computed in Eq. (1).

Problem 2.45

To proceed with the Fourier series expansion of the square wave specified in Fig. 2.48, we note the following by inspection of the square wave:

- The square wave is symmetric about the origin $t = 0$.
- The average value is therefore zero.
- The expansion consists of cosine terms only.

Accordingly, we may write

$$g(t) = \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t)$$

where $f_0 = 1/T_0 = 1/4$. The coefficient a_n is defined by

$$a_n = \frac{1}{T}\int_{-T_0/2}^{T_0/2} g(t)\cos(2\pi n f_0 t)dt$$

$$= \frac{1}{4}\int_{-2}^{-1}(-1)\cos\left(\frac{\pi n t}{2}\right)dt + \frac{1}{4}\int_{-1}^{1}(1)\cos\left(\frac{\pi n t}{2}\right)dt$$

$$+ \frac{1}{4}\int_{1}^{2}(-1)\cos\left(\frac{\pi n t}{2}\right)dt$$

$$= \frac{1}{2}\int_{0}^{1}\cos\left(\frac{\pi n t}{2}\right)dt - \int_{1}^{2}\cos\left(\frac{\pi n t}{2}\right)dt$$

$$= \frac{1}{2} \cdot \frac{2}{n\pi}\left[\sin\left(\frac{\pi n t}{2}\right)\Big|_{t=0}^{1} - \sin\left(\frac{\pi n t}{2}\right)\Big|_{t=1}^{2}\right]$$

$$= \frac{1}{n\pi}\left[2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right]$$

Since $\sin(n\pi) = 0$ for integer values of n, the formula for a_n reduces to

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= \text{sinc}\left(\frac{n}{2}\right), \qquad n = 1,3,5,...$$

The Fourier series expansion of the square wave is thus defined by

$$g(t) = \frac{2}{\pi}\left[\cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \ldots\right]$$

Correspondingly, the Fourier transform of the square wave is defined by

$$G(f) = \frac{1}{\pi}\left[\delta(f - f_0) + \delta(f + f_0) - \frac{1}{3}\delta(f - 3f_0) - \frac{1}{3}\delta(f + 3f_0)\right.$$

$$\left. + \frac{1}{5}\delta(f - 5f_0) + \frac{1}{5}\delta(f - 5f_0) - \ldots\right] \qquad (1)$$

Next, we make use of the following formula. Given a periodic signal with the Fourier series expansion

$$G(f) = \sum_{n=-\infty}^{\infty} c_n[\delta(f - nf_0) + \delta(f - nf_0)] \qquad (2)$$

we may express the power spectral density of the signal by (see the solution to Problem 2.46)

$$S_g(f) = \sum_{n=-\infty}^{\infty} |c_n|^2[\delta(f - nf_0) + \delta(f - nf_0)] \qquad (3)$$

Hence, in light of Eqs. (2) and (3), we may use Eq. (1) pertaining to the square wave of Fig. 2.48 to write

$$S_g(f) = \frac{1}{\pi^2}[\delta(f - f_0) + \delta(f + f_0) - \frac{1}{9}\delta(f - 3f_0) - \frac{1}{9}\delta(f + 3f_0)$$

$$+ \frac{1}{25}\delta(f - 5f_0) + \frac{1}{25}\delta(f + 5f_0) - \ldots] \qquad (4)$$

The inverse Fourier transform of $S_g(f)$ defined in Eq. (4) yields the autocorrelation function

$$R_x(\tau) = \frac{2}{\pi^2}\left[\cos(2\pi f_0 t) - \frac{1}{9}\cos(6\pi f_0 t) + \frac{1}{25}\cos(10\pi f_0 t) - \ldots\right] \qquad (5)$$

which follows the symmetric triangular waveform plotted in Fig. 1.

