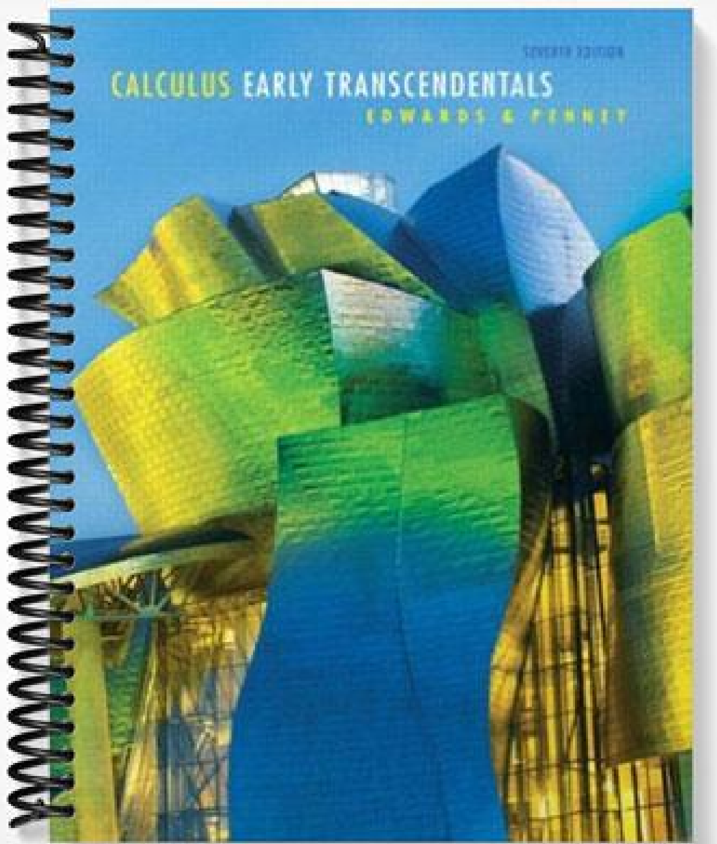
SOLUTIONS MANUAL



Section 2.1

2.1.1: $f(x) = 0 \cdot x^2 + 0 \cdot x + 5$, so $m(a) = 0 \cdot 2 \cdot a + 0 \equiv 0$. In particular, $m(2) = 0$, so the tangent line has equation $y - 5 = 0 \cdot (x - 0)$; that is, $y \equiv 5$.

2.1.2: $f(x) = 0 \cdot x^2 + 1 \cdot x + 0$, so $m(a) = 0 \cdot 2 \cdot a + 1 \equiv 1$. In particular, $m(2) = 1$, so the tangent line has equation $y - 2 = 1 \cdot (x - 2)$; that is, $y = x$.

2.1.3: Because $f(x) = 1 \cdot x^2 + 0 \cdot x + 0$, the slope-predictor is $m(a) = 2 \cdot 1 \cdot a + 0 = 2a$. Hence the line L tangent to the graph of f at $(2, f(2))$ has slope $m(2) = 4$. So an equation of L is $y - f(2) = 4(x - 2)$; that is, $y = 4x - 4$.

2.1.4: Because $f(x) = -2x^2 + 0 \cdot x + 1$, the slope-predictor for f is $m(a) = 2 \cdot (-2)a + 0 = -4a$. Thus the line L tangent to the graph of f at $(2, f(2))$ has slope $m(2) = -8$ and equation $y - f(2) = -8(x - 2)$; that is, $y = -8x + 9$.

2.1.5: Because $f(x) = 0 \cdot x^2 + 4x - 5$, the slope-predictor for f is $m(a) = 2 \cdot 0 \cdot a + 4 = 4$. So the line tangent to the graph of f at $(2, f(2))$ has slope 4 and therefore equation $y - 3 = 4(x - 2)$; that is, $y = 4x - 5$.

2.1.6: Because $f(x) = 0 \cdot x^2 - 3x + 7$, the slope-predictor for f is $m(a) = 2 \cdot 0 \cdot a - 3 = -3$. So the line tangent to the graph of f at $(2, f(2))$ has slope -3 and therefore equation $y - 1 = -3(x - 2)$; that is, $y = -3x + 7$.

2.1.7: Because $f(x) = 2x^2 - 3x + 4$, the slope-predictor for f is $m(a) = 2 \cdot 2 \cdot a - 3 = 4a - 3$. So the line tangent to the graph of f at $(2, f(2))$ has slope 5 and therefore equation $y - 6 = 5(x - 2)$; that is, $y = 5x - 4$.

2.1.8: Because $f(x) = (-1) \cdot x^2 - 3x + 5$, the slope-predictor for f is $m(a) = 2 \cdot (-1) \cdot a - 3 = -2a - 3$. So the line tangent to the graph of f at $(2, f(2))$ has slope -7 and therefore equation $y + 5 = -7(x - 2)$; that is, $y = -7x + 9$.

2.1.9: Because $f(x) = 2x^2 + 6x$, the slope-predictor for f is $m(a) = 4a + 6$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = 14$ and therefore equation $y - 20 = 14(x - 2)$; that is, $y = 14x - 8$.

2.1.10: Because $f(x) = -3x^2 + 15x$, the slope-predictor for f is $m(a) = -6a + 15$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = 3$ and therefore equation $y - 18 = 3(x - 2)$; that is, $y = 3x + 12$.

2.1.11: Because $f(x) = -\frac{1}{100}x^2 + 2x$, the slope-predictor for f is $m(a) = -\frac{2}{100}a + 2$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = -\frac{1}{25} + 2 = \frac{49}{25}$ and therefore equation $y - \frac{99}{25} = \frac{49}{25}(x - 2)$; that is, $25y = 49x + 1$.

2.1.12: Because $f(x) = -9x^2 - 12x$, the slope-predictor for f is $m(a) = -18a - 12$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = -48$ and therefore equation $y + 60 = -48(x - 2)$; that is, $y = -48x + 36$.

2.1.13: Because $f(x) = 4x^2 + 1$, the slope-predictor for f is $m(a) = 8a$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = 16$ and therefore equation $y - 17 = 16(x - 2)$; that is, $y = 16x - 15$.

2.1.14: Because $f(x) = 24x$, the slope-predictor for f is $m(a) = 24$. So the line tangent to the graph of f at $(2, f(2))$ has slope $m(2) = 24$ and therefore equation $y - 48 = 24(x - 2)$; that is, $y = 24x$.

2.1.15: If $f(x) = -x^2 + 10$, then the slope-predictor for f is $m(a) = -2a$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 0$. So the tangent line is horizontal at the point $(0, 10)$ and at no other point of the graph of f.

2.1.16: If $f(x) = -x^2 + 10x$, then the slope-predictor for f is $m(a) = -2a + 10$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 5$. So the tangent line is horizontal at the point $(5, 25)$ and at no other point of the graph of f.

2.1.17: If $f(x) = x^2 - 2x + 1$, then the slope-predictor for f is $m(a) = 2a - 2$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 1$. So the tangent line is horizontal at the point $(1, 0)$ and at no other point of the graph of f.

2.1.18: If $f(x) = x^2 + x - 2$, then the slope-predictor for f is $m(a) = 2a + 1$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = -\frac{1}{2}$. So the tangent line is horizontal at the point $\left(-\frac{1}{2}, -\frac{9}{4}\right)$ and at no other point of the graph of f.

2.1.19: If $f(x) = -\frac{1}{100}x^2 + x$, then the slope-predictor for f is $m(a) = -\frac{1}{50}a + 1$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 50$. So the tangent line is horizontal at the point $(50, 25)$ and at no other point of the graph of f.

2.1.20: If $f(x) = -x^2 + 100x$, then the slope-predictor for f is $m(a) = -2a + 100$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 50$. So the tangent line is horizontal at the point $(50, 2500)$ and at no other point of the graph of f.

2.1.21: If $f(x) = x^2 - 2x - 15$, then the slope-predictor for f is $m(a) = 2a - 2$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 1$. So the tangent line is horizontal at the point $(1, -16)$ and at no other point of the graph of f.

2.1.22: If $f(x) = x^2 - 10x + 25$, then the slope-predictor for f is $m(a) = 2a - 10$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 5$. So the tangent line is horizontal at the point $(5, 0)$ and at no other point of the graph of f.

2.1.23: If $f(x) = -x^2 + 70x$, then the slope-predictor for f is $m(a) = -2a + 70$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 35$. So the tangent line is horizontal at the point (35, 1225) and at no other point of the graph of f.

2.1.24: If $f(x) = x^2 - 20x + 100$, then the slope-predictor for f is $m(a) = 2a - 20$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 10$. So the tangent line is horizontal at the point (10, 0) and at no other point of the graph of f.

2.1.25: If $f(x) = x^2$, then the slope-predictor for f is $m(a) = 2a$. So the line tangent to the graph of f at the point $P(-2, 4)$ has slope $m(-2) = -4$ and the normal line at P has slope $\frac{1}{4}$. Hence an equation for the line tangent to the graph of f at P is $y - 4 = -4(x + 2)$; that is, $y = -4x - 4$. An equation for the line normal to the graph of f at P is $y - 4 = \frac{1}{4}(x + 2)$; that is, $4y = x + 18$.

2.1.26: If $f(x) = -2x^2 - x + 5$, then the slope-predictor for f is $m(a) = -4a - 1$. So the line tangent to the graph of f at the point $P(-1, 4)$ has slope $m(-1) = 3$ and the normal line at P has slope $-\frac{1}{3}$. Hence an equation for the line tangent to the graph of f at P is $y - 4 = 3(x + 1)$; that is, $y = 3x + 7$. An equation for the line normal to the graph of f at P is $y - 4 = -\frac{1}{3}(x + 1)$; that is, $x + 3y = 11$.

2.1.27: If $f(x) = 2x^2 + 3x - 5$, then the slope-predictor for f is $m(a) = 4a + 3$. So the line tangent to the graph of f at the point $P(2, 9)$ has slope $m(2) = 11$ and the normal line at P has slope $-\frac{1}{11}$. Hence an equation for the line tangent to the graph of f at P is $y - 9 = 11(x - 2)$; that is, $y = 11x - 13$. An equation for the line normal to the graph of f at P is $y - 9 = -\frac{1}{11}(x - 2)$; that is, $x + 11y = 101$.

2.1.28: If $f(x) = x^2$, then the slope-predictor for f is $m(a) = 2a$. Hence the line L tangent to the graph of f at the point (x_0, y_0) has slope $m(x_0) = 2x_0$. Because $y_0 = x_0^2$, an equation of L is $y - x_0^2 = 2x_0(x - x_0)$. To find where L meets the x-axis, we substitute $y = 0$ in the equation of L and solve for x:

$$0 - x_0^2 = 2x_0(x - x_0);$$

$$x - x_0 = -\tfrac{1}{2}x_0 \qquad (\text{if } x_0 \neq 0);$$

$$x = x_0 - \tfrac{1}{2}x_0 = \tfrac{1}{2}x_0.$$

Therefore if $x_0 \neq 0$, L meets the x-axis at the point $\left(\frac{1}{2}x_0, 0\right)$. If $x_0 = 0$, then L *is* the x-axis and therefore meets the x-axis at $\left(\frac{1}{2}x_0, 0\right) = (0, 0)$ as well as at every other point.

2.1.29: If the ball has height $y(t) = -16t^2 + 96t$ (feet) at time t (s), then the slope-predictor for y is $m(a) = -32a + 96$. Assuming that the maximum height of the ball occurs at the point on the graph of y where the tangent line is horizontal, we find that point by solving $m(a) = 0$ and find that $a = 3$. So the highest point on the graph of y is the point $(3, y(3)) = (3, 144)$. Therefore the ball reaches a maximum height of 144 (ft).

2.1.30: The slope-predictor for $A(x) = -x^2 + 50x$ is $m(a) = -2a + 50$. The highest point on the graph of A occurs where the tangent line is horizontal; that is, when $2a = 50$, so that $a = 25$. (We know it's the high point rather than the low point because the graph of $y = A(x)$ is a parabola that opens downward.) So the highest point on the graph of A is the point $(25, A(25)) = (25, 625)$. Because $a = 25$ is in the domain $[0, 50]$ of the function A, the maximum possible area of the rectangle is 625 (ft^2).

2.1.31: If the two positive numbers x and y have sum 50, then $y = 50 - x$, $x > 0$, and $x < 50$ (because $y > 0$). So the product of two such numbers is given by

$$p(x) = x(50 - x), \qquad 0 < x < 50.$$

The graph of $p(x) = -x^2 + 50x$ has a highest point because the graph of $y = p(x)$ is a parabola that opens downward. The slope-predictor for the function p is $m(a) = -2a + 50$. The highest point on the graph of p will occur when the tangent line is horizontal, so that $m(a) = 0$. This leads to $a = 25$, which does lie in the domain of p. Therefore the highest point on the graph of p is $(25, p(25)) = (25, 625)$. Hence the maximum possible value of $p(x)$ is 625. So the maximum possible product of two positive numbers with sum 50 is 625.

2.1.32: If $y = f(x) = -\frac{1}{625}x^2 + x$, then the slope predictor for f is $m(a) = -\frac{2}{625}a + 1$. (a) The projectile hits the ground at that point x for which $f(x) = 0$; that is, $x^2 = 625x$, so that $x = 0$ (which we reject; this is where the projectile *leaves* the ground) or $x = 625$. Because the projectile travels from $x = 0$ to $x = 625$, the horizontal distance it travels is 625 (ft). (b) To find the maximum height of the projectile, we find where the line tangent to the graph of f is horizontal. This occurs when $m(a) = 0$, so that $a = 312.5$. So the maximum height of the projectile is $f(312.5) = 156.25$ (ft). (It's a maximum rather than a minimum because the graph of $y = f(x)$ is a parabola that opens downwards *and* $x = 312.5$ does lie in the domain $[0, 625]$ of the function f.)

2.1.33: Suppose that the "other" line L is tangent to the parabola at the point (a, a^2). The slope-predictor for $y = f(x) = x^2$ is $m(a) = 2a$, so the line L has slope $m(a) = 2a$. (Note that a changes from a variable to a constant in the last sentence. This is dangerous but the notation has forced this situation upon us.) Using the two-point formula for slope, we can compute the slope of L in another way and equate our two results:

$$\begin{aligned}\frac{a^2 - 0}{a - 3} &= 2a;\\ a^2 &= 2a(a-3);\\ a &= 2a - 6; \qquad \text{(because } a \neq 0\text{)};\\ a &= 6.\end{aligned}$$

Therefore L has slope $m(6) = 12$. Because L passes through $(3, 0)$, an equation of L is $y - 0 = 12(x - 3)$; that is, $y = 12x - 36$.

2.1.34: If $y = f(x) = -x^2 + 4x$, then the slope-predictor for f is $m(a) = -2a + 4$. Suppose that the line L passes through the point $P(2, 5)$ and is tangent to the graph of f. Let $Q(c, f(c)) = (c, 4c - c^2)$ be the point of tangency. We can use the two points P and Q to compute the slope of L. We can also use the slope-predictor. We do so and equate the results:

$$\begin{aligned} \frac{4c - c^2 - 5}{c - 2} &= -2c + 4; \\ 4c - c^2 - 5 &= (c-2)(-2c+4) = -2c^2 + 8c - 8; \\ c^2 - 4c + 3 &= 0; \\ (c-1)(c-3) &= 0. \end{aligned}$$

Therefore $c = 1$ or $c = 3$. We have discovered that there are two points at which L may be tangent to the graph of f: $(1, f(1)) = (1, 3)$ and $(3, f(3)) = (3, 3)$. Thus one tangent line has slope 2 and the other has slope -2; their equations may be written as

$$y - 5 = 2(x - 2) \qquad \text{and} \qquad y - 5 = -2(x - 2).$$

2.1.35: Suppose that (a, a^2) is the point on the graph of $y = x^2$ closest to $(3, 0)$. Let L be the line segment from $(3, 0)$ to (a, a^2). Under the plausible assumption that L is normal to the tangent line at (a, a^2), we infer that the slope m of L is $-1/(2a)$ because the slope of the tangent line is $2a$. Because we can also compute m by using the two points known to lie on it, we find that

$$m = -\frac{1}{2a} = \frac{a^2 - 0}{a - 3}.$$

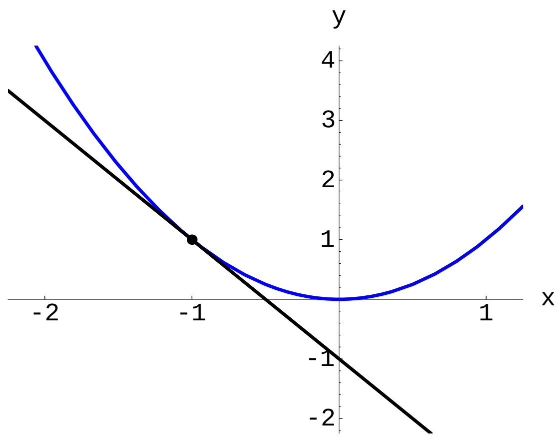
This leads to the equation $0 = 2a^3 + a - 3 = (a - 1)(2a^2 + 2a + 3)$, which has $a = 1$ as its only real solution (note that the discriminant of $2a^2 + 2a + 3$ is negative). Intuitively, it's clear that there *is* a point on the graph nearest $(3\,,0)$, so we have found it: That point is $(1, 1)$.

Alternatively, if (x, x^2) is an arbitrary point on the given parabola, then the distance from (x, x^2) to $(3, 0)$ is the square root of $f(x) = (x^2 - 0)^2 + (x - 3)^2 = x^4 + x^2 - 6x + 9$. A positive quantity is minimized when its square is minimized, so we minimize the distance from (x, x^2) to $(3, 0)$ by minimizing $f(x)$. The slope-predictor for f is $m(a) = 4a^3 + 2a - 6 = 2(a - 1)(2a^2 + 2x + 3)$, and (as before) the equation $m(a) = 0$ has only one real solution, $a = 1$. Again appealing to intuition for the existence of a point on the parabola nearest to $(3, 0)$, we see that it can only be the point $(1, 1)$. In Chapter 3 we will see how the existence of the closest point can be established without an appeal to the intuition.

2.1.36: Given: $f(x) = x^2$ and $a = -1$. We computed

$$\frac{f(a+h) - f(a-h)}{2h} \tag{1}$$

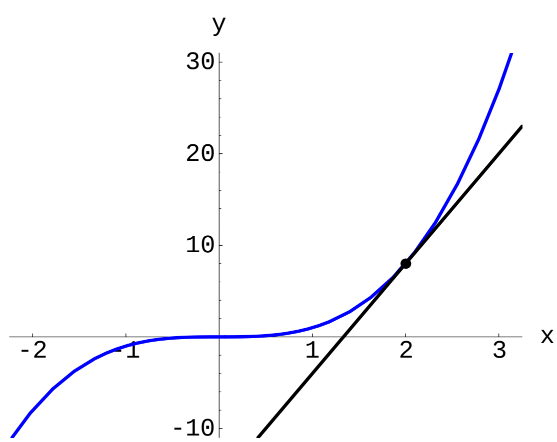
for $h = 10^{-1},\ 10^{-2},\ \ldots,$ and 10^{-10}. The values of the expression in (1) were all -2.00000000000000000000 (to twenty places). The numerical evidence overwhelmingly suggests that the slope of the tangent line is exactly -2 and thus that it has equation $y = -2x - 1$. The graph of this line and $y = f(x)$ are shown next.