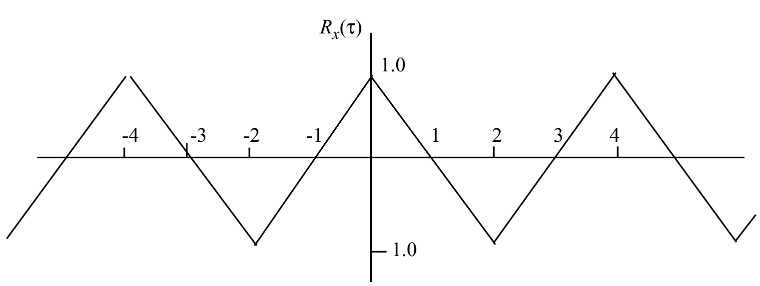


Figure 1

The average power of the square wave is defined by

$$
\begin{aligned}
P_g &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g^2(t)dt \\
&= \frac{1}{4}\left[\int_{-2}^{2}(1)^2 dt\right] \\
&= 1
\end{aligned}
$$

Additional Note: A follow-up exercise for the reader is to demonstrate that the triangular wave of Fig. 1 has the Fourier series representation given in Eq. (5); see Eq. (4) of the solution to Problem 2.49.

Problem 2.46

We are given the two periodic signals

$$
g_{p1}(t) = \sum_{n=-\infty}^{\infty} c_{2,n}\exp\left(-\frac{j2\pi nt}{T_0}\right)
$$

$$
g_{p2}(t) = \sum_{n=-\infty}^{\infty} c_{2,n}\exp\left(-\frac{j2\pi nt}{T_0}\right)
$$

Applying the formula for autocorrelation function

$$
R_{12}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{p1}(t)g_{p2}^{*}(t-\tau)dt
$$

we get

$$
\begin{aligned}
R_{12}(\tau) &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left\{\sum_{n=-\infty}^{\infty} c_{1,n}\exp\left(-\frac{j2\pi nt}{T_0}\right)\sum_{m=-\infty}^{\infty} c_{2,n}^{*}\exp\left(\frac{j2\pi nt}{T_0}(t-\tau)\right)\right\}dt \\
&= \frac{1}{T_0}\sum_{m=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{1,n}c_{2,n}^{*}\exp(-j2\pi m\tau/T_0)\int_{-T_0/2}^{T_0/2}\exp\left(\frac{j2\pi}{T_0}(m-n)t\right)dt \qquad (1)
\end{aligned}
$$

Consider the integral

$$
\begin{aligned}
\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\exp\left(\frac{j2\pi}{T_0}(m\text{-}n)t\right)dt &= \frac{1}{j2\pi(m\text{-}n)}[\exp(j\pi(m\text{-}n)) - \exp(-j\pi(m\text{-}n))] \\
&= \frac{1}{\pi(m\text{-}n)}\sin(\pi(m\text{-}n)) \\
&= \operatorname{sinc}(m\text{-}n) \\
&= \begin{cases} 1 & \text{for } m = n \\ 2 & \text{otherwise} \end{cases} \qquad (2)
\end{aligned}
$$

Using Eq. (2) in (1), we get the simplified result

$$R_{12}(\tau) = \sum_{n=-\infty}^{\infty} c_{1,n} c_{2,n}^{*} \exp(-j2\pi n t / T_0) \tag{3}$$

Finally, taking the Fourier transform of $R_{12}(\tau)$ defined in Eq. (3), we get

$$S_{12}(f) = \mathbf{F}[R_{12}(\tau)]$$

$$= \sum_{n=-\infty}^{\infty} c_{1,n} c_{2,n}^{*} \delta\left(f - \frac{n}{T_0}\right)$$

We may thus finally write

$$R_{12}(\tau) \rightleftharpoons \sum_{n=-\infty}^{\infty} c_{1,n} c_{2,n}^{*} \delta\left(f - \frac{n}{T_0}\right) \tag{4}$$

Note: For the special case of $g_{p1}(t) = g_{p2}(t) = g_p(t)$, Eq. (4) reduces to the simple form

$$R_{12}(\tau) \rightleftharpoons \sum_{n=-\infty}^{\infty} |c_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$

Problem 2.47

(a) We are given the autocorrelation formula

$$R_{g_p}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t) g_p^{*}(t-\tau)dt \tag{1}$$

For the problem at hand, $g_p(t)$ is defined by

$$g_p(t) = A\cos(2\pi f_c t + \theta) \tag{2}$$

The use of Eq. (2) in (1) yields

$$R_{g_p}(\tau) = \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2} \cos(2\pi f_c t + \theta)\cos(2\pi f_c t - 2\pi f_c \tau + \theta)dt$$

$$= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(4\pi f_c t - 2\pi f_c \tau + 2\theta) + \cos(2\pi f_c \tau)dt$$

$$= \frac{A^2}{2}\cos(2\pi f_c \tau) \tag{3}$$

The waveform of $R_{g_p}(\tau)$ is plotted in Fig. 1. Examining this figure, we observe two important points:

1. Information on the phase θ of the sinusoidal wave $g_p(t)$ is lost in calculating the autocorrelation function.

2. The autocorrelation function $R_{g_p}(\tau)$ has the same waveform as the input sinusoidal signal $g_p(t)$.

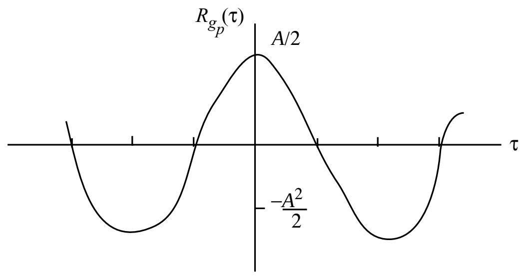


Figure 1

(b) From Fig. 1, we readily find that

$$R_{g_p}(0) = \frac{A^2}{2}$$

which is an intuitively satisfying result.

Problem 2.48

(a) The square wave $g_p(t)$ has the waveform plotted in Fig. 1.

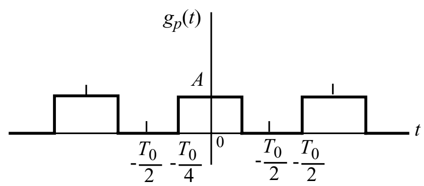


Figure 1

To apply the formula for the autocorrelation function

$$R_{g_p}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t) g_p^*(t-\tau)dt \tag{1}$$

we proceed in stages, as pictured in Fig. 2.

1. For $\tau = 0$. $g_p(t)$ and $g_p^*(t-\tau)$ overlap each other completely. Hence,

$$R_{12}(0) = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A^2 dt$$

$$= \frac{A^2}{2} \tag{2}$$

2. For $0 < \tau < T_0/4$, we find from Fig. 2(a) that

$$R_{g_p}(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{4}+\tau}^{T_0/4} A^2 dt$$

$$= \frac{A^2}{T_0}\left(\frac{T_0}{2} - \tau\right) \tag{3}$$

3. For $T_0/4 < \tau < T_0/2$, we find from Fig. 2(b) that

$$R_{g_p}(\tau) = \frac{1}{T_0}\int_{-\frac{3T_0}{4}+\tau}^{-T_0/4} A^2 dt$$

$$= \frac{A^2}{T_0}\left(\frac{T_0}{2} - \tau\right) \tag{4}$$

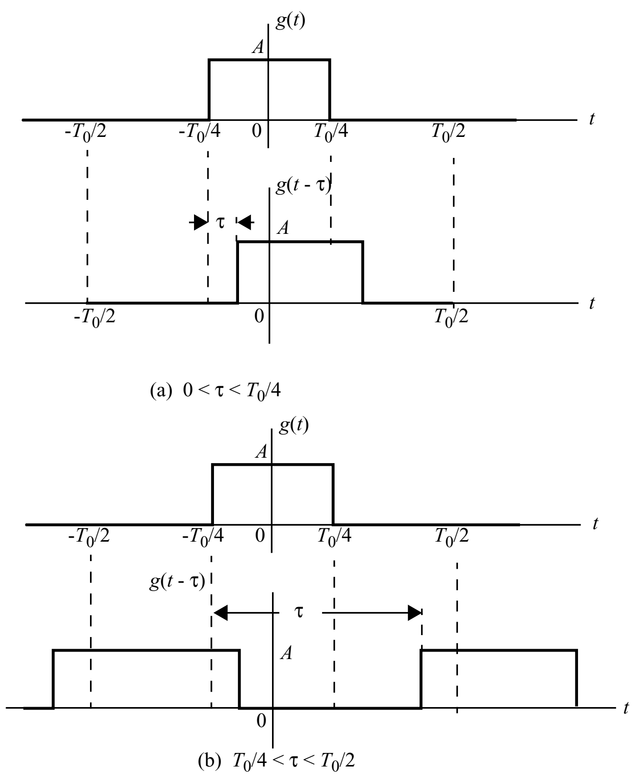


Figure 2

Piercing these results together, we get the picture depicted in Fig. 3 for $R_{g_p}(\tau)$, where we have used the following facts:

1. $R_{g_p}(\tau)$ is periodic with exactly the same period as the square wave $g_p(t)$.
2. $R_{g_p}(\tau)$ is an even function of the lag τ.