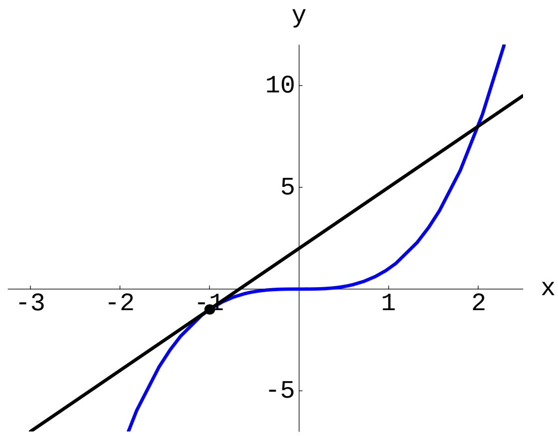
2.1.37: Given: $f(x) = x^3$ and $a = 2$. We computed

$$\frac{f(a+h) - f(a-h)}{2h} \tag{1}$$

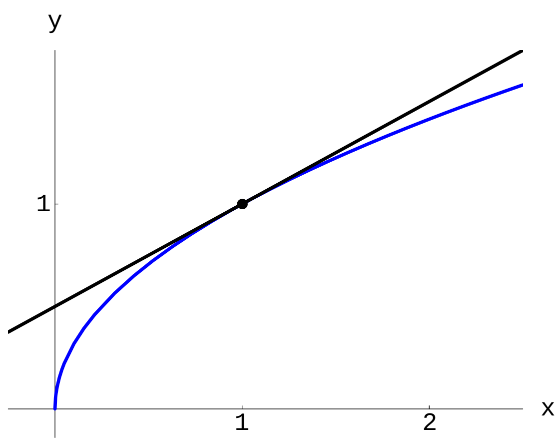
for $h = 10^{-1},\ 10^{-2},\ \ldots,\ 10^{-10}$. The values of the expression in (1) were $12.01,\ 12.0001,\ 12.000001,\ \ldots,$ 12.00000000000000000001. The numerical evidence overwhelmingly suggests that the slope of the tangent line is 12 and thus that it has equation $y = 12x - 16$. The graph of this line and $y = f(x)$ are shown next.



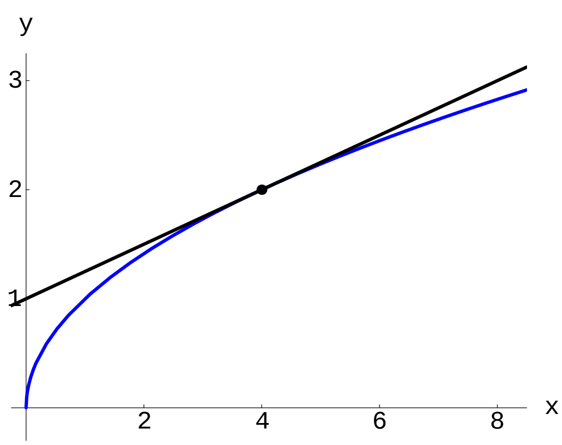
2.1.38: Using the techniques in the previous solution produced strong evidence that the slope of the tangent line is 3, so that its equation is $y = 3x + 2$. The graphs of $f(x) = x^3$ and the tangent line are shown next.



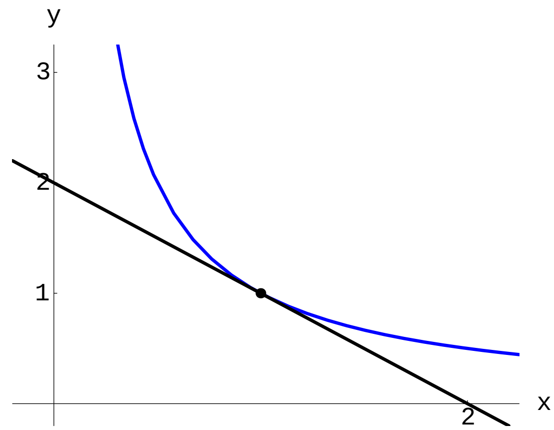
2.1.39: The numerical evidence suggests that the slope of the tangent line is $\frac{1}{2}$, so that its equation is $y = \frac{1}{2}(x+1)$. The graph of the tangent line and the graph of $f(x) = \sqrt{x}$ are shown next.



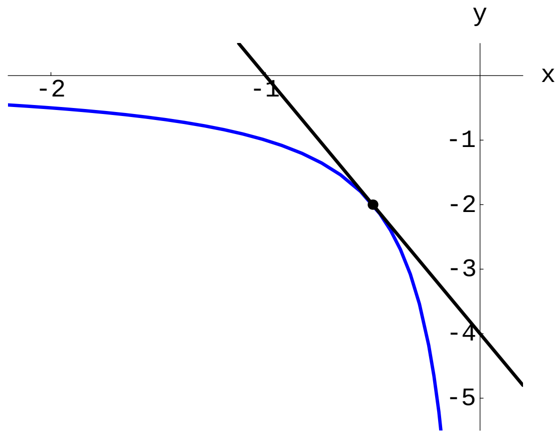
2.1.40: The numerical evidence suggests that the slope of the tangent line is $\frac{1}{4}$, so that its equation is $y = \frac{1}{4}(x+4)$. The graph of the tangent line and the graph of $f(x) = \sqrt{x}$ are shown next.



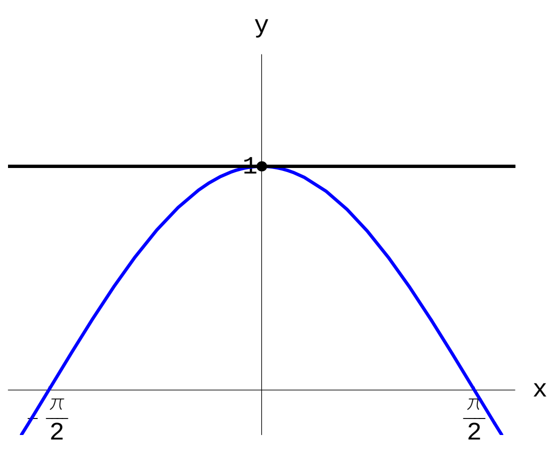
2.1.41: The numerical evidence suggests that the slope of the tangent line is -1, so that its equation is $y = -x + 2$. The graph of the tangent line and the graph of $f(x) = 1/x$ are shown next.



2.1.42: The numerical evidence suggests that the slope of the tangent line is -4, so that its equation is $y = -4x - 4$. The graph of the tangent line and the graph of $f(x) = 1/x$ are shown next.

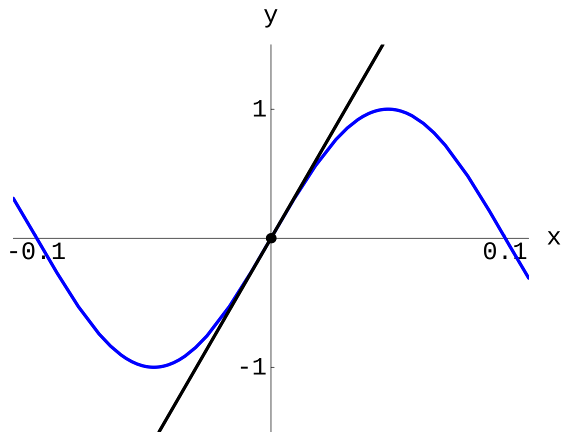


2.1.43: The numerical evidence suggests that the slope of the tangent line is 0, so that its equation is $y = 1$. The graph of the tangent line and the graph of $f(x) = \cos x$ are shown next.



2.1.44: The numerical evidence suggests that the slope of the tangent line is 10π, so that its equation is

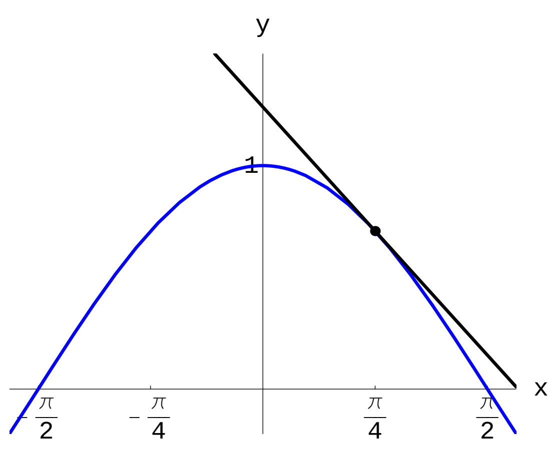
$y = 10\pi x$. The graph of the tangent line and the graph of $f(x) = \sin 10\pi x$ are shown next.



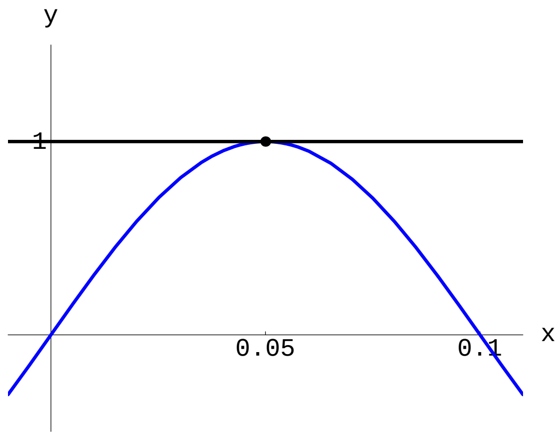
2.1.45: The numerical evidence suggests that the slope of the tangent line is $-1/\sqrt{2}$, so that its equation is

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right).$$

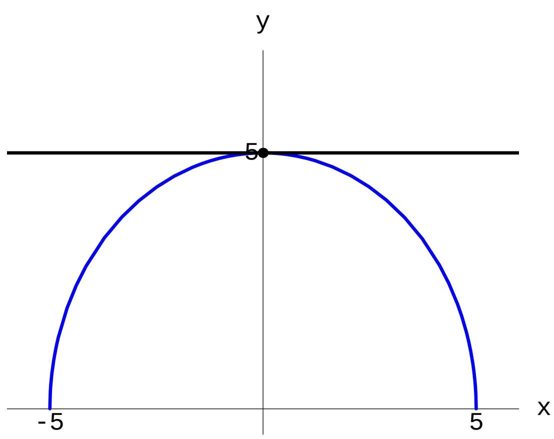
The graph of the tangent line and the graph of $f(x) = \cos x$ are shown next.



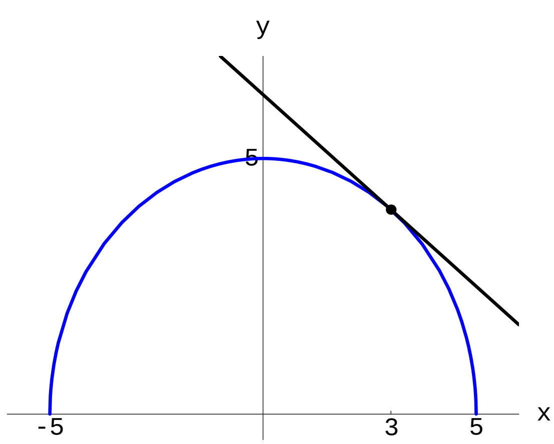
2.1.46: The numerical evidence suggests that the tangent line is horizontal, so that its equation is $y \equiv 1$. The graph of the tangent line and the graph of $f(x) = \sin 10\pi x$ are shown next.



2.1.47: The numerical evidence suggests that the tangent line is horizontal, so that its equation is $y \equiv 5$. The graph of the tangent line and the graph of $f(x) = \sqrt{25 - x^2}$ are shown next.



2.1.48: The numerical evidence suggests that the tangent line has slope $-\frac{3}{4}$, so that its equation is $3x + 4y = 25$. The graph of the tangent line and the graph of $f(x) = \sqrt{25 - x^2}$ are shown next.



Section 2.2

2.2.1: $\lim_{x\to 3}(3x^2+7x-12) = 3\left(\lim_{x\to 3} x\right)^2 + 7\left(\lim_{x\to 3} x\right) - \lim_{x\to 3} 12 = 3\cdot 3^2 + 7\cdot 3 - 12 = 36.$

2.2.2: $\lim_{x\to -2}\left(x^3-3x^2+5\right) = \lim_{x\to -2} x^3 - 3\lim_{x\to -2} x^2 + \lim_{x\to -2} 5 = -15.$

2.2.3: $\lim_{x\to 1}\left(x^2-1\right)\left(x^7+7x-4\right) = \lim_{x\to 1}\left(x^2-1\right)\cdot\lim_{x\to 1}\left(x^7+7x-4\right) = 0\cdot 4 = 0.$

2.2.4: $\lim_{x\to -2}\left(x^3-3x+3\right)\left(x^2+2x+5\right) = \lim_{x\to -2}\left(x^3-3x+3\right)\cdot\lim_{x\to -2}\left(x^2+2x+5\right) = 1\cdot 5 = 5.$

2.2.5: $\lim_{x\to 1}\dfrac{x+1}{x^2+x+1} = \dfrac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(x^2+x+1)} = \dfrac{2}{3}.$

2.2.6: $\lim_{t\to -2}\dfrac{t+2}{t^2+4} = \dfrac{\lim_{t\to -2}(t+2)}{\lim_{t\to -2}(t^2+4)} = \dfrac{0}{8} = 0.$

2.2.7: $\lim_{x\to 3}\dfrac{\left(x^2+1\right)^3}{\left(x^3-25\right)^3} = \dfrac{\lim_{x\to 3}\left(x^2+1\right)^3}{\lim_{x\to 3}\left(x^3-25\right)^3} = \dfrac{\left(\lim_{x\to 3}\left(x^2+1\right)\right)^3}{\left(\lim_{x\to 3}\left(x^3-25\right)\right)^3} = \dfrac{10^3}{2^3} = \dfrac{1000}{8} = 125.$

2.2.8: $\lim_{z\to -1}\dfrac{\left(3z^2+2z+1\right)^{10}}{\left(z^3+5\right)^5} = \dfrac{\lim_{z\to -1}\left(3z^2+2z+1\right)^{10}}{\lim_{z\to -1}\left(z^3+5\right)^5} = \dfrac{\left(\lim_{z\to -1}\left(3z^2+2z+1\right)\right)^{10}}{\left(\lim_{z\to -1}\left(z^3+5\right)\right)^5} = \dfrac{2^{10}}{4^5} = 1.$

2.2.9: $\lim_{x\to 1}\sqrt{4x+5} = \sqrt{\lim_{x\to 1}(4x+5)} = \sqrt{9} = 3.$

2.2.10: $\lim_{y\to 4}\sqrt{27-\sqrt{y}} = \sqrt{\lim_{y\to 4}\left(27-\sqrt{y}\,\right)} = \sqrt{25} = 5.$

2.2.11: $\lim_{x\to 3}\left(x^2-1\right)^{3/2} = \left(\lim_{x\to 3}\left(x^2-1\right)\right)^{3/2} = 8^{3/2} = 16\sqrt{2}.$

2.2.12: $\lim_{t\to -4}\sqrt{\dfrac{t+8}{25-t^2}} = \dfrac{\sqrt{\lim_{t\to -4}(t+8)}}{\sqrt{\lim_{t\to -4}\left(25-t^2\right)}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}.$

2.2.13: $\lim_{z\to 8}\dfrac{z^{2/3}}{z-\sqrt{2z}} = \dfrac{\lim_{z\to 8} z^{2/3}}{\lim_{z\to 8}\left(z-\sqrt{2z}\,\right)} = \dfrac{4}{4} = 1.$

2.2.14: $\lim_{t\to 2}\sqrt[3]{3t^3+4t-5} = \sqrt[3]{\lim_{t\to 2}\left(3t^3+4t-5\right)} = 3.$

2.2.15: $\lim_{w\to 0}\sqrt{(w-2)^4} = \sqrt{\lim_{w\to 0}(w-2)^4} = \sqrt{(-2)^4} = 4.$

2.2.16: $\lim_{t\to -4} \sqrt[3]{(t+1)^6} = \sqrt[3]{\lim_{t\to -4} (t+1)^6} = 9.$

2.2.17: $\lim_{x\to -2} \sqrt[3]{\dfrac{x+2}{(x-2)^2}} = \sqrt[3]{\lim_{x\to -2} \dfrac{(x+2)}{(x-2)^2}} = 0.$

2.2.18: $\lim_{y\to 5} \left(\dfrac{2y^2+2y+4}{6y-3}\right)^{1/3} = \left(\dfrac{64}{27}\right)^{1/3} = \dfrac{4}{3}.$

2.2.19: $\lim_{x\to -1} \dfrac{x+1}{x^2-x-2} = \lim_{x\to -1} \dfrac{x+1}{(x+1)(x-2)} = \lim_{x\to -1} \dfrac{1}{x-2} = -\dfrac{1}{3}.$

2.2.20: $\lim_{t\to 3} \dfrac{t^2-9}{t-3} = \lim_{t\to 3} \dfrac{(t-3)(t+3)}{t-3} = \lim_{t\to 3} (t+3) = 6.$

2.2.21: $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-4x+3} = \lim_{x\to 1} \dfrac{(x+2)(x-1)}{(x-3)(x-1)} = \lim_{x\to 1} \dfrac{x+2}{x-3} = -\dfrac{3}{2}.$

2.2.22: $\lim_{y\to -1/2} \dfrac{4y^2-1}{4y^2+8y+3} = \lim_{y\to -1/2} \dfrac{(2y-1)(2y+1)}{(2y+3)(2y+1)} = \lim_{y\to -1/2} \dfrac{2y-1}{2y+3} = -\dfrac{2}{2} = -1.$

2.2.23: $\lim_{t\to -3} \dfrac{t^2+6t+9}{t^2-9} = \lim_{t\to -3} \dfrac{(t+3)(t+3)}{(t+3)(t-3)} = \lim_{t\to -3} \dfrac{t+3}{t-3} = 0.$

2.2.24: $\lim_{x\to 2} \dfrac{x^2-4}{3x^2-2x-8} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(3x+4)} = \lim_{x\to 2} \dfrac{x+2}{3x+4} = \dfrac{2}{5}.$

2.2.25: $\lim_{z\to -2} \dfrac{(z+2)^2}{z^4-16} = \lim_{z\to -2} \dfrac{(z+2)(z+2)}{(z+2)(z-2)(z^2+4)} = \lim_{z\to -2} \dfrac{z+2}{(z-2)(z^2+4)} = 0.$

2.2.26: $\lim_{t\to 3} \dfrac{t^3-9t}{t^2-9} = \lim_{t\to 3} \dfrac{t(t^2-9)}{t^2-9} = 3.$

2.2.27: $\lim_{x\to 1} \dfrac{x^3-1}{x^4-1} = \lim_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)(x^2+1)} = \lim_{x\to 1} \dfrac{x^2+x+1}{(x+1)(x^2+1)} = \dfrac{3}{4}.$