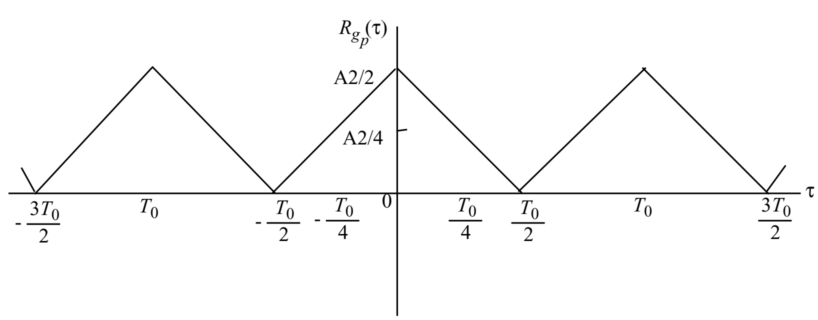


Figure 3

Notes on Problems 2.47 and 2.48

In defining the formula for the autocorrelation function $R_{g_p}(\tau)$ for a periodic signal $g_p(t)$ in Problem 2.47, the $g_p(t)$ is formulated in its most general form: complex fourier series expansion, assuming that $g_p(t)$ is complex valued. However, in presenting the solutions to Problems 2.47 and 2.48, we did not make use of the complex Fourier series for $g_p(t)$. The reason for doing so is two-fold:

- The sinusoidal wave in Problem 2.47 and square wave in Problem 2.48 are real valued.
- The solutions were handled by making direct use of the rather simple forms of both input signals $g_p(t)$.

Problem 2.49

We first note that for periodic signals, the power spectral density and autocorrelation function form a Fourier-transform pair:

$R_{g_p}(\tau) \rightleftharpoons S_{g_p}(f)$

(a) For the sinusoidal wave

$g_p(t) = A\cos(2\pi f_c t + \theta)$

the autocorrelation function was derived in the solution to Problem 2.47 and repeated here for convenience of presentation:

$$R_{g_p}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) \tag{1}$$

Hence, applying the Fourier transform to $R_{g_p}(\tau)$ yields the power spectral density

$$S_{g_p}(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] \tag{2}$$

(b) For the square wave of Problem 2.48, the autocorrelation function $R_{g_p}(\tau)$ is pictured in Fig. 3 in the solution to that problem. Specifically, one period of $R_{g_p}(\tau)$ is defined by

$$R_{g_p}(\tau) = \begin{cases} \dfrac{A^2 T_0}{4}\left(1 - \dfrac{2\tau}{T_0}\right) & \text{for } 0 < \tau < \dfrac{T_0}{2} \\ \dfrac{A^2 T_0}{4}\left(1 + \dfrac{2\tau}{T_0}\right) & \text{for } -\dfrac{T_0}{2} < \tau < 0 \end{cases} \tag{3}$$

From the Fourier-transform pairs table of Appendix 6, we find that the Fourier transform of the triangular pulse

$$g(t) = \begin{cases} 1 - \dfrac{|t|}{T}, & |t| \le T \\ 0, & |t| \ge 0 \end{cases}$$

is $T\text{sinc}^2(fT)$. Adapting this pair to the problem at hand, we may state that the Fourier transform of the triangular pulse defined in Eq. (3) is

$$G(f) = \frac{A^2 T_0^2}{8} \text{sinc}^2\left(\frac{f T_0}{2}\right)$$

Finally, applying Eq. (2.88) in the text book to the problem at hand, we get the desired result

$$S_{g_p}(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{A^2 T_0^2}{8} \text{sinc}^2\left(\frac{n f_0 T_0}{2}\right) \delta(f - n f_0)$$

$$= \frac{A^2 T_0^2}{8} \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2}\right) \delta(f - n f_0) \qquad (4)$$

Problem 2.50

(a) In the flow graph of Fig. 2.47 in the text book for the 8-point FFT algorithm, the incoming data sequence g_n is in normal order, but the transform sequence produced by the algorithm is in bit-reversed order. In Fig. 1, we show another implementation of the FFT algorithm, in which the transform sequence is in normal order, but the incoming data sequence is in bit-reversed order. In other words, the new flow graph of Fig. 1 portrays the decimation-in-time version of the FFT algorithm.