2.2.28: $\lim_{y\to -3} \dfrac{y^3+27}{y^2-9} = \lim_{y\to -3} \dfrac{(y+3)(y^2-3y+9)}{(y+3)(y-3)} = \lim_{y\to -3} \dfrac{y^2-3y+9}{y-3} = -\dfrac{27}{6} = -\dfrac{9}{2}.$

2.2.29: $\lim_{x\to 3} \dfrac{\dfrac{1}{x}-\dfrac{1}{3}}{x-3} = \lim_{x\to 3} \left(\dfrac{3-x}{3x}\right)\left(\dfrac{1}{x-3}\right) = \lim_{x\to 3} \dfrac{-1}{3x} = -\dfrac{1}{9}.$

2.2.30: $\lim_{t\to 0} \dfrac{\dfrac{1}{2+t}-\dfrac{1}{2}}{t} = \lim_{t\to 0} \left(\dfrac{2-(2+t)}{2(2+t)}\right)\left(\dfrac{1}{t}\right) = \lim_{t\to 0} \left(\dfrac{2-2-t)}{2(2+t)}\right)\left(\dfrac{1}{t}\right) = \lim_{t\to 0} \dfrac{-1}{2(2+t)} = -\dfrac{1}{4}.$

2.2.31: $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim_{x\to 4} \sqrt{x}+2 = 4.$

2.2.32: $\lim_{x\to 9} \dfrac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \dfrac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})} = \lim_{x\to 9} \dfrac{1}{3+\sqrt{x}} = \dfrac{1}{6}.$

2.2.33: $$\lim_{t\to 0}\frac{\sqrt{t+4}-2}{t} = \lim_{t\to 0}\left(\frac{\sqrt{t+4}-2}{t}\right)\cdot\left(\frac{\sqrt{t+4}+2}{\sqrt{t+4}+2}\right)$$
$$= \lim_{t\to 0}\frac{t+4-4}{t\left(\sqrt{t+4}+2\right)}$$
$$= \lim_{t\to 0}\frac{t}{t\left(\sqrt{t+4}+2\right)} = \lim_{t\to 0}\frac{1}{\sqrt{t+4}+2} = \frac{1}{4}.$$

2.2.34: $$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{9+h}}-\frac{1}{3}\right) = \lim_{h\to 0}\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}$$
$$= \lim_{h\to 0}\left(\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}\right)\cdot\left(\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}}\right)$$
$$= \frac{9-(9+h)}{3h\sqrt{9+h}\left(3+\sqrt{9+h}\,\right)} = \lim_{h\to 0}\frac{-1}{3\sqrt{9+h}\left(3+\sqrt{9+h}\,\right)} = -\frac{1}{54}.$$

2.2.35: $$\lim_{x\to 4}\frac{x^2-16}{2-\sqrt{x}} = \lim_{x\to 4}\frac{(x+4)(\sqrt{x}-2)(\sqrt{x}+2)}{2-\sqrt{x}} = \lim_{x\to 4}\left[-(x+4)(\sqrt{x}+2)\right] = -32.$$

2.2.36: $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)\cdot\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$
$$= \lim_{x\to 0}\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\,\right)}$$
$$= \lim_{x\to 0}\frac{2}{\left(\sqrt{1+x}+\sqrt{1-x}\,\right)} = 1.$$

2.2.37: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^3-x^3}{h} = \dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = 3x^2+3xh+h^2 \to 3x^2$ as $h\to 0$. When $x=2$, $y=f(2)=x^3=8$ and the slope of the tangent line to this curve at $x=2$ is $3x^2=12$, so an equation of this tangent line is $y=12x-16$.

2.2.38: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\left(\dfrac{1}{x+h}\right)-\left(\dfrac{1}{x}\right)}{h} = \dfrac{x-(x+h)}{hx(x+h)} = \dfrac{-1}{x(x+h)} \to -\dfrac{1}{x^2}$ as $h\to 0$. When $x=2$, $y=f(2)=\frac{1}{2}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{4}$, so an equation of this tangent line is $y-\frac{1}{2}=-\frac{1}{4}(x-2)$; that is, $y=-\frac{1}{4}(x-4)$.

2.2.39: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \dfrac{x^2-(x+h)^2}{hx^2(x+h)^2} = \dfrac{-2x-h}{x^2(x+h)^2} \to -\dfrac{2}{x^3}$ as $h\to 0$. When $x=2$, $y=f(2)=\frac{1}{4}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{4}$, so an equation of this tangent line is $y-\frac{1}{4}=-\frac{1}{4}(x-2)$; that is, $y=-\frac{1}{4}(x-3)$.

2.2.40: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\left(\dfrac{1}{x+h+1}\right)-\left(\dfrac{1}{x+1}\right)}{h} = \dfrac{x+1-x-h-1}{h(x+1)(x+h+1)} = \dfrac{-1}{(x+1)(x+h+1)}$.
This approaches $-\dfrac{1}{(x+1)^2}$ as h approaches 0. When $x=2$, $y=f(2)=\frac{1}{3}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{9}$, so an equation of this tangent line is $y-\frac{1}{3}=-\frac{1}{9}(x-2)$; that is, $y=-\frac{1}{9}(x-5)$.

2.2.41: $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\left(\dfrac{2}{x+h-1}\right)-\left(\dfrac{2}{x-1}\right)}{h} = \dfrac{2(x-1-x-h+1)}{h(x-1)(x+h-1)} = \dfrac{-2}{(x-1)(x+h-1)}$.

This approaches $\dfrac{-2}{(x-1)^2}$ as h approaches 0. When $x = 2$, $y = f(2) = 2$ and the slope of the line tangent to this curve at $x = 2$ is -2, so an equation of this tangent line is $y-2 = -2(x-2)$; alternatively, $y = -2(x-3)$.

2.2.42: $$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{x+h}{x+h-1}\right) - \left(\dfrac{x}{x-1}\right)}{h} = \frac{(x-1)(x+h) - x^2 - xh + x}{h(x-1)(x+h-1)}$$
$$= \frac{-1}{(x-1)(x+h-1)} \to \frac{-1}{(x-1)^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = 2$ and the slope of the line tangent to this curve at $x = 2$ is -1, so an equation of this tangent line is $y - 2 = -1(x-2)$; that is, $y = -x + 4$.

2.2.43: $$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{1}{\sqrt{x+h+2}}\right) - \left(\dfrac{1}{\sqrt{x+2}}\right)}{h}$$
$$= \left(\frac{\sqrt{x+2} - \sqrt{x+h+2}}{h\sqrt{x+2}\sqrt{x+h+2}}\right) \cdot \left(\frac{\sqrt{x+2} + \sqrt{x+h+2}}{\sqrt{x+2} + \sqrt{x+h+2}}\right)$$
$$= \frac{-h}{h\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2} + \sqrt{x+h+2}\,\right)} \to \frac{-1}{(x+2)\left(2\sqrt{x+2}\,\right)}$$

as $h \to 0$. When $x = 2$, $y = f(2) = \frac{1}{2}$ and the slope of the line tangent to this curve at $x = 2$ is $-\frac{1}{16}$, so an equation of this tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-2)$; that is, $y = -\frac{1}{16}(x-10)$.

2.2.44: $$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + \dfrac{3}{x+h} - x^2 - \dfrac{3}{x}}{h}$$
$$= (2x+h) + \frac{-3}{x(x+h)} \to 2x - \frac{3}{x^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = \frac{11}{2}$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{13}{4}$, so an equation of this tangent line is $y - \frac{11}{2} = \frac{13}{4}(x-2)$.

2.2.45: $$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}$$
$$= \left(\frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}\right) \cdot \left(\frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}\right)$$
$$= \frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} \to \frac{1}{\sqrt{2x+5}} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = 3$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{1}{3}$, so an equation of this tangent line is $y - 3 = \frac{1}{3}(x-2)$; if you prefer, $y = \frac{1}{3}(x+7)$.

2.2.46: $$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{(x+h)^2}{x+h+1}\right) - \left(\dfrac{x^2}{x+1}\right)}{h} = \frac{(x+1)\,(x+h)^2 - (x+h+1)\left(x^2\right)}{h(x+1)(x+h+1)}$$
$$= \frac{x^2 + xh + 2x + h}{(x+1)(x+h+1)} \to \frac{x^2+2x}{(x+1)^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = \frac{4}{3}$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{8}{9}$, so an equation of this tangent line is $y - \frac{4}{3} = \frac{8}{9}(x-2)$; that is, $9y = 8x - 4$.

2.2.47:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	2.01	2.001	2.	2.	2.
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	1.99	1.999	2.	2.	2.

The limit appears to be 2.

2.2.48:

x	$1+10^{-2}$	$1+10^{-4}$	$1+10^{-6}$	$1+10^{-8}$	$1+10^{-10}$
$f(x)$	4.0604	4.0006	4.00001	4.	4.
x	$1-10^{-2}$	$1-10^{-4}$	$1-10^{-6}$	$1-10^{-8}$	$1-10^{-10}$
$f(x)$	3.9404	3.9994	3.99999	4.	4.

The limit appears to be 4.

2.2.49:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.16662	0.166666	0.166667	0.166667	0.166667
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	1.66713	0.166667	0.166667	0.166667	0.166667

The limit appears to be $\frac{1}{6}$.

2.2.50:

x	$4+10^{-2}$	$4+10^{-4}$	$4+10^{-6}$	$4+10^{-8}$	$4+10^{-10}$
$f(x)$	3.00187	3.00002	3.	3.	3.
x	$4-10^{-2}$	$4-10^{-4}$	$4-10^{-6}$	$4-10^{-8}$	$4-10^{-10}$
$f(x)$	2.99812	2.99998	3.	3.	3.

The limit appears to be 3.

2.2.51:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	−0.37128	−0.374963	−0.375	−0.375	−0.375
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	−0.378781	−0.375038	−0.375	−0.375	−0.375

The limit appears to be $-\frac{3}{8}$.

2.2.52:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	-0.222225	-0.222222	-0.222222	-0.222222	-0.222222
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	-0.222225	-0.222222	-0.222222	-0.222222	-0.222222

The limit appears to be $-\frac{2}{9}$.

2.2.53:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.999983	1.	1.	1.	1.
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.999983	1.	1.	1.	1.

The limit appears to be 1.

2.2.54:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.499996	0.5	0.5	0.499817	0
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.499996	0.5	0.5	0.499817	0

We must beware of round-off errors. The actual limit is $\frac{1}{2}$.

2.2.55:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.166666	0.166667	0.166667	0.166667	0.166667
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.166666	0.166667	0.166667	0.166667	0.166667

The limit appears to be $\frac{1}{6}$.

2.2.56:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	1.04723	1.00092	1.00001	1.	1.
x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.954898	0.999079	0.999986	1.	1.

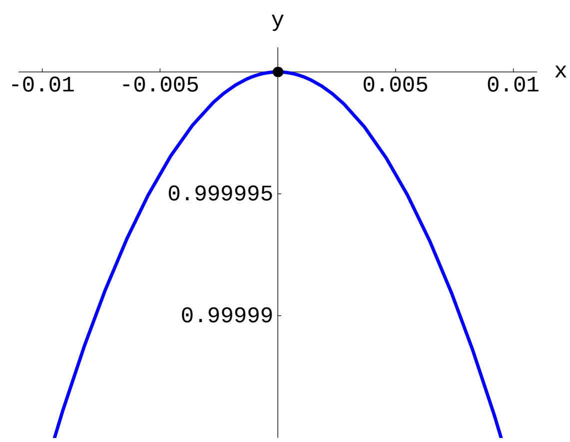
The limit appears to be 1.

2.2.57:

x	2^{-1}	2^{-5}	2^{-10}	2^{-15}	2^{-20}
$f(x)$	2.25	2.67699	2.71696	2.71824	2.71828
x	-2^{-1}	-2^{-5}	-2^{-10}	-2^{-15}	-2^{-20}
$f(x)$	4.	2.76210	2.71961	2.71832	2.71828

Perhaps the limit is the special number $e \approx 2.71828$ that was mentioned in the subsection on natural logarithms in Section 1.4. However, the numerical data here are insufficient to be convincing.

2.2.58: The graph of $y = (\sin x)/x$ on the interval $[-0.01, 0.01]$ is next.

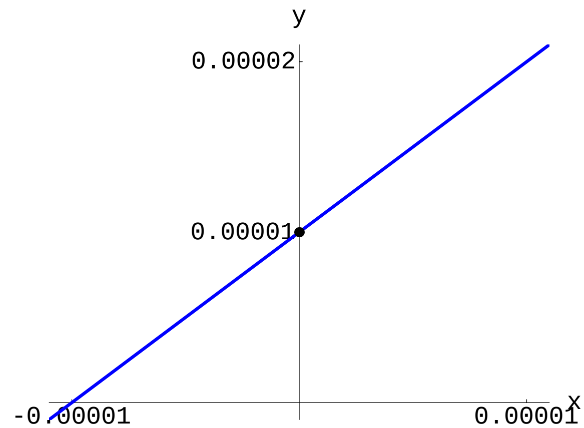


2.2.59: $\displaystyle\lim_{x\to 0} \frac{x - \tan x}{x^3} = -\frac{1}{3}.$ Answer: -0.3333.

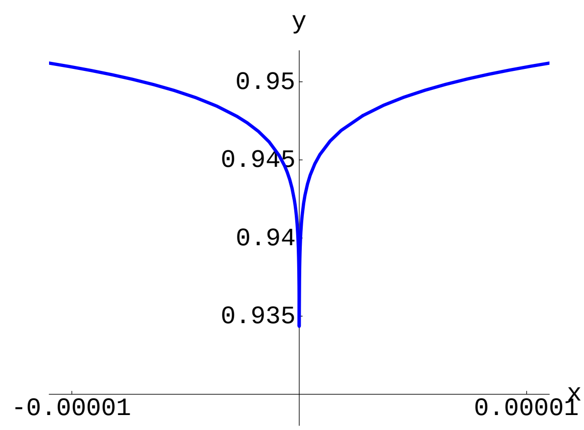
2.2.60: $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{\tan 5x} = \frac{2}{5}.$

2.2.61: $\sin\left(\dfrac{\pi}{2^{-n}}\right) = \sin\left(2\pi \cdot 2^{(n-1)}\right) = 0$ for every positive integer n. Therefore $\displaystyle\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$, if it were to exist, would be 0. Notice however that $\sin\left(3^n \cdot \dfrac{\pi}{2}\right)$ alternates between $+1$ and -1 for $n = 1, 2, 3, \ldots$. Therefore $\displaystyle\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist.

2.2.62: The graph of $f(x) = \sin x + 10^{-5} \cos x$ on the interval $[-0.00001, 0.00001]$ is shown next. The graph makes it clear that the limit is certainly not zero and almost certainly is 10^{-5}.



2.2.63: The graph of $f(x) = (\log_{10}(1/|x|))^{-1/32}$ is shown next, as well as a table of values of $f(x)$ for x very close to zero. The table was generated by *Mathematica*,, but virtually any computer algebra system will produce similar results.



x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
10^{-1}	1.0000	10^{-6}	0.9455	10^{-11}	0.9278	10^{-16}	0.9170
10^{-2}	0.9786	10^{-7}	0.9410	10^{-12}	0.9253	10^{-17}	0.9153
10^{-3}	0.9663	10^{-8}	0.9371	10^{-13}	0.9230	10^{-18}	0.9136
10^{-4}	0.9576	10^{-9}	0.9336	10^{-14}	0.9208	10^{-19}	0.9121
10^{-5}	0.9509	10^{-10}	0.9306	10^{-15}	0.9189	10^{-20}	0.9106

2.2.64: The slope of the line tangent to the graph of $y = 10^x$ at the point (0, 1) is

$$L = \lim_{h\to 0} \frac{10^{0+h} - 10^0}{h} = \lim_{h\to 0} \frac{10^h - 1}{h}.$$

With $h = 0.1,\ 0.01,\ 0.001,\ \ldots,\ 0.000001$, a calculator reports that the corresponding values of $(10^h - 1)/h$ are (approximately) $2.58925,\ 2.32930,\ 2.30524,\ \ldots,$ and 2.30259. This is fair evidence that $L = \ln 10 \approx$

2.302585. The slope-predictor for $y = 10^x$ is

$$m(x) = \lim_{h \to 0} \frac{10^{x+h} - 10^x}{h} = 10^x \cdot \left(\lim_{h \to 0} \frac{10^h - 1}{h} \right) = L \cdot 10^x.$$

The line tangent to the graph of $y = 10^x$ at the point $P(a, 10^a)$ has predicted equation

$$y - 10^a = L \cdot 10^a \cdot (x - a).$$

To see the graph of $y = 10^x$ near P and the line predicted to be tangent to that graph at P, enter the *Mathematica* commands

```
a = 2;      (* or any other value you please *)

Plot[ { 10∧x, 10∧a + (10∧a)*Log[10]*(x - a) }, { x, a - 1, a + 1 }];
```

Section 2.3

2.3.1: $\theta \cdot \dfrac{\theta}{\sin\theta} \to 0 \cdot 1 = 0$ as $\theta \to 0$.

2.3.2: $\dfrac{\sin\theta}{\theta} \cdot \dfrac{\sin\theta}{\theta} \to 1 \cdot 1 = 1$ as $\theta \to 0$.

2.3.3: Multiply numerator and denominator by $1 + \cos\theta$ (the *conjugate* of the numerator) to obtain

$$\lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta^2(1 + \cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{1 + \cos\theta} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

2.3.4: $\dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta} = \dfrac{\sin\theta}{\theta} \cdot \dfrac{1}{\cos\theta} \to 1 \cdot \dfrac{1}{1} = 1$ as $\theta \to 0$.

2.3.5: Divide each term in numerator and denominator by t. Then it's clear that the denominator is approaching zero whereas the numerator is not, so the limit does not exist. Because the numerator is positive and the denominator is approaching zero through negative values, the answer $-\infty$ is also correct.

2.3.6: As $\theta \to 0$, so does $\omega = \theta^2$, and

$$\frac{\sin 2\omega}{\omega} = \frac{2\sin\omega\cos\omega}{\omega} = \frac{\sin\omega}{\omega} \cdot 2\cos\omega \to 1 \cdot 2 \cdot 1 = 2.$$

2.3.7: Let $z = 5x$. Then $z \to 0$ as $x \to 0$, and $\dfrac{\sin 5x}{x} = \dfrac{5\sin z}{z} \to 5 \cdot 1 = 5$.

2.3.8: $\dfrac{\sin 2z}{z\cos 3z} = \dfrac{2\sin z\cos z}{z\cos 3z} = \dfrac{2\cos z}{\cos 3z} \cdot \dfrac{\sin z}{z} \to 2 \cdot \dfrac{1}{1} \cdot 1 = 2$ as $z \to 0$.

2.3.9: This limit does not exist because $\sqrt{x}$ is not defined for x near 0 if $x < 0$. But

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \left(\sqrt{x}\right)\cdot\left(\frac{\sin x}{x}\right) = 0 \cdot 1 = 0.$$

2.3.10: Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (inside the front cover), we obtain

$$\lim_{x\to 0} \frac{1-\cos 2x}{x} = \lim_{x\to 0} \frac{2(1-\cos 2x)}{2x} = \lim_{x\to 0} \frac{2\sin^2 x}{x} = \lim_{x\to 0} (2\sin x)\frac{\sin x}{x} = 2\cdot 0\cdot 1 = 0.$$

Alternatively, you could multiply numerator and denominator by $1+\cos 2x$ (the conjugate of the numerator).

2.3.11: Let $x = 3z$. Then $x \to 0$ is equivalent to $z \to 0$, and therefore

$$\lim_{x\to 0} \frac{1}{x} \sin \frac{x}{3} = \lim_{z\to 0} \frac{1}{3z} \sin z = \lim_{z\to 0} \frac{1}{3} \cdot \frac{\sin z}{z} = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

2.3.12: Let $x = 3\theta$. Then $\theta = \frac{1}{3}x$ and $\theta \to 0$ is equivalent to $x \to 0$. Hence

$$\lim_{\theta\to 0} \frac{(\sin 3\theta)^2}{\theta^2 \cos\theta} = \lim_{x\to 0} \frac{(\sin x)^2}{\frac{1}{9}x^2 \cos\left(\frac{1}{3}x\right)} = \lim_{x\to 0} 9 \cdot \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos\left(\frac{1}{3}x\right)} = 9\cdot 1\cdot 1\cdot \frac{1}{1} = 9.$$

2.3.13: Multiply numerator and denominator by $1 + \cos x$ (the conjugate of the numerator) to obtain

$$\lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{(\sin x)(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{(\sin x)(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{(\sin x)(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{1+\cos x} = \frac{0}{1+1} = 0.$$

2.3.14: By Problem 4, $(\tan x)/x \to 1$ as $x \to 0$. This observation implies that

$$\lim_{x\to 0} \frac{\tan kx}{kx} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{kx}{\tan kx} = 1$$

for any nonzero constant k. Hence

$$\lim_{x\to 0} \frac{\tan 3x}{\tan 5x} = \lim_{x\to 0} \frac{3}{5} \cdot \frac{\tan 3x}{3x} \cdot \frac{5x}{\tan 5x} = \frac{3}{5} \cdot 1 \cdot 1 = \frac{3}{5}.$$

2.3.15: Recall that $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$. Hence

$$\lim_{x\to 0} x \sec x \csc x = \lim_{x\to 0} \frac{1}{\cos x} \cdot \frac{x}{\sin x} = \frac{1}{1} \cdot 1 = 1.$$

We also used the fact that

$$\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\dfrac{\sin x}{x}} = \frac{1}{1} = 1.$$

2.3.16: $\displaystyle\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=\lim_{\theta\to 0}\frac{2\sin\theta\cos\theta}{\theta}=\lim_{\theta\to 0}(2\cos\theta)\cdot\frac{\sin\theta}{\theta}=2\cdot 1\cdot 1=2.$

2.3.17: Multiply numerator and denominator by $1+\cos\theta$ (the conjugate of the numerator) to obtain

$$\begin{aligned}\lim_{\theta\to 0}\frac{(1-\cos\theta)(1+\cos\theta)}{(\theta\sin\theta)(1+\cos\theta)}&=\lim_{\theta\to 0}\frac{\sin^2\theta}{(\theta\sin\theta)(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta(1+\cos\theta)}\\&=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{1}{1+\cos\theta}=1\cdot\frac{1}{1+1}=\frac{1}{2}.\end{aligned}$$

2.3.18: $\displaystyle\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\sin\theta=1\cdot 0=0.$

2.3.19: $\displaystyle\lim_{z\to 0}\frac{\tan z}{\sin 2z}=\lim_{z\to 0}\frac{\sin z}{(\cos z)(2\sin z\cos z)}=\lim_{z\to 0}\frac{1}{2\cos^2 z}=\frac{1}{2\cdot 1^2}=\frac{1}{2}.$

2.3.20: $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{3x}=\lim_{x\to 0}\frac{2}{3}\cdot\frac{\tan 2x}{2x}=\frac{2}{3}\cdot 1=\frac{2}{3}$ (with the aid of Problem 4).

2.3.21: $\displaystyle\lim_{x\to 0}x\cot 3x=\lim_{x\to 0}\frac{x\cos 3x}{\sin 3x}=\lim_{x\to 0}\frac{3x}{\sin 3x}\cdot\frac{\cos 3x}{3}=1\cdot\frac{1}{3}=\frac{1}{3}$ (see Problem 15, last line).

2.3.22: $\displaystyle\lim_{x\to 0}\frac{x-\tan x}{\sin x}=\lim_{x\to 0}\frac{\left(\dfrac{x}{x}-\dfrac{\tan x}{x}\right)}{\left(\dfrac{\sin x}{x}\right)}=\frac{1-1}{1}=0$ (with the aid of Problem 4).

2.3.23: Let $x=\frac{1}{2}t$. Then $x\to 0$ is equivalent to $t\to 0$, so

$$\lim_{t\to 0}\frac{\sin\left(\dfrac{t}{2}\right)}{\dfrac{t}{2}}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Therefore

$$\lim_{t\to 0}\frac{1}{t^2}\sin^2\left(\frac{t}{2}\right)=\lim_{t\to 0}\frac{1}{4}\cdot\frac{4}{t^2}\sin^2\left(\frac{t}{2}\right)=\lim_{t\to 0}\frac{1}{4}\cdot\left[\frac{\sin\left(\dfrac{t}{2}\right)}{\dfrac{t}{2}}\right]^2=\frac{1}{4}\cdot 1^2=\frac{1}{4}.$$

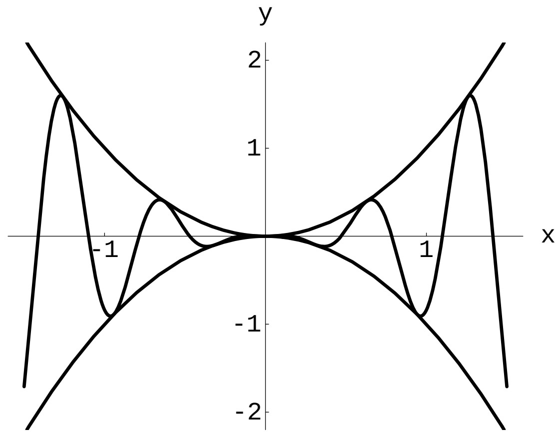
2.3.24: Because $\dfrac{\sin x}{x}\to 1$ as $x\to 0$, it follows that

$$\lim_{x\to 0}\frac{\sin kx}{kx}=1\qquad\text{and}\qquad\lim_{x\to 0}\frac{kx}{\sin kx}=1$$

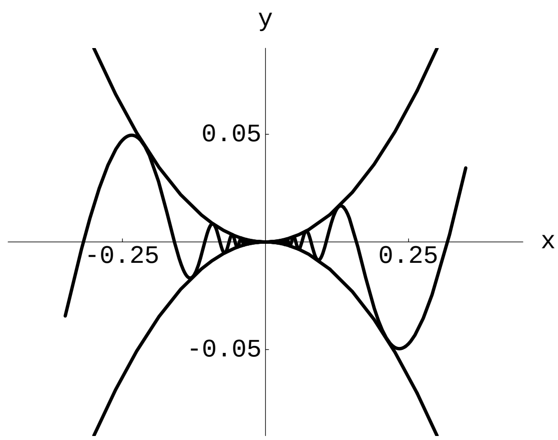
for any nonzero constant k. Hence

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\lim_{x\to 0}\frac{2}{5}\cdot\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x}=\frac{2}{5}\cdot 1\cdot 1=\frac{2}{5}.$$

2.3.25: Because $-1 \leqq \cos 10x \leqq 1$ for all x, $-x^2 \leqq x^2 \cos 10x \leqq x^2$ for all x. But both $-x^2$ and x^2 approach zero as $x \to 0$. Therefore $\lim_{x \to 0} x^2 \cos 10x = 0$. The second inequality is illustrated next.



2.3.26: Because $-1 \leqq \sin x \leqq 1$ for all x, also $-x^2 \leqq x^2 \sin \dfrac{1}{x} \leqq x^2$ for all $x \neq 0$. Because both $-x^2$ and x^2 approach zero as $x \to 0$, it follows from the squeeze law that $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$. The second inequality is illustrated next.



2.3.27: First, $-1 \leqq \cos x \leqq 1$ for all x. Therefore

$$-x^2 \leqq x^2 \cos \frac{1}{\sqrt[3]{x}} \leqq x^2$$

for all $x \neq 0$. Finally, both x^2 and $-x^2$ approach zero as $x \to 0$. Therefore, by the squeeze law,

$$\lim_{x \to 0} x^2 \cos \frac{1}{\sqrt[3]{x}} = 0.$$

2.3.28: Because $-1 \leqq \sin x \leqq 1$ for all x,

$$-\left|\sqrt[3]{x}\right| \leqq \sqrt[3]{x} \sin \frac{1}{x} \leqq \left|\sqrt[3]{x}\right|$$

for all $x \neq 0$. Because both $-|\sqrt[3]{x}\,|$ and $|\sqrt[3]{x}\,|$ approach zero as $x \to 0$, it follows from the squeeze law that

$$\lim_{x\to 0} \sqrt[3]{x} \sin\frac{1}{x} = 0.$$

2.3.29: $\displaystyle\lim_{x\to 0^+} \left(3 - \sqrt{x}\,\right) = 3 - \sqrt{\lim_{x\to 0^+} x} = 3 - 0 = 3.$

2.3.30: $\displaystyle\lim_{x\to 0^+} \left(4 + 3x^{3/2}\right) = 4 + 3 \cdot \left(\lim_{x\to 0^+} x\right)^{3/2} = 4 + 3 \cdot 0 = 4.$

2.3.31: $\displaystyle\lim_{x\to 1^-} \sqrt{x-1}$ does not exist because if $x < 1$, then $x - 1 < 0$.

2.3.32: Because $x \to 4^-$, $x < 4$, so that $\sqrt{4-x}$ is defined for all such x. Therefore the limit exists and $\displaystyle\lim_{x\to 4^-} \sqrt{4-x} = \sqrt{4-4} = 0.$

2.3.33: Because $x \to 2^+$, $x > 2$, so that $x^2 > 4$. Hence $\sqrt{x^2-4}$ is defined for all such x and $\displaystyle\lim_{x\to 2^+} \sqrt{x^2-4} = \sqrt{4-4} = 0.$

2.3.34: Because $x \to 3^+$, $x > 3$, so that $9 - x^2 < 0$ for all such x. Thus the given limit does not exist.

2.3.35: Because $x \to 5^-$, $x < 5$ and $x > 0$ for x sufficiently close to 5. Therefore $x(5-x) > 0$ for such x, so that $\sqrt{x(5-x)}$ exists for such x. Therefore

$$\lim_{x\to 5^-} \sqrt{x(5-x)} = \sqrt{\left(\lim_{x\to 5^-} x\right)\left(5 - \lim_{x\to 5^-} x\right)} = \sqrt{5 \cdot 0} = 0.$$

2.3.36: As $x \to 2^-$, $x < 2$, and $x > -2$ for x sufficiently close to 2. For such x, $4 - x^2 > 0$, so that $\sqrt{4-x^2}$ exists. Therefore $\displaystyle\lim_{x\to 2^-} \sqrt{4-x^2} = \sqrt{4-2^2} = \sqrt{0} = 0.$

2.3.37: As $x \to 4^+$, $x > 4$, so that both $4x$ and $x - 4$ are positive. Hence the radicand is positive and the square root exists. But the denominator in the radicand is approaching zero through positive values while the numerator is approaching 16. So the fraction is approaching $+\infty$. Therefore

$$\lim_{x\to 4^+} \sqrt{\frac{4x}{x-4}} = +\infty.$$

It is also correct to say that this limit does not exist.

2.3.38: First, $6 - x - x^2 = (3+x)(2-x)$, so that as $x \to -3^+$, $x > -3$, and thus $3 + x > 3 - 3 = 0$. Also $x < 2$ if x is sufficiently close to -3, so that $2 - x > 0$. Therefore $(3+x)(2-x) > 0$, and so the square root is defined. Finally,

$$\lim_{x\to -3^+} \sqrt{6-x-x^2} = \lim_{x\to -3^+} \sqrt{(3+x)(2-x)} = \sqrt{0 \cdot 5} = \sqrt{0} = 0.$$

2.3.39: If $x < 5$, then $x - 5 < 0$, so $\dfrac{x-5}{|x-5|} = \dfrac{x-5}{-(x-5)} = -1$. Therefore the limit is -1.

2.3.40: If $-4 < x < 4$, then $16 - x^2 > 0$, so $\dfrac{16-x^2}{\sqrt{16-x^2}} = \sqrt{16-x^2} \to 0$ as $x \to -4^+$.

2.3.41: If $x > 3$, then $x^2 - 6x + 9 = (x-3)^2 > 0$ and $x - 3 > 0$, so $\dfrac{\sqrt{x^2-6x+9}}{x-3} = \dfrac{|x-3|}{x-3} = \dfrac{x-3}{x-3} \to 1$ as $x \to 3^+$.

2.3.42: $\dfrac{x-2}{x^2-5x+6} = \dfrac{x-2}{(x-2)(x-3)} = \dfrac{1}{x-3} \to -1$ as $x \to 2^+$. Indeed, the two-sided limit exists and is equal to -1.

2.3.43: If $x > 2$ then $x - 2 > 0$, so $\dfrac{2-x}{|x-2|} = \dfrac{2-x}{x-2} = -1$. Therefore the limit is also -1.

2.3.44: If $x < 7$ then $x - 7 < 0$, so $\dfrac{7-x}{|x-7|} = \dfrac{7-x}{-(x-7)} = 1$. So the limit is 1.

2.3.45: $\dfrac{1-x^2}{1-x} = \dfrac{(1+x)(1-x)}{1-x} = 1 + x$, so the limit is 2.

2.3.46: As $x \to 0^-$, $x < 0$, so that $x - |x| = x - (-x) = 2x$. Therefore

$$\lim_{x\to 0^-} \frac{x}{x-|x|} = \lim_{x\to 0^-} \frac{x}{2x} = \lim_{x\to 0^-} \frac{1}{2} = \frac{1}{2}.$$

2.3.47: Recall first that $\sqrt{z^2} = |z|$ for every real number z. Because $x \to 5^+$, $x > 5$, so $5 - x < 0$. Therefore $\sqrt{(5-x)^2} = |5-x| = -(5-x) = x - 5$. Therefore

$$\lim_{x\to 5^+} \frac{\sqrt{(5-x)^2}}{5-x} = \lim_{x\to 5^+} \frac{x-5}{5-x} = \lim_{x\to 5^+} (-1) = -1.$$

2.3.48: Recall that $\sqrt{z^2} = |z|$ for every real number z. Because $x \to -4^-$, we know that $x < -4$. Hence $4 + x < 4 + (-4) = 0$. Therefore

$$\lim_{x\to -4^-} \frac{4+x}{\sqrt{(4+x)^2}} = \lim_{x\to -4^-} \frac{4+x}{|4+x|} = \lim_{x\to -4^-} \frac{4+x}{-(4+x)} = \lim_{x\to -4^-} (-1) = -1.$$

2.3.49: The right-hand and left-hand limits both fail to exist at $a = 1$. The behavior of f near a is best described by observing that

$$\lim_{x\to 1^+} \frac{1}{x-1} = +\infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

2.3.50: The right-hand and left-hand limits both fail to exist at $a = 3$. The behavior of f near a is best described by observing that

$$\lim_{x\to 3^+} \frac{2}{3-x} = -\infty \qquad \text{and} \qquad \lim_{x\to 3^-} \frac{2}{3-x} = +\infty.$$

2.3.51: The right-hand and left-hand limits both fail to exist at $a = -1$. The behavior of f near a is best described by observing that

$$\lim_{x\to -1^+} \frac{x-1}{x+1} = -\infty \qquad \text{and} \qquad \lim_{x\to -1^-} \frac{x-1}{x+1} = +\infty.$$

2.3.52: The right-hand and left-hand limits both fail to exist at $a = 5$. The behavior of f near a is best described by observing that

$$\lim_{x\to 5^+} \frac{2x-5}{5-x} = -\infty \qquad \text{and} \qquad \lim_{x\to 5^-} \frac{2x-5}{5-x} = +\infty.$$

2.3.53: The right-hand and left-hand limits both fail to exist at $a = -2$. If x is slightly greater than -2, then $1 - x^2$ is close to $1 - 4 = -3$, while $x + 2$ is a positive number close to zero. In this case $f(x)$ is a large negative number. Similarly, if x is slightly less than -2, then $1 - x^2$ is close to -3, while $x + 2$ is a negative number close to zero. In this case $f(x)$ is a large positive number. The behavior of f near -2 is best described by observing that

$$\lim_{x\to -2^+} \frac{1-x^2}{x+2} = -\infty \qquad \text{and} \qquad \lim_{x\to -2^-} \frac{1-x^2}{x+2} = +\infty.$$

2.3.54: The right-hand and left-hand limits fail to exist at $a = 5$. If x is close to 5 but $x \neq 5$, then $x - 5$ is close to zero, so that $(x-5)^2$ is a *positive* number still very close to zero. Its reciprocal is therefore a very large positive number. That is,

$$\lim_{x\to 5^+} \frac{1}{(x-5)^2} = \lim_{x\to 5^-} \frac{1}{(x-5)^2} = +\infty. \tag{1}$$

Unlike the previous problems of this sort, we may in this case also write

$$\lim_{x\to 5} \frac{1}{(x-5)^2} = +\infty.$$

Nevertheless, Eq. (1) implies that neither the left-hand nor the right-hand limit of $f(x)$ *exists* (is a real number) at $x = 5$.