The flow graph of Fig. 1 shown here is obtained from the flow graph of Fig. 2.46 in the text as follows:

- All the nodes that are horizontally adjacent to G_4 in Fig. 2.46 are interchanged with all the nodes that are horizontally adjacent to G_1.
- In a similar way, all the nodes that are horizontally adjacent to G_6 in Fig. 2.46 are interchanged with the nodes that are horizontally adjacent to G_3.
- The nodes that are horizontally adjacent to G_0, G_2, G_5 and G_7 are left unchanged.

(b) Comparing the decimation-in-frequency version of the FFT algorithm in Fig. 2.46 of the text with the decimation-in-time version of the algorithm presented here as Fig. 1, we may make the following observations:

- In the decimation-in-frequency version, the input data sequence is in normal order and the output transform sequence is in bit-reversed order. The reverse applies to the decimation-in-time version: the input data sequence is in bit-reversed order and the output transform sequence is in normal order.
- Both versions of the FFT algorithm contain the same number of butterflies, but in reversed order.
- The computational complexity is the same for both versions of the algorithm.

In the final analysis, it is merely a matter of choice as to which particular version of the FFT algorithm is used to compute the discrete Fourier transform in practice.

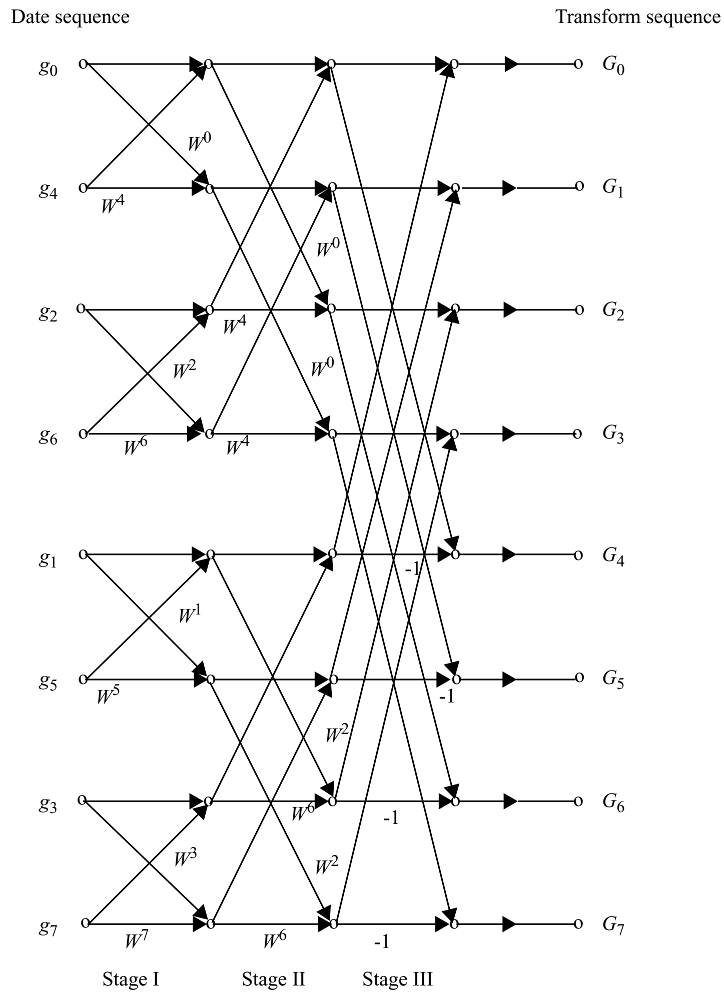


Figure 1: Decimation-in-time FFT algorithm

Problem 2.51

(a) According to Schwarz's inequality (see Appendix 5)

$$\left\{\int_{-\infty}^{\infty}[g_1^*(t)g_2(t)+g_1(t)g_2^*(t)]dt\right\}^2 \le 4\int_{-\infty}^{\infty}|g_1(t)|^2dt\int_{-\infty}^{\infty}|g_2(t)|^2dt \qquad (1)$$