2.3.55: The left-hand and right-hand limits both fail to exist at $x = 1$. To simplify $f(x)$, observe that

$$f(x) = \frac{|1-x|}{(1-x)^2} = \frac{|1-x|}{|1-x|^2} = \frac{1}{|1-x|}.$$

Therefore we can describe the behavior of $f(x)$ near $a = 1$ in this way:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{|1-x|} = +\infty.$$

2.3.56: Because $x^2 + 6x + 9 = (x+3)^2$, the denominator in $f(x)$ is zero when $x = -3$, and so the left-hand and right-hand limits fail to exist at $a = -3$. When x is close to -3 but $x \neq -3$, $(x+3)^2$ is a *positive* number very close to zero, while the numerator $x+1$ is close to -2. Therefore $f(x)$ is a very large negative number. That is,

$$\lim_{x \to -3} \frac{x+1}{x^2+6x+9} = -\infty.$$

2.3.57: First simplify $f(x)$: If $x^2 \neq 4$ (that is, if $x \neq \pm 2$), then

$$f(x) = \frac{x-2}{4-x^2} = \frac{x-2}{(2+x)(2-x)} = \frac{-1}{2+x}.$$

So even though $f(2)$ does not exist, there is no real problem with the limit of $f(x)$ as $x \to 2$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{-1}{2+x} = -\frac{1}{4}.$$

But the left-hand and right-hand limits of $f(x)$ fail to exist at $x = -2$, because

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{-1}{2+x} = -\infty \quad \text{and} \quad \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{-1}{2+x} = +\infty.$$

2.3.58: First simplify:

$$f(x) = \frac{x-1}{x^2-3x+2} = \frac{x-1}{(x-1)(x-2)} = \frac{1}{x-2}$$

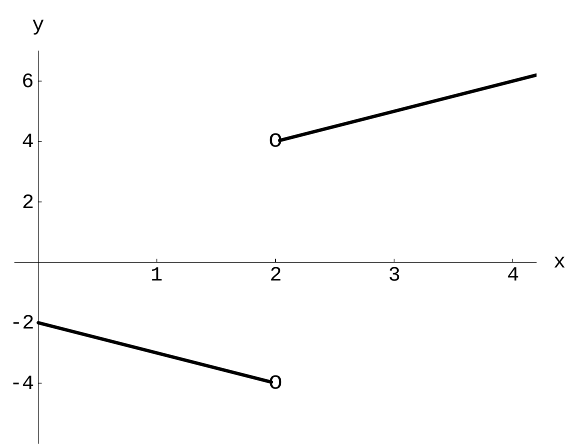
if $x \neq 1$ and $x \neq 2$. But even though $f(1)$ is undefined,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1.$$

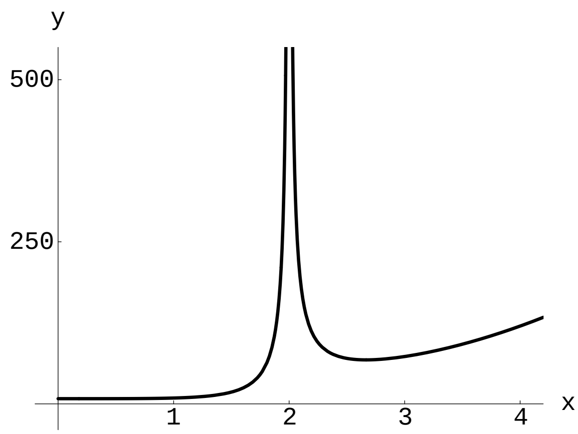
But the one-sided limits fail to exist at $x = 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x-2} = +\infty \quad \text{and} \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x-2} = -\infty.$$

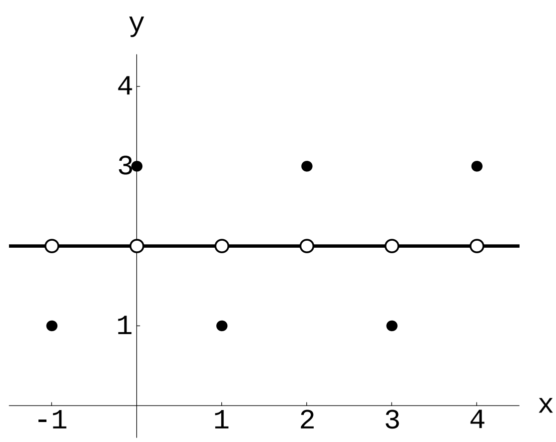
2.3.59: $\displaystyle\lim_{x \to 2^+} \frac{x^2-4}{|x-2|} = 4$ and $\displaystyle\lim_{x \to 2^-} \frac{x^2-4}{|x-2|} = -4$. The two-sided limit does not exist. The graph is shown next.



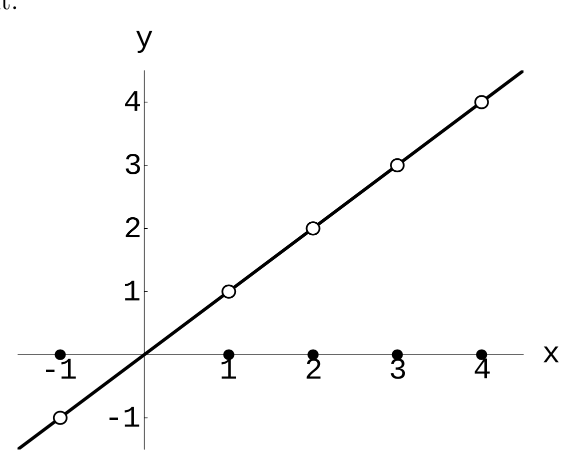
2.3.60: Because $\lim_{x\to 2^+} \dfrac{x^4 - 8x + 16}{|x-2|} = +\infty$ and $\lim_{x\to 2^-} \dfrac{x^4 - 8x + 16}{|x-2|} = +\infty$, the two-sided limit also fails to exist. The graph is shown next.



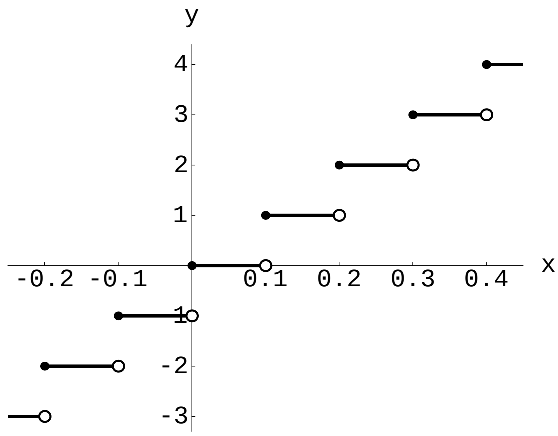
2.3.61: If x is an even integer then $f(x) = 3$, if x is an odd integer then $f(x) = 1$, and $\lim_{x\to a} f(x) = 2$ for *all* real number values of a. The graph of f is shown next.



2.3.62: If n is any integer then $f(x) \to n$ as $x \to n$. Note: $\lim_{x\to a} f(x) = a$ for *all* real number values of a. The graph of f is shown next.



2.3.63: For any integer n, $\lim_{x \to n^-} f(x) = 10n - 1$ and $\lim_{x \to n^+} f(x) = 10n$. Note: $\lim_{x \to a} f(x)$ exists if and only if $10a$ is not an integer. The graph is shown next.



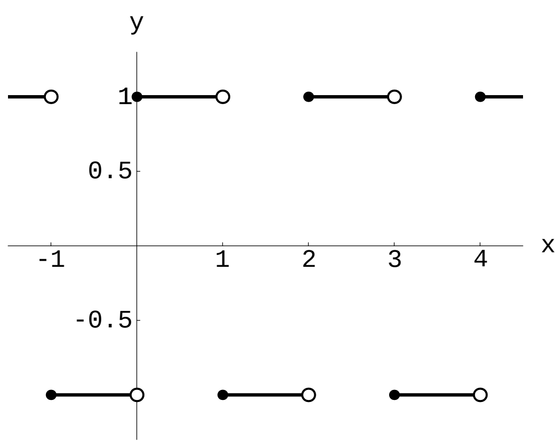
2.3.64: If n is an odd integer, then $f(x) = n - 1$, an even integer, for $n - 1 \leqq x < n$ and $f(x) = n$, an odd integer, for $n \leqq x < n + 1$. Therefore

$$\lim_{x \to n^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = -1.$$

Similarly, if n is an even integer, then

$$\lim_{x \to n^-} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = 1.$$

Finally, $\lim_{x \to a} f(x)$ exists if and only if a is *not* an integer. The graph of f is shown next.



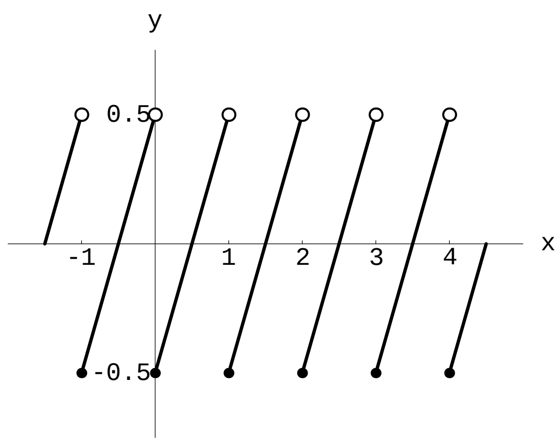
2.3.65: If n is an integer and $n < x < n + 1$, then write $x = n + t$ where $0 < t < 1$. Then $f(x) = n + t - n - \frac{1}{2} = t - \frac{1}{2}$. Moreover, $x \to n^+$ is equivalent to $t \to 0^+$. Therefore

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} \left(t - \tfrac{1}{2}\right) = \lim_{t \to 0^+} \left(t - \tfrac{1}{2}\right) = -\frac{1}{2}.$$

Similar reasoning, with $n - 1 < x < n$, shows that if n is an integer, then

$$\lim_{x \to n^-} f(x) = \frac{1}{2}.$$

Finally, if a is a real number other than an integer, then $\lim_{x \to a} f(x)$ exists. The graph of f is next.

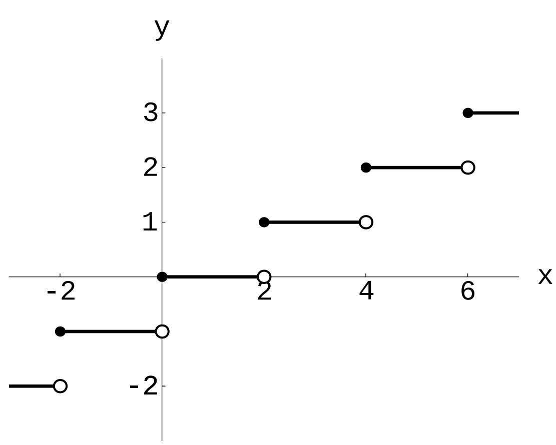


2.3.66: Given the real number x, there is a [unique] integer n such that $2n \leqq x < 2n + 2$. Thus $n \leqq \frac{1}{2}x < n + 1$, and in this case $f(x) = n$. So if $m = 2n$ is an even integer, then $f(x) \to m$ as $x \to m^+$ and $x \to a$ for every real number a strictly between $2n$ and $2n + 2$. But if $2n - 2 < x < 2n$, then $n - 1 < \frac{1}{2}x < n$, so that $f(x) = n - 1$; in this case $f(x) \to n - 1$ as $x \to m^-$. Therefore:

If k is an odd integer, then $\lim_{x \to k} f(x) = \frac{1}{2}(k - 1)$.

If k is an even integer, then $\lim_{x \to k^+} f(x) = \frac{1}{2}k$ and $\lim_{x \to k^-} f(x) = \frac{1}{2}(k - 2)$.

Finally, $\lim_{x \to a} f(x)$ exists if and only if a is not an even integer. The graph of f is next.



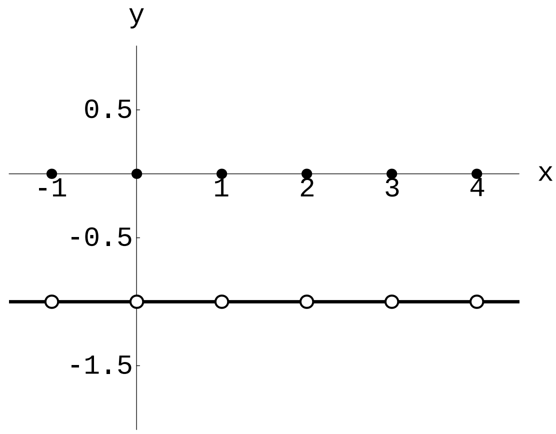
2.3.67: If x is an integer, then $f(x) = x - x = 0$. If x is not an integer, choose the [unique] integer n such that $n < x < n + 1$. Then $-(n + 1) < -x < -n$, so $f(x) = n - (n + 1) = -1$. Therefore

$$\lim_{x \to a} f(x) = \lim_{x \to a} (-1) = -1$$

for every real number a. In particular, for every integer n,

$$\lim_{x\to n^-} f(x) = -1 \qquad \text{and} \qquad \lim_{x\to n^+} f(x) = -1.$$

The graph of f is shown next.



2.3.68: If n is a positive integer, then

$$\lim_{x\to n^-} f(x) = \frac{n-1}{n} \qquad \text{and} \qquad \lim_{x\to n^+} f(x) = 1.$$

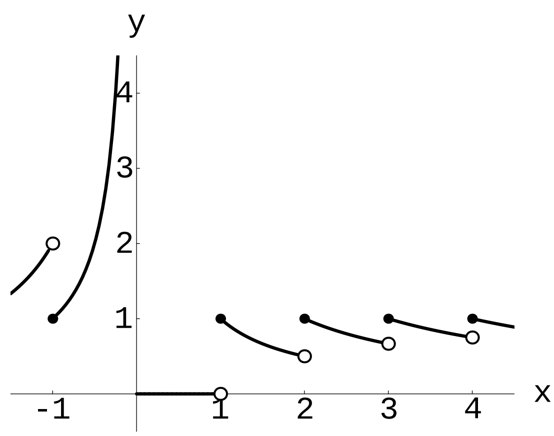
For any integer $n < 0$,

$$\lim_{x\to n^-} f(x) = \frac{n+1}{n} \qquad \text{and } \lim_{x\to n^+} f(x) = 1.$$

Also,

$$\lim_{x\to 0^-} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 0.$$

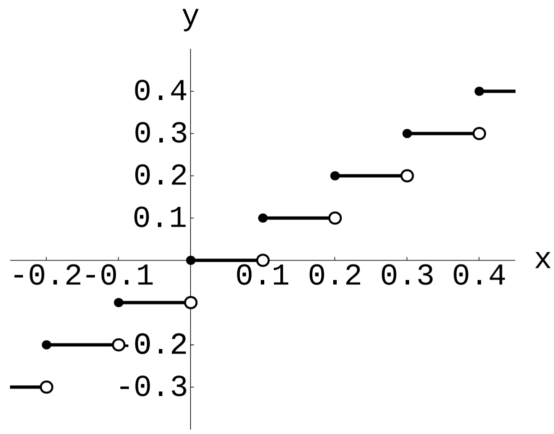
Note: $\lim_{x\to a} f(x)$ exists if and only if a is not an integer. The graph of f is next.



2.3.69: The values of a for which $\lim_{x\to a} g(x)$ exists are those real numbers not integral multiples of $\frac{1}{10}$. If b is an integral multiple of $\frac{1}{10}$, then

$$\lim_{x\to b^-} g(x) = b - \tfrac{1}{10} \qquad \text{and} \qquad \lim_{x\to b^+} g(x) = b.$$

The graph of g is shown next.



2.3.70: Let $f(x) = \operatorname{sgn}(x)$ and $g(x) = -\operatorname{sgn}(x)$. Clearly neither $f(x)$ nor $g(x)$ has a limit as $x \to 0$ (for example, $f(x) \to 1$ as $x \to 1^+$ but $f(x) \to -1$ as $x \to 1^-$). But

$$f(x) + g(x) = \begin{cases} 1 - 1 & \text{if } x > 0, \\ -1 + 1 & \text{if } x < 0, \\ 0 + 0 & \text{if } x = 0, \end{cases}$$

so that $f(x) + g(x) \equiv 0$, and therefore $f(x) + g(x) \to 0$ as $x \to 0$. Also,

$$f(x) \cdot g(x) = \begin{cases} -1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Therefore $\lim_{x \to 0} f(x) \cdot g(x) = -1$.

2.3.71: Because $-x^2 \leqq f(x) \leqq x^2$ for all x and because $-x^2 \to 0$ and $x^2 \to 0$ as $x \to 0$, it follows from the squeeze law for limits that $\lim_{x \to 0} f(x) = 0 = f(0)$.

2.3.72: As $x \to 0^+$, $1/x \to +\infty$, so $1 + 2^{1/x} \to +\infty$ as well. Therefore

$$\lim_{x \to 0^+} \frac{1}{1 + 2^{1/x}} = 0.$$

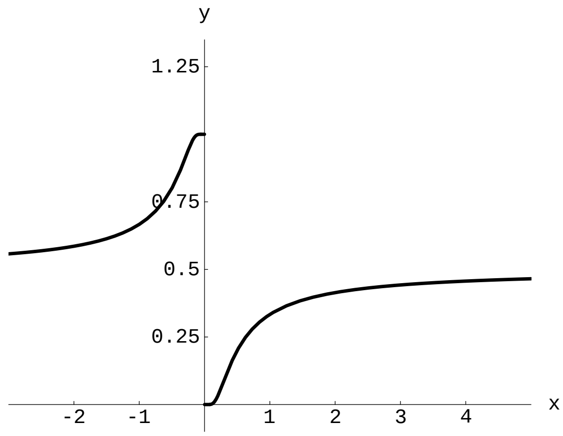
As $x \to 0^-$, $1/x \to -\infty$, so $2^{1/x} \to 0$. Consequently,

$$\lim_{x \to 0^-} \frac{1}{1 + 2^{1/x}} = 1.$$

Therefore $\lim_{x \to 0} f(x)$ does not exist. This function approaches its one-sided limits at $x = 0$ very rapidly. For example,

$$f(0.01) \approx 7.888609052 \times 10^{-31} \qquad \text{and} \qquad f(-0.01) \approx 1 - 7.888601052 \times 10^{-31}.$$

The graph of f for x near zero is next.



2.3.73: Given: $f(x) = x \cdot [\![1/x]\!]$. Let's first study the right-hand limit of $f(x)$ at $x = 0$. We need consider only values of x in the interval (0, 1), and if $0 < x < 1$ then

$$1 < \frac{1}{x}, \qquad \text{so that} \qquad n \leqq \frac{1}{x} < n + 1$$

for some [unique] positive integer n. Moreover, if so then

$$\frac{1}{n+1} < x \leqq \frac{1}{n}.$$

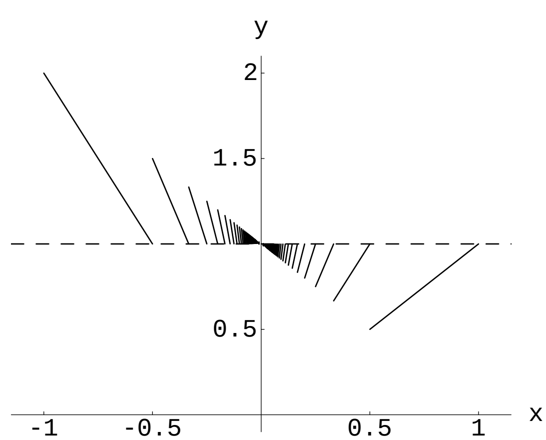
Therefore $f(x) = x \cdot n$, so that

$$\frac{n}{n+1} < f(x) \leqq \frac{n}{n} = 1. \tag{1}$$

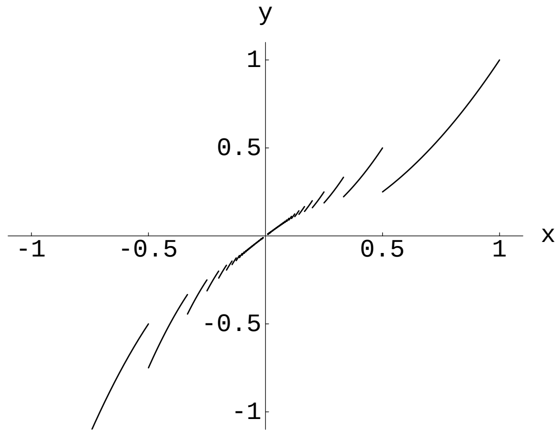
As $x \to 0^+$, $n \to \infty$, so the bounds on $f(x)$ in (1) both approach 1. Therefore

$$\lim_{x \to 0^+} f(x) = 1.$$

A similar (but slightly more delicate) argument shows that $f(x) \to 1$ as $x \to 0^-$ as well. Therefore $\lim_{x \to 0} f(x)$ exists and is equal to 1. The graph of f is next.