Define

$$g_1(t) = tg(t)$$

$$g_2(t) = \frac{dg(t)}{dt}$$

Then, for the left-hand side of the inequality (1) we may write

$$\int_{-\infty}^{\infty}\left[tg^*(t)\frac{dg(t)}{dt}+tg(t)\frac{dg^*(t)}{dt}\right]dt = \int_{-\infty}^{\infty} t\frac{d}{dt}|g(t)|^2 dt$$

Integrating by parts:

$$\int_{-\infty}^{\infty} t\frac{d}{dt}|g(t)|^2 dt = [|g(t)|^2]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}|g(t)|^2 dt$$

Assuming that $|g(t)| \to 0$ faster than $1/\sqrt{t}$ as $|t| \to \infty$, we get

$$\int_{-\infty}^{\infty} t\frac{d}{dt}|g(t)|^2 dt = \int_{-\infty}^{\infty}|g(t)|^2 dt$$

Therefore, using Schwarz's inequality we have

$$\left[\int_{-\infty}^{\infty}|g(t)|^2 dt\right]^2 \le 4\int_{-\infty}^{\infty} t^2|g(t)|^2 dt\int_{-\infty}^{\infty}\left|\frac{dg(t)}{dt}\right|^2 dt$$

We now recognize that

$\frac{dg(t)}{dt} \rightleftharpoons j2\pi f G(f)$ from the differentiation property of the Fourier transform

$\int_{-\infty}^{\infty}|g(t)|^2 dt = \int_{-\infty}^{\infty}|G(f)|^2 df$ from Rayleigh's energy theorem

Hence,

$$\frac{\int_{-\infty}^{\infty} t^2|g(t)|^2 dt}{\int_{-\infty}^{\infty}|g(t)|^2 dt}\cdot\frac{\int_{-\infty}^{\infty}(2\pi f)^2|G(f)|^2 df}{\int_{-\infty}^{\infty}|G(f)|^2 df} \ge \frac{1}{4}$$

Using the root mean-square (rms) definitions of bandwidth and duration, we may write

$$4\pi^2 T_{\text{rms}}^2 W_{\text{rms}}^2 \ge \frac{1}{4}$$

That is,

$$T_{\text{rms}} W_{\text{rms}} \ge \frac{1}{4\pi}$$

(b) For the Gaussian pulse $g(t) = \exp(-\pi t^2)$, we have $G(f) = \exp(--\pi f^2)$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}|g(t)|^2 dt &= \int_{-\infty}^{\infty}\exp(-2\pi t^2)dt \\ &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\exp(-2\pi t^2)dt \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} t^2|g(t)|^2 dt &= \int_{-\infty}^{\infty} t^2\exp(-2\pi t^2)dt \\ &= \left[-\frac{t}{4\pi}\exp(-2\pi t^2)\right]_{-\infty}^{\infty} + \frac{t}{4\pi}\int_{-\infty}^{\infty}\exp(-2\pi t^2)dt\end{aligned}$$

$$= \frac{1}{4\sqrt{2}\pi}$$

Therefore, we may write

$$T_{\text{rms}} = \left(\frac{\sqrt{2}}{4\sqrt{2}\pi}\right)^{1/2} = \frac{1}{\sqrt{2\pi}}$$

Similarly,

$$\int_{-\infty}^{\infty} |G(f)|^2 df = \frac{1}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} f^2 |G(f)|^2 df = \frac{1}{4\sqrt{2}\pi}$$

$$W_{\text{rms}} = \frac{1}{2\sqrt{\pi}}$$

We thus find that for the Gaussian pulse $\exp(-\pi t^2)$,

$$T_{\text{rms}} W_{\text{rms}} = \frac{1}{4\pi}$$

and the relation connecting T_{rms} and W_{rms} is therefore satisfied with the equality sign.

Problem 2.52

(a) $g(t) = \dfrac{\sin t}{t}$

The Hilbert transform of $\sin t/t$ is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau(t-\tau)} d\tau$$

$$= \frac{1}{\pi t}\int_{-\infty}^{\infty} \left(\frac{1}{\tau} + \frac{1}{t-\tau}\right) \sin\tau \, d\tau$$

$$= \frac{1}{\pi t}\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau} d\tau + \frac{1}{\pi t}\int_{-\infty}^{\infty} \frac{\sin\tau}{t-\tau} d\tau \qquad (10$$

We now note that (see the mathematical tables of Appendix 6)

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t) dt = 1$$

Therefore, for the first integral of Eq. (1) we have

$$\int_{-\infty}^{\infty} \frac{\sin\tau}{\tau} d\tau = \pi$$

Next, for the second integral of Eq. (1) we write

$$\int_{-\infty}^{\infty} \frac{\sin\tau}{t-\tau} d\tau = \int_{-\infty}^{\infty} \frac{\sin(t-\tau)}{\tau} d\tau$$

$$= \sin t \int_{-\infty}^{\infty} \frac{\cos \tau}{\tau} d\tau - \cos t \int_{-\infty}^{\infty} \frac{\sin(\tau)}{\tau} d\tau$$

$$= -\pi \cos t$$

We thus obtain the Hilbert transform

$$\hat{g}(t) = \frac{1}{t}(1 - \cos t)$$

(b) $g(t) = \text{rect}(t)$

$$= \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$

The Hilbert transform of rect(t) is given by

$$\hat{g}(t) = \frac{1}{\pi} P \int_{-1/2}^{1/2} \frac{1}{t - \tau} d\tau$$

where P denotes the "principal value of". When $t < -1/2$ the singularity in the integrand is below the range of integration and the significant values of $t - \tau$ are negative. We then have

$$\hat{g}(t) = -\frac{1}{\pi}[\ln(t - \tau)]_{-1/2}^{1/2}$$

$$= -\frac{1}{\pi} \ln\left(\frac{t - 1/2}{t + 1/2}\right), \qquad t < -\frac{1}{2} \tag{2}$$

where ln denotes the natural

When $t > 1/2$, the singularity is above the range of integration and the significant values of $t - \tau$ are positive. The corresponding value of $\hat{g}(t)$ is

$$\hat{g}(t) = -\frac{1}{\pi}[\ln(t - \tau)]_{-1/2}^{1/2}$$

$$= -\frac{1}{\pi} \ln\left(\frac{t - 1/2}{t + 1/2}\right), \qquad t > \frac{1}{2}$$

For the case when $-\frac{1}{2} < t < \frac{1}{2}$, we write

$$\hat{g}(t) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\int_{-1/2}^{t-\varepsilon} \frac{d\tau}{t-\tau} + \int_{t-\varepsilon}^{1/2} \frac{d\tau}{t-\tau} \right]$$

$$= \frac{1}{\pi} \lim_{\varepsilon \to 0} \{ [-\ln(t - \tau)]_{-1/2}^{t-\varepsilon} + [-\ln(t - \tau)]_{t+\varepsilon}^{1/2} \}$$

$$= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[-\ln\left(\frac{\varepsilon}{t + 1/2}\right) - \ln\left(-\frac{t - 1/2}{\varepsilon}\right) \right]$$

$$= -\frac{1}{\pi}\ln\left|\frac{\frac{1}{2}-t}{\frac{1}{2}+6}\right|, \qquad -\frac{1}{2} < t < \frac{1}{2} \tag{3}$$

We finally combine the results of Eqs. (2) and (3) by expressing the Hilbert transform $\hat{g}(t)$ for all t as follows

$$\hat{g}(t) = -\frac{1}{\pi}\ln\left|\frac{t-(1/2)}{t+(1/2)}\right|$$

(c) $g(t) = \delta(t)$

The Hilbert transform of the delta function $\delta(t)$ is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\tau)}{t-\tau}d\tau$$

Using the sifting property of the delta function, we get the desired result

$$\hat{g}(t) = \frac{1}{\pi t}$$

(d) $g(t) = \dfrac{1}{1+t^2}$

The Hilbert transform of $1/(t+t^2)$ is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+\tau^2)(t-\tau)}d\tau$$

$$= \frac{1}{\pi(1+t^2)}\int_{-\infty}^{\infty}\left(\frac{t+\tau}{1+\tau^2}+\frac{1}{t-\tau}\right)d\tau \tag{4}$$