2.3.74: Here, $f(x)$ is obtained from the function in Problem 73 by multiplication by x. Therefore, because the function in Problem 73 had limit 1 as $x \to 0$, the product rule for limits implies that $f(x) \to 0 \cdot 1 = 0$ as $x \to 0$. The graph of f near zero is next.



2.3.75: Given $\epsilon > 0$, let $\delta = \epsilon/7$. Suppose that

$$0 < |x - (-3)| < \delta.$$

Then

$$|x + 3| < \frac{\epsilon}{7};$$

$$|7x + 21| < \epsilon;$$

$$|7x - 9 + 30| < \epsilon;$$

$$|(7x - 9) - (-30)| < \epsilon.$$

Therefore, by definition, $\lim_{x \to -3} (7x - 9) = -30$.

2.3.76: Given $\epsilon > 0$, let $\delta = \epsilon/17$. Suppose that $0 < |x - 5| < \delta$. Then

$$|17x - 85| < 17\delta;$$

$$|(17x - 35) - 50| < \epsilon.$$

Therefore, by definition, $\lim_{x \to 5} (17x - 35) = 50$.

2.3.77: Definition: We say that the number L is the **right-hand limit** of the function f at $x = a$ provided that, for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < |x - a| < \delta$ and $x > a$, then $|f(x) - L| < \epsilon$.

To prove that $\lim_{x \to 0^+} \sqrt{x} = 0$, suppose that $\epsilon > 0$ is given. Let $\delta = \epsilon^2$. Suppose that $|x - 0| < \delta$ and that $x > 0$. Then $0 < x < \delta = \epsilon^2$. Hence $\sqrt{x} < \epsilon$, and therefore

$$|\sqrt{x} - 0| < \epsilon.$$

So, by definition, $\lim_{x \to 0^+} \sqrt{x} = 0$.

2.3.78: Let $\epsilon > 0$ be given. Let $\delta = \sqrt{\epsilon}$. Suppose that $0 < |x - 0| < \delta$. Then $0 < x^2 < \delta^2 = \epsilon$. Hence $|x^2 - 0| < \epsilon$. Therefore, by definition,

$$\lim_{x \to 0} x^2 = 0.$$

2.3.79: Suppose that $\epsilon > 0$ is given. Let δ be the minimum of the two numbers 1 and $\epsilon/5$ and suppose that $0 < |x - 2| < \delta$. Then

$$|x - 2| < 1;$$

$$-1 < x - 2 < 1;$$

$$3 < x + 2 < 5;$$

$$|x + 2| < 5.$$

Therefore

$$|x^2 - 4| = |x + 2| \cdot |x - 2| < 5 \cdot \delta \leqq 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Hence, by definition, $\lim_{x \to 2} x^2 = 4$.

2.3.80: Given $\epsilon > 0$, choose δ to be the minimum of 1 and $\epsilon/10$. Suppose that $0 < |x - 7| < \delta$. Then

$$|x - 7| < 1;$$

$$-1 < x - 7 < 1;$$

$$8 < x + 2 < 10;$$

$$|x + 2| < 10.$$

Therefore

$$|(x^2 - 5x - 4) - 10| = |x + 2| \cdot |x - 7| < 10 \cdot \delta \leqq 10 \cdot \frac{\epsilon}{10} = \epsilon.$$

Thus, by definition, $\lim_{x \to 7} (x^2 - 5x - 4) = 10$.

2.3.81: Given $\epsilon > 0$, let δ be the minimum of 1 and $\epsilon/29$. Suppose that $0 < |x - 10| < \delta$. Then

$$0 < |x - 10| < 1;$$

$$-1 < x - 10 < 1;$$

$$-2 < 2x - 20 < 2;$$

$$25 < 2x + 7 < 29;$$

$$|2x + 7| < 29.$$

Thus

$$|(2x^2 - 13x - 25) - 45| = |2x + 7| \cdot |x - 10| < 29 \cdot \delta \leqq 29 \cdot \frac{\epsilon}{29} = \epsilon.$$

Therefore, by definition, $\lim_{x \to 10} (2x^2 - 13x - 25) = 45$.

2.3.82: Given $\epsilon > 0$, choose δ to be the minimum of 1 and $\epsilon/19$. Suppose that $0 < |x - 2| < \delta$. Then

$$0 < |x - 2| < 1;$$

$$-1 < x - 2 < 1;$$

$$1 < x < 3;$$

$$1 < x^2 < 9 \qquad \text{and} \qquad 2 < 2x < 6;$$

$$3 < x^2 + 2x < 15;$$

$$7 < x^2 + 2x + 4 < 19;$$

$$|x^2 + 2x + 4| < 19.$$

Consequently,

$$|x^3 - 8| = |x^2 + 2x + 4| \cdot |x - 2| < 19 \cdot \delta \leqq 19 \cdot \frac{\epsilon}{19} = \epsilon.$$

Therefore, by definition, $\lim_{x \to 2} x^3 = 8$.

2.3.83: In Problem 78 we showed that if $a = 0$, then

$$\lim_{x \to a} x^2 = \lim_{x \to 0} x^2 = 0 = 0^2 = a^2,$$

so the result we are to prove here holds when $a = 0$. Next case: Suppose that $a > 0$. Let $\epsilon > 0$ be given. Choose δ to be the minimum of the numbers 1 and $\epsilon/(2a+1)$. Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. Then

$$|x - a| < 1;$$

$$-1 < x - a < 1;$$

$$2a - 1 < x + a < 2a + 1;$$

$$|x + a| < 2a + 1.$$

Thus

$$|x^2 - a^2| = |x + a| \cdot |x - a| < (2a + 1) \cdot \frac{\epsilon}{2a + 1} = \epsilon.$$

Therefore, by definition, $\lim_{x \to a} x^2 = a^2$ if $a > 0$.

Final case: $a < 0$. Given $\epsilon > 0$, let

$$\delta = \min\left\{1, \frac{\epsilon}{|2a - 1|}\right\}.$$

Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. The

$$|x - a| < 1;$$

$$-1 < x - a < 1;$$

$$2a - 1 < x + a < 2a + 1;$$

$$|x + a| < |2a - 1|$$

(because $|2a - 1| > |2a + 1|$ if $a < 0$). It follows that

$$|x^2 - a^2| = |x + a| \cdot |x - a| < |2a - 1| \cdot \frac{\epsilon}{|2a - 1|} = \epsilon.$$

Therefore, by definition, $\lim_{x \to a} x^2 = a^2$ if $a < 0$.

2.3.84: Suppose that $\epsilon > 0$ is given. Case (1): $a = 0$. Let $\delta = \sqrt[3]{\epsilon}$ and proceed much as in the solution of Problem 78. Case (2): $a > 0$. Let

$$\delta = \min\left\{\frac{a}{2}, \frac{4\epsilon}{19a^2}\right\}.$$

Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. Then:

$$|x - a| < \frac{a}{2};$$

$$-\frac{a}{2} < x - a < \frac{a}{2};$$

$$\frac{a}{2} < x < \frac{3a}{2};$$

$$\frac{a^2}{4} < x^2 < \frac{9a^2}{4} \qquad \text{(because } x > 0\text{);}$$

$$\begin{aligned}
&\frac{a^2}{2} < ax < \frac{3a^2}{2}; \\
&\frac{3a^2}{4} < x^2 + ax < \frac{15a^2}{4}; \\
&\frac{7a^2}{4} < x^2 + ax + a^2 < \frac{19a^2}{4}; \\
&|x^2 + ax + a^2| < \frac{19a^2}{4}.
\end{aligned}$$

Therefore

$$|x^3 - a^2| = |x^2 + ax + a^2| \cdot |x - a| < \frac{19a^2}{4} \cdot \frac{4\epsilon}{19a^2} = \epsilon.$$

Thus, by definition, $\lim\limits_{x \to a} x^2 = a^2$ if $a > 0$. Case (3), in which $a < 0$, is similar.

Section 2.4

2.4.1: Suppose that a is a real number. Then

$$\begin{aligned}
\lim_{x \to a} f(x) = \lim_{x \to a} (2x^5 - 7x^2 + 13) &= \left(\lim_{x \to a} 2x^5\right) - \left(\lim_{x \to a} 7x^2\right) + \left(\lim_{x \to a} 13\right) \\
&= \left(\lim_{x \to a} 2\right)\left(\lim_{x \to a} x\right)^5 - \left(\lim_{x \to a} 7\right)\left(\lim_{x \to a} x\right)^2 + 13 = 2a^5 - 7a^2 + 13 = f(a).
\end{aligned}$$

Therefore f is continuous at x for every real number x.

2.4.2: Suppose that a is a real number. Then

$$\begin{aligned}
\lim_{x \to a} f(x) &= \left(\lim_{x \to a} 7x^3\right) - \left(\lim_{x \to a} (2x+1)^5\right) = \left(\lim_{x \to a} 7\right)\left(\lim_{x \to a} x\right)^3 - \left(\lim_{x \to a} (2x+1)\right)^5 \\
&= 7a^3 - \left[\left(\lim_{x \to a} 2\right)\left(\lim_{x \to a} x\right) + \left(\lim_{x \to a} 1\right)\right]^5 = 7a^3 - (2a+1)^5 = f(a).
\end{aligned}$$

Therefore f is continuous at x for every real number x.

2.4.3: Suppose that a is a real number. Then

$$\begin{aligned}
\lim_{x \to a} g(x) = \lim_{x \to a} \frac{2x - 1}{4x^2 + 1} &= \frac{\lim\limits_{x \to a} (2x - 1)}{\lim\limits_{x \to a} (4x^2 + 1)} \\
&= \frac{\left(\lim\limits_{x \to a} 2\right)\left(\lim\limits_{x \to a} x\right) - \left(\lim\limits_{x \to a} 1\right)}{\left(\lim\limits_{x \to a} 4\right)\left(\lim\limits_{x \to a} x\right)^2 + \left(\lim\limits_{x \to a} 1\right)} = \frac{2a - 1}{4a^2 + 1} = g(a).
\end{aligned}$$

Therefore g is continuous at x for every real number x.

2.4.4: Suppose that a is a fixed real number. Then

$$\lim_{x\to a} g(x) = \frac{\lim_{x\to a} x^3}{\lim_{x\to a} x^2 + 2\lim_{x\to a} x + 5}$$

$$= \frac{(\lim_{x\to a} x)^3}{(\lim_{x\to a} x)^2 + 2\lim_{x\to a} x + 5} = \frac{a^3}{a^2 + 2a + 5} = g(a).$$

Therefore g is continuous at x for all real x.

2.4.5: Suppose that a is a fixed real number. Then $a^2 + 4a + 5 = (a+2)^2 + 1 > 0$, so $h(a)$ is defined. Moreover,

$$\lim_{x\to a} h(x) = \lim_{x\to a} \sqrt{x^2 + 4x + 5} = \left(\lim_{x\to a} (x^2 + 4x + 5)\right)^{1/2}$$

$$= \left[\left(\lim_{x\to a} x\right)^2 + \left(\lim_{x\to a} 4\right)\left(\lim_{x\to a} x\right) + \left(\lim_{x\to a} 5\right)\right]^{1/2} = \sqrt{a^2 + 4a + 5} = h(a).$$

Therefore, by definition, h is continuous at $x = a$. Because a is arbitrary, h is continuous at x for every real number x.

2.4.6: Suppose that a is a real number. Then $h(a)$ exists because $x^{1/3}$ is defined for every real number x. Moreover,

$$\lim_{x\to a} h(x) = \lim_{x\to a} (1 - 5x)^{1/3} = \left(\lim_{x\to a} (1 - 5x)\right)^{1/3}$$

$$= \left[\left(\lim_{x\to a} 1\right) - \left(\lim_{x\to a} 5\right)\left(\lim_{x\to a} x\right)\right]^{1/3} = (1 - 5a)^{1/3} = h(a).$$

Therefore h is continuous at $x = a$, and thus continuous at every real number x.

2.4.7: Suppose that a is a real number. Then $1 + \cos^2 a \neq 0$, so that $f(a)$ is defined. Note also that

$$\lim_{x\to a} \sin x = \sin a \qquad \text{and} \qquad \lim_{x\to a} \cos x = \cos a$$

because the sine and cosine functions are continuous at every real number (Theorem 1). Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1 - \sin x}{1 + \cos^2 x} = \frac{\lim_{x\to a} (1 - \sin x)}{\lim_{x\to a} (1 + \cos^2 x)}$$

$$= \frac{\left(\lim_{x\to a} 1\right) - \left(\lim_{x\to a} \sin x\right)}{\left(\lim_{x\to a} 1\right) + \left(\lim_{x\to a} \cos x\right)^2} = \frac{1 - \sin a}{1 + \cos^2 a} = f(a).$$

Therefore f is continuous at a. Because a is arbitrary, f is continuous at x for every real number x.

2.4.8: Suppose that a is a real number. Then $0 \leqq \sin^2 a \leqq 1$, so that $1 - \sin^2 a \geqq 0$, and thus $g(a) = (1 - \sin^2 a)^{1/4}$ exists. Because the sine function is continuous on the set of all real numbers (Theorem 1), we know also that $\sin x \to \sin a$ as $x \to a$. Therefore

$$\begin{aligned}\lim_{x \to a} g(x) = \lim_{x \to a} (1 - \sin^2 x)^{1/4} &= \left(\lim_{x \to a} (1 - \sin^2 x)\right)^{1/4} \\ &= \left[\left(\lim_{x \to a} 1\right) - \left(\lim_{x \to a} \sin x\right)^2\right]^{1/4} = \left(1 - \sin^2 a\right)^{1/4} = g(a).\end{aligned}$$

Therefore g is continuous at a for every real number a.

2.4.9: If $a > -1$, then $f(a)$ exists because $a \neq -1$. Moreover,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{x+1} = \frac{\displaystyle\lim_{x \to a} 1}{\left(\displaystyle\lim_{x \to a} x\right) + \left(\displaystyle\lim_{x \to a} 1\right)} = \frac{1}{a+1} = f(a).$$

Therefore f is continuous on the interval $x > -1$.

2.4.10: If $-2 < a < 2$, then $f(a)$ exists because $a^2 - 4 \neq 0$. Moreover,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x-1}{x^2-4} = \frac{\left(\displaystyle\lim_{x \to a} x\right) - \left(\displaystyle\lim_{x \to a} 1\right)}{\left(\displaystyle\lim_{x \to a} x\right)^2 - \left(\displaystyle\lim_{x \to a} 4\right)} = \frac{a-1}{a^2-4} = f(a).$$

Therefore f is continuous at x if $-2 < x < 2$.

2.4.11: Because $-\frac{3}{2} \leqq t \leqq \frac{3}{2}$, $0 \leqq 4t^2 \leqq 9$, so that the radicand in $g(t)$ is never negative. Therefore $g(a)$ is defined for every real number a in the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$, and

$$\begin{aligned}\lim_{t \to a} g(t) = \lim_{t \to a} (9 - 4t^2)^{1/2} &= \left(\lim_{t \to a} (9 - 4t^2)\right)^{1/2} \\ &= \left[\left(\lim_{t \to a} 9\right) - 4\left(\lim_{t \to a} t\right)^2\right]^{1/2} = (9 - 4a^2)^{1/2} = g(a).\end{aligned}$$

Therefore g is continuous at a for every real number a in $\left[-\frac{3}{2}, \frac{3}{2}\right]$.

2.4.12: If $1 \leqq z \leqq 3$, then $0 \leqq z - 1 \leqq 2$, $-3 \leqq -z \leqq -1$, and $0 \leqq 3 - z \leqq 2$. So the radicand in $h(z)$ is nonnegative for such values of z, and therefore if $1 \leqq a \leqq 3$ then

$$\begin{aligned}\lim_{z \to a} h(z) = \lim_{z \to a} [(z-1)(3-z)]^{1/2} &= \left[\lim_{z \to a} (z-1)(3-z)\right]^{1/2} \\ &= \left[\left(\left[\lim_{z \to a} z\right] - \left[\lim_{z \to a} 1\right]\right)\left(\left[\lim_{z \to a} 3\right] - \left[\lim_{z \to a} z\right]\right)\right]^{1/2} = [(a-1)(3-a)]^{1/2} = h(a).\end{aligned}$$

Therefore h is continuous at a for all real numbers a in the interval $[1, 3]$.

2.4.13: If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then $\cos x \neq 0$, so $f(x)$ is defined for all such x. In addition, $\cos x \to \cos a$ as $x \to a$ because the cosine function is continuous everywhere (Theorem 1). Therefore

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x}{\cos x} = \frac{\displaystyle\lim_{x \to a} x}{\displaystyle\lim_{x \to a} \cos x} = \frac{a}{\cos a} = f(a).$$

Therefore f is continuous at x if $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

2.4.14: If $-\frac{1}{6}\pi < t < \frac{1}{6}\pi$, then $-\frac{1}{2} < \sin t < \frac{1}{2}$, so that $1 - 2\sin t > 0$ for such values of t. Therefore $g(a)$ is defined if $-\frac{1}{6}\pi < a < \frac{1}{6}\pi$. Moreover, for such values of a, we have

$$\begin{aligned} \lim_{t \to a} g(t) &= \lim_{t \to a} (1 - 2\sin t)^{1/2} = \left(\lim_{t \to a} (1 - 2\sin t) \right)^{1/2} \\ &= \left[\left(\lim_{t \to a} 1 \right) - \left(\lim_{t \to a} 2 \right) \left(\lim_{t \to a} \sin t \right) \right]^{1/2} = (1 - 2\sin a)^{1/2} = g(a). \end{aligned}$$

Therefore g is continuous at a for each real number a in the interval $\left(-\frac{1}{6}\pi, \frac{1}{6}\pi\right)$.

2.4.15: The root law of Section 2.2 implies that $g(x) = \sqrt[3]{x}$ is continuous on the set $\boldsymbol{R}$ of all real numbers. We know that the polynomial $h(x) = 2x$ is continuous on $\boldsymbol{R}$ (Section 2.4, page 88). Hence the sum $f(x) = h(x) + g(x)$ is continuous on $\boldsymbol{R}$.

2.4.16: The polynomial $f(x) = x^2$ is continuous on $\boldsymbol{R}$ (the set of all real numbers) and the quotient

$$h(x) = \frac{1}{x}$$

of continuous functions is continuous where its denominator is not zero. Hence the sum $g(x) = f(x) + h(x)$ is continuous on its domain, the set of all nonzero real numbers.