But (see the mathematical tables of Appendix 6)

$$\int_{-\infty}^{\infty}\frac{1}{1+\tau^2}d\tau = \pi$$

$$\int_{-\infty}^{\infty}\frac{\tau}{1+\tau^2}d\tau = 0$$

$$\int_{-\infty}^{\infty}\frac{1}{(t-\tau)}d\tau = 0$$

Therefore, Eq. (4) reduces to

$$\hat{g}(t) = \frac{t}{1+\tau^2}$$

Problem 2.53

We are given the one-sided frequency function

$$G(f) = \begin{cases} \exp(-f), & f > 0 \\ \frac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases} \tag{1}$$

Applying the inverse Fourier transform to Eq. (1) yields the corresponding time function

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df \\ &= \int_{0}^{\infty} \exp(-f)\exp(j2\pi ft)df \\ &= \int_{0}^{\infty} \exp(f(j2\pi t - 1))df \\ &= \frac{1}{-1 + j2\pi t}\exp(f(j2\pi t - 1))\Big|_{f=0}^{\infty} \\ &= \frac{1}{1 - j2\pi t} \end{aligned} \tag{2}$$

Expressing $g(t)$ in terms of its real and imaginary parts:

$$g(t) = \frac{1}{1 + (2\pi t)^2} + j\frac{2\pi t}{1 + (2\pi t)^2}$$

Hence, for the real and imaginary parts of $g(t)$ we may write

$$g_r(t) = \frac{1}{1 + (2\pi t)^2} \tag{3}$$

$$g_i(t) = \frac{2\pi t}{1 + (2\pi t)^2} \tag{4}$$

According to the last entry of the table in Problem 2.52, we find that $t/(1 + t^2)$ is the Hilbert transform of $1/(1 + t^2)$. From this pair, we readily see that the imaginary part $g_i(t)$ defined in Eq. (4) is indeed the Hilbert transform of the real part $g_r(t)$ defined in Eq. (3).

Problem 2.54

From the hilbert transform

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau}d\tau$$

we first note that the Hilbert transform $\hat{g}(t)$ is obtained by convolving the time function $g(t)$ with $1/(\pi t)$. Next, we note that the Fourier transform $1/(\pi t)$ is defined by

$$\mathbf{F}\left[\frac{1}{\pi t}\right] = \frac{1}{j}\mathrm{sgn}(f)$$

Accordingly, the process of Hilbert transformation is equivalent to passing the time function through a linear time-invariant system whose transfer function is defined by

⇌

$$H(f) = \frac{1}{j}\mathrm{sgn}(f) \tag{1}$$

from which we readily find that

$$\begin{aligned}\hat{G}(f) &= H(f)G(f) \\ &= \frac{1}{j}\mathrm{sgn}(f)G(f)\end{aligned}$$

In particular, Eq. (1) shows that the Hilbert transformer satisfies the following pair of conditions:

(a) Magnitude response

$|H(f)| = 1$ for all f

(b) Phase response

$$\arg[H(f)] = \begin{cases} -90^{\circ} & \text{for } f > 0 \\ +90^{\circ} & \text{for } f < 0 \end{cases}$$

which are the conditions specified under parts (a) and (b) of the problem statement.

To address part (c) of the statement, we note that the impulse response of the Hilbert transformer, namely,

$$\begin{aligned}h(t) &= \mathbf{F}^{-1}[H(f)] \\ &= \frac{1}{\pi t}\end{aligned}$$

is noncausal. Specifically, it has the dependence on time pictured in Fig. 1. This figure shows that $h(t)$ is infinitely large at $t = 0$ and nonzero for $t < 0$. The impulse response $h(t)$ is not physically realizable for the simple reason that it requires the impulse response to be infinitely large at time $t = 0$. Note also that it violates causality since $h(t)$ is nonzero for negative time.

Another way of stressing the fact that the Hilbert transform is non-physically realizable is to recognize that it is impossible to build a linear system that has a constant magnitude response for all frequencies, and a constant phase response of -90° for all positive frequencies and +90° for negative frequencies. We could approximately satisfy these two requirements over a finite band of frequencies, but not for a frequency band of infinite extent.