2.4.17: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than -3.

2.4.18: Because $f(t)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 5.

2.4.19: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers.

2.4.20: Because $g(z)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), g is continuous wherever its denominator is nonzero. Therefore g is continuous on its domain, the set of all real numbers other than ± 1.

2.4.21: Note that $f(x)$ is not defined at $x = 5$, so it is not continuous there. Because $f(x) = 1$ for $x > 5$ and $f(x) = -1$ for $x < 5$, f is a polynomial on the interval $(5, +\infty)$ and a [another] polynomial on the interval $(-\infty, 5)$. Therefore f is continuous on its domain, the set of all real numbers other than 5.

2.4.22: Because $h(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), h is continuous wherever its denominator is nonzero. Therefore h is continuous on its domain, the set of all real numbers.

2.4.23: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 2.

2.4.24: Because $g(t) = 4 + t^4$ is a polynomial, it is continuous everywhere (and never negative). Because $h(t) = \sqrt[4]{t}$ is a root function, it is continuous wherever $t \geqq 0$. Therefore (by Theorem 2) the composition $f(t) = h(g(t))$ is continuous everywhere.

2.4.25: Let

$$h(x) = \frac{x+1}{x-1}.$$

Because $h(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), h is continuous wherever its denominator is nonzero. Therefore h is continuous on its domain, the set of all real numbers other than 1. Now let $g(x) = \sqrt[3]{x}$. By the root rule of Section 2.2, g is continuous everywhere. Therefore the composition $f(x) = g(h(x))$ is continuous on the set of all real numbers other than 1.

2.4.26: Here, $F(u) = g(h(u))$ where $g(u) = \sqrt[3]{u}$ and $h(u) = 3 - u^3$. Now g is continuous everywhere by the root rule of Section 2.2; h is continuous everywhere because $h(u)$ is a polynomial. Therefore the composition $F(u) = g(h(u))$ of continuous functions is continuous where defined; namely, on the set $\boldsymbol{R}$ of all real numbers.

2.4.27: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 0 and 1.

2.4.28: The domain of f is the interval $-3 \leqq z \leqq 3$, and on that domain $f(z)$ is the composition of continuous functions, thus f is continuous there. Because $f(z)$ is not defined if $|z| > 3$, it is not continuous for $z < -3$ nor for $z > 3$. But it is still correct to say simply that "f is continuous" (see the definition of **continuous** on page 88).

2.4.29: Let $h(x) = 4 - x^2$. Then h is continuous everywhere because $h(x)$ is a polynomial. The root function $g(x) = \sqrt{x}$ is continuous for $x \geqq 0$ by the root rule of Section 2.2. Hence $g(h(x)) = \sqrt{4 - x^2}$ is continuous wherever $x^2 \leqq 4$; that is, on the interval $[-2, 2]$. The quotient

$$f(x) = \frac{x}{\sqrt{4 - x^2}} = \frac{x}{g(h(x))}$$

is continuous wherever the numerator is continuous (that's everywhere) and the denominator is both continuous and nonzero (that's the open interval $(-2, 2)$). Therefore f is continuous on the open interval $(-2, 2)$. That is, f is continuous on its domain.

2.4.30: Because $f(x)$ is formed by the composition and quotient of continuous functions (polynomials and root functions), it will be continuous wherever the denominator in the fraction is nonzero and the fraction is nonnegative. So continuity of f will occur when both

$$4 - x^2 \neq 0 \qquad \text{and} \qquad \frac{1 - x^2}{4 - x^2} \geqq 0.$$

The first inequality is equivalent to $x \neq \pm 2$ and the second will hold when $1 - x^2$ and $4 - x^2$ have the same sign (both positive or both negative) or the numerator is zero. If both are positive, then $x^2 < 1$ and $x^2 < 4$, so that $-1 < x < 1$. If both are negative, then $x^2 > 1$ and $x^2 > 4$, so that $x^2 > 4$; that is, $x < -2$ or $x > 2$. Finally, $1 - x^2 = 0$ when $x = \pm 1$. Therefore f is continuous on its domain, $(-\infty, -2) \cup [-1, 1] \cup (2, +\infty)$.

2.4.31: Because $f(x)$ is the quotient of continuous functions, it is continuous where its denominator is nonzero; that is, if $x \neq 0$. Thus f is continuous on its domain and not continuous at $x = 0$ (because it is undefined there).

2.4.32: Given:

$$g(\theta) = \frac{\theta}{\cos \theta}.$$

Because $g(\theta)$ is the quotient of continuous functions, it is continuous wherever its denominator is nonzero; that is, at every real number x not an odd integral multiple of $\pi/2$. That is, g is discontinuous (because it is undefined) at

$$\ldots, \; -\frac{5\pi}{2}, \; -\frac{3\pi}{2}, \; -\frac{\pi}{2}, \; \frac{\pi}{2}, \; \frac{3\pi}{2}, \; \frac{5\pi}{2}, \; \frac{7\pi}{2}, \; \ldots.$$

Therefore g is continuous on its domain, the set

$$\cdots \cup \left(-\tfrac{5}{2}\pi, \, -\tfrac{3}{2}\pi\right) \cup \left(-\tfrac{3}{2}\pi, \, -\tfrac{1}{2}\pi\right) \cup \left(-\tfrac{1}{2}\pi, \, \tfrac{1}{2}\pi\right) \cup \left(\tfrac{1}{2}\pi, \, \tfrac{3}{2}\pi\right) \cup \left(\tfrac{3}{2}\pi, \, \tfrac{5}{2}\pi\right) \cup \left(\tfrac{5}{2}\pi, \, \tfrac{7}{2}\pi\right) \cup \cdots.$$

2.4.33: Given:

$$f(x) = \frac{1}{\sin 2x}.$$

The numerator in $f(x)$ is a polynomial, thus continuous everywhere. The denominator is the composition of a function continuous on the set of all real numbers (the sine function) with another continuous function (a polynomial), hence is also continuous everywhere. Thus because $f(x)$ is the quotient of continuous functions, it is continuous wherever its denominator is nonzero; that is, its only discontinuities occur when $\sin 2x = 0$. Thus f is continuous at every real number other than an integral multiple of $\pi/2$.

2.4.34: Because $f(x) = \sqrt{\sin x}$ is the composition of continuous functions, it is continuous wherever it is defined; that is, wherever $\sin x \geqq 0$. Hence f is continuous on its domain, the set

$$\cdots \cup [-4\pi,\ -3\pi] \cup [-2\pi,\ -\pi] \cup [0,\ \pi] \cup [2\pi,\ 3\pi] \cup [4\pi,\ 5\pi] \cup \cdots .$$

2.4.35: Given: $f(x) = \sin|x|$. The sine function is continuous on the set of all real numbers, as is the absolute value function. Therefore their composition f is continuous on the set $\boldsymbol{R}$ of all real numbers.

2.4.36: Given:

$$G(u) = \frac{1}{\sqrt{1 + \cos u}}.$$

Because $G(u)$ is the sum, composition, and quotient of continuous functions, it is continuous where it is defined. There is no obstruction to computing $\sqrt{1+\cos u}$ because $1 + \cos u \geqq 0$ for every real number u. Hence G will be undefined, and thus not continuous, exactly when its denominator is zero, which is exactly when $1 + \cos u = 0$. Therefore G is continuous except at the odd integral multiples of π. Put another way, G is continuous on the union of open intervals of the form $([2n-1]\pi,\ [2n+1]\pi)$ where n runs through all integral values.

2.4.37: The function

$$f(x) = \frac{x}{(x+3)^3}$$

is not continuous when $x = -3$. This discontinuity is not removable because $f(x) \to -\infty$ as $x \to -3^+$, so that the limit of $f(x)$ at $x = -3$ does not exist.

2.4.38: The function

$$f(t) = \frac{t}{t^2 - 1}$$

is not continuous when $t = \pm 1$ because $t^2 - 1 = 0$ then. These discontinuities are not removable because $f(t) \to +\infty$ as $t \to 1^+$ and as $t \to -1^+$, so that f has no limit at either $t = 1$ or $t = -1$.

2.4.39: First simplify $f(x)$:

$$f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2} \quad \text{if} \quad x \neq 2.$$

Now $f(x)$ is not defined at $x = \pm 2$ because $x^2 - 4 = 0$ for such x. The discontinuity at -2 is not removable because $f(x) \to +\infty$ as $x \to -2^+$. But $f(x) \to \frac{1}{4}$ as $x \to 2$, so the discontinuity at $x = 2$ is removable; f can be made continuous at $x = 2$ by defining its value there to be its limit there, $\frac{1}{4}$.

2.4.40: First try to simplify the formula of G:

$$G(u) = \frac{u+1}{u^2 - u - 6} = \frac{u+1}{(u-3)(u+2)}.$$

This computation shows that G is not continuous at $u = 3$ and at $u = -2$. It also shows that these discontinuities are not removable because $G(u) \to +\infty$ as $u \to 3^+$ and as $u \to -2^+$, so that G has no limit at either of its discontinuities.

2.4.41: Given:

$$f(x) = \frac{1}{1 - |x|}.$$

The function f is not continuous at ± 1 because its denominator is zero if $x = -1$ and if $x = 1$. Because $f(x) \to +\infty$ as $x \to 1^-$ and as $x \to -1^+$ (consider separately the cases $x > 0$ and $x < 0$), these discontinuities are not removable; $f(x)$ has no limit at -1 or at 1.

2.4.42: If $x > 1$, then

$$h(x) = \frac{|x-1|}{(x-1)^3} = \frac{x-1}{(x-1)^3} = \frac{1}{(x-1)^2}.$$

Therefore h is discontinuous at $x = 1$ and, because $h(x) \to +\infty$ as $x \to 1^+$, this discontinuity is not removable.

2.4.43: If $x > 17$, then $x - 17 > 0$, so that

$$f(x) = \frac{x-17}{|x-17|} = \frac{x-17}{x-17} = 1.$$

But if $x < 17$, then $x - 17 < 0$, and thus

$$f(x) = \frac{x-17}{|x-17|} = \frac{x-17}{-(x-17)} = -1.$$

Therefore $h(x)$ has no limit as $x \to 17$ because its left-hand and right-hand limits there are unequal. Thus the discontinuity at $x = 17$ is not removable.

2.4.44: First simplify:

$$g(x) = \frac{x^2 + 5x + 6}{x+2} = \frac{(x+2)(x+3)}{x+2} = x + 3 \quad \text{if} \quad x \neq -2.$$

Therefore, although g is discontinuous at $x = -2$ (because it is not defined there), this discontinuity is removable; simply define $g(-2)$ to be 1, the limit of $g(x)$ as $x \to -2$.

2.4.45: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 0$ as $x \to 0^+$ and as $x \to 0^-$, so defining $f(0)$ to be 0, the limit of $f(x)$ at $x = 0$, will make f continuous there.

2.4.46: Although $f(x)$ is not continuous at $x = 1$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 2$ as $x \to 1^+$ and as $x \to 1^-$, so defining $f(1)$ to be 2, the limit of $f(x)$ at $x = 1$, will make f continuous there.

2.4.47: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 1$ as $x \to 0^+$ and as $x \to 0^-$, so defining $f(0)$ to be 1, the limit of $f(x)$ at $x = 0$, will make f continuous there.

2.4.48: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos x}{x} = \lim_{x\to 0^-} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0^-} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \frac{0}{1+1} = 0,$$

and it is clear that $f(x) \to 0$ as $x \to 0^+$. So defining $f(0)$ to be 0, the limit of $f(x)$ at $x = 0$, will make f continuous there.

2.4.49: The given function is clearly continuous for all x *except possibly* for $x = 0$. For continuity at $x = 0$, the left-hand and right-hand limits of $f(x)$ must be the same there. But

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+c) = c$$

and $f(x) \to 4$ as $x \to 0^+$. So continuity of f at $x = 0$ can occur only if $c = 4$. Moreover, if $c = 4$, then (as we have seen) $f(x) \to 4$ as $x \to 0$ and $f(0) = 4$, so f will be continuous at $x = 0$ if and only if $c = 4$. Answer: $c = 4$.

2.4.50: Clearly f is continuous if $x \neq 3$, for if $x < 3$ or if $x > 3$, then $f(x)$ is a polynomial, regardless of the value of c. For continuity at $x = 3$, we require that the one-sided limits of $f(x)$ at $x = 3$ be equal. But $f(x) \to 6 + c$ as $x \to 3^-$ and $f(x) \to 2c - 3$ as $x \to 3^+$. Equality of the one-sided limits is equivalent to

$$6 + c = 2c - 3; \quad \text{that is,} \quad c = 9.$$

Finally, if $c = 9$, then the two-sided limit of $f(x)$ at $x = 3$ is 15 and $f(3) = 2 \cdot 3 + 9 = 15$, so f will be continuous at $x = 3$ if $c = 9$. Answer: $c = 9$.

2.4.51: Note that f is continuous at x if $x \neq 0$, because $f(x)$ is a polynomial for $x < 0$ and for $x > 0$ regardless of the value of c. To be continuous at $x = 0$, it's necessary that the left-hand and right-hand limits exist and are equal there. Now

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (c^2 - x^2) = c^2 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2(x-c)^2 = 2c^2,$$

and therefore continuity at $x = 0$ will hold if and only if $c^2 = 2c^2$; that is, if $c = 0$. And if so, then $f(0) = \lim_{x\to 0} f(x)$ as well, so f will be continuous at $x = 0$. Answer: $c = 0$.

2.4.52: Note that f is continuous if $x < \pi$ because $f(x)$ is a polynomial for such x; also, f is continuous for $x > \pi$ because (regardless of the value of c) $f(x)$ is a constant multiple of a continuous function. For continuity of f at $x = \pi$, the left-hand and right-hand limits must be equal there. But

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} (c^3 - x^3) = c^3 - \pi^3 \qquad \text{and} \qquad \lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} c\sin x = \lim_{x\to\pi^+} c\sin\pi = 0.$$

So continuity of f at $x = \pi$ requires $c^3 - \pi^3 = 0$; that is, $c = \pi$. And if so, then $f(\pi) = \pi^3 - \pi^3 = 0 = \lim_{x\to\pi} f(x)$, so f will be continuous at $x = \pi$. Answer: $c = \pi$.

2.4.53: Let $f(x) = x^2 - 5$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [2, 3]. Also $f(2) = -1 < 0 < 4 = f(3)$, so $f(c) = 0$ for some number c in [2, 3]. That is, $c^2 - 5 = 0$. Hence the equation $x^2 - 5 = 0$ has a solution in [2, 3].

2.4.54: Let $f(x) = x^3 + x + 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval $[-1, 0]$. Also $f(-1) = -1 < 0 < 1 = f(0)$, so $f(c) = 0$ for some number c in $[-1, 0]$. That is, $c^3 + c + 1 = 0$. Hence the equation $x^3 + x + 1 = 0$ has a solution in $[-1, 0]$.

2.4.55: Let $f(x) = x^3 - 3x^2 + 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [0, 1]. Also $f(0) = 1 > 0 > -1 = f(1)$, so $f(c) = 0$ for some number c in [0, 1]. That is, $c^3 - 3c^2 + 1 = 0$. Hence the equation $x^3 - 3x^2 + 1 = 0$ has a solution in [0, 1].

2.4.56: Let $f(x) = x^3 - 5$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [1, 2]. Also $f(1) = -4 < 0 < 3 = f(2)$, so $f(c) = 0$ for some number c in [1, 2]. That is, $c^3 - 5 = 0$. Hence the equation $x^3 = 5$ has a solution in [1, 2].

2.4.57: Let $f(x) = x^4 + 2x - 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [0, 1]. Also $f(0) = -1 < 0 < 2 = f(1)$, so $f(c) = 0$ for some number c in [0, 1]. That is, $c^4 + 2c - 1 = 0$. Hence the equation $x^4 + 2x - 1 = 0$ has a solution in [0, 1].

2.4.58: Let $f(x) = x^5 - 5x^3 + 3$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval $[-3, -2]$. Also $f(-3) = -105 < 0 < 11 = f(-2)$, so $f(c) = 0$ for some number c in $[-3, -2]$. That is, $c^5 - 5c^3 + 3 = 0$. Hence the equation $x^5 - 5x^3 + 3 = 0$ has a solution in $[-3, -2]$.

2.4.59: Given: $f(x) = x^3 - 4x + 1$. Values of $f(x)$:

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

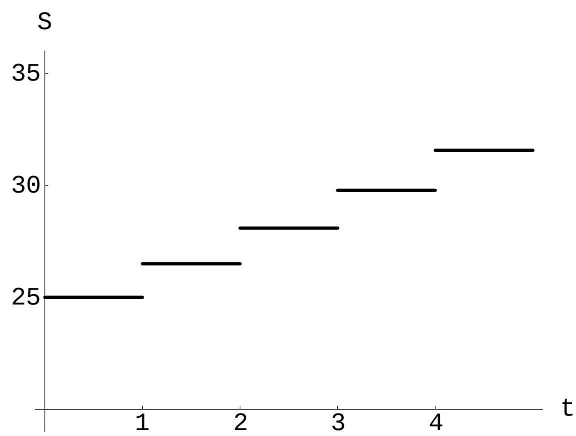
So $f(x_i) = 0$ for x_1 in $(-3, -2)$, x_2 in $(0, 1)$, and x_3 in $(1, 2)$. Because these intervals do not overlap, the equation $f(x) = 0$ has at least three real solutions. Because $f(x)$ is a polynomial of degree 3, that equation also has at most three real solutions. Therefore the equation $x^3 - 4x + 1 = 0$ has exactly three real solutions.

2.4.60: Given: $f(x) = x^3 - 3x^2 + 1$. Values of $f(x)$:

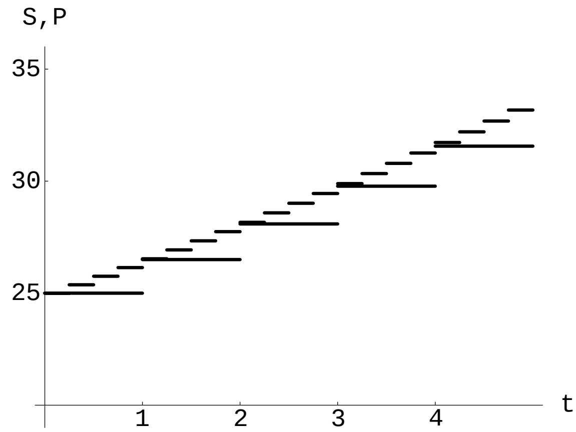
x	-3	-2	-1	0	1	2	3
$f(x)$	-53	-19	-3	1	-1	-3	1

So $f(x_i) = 0$ for x_1 in $(-1, 0)$, x_2 in $(0, 1)$, and x_3 in $(2, 3)$. Because these intervals do not overlap, the equation $f(x) = 0$ has at least three real solutions. Because $f(x)$ is a polynomial of degree 3, that equation also has at most three real solutions. Therefore it has exactly three real solutions.

2.4.61: At time t, $[\![t]\!]$ years have elapsed, and at that point your starting salary has been multiplied by 1.06 exactly t times. Thus it is $S(t) = 25 \cdot (1.06)^{[\![t]\!]}$. Of course S is discontinuous exactly when t is an integer between 1 and 5. The graph is next.

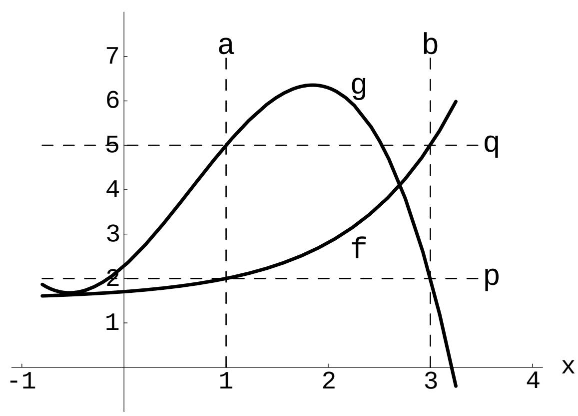


2.4.62: The salary function is $P(t) = 25 \cdot (1.015)^{[\![4t]\!]}$. It is discontinuous at the end of every three-month period; that is, at integral multiples of $\frac{1}{4}$. You will accumulate more money with three-month raises of 1.5% than with yearly raises of 6%; the total salary received by the end of the first five years with yearly raises would be \$140,930 but with quarterly raises it would be \$144,520. The graphs of the function $S(t)$ of Problem 61 and $P(t)$ are shown next. Although the graphs do not make it perfectly clear, it turns out that $P(t) > S(t)$ if $t \geqq \frac{1}{4}$, so that the quarterly raise is better for you financially after the first three months and continues to outpace the yearly raise as long as you keep the job.



2.4.63: The next figure shows the graphs of two such functions f and g, with $[a, b] = [1, 3]$, $p = 2$, and $q = 5$. Because f and g are continuous on $[a, b]$, so is $h = f - g$. Because $p \neq q$, $h(a) = p - q$ and $h(b) = q - p$ have opposite signs, so that 0 is an intermediate value of the continuous function h. Therefore $h(c) = 0$ for some number c in (a, b). That is, $f(c) = g(c)$. This concludes the proof. To construct the figure, we used (the given coefficients are approximate)

$$f(x) = 1.53045 + (0.172739)e^x \qquad \text{and} \qquad g(x) = 2.27857 + (2.05)x + (1.36429)x^2 - (0.692857)x^3.$$



2.4.64: Let $f(t)$ denote your distance from Estes Park during your trip today, with f measured in kilometers and t in hours, $t = 0$ corresponding to time 1 P.M. Let $g(t)$ denote your distance from Estes Park during your trip tomorrow, with g in kilometers, t in hours, and $t = 0$ corresponding to 1 P.M. tomorrow. Assuming that both f and g are continuous, we use the facts that $f(0) = 0$, $f(1) = M$ (where M is the distance from Estes Park to Grand Lake), $g(0) = M$, and $g(1) = 0$ and apply the result of Problem 63 to conclude that $f(c) = g(c)$ for some number c in $(0, 1)$. That is, at time $t = c$ tomorrow you will be at exactly the same spot (at distance $g(c)$ from Estes Park) as you will be at the same time $t = c$ today at distance $f(c) = g(c)$ from Estes Park.

The 1999 *National Geographic Road Atlas* indicates that $M \approx 101$ (km). Making this trip in a single hour is unforgivable given the magnificent scenery (and probably impossible as well given the dozens of tight turns on the highway).

2.4.65: Given $a > 0$, let $f(x) = x^2 - a$. Then f is continuous on $[0,\, a+1]$ because $f(x)$ is a polynomial. Also $f(a+1) > 0$ because

$$f(a+1) = (a+1)^2 - a = a^2 + a + 1 > 1 > 0.$$

So $f(0) = -a < 0 < f(a+1)$. Therefore, because f has the intermediate value property on the interval $[0,\, a+1]$, there exists a number r in $(0,\, a+1)$ such that $f(r) = 0$. That is, $r^2 - a = 0$, so that $r^2 = a$. Therefore a has a square root.

Our proof shows that a has a positive square root. Can you modify it to show that a also has a negative square root? Do you see why we used the interval $[0,\, a+1]$ rather than the simpler $[0,\, a]$?

2.4.66: Clearly $a = 0$ has a cube root. Suppose first that $a > 0$. Let $f(x) = x^3 - a$. Then f has the intermediate value property on $[0,\, a+1]$ because $f(x)$ is a polynomial. Moreover,

$$f(a+1) = (a+1)^3 - a = a^3 + 3a^2 + 2a + 1 > 1 > 0 > -a = f(0).$$

Therefore there exist a number c in $[0,\, a+1]$ such that $f(c) = 0$. That is, $c^3 - a = 0$, so that $c^3 = a$. Thus the positive real number a has a cube root. Moreover, $(-c)^3 = -(c^3) = -a$, so that every negative real number has a cube root as well.

2.4.67: Given the real number a, we need to show that

$$\lim_{x \to a} \cos x = \cos a.$$

Let $h = x - a$, so that $x = a + h$. Then $x \to a$ is equivalent to $h \to 0$; also, $\cos x = \cos(a+h)$. Thus

$$\begin{aligned}\lim_{x \to a} \cos x &= \lim_{h \to 0} \cos(a+h)\\ &= \lim_{h \to 0} (\cos a \cos h - \sin a \sin h) = (\cos a) \cdot 1 - (\sin a) \cdot 0 = \cos a.\end{aligned}$$

Therefore the cosine function is continuous at $x = a$ for every real number a.

2.4.68: If x is not an integer, choose that [unique] integer $n = [\![x]\!]$ such that $n < x < n+1$. Then $f(x) = x + n$ on the interval $(n,\, n+1)$, thus $f(x)$ is effectively a polynomial on that interval. So f is continuous at x. But if m is an integer, then

$$\lim_{x \to m^-} f(x) = \lim_{x \to m^-} (x + [\![x]\!]) = \lim_{x \to m^-} (x + m - 1) = m + m - 1 = 2m - 1,$$

whereas

$$\lim_{x \to m^+} f(x) = \lim_{x \to m^+} (x + [\![x]\!]) = \lim_{x \to m^+} (x + m) = m + m = 2m.$$

Because the left-hand and right-hand limits of $f(x)$ differ at m, f is not continuous there. Thus f is discontinuous at each integer and continuous at every other real number.

2.4.69: Suppose that a is a real number. We appeal to the formal definition of the limit in Section 2.2 (page 74) to show that $f(x)$ has no limit as $x \to a$. Suppose by way of contradiction that $f(x) \to L$ as $x \to a$. Then, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ for every number x such that $0 < |x - a| < \delta$. So this statement must hold if $\epsilon = \frac{1}{4}$.

Case 1: $L = 0$. Then there must exist a number $\delta > 0$ such that

$$|f(x) - 0| < \tfrac{1}{4}$$

if $0 < |x - a| < \delta$. But, regardless of the value of δ, there exist irrational values of x satisfying this last inequality. (We'll explain why in a moment.) Choose such a number x. Then

$$|f(x) - 0| = |1 - 0| = 1 < \tfrac{1}{4}.$$

This is impossible. So $L \neq 0$.

Case 2: $L = 1$. Proceed exactly as in Case 1, except choose a *rational* value of x such that $0 < |x - a| < \delta$. Then

$$|f(x) - 1| = |0 - 1| = 1 < \tfrac{1}{4}.$$

This, too, is impossible. So $L \neq 1$.

Case 3: L is neither 0 nor 1. Let $\epsilon = \frac{1}{3}|L|$. Note that $\epsilon > 0$. Then suppose that there exists $\delta > 0$ such that,

$$\text{if} \qquad 0 < |x - a| < \delta, \qquad \text{then} \qquad |f(x) - L| < \epsilon. \tag{1}$$

Choose a rational number x satisfying the left-hand inequality. Then

$$|f(x) - L| = |0 - L| = |L| = 3\epsilon.$$

It follows from (1) that $3\epsilon < \epsilon$, which is impossible because $\epsilon > 0$.

In summary, L cannot be 0, nor can it be 1, nor can it be any other real number. Therefore $f(x)$ has no limit as $x \to a$. Consequently f is not continuous at $x = a$.

In this proof we relied heavily on the fact that if a is any real number, then we can find both rational and irrational numbers arbitrarily close to a. Rather than providing a formal proof, we illustrate how to do this in the case that

$$a = 1.23456789101112131415\cdots.$$

(It doesn't matter whether a is rational or irrational.) To produce rational numbers arbitrarily close to a, use

$$1.2,\ 1.23,\ 1.234,\ 1.2345,\ 1.23456,\ 1.234567,\ \dots\ . \tag{2}$$

The numbers in (2) are all rational because they all have terminating decimal expansions, and the nth number in (2) differs from a by less than 10^{-n}, so there are rational numbers arbitrarily close to a. To get irrational numbers with the same properties, use

$$1.20100100010000100000100\cdots,\ 1.23010010001000010000100\cdots,$$

$$1.234010010001000010000100\cdots,\ 1.2345010010001000010000100\cdots,\ \dots\ .$$

These numbers are irrational because every one of them has a nonrepeating decimal expansion.

2.4.70: You can modify the argument in the solution of Problem 69 to show that $f(x)$ has no limit at $x = a$ if $a \neq 0$. Simply use a^2 in place of 1 in that argument. Because $0 \leqq f(x) \leqq x^2$ for all x, and because $0 \to 0$ and $x^2 \to 0$ as $x \to 0$, it follows from the squeeze theorem that

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

Therefore f is continuous at $x = 0$.

2.4.71: Let $g(x) = x - \cos x$. Then $g(x)$ is the sum of continuous functions, thus continuous everywhere, and in particular on the interval $\left[0,\ \frac{1}{2}\pi\right]$. So g has the intermediate value property there. Also $g(0) = -1 < 0$ and $g(1) = 1 > 0$. Therefore there exists a number c in $\left(0,\ \frac{1}{2}\pi\right)$ such that $g(c) = 0$. Thus $c - \cos x = 0$, so that $c = \cos c$. Thus the equation $x = \cos x$ has a solution in $\left(0,\ \frac{1}{2}\pi\right)$. (This solution is approximately 0.7390851.)

2.4.72: Let $h(x) = x + 5\cos x$. Then $h(x)$ is the sum and product of continuous functions, thus is continuous everywhere, including the intervals $[-\pi,\ 0]$, $[0,\ \pi]$, and $[\pi,\ 2\pi]$. Moreover,

$$h(-\pi) = -\pi - 5 < 0 < 5 = h(0), \qquad h(0) = 5 > 0 > \pi - 5 = h(\pi),$$

$$\text{and} \qquad h(\pi) = \pi - 5 < 0 < 2\pi + 5 = h(2\pi).$$

By the intermediate value property of continuous functions, $h(a) = 0$ for some number a in $I = (-\pi,\ 0)$, $h(b) = 0$ for some number b in $J = (0,\ \pi)$, and $h(c) = 0$ for some number c in $K = (\pi,\ 2\pi)$. Because no two of I, J, and K have any points in common, the numbers a, b, and c are distinct. Therefore the equation $h(x) = 0$ three distinct solutions; in other words, the equation $x = -5\cos x$ has three distinct solutions. (We have *not* shown that there are no additional solutions, but this was not required.) Finally, $a \approx -1.30644$, $b \approx 1.97738$, and $c \approx 3.83747$.

2.4.73: Because

$$\lim_{x\to 0^+} 2^{1/x} = +\infty,$$

f is not right continuous at $x = 0$. Because

$$\lim_{x\to 0^-} 2^{1/x} = \lim_{u\to -\infty} 2^u = \lim_{z\to +\infty} \frac{1}{2^z} = 0 = f(0),$$

f is left continuous at $x = 0$.

2.4.74: Because

$$\lim_{x\to 0} 2^{-1/x^2} = 0 = f(0),$$

the function f is both left and right continuous—thus continuous—at $x = 0$.

2.4.75: Because

$$\lim_{x\to 0^+} \frac{1}{1+2^{1/x}} = 0 \neq f(0),$$

f is not right continuous at $x = 0$. But

$$\lim_{x\to 0^-} \frac{1}{1+2^{1/x}} = \frac{1}{1+0} = 1 = f(0),$$

f is left continuous at $x = 0$.

2.4.76: Because

$$\lim_{x\to 0} \frac{1}{1+2^{-1/x^2}} = \frac{1}{1+0} = 1 = f(0),$$

the function f is both left and right continuous at $x = 0$—thus it is continuous there.

2.4.77: We consider only the discontinuity at $x = a = \pi/2$; the behavior of f is the same near all of its discontinuities (the odd integral multiples of a). Because

$$\lim_{x\to a^+} \frac{1}{1+2^{\tan x}} = \frac{1}{1+0} = 1 = f(1),$$

the function f is right continuous at $x = a$. But

$$\lim_{x\to a^-} \frac{1}{1+2^{\tan x}} = 0 \neq f(0),$$

so f is not left continuous at $x = a$.

2.4.78: We consider only the discontinuities at $x = 0$ and $x = \pi$, because the behavior of f at every even integral multiple of π is the same as its behavior at $x = 0$, and its behavior at every odd integral multiple of π is the same as its behavior at $x = \pi$. We first note that

$$\lim_{x\to 0^+} \frac{1}{1+2^{1/\sin x}} = 0 = f(0),$$

so that f is right continuous at $x = 0$. But

$$\lim_{x\to 0^-} \frac{1}{1+2^{1/\sin x}} = 1 \neq f(0),$$

and thus f is not left continuous at $x = 0$. The situation is reversed at π, as one might gather from examining the graph of the sine function:

$$\lim_{x\to \pi^-} \frac{1}{1+2^{1/\sin x}} = 0 = f(\pi),$$

so that f is left continuous at $x = \pi$, but

$$\lim_{x\to \pi^+} \frac{1}{1+2^{1/\sin x}} = 1 \neq f(\pi).$$

Therefore f is not right continuous at $x = \pi$.

Chapter 2 Miscellaneous Problems

2.M.1: $\displaystyle\lim_{x\to 0}(x^2-3x+4) = \left(\lim_{x\to 0} x\right)^2 - 3\cdot\left(\lim_{x\to 0} x\right) + 4 = 0^2 - 3\cdot 0 + 4 = 4.$

2.M.2: $\displaystyle\lim_{x\to -1}(3-x+x^3) = 3 - \left(\lim_{x\to -1} x\right) + \left(\lim_{x\to -1} x\right)^3 = 3-(-1)+(-1)^3 = 3.$

2.M.3: $\displaystyle\lim_{x\to 2}(4-x^2)^{10} = \left[4-\left(\lim_{x\to 2} x\right)^2\right]^{10} = \left(4-2^2\right)^{10} = 0^{10} = 0.$

2.M.4: $\displaystyle\lim_{x\to 1}(x^2+x-1)^{17} = \left[\left(\lim_{x\to 1} x\right)^2 + \left(\lim_{x\to 1} x\right) - 1\right]^{17} = (1^2+1-1)^{17} = 1.$

2.M.5: $\displaystyle\lim_{x\to 2}\frac{1+x^2}{1-x^2} = \frac{1+\left(\lim_{x\to 2} x\right)^2}{1-\left(\lim_{x\to 2} x\right)^2} = \frac{1+2^2}{1-2^2} = \frac{1+4}{1-4} = -\frac{5}{3}.$

2.M.6: $\displaystyle\lim_{x\to 3}\frac{2x}{x^2-x-3} = \frac{2\cdot\left(\lim_{x\to 3} x\right)}{\left(\lim_{x\to 3} x\right)^2 - \left(\lim_{x\to 3} x\right) - 3} = \frac{2\cdot 3}{3^2-3-3} = \frac{6}{3} = 2.$

2.M.7: $\displaystyle\frac{x^2-1}{1-x} = -\frac{(x+1)(x-1)}{x-1} = -(x+1) \to -2$ as $x \to 1$.

2.M.8: $\displaystyle\frac{x+2}{x^2+x-2} = \frac{x+2}{(x+2)(x-1)} = \frac{1}{x-1} \to -\frac{1}{3}$ as $x \to -2$.

2.M.9: $\displaystyle\frac{t^2+6t+9}{9-t^2} = -\frac{(t+3)^2}{(t+3)(t-3)} = -\frac{t+3}{t-3} \to -\frac{-3+3}{-3-3} = 0$ as $t \to -3$.

2.M.10: $\displaystyle\frac{4x-x^3}{3x+x^2} = \frac{x(4-x^2)}{x(3+x)} = \frac{4-x^2}{3+x} \to \frac{4-0}{3+0} = \frac{4}{3}$ as $x \to 0$.

2.M.11: $\lim_{x\to 3}(x^2-1)^{2/3} = \left[\left(\lim_{x\to 3} x\right)^2 - 1\right]^{2/3} = (3^2-1)^{2/3} = 8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4.$

2.M.12: $\lim_{x\to 2}\left(\dfrac{2x^2+1}{2x}\right)^{1/2} = \left[\dfrac{2\cdot\left(\lim_{x\to 2} x\right)^2+1}{2\cdot\left(\lim_{x\to 2} x\right)}\right]^{1/2} = \left(\dfrac{2\cdot 4+1}{2\cdot 2}\right)^{1/2} = \left(\dfrac{9}{4}\right)^{1/2} = \dfrac{3}{2}.$

2.M.13: $\lim_{x\to 3}\left(\dfrac{5x+1}{x^2-8}\right)^{3/4} = \left[\dfrac{5\cdot\left(\lim_{x\to 3} x\right)+1}{\left(\lim_{x\to 3} x\right)^2-8}\right]^{3/4} = \left(16^{1/4}\right)^3 = 8.$

2.M.14: $\dfrac{x^4-1}{x^2+2x-3} = \dfrac{(x^2+1)(x+1)(x-1)}{(x+3)(x-1)} = \dfrac{(x^2+1)(x+1)}{x+3} \to \dfrac{2\cdot 2}{4} = 1$ as $x\to 1$.

2.M.15: First multiply numerator and denominator by $\sqrt{x+2}+3$ (the *conjugate* of the numerator) to obtain

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7} = \lim_{x\to 7}\frac{x+2-9}{(x-7)\left(\sqrt{x+2}+3\right)} = \lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}$$

$$= \lim_{x\to 7}\frac{1}{\sqrt{x+2}+3} = \frac{\lim_{x\to 7} 1}{\lim_{x\to 7}\left(\sqrt{x+2}+3\right)} = \frac{1}{\left(2+\lim_{x\to 7} x\right)^{1/2} + \lim_{x\to 7} 3} = \frac{1}{3+3} = \frac{1}{6}.$$

2.M.16: Note that $x>1$ as $x\to 1^+$, so that $\sqrt{x^2-1}$ is defined for such x. Therefore

$$\lim_{x\to 1^+}\left(x-\sqrt{x^2-1}\right) = \left(\lim_{x\to 1^+} x\right) - \left[\left(\lim_{x\to 1^+} x\right)^2 - \left(\lim_{x\to 1^+} 1\right)\right]^{1/2} = 1-\sqrt{1^2-1} = 1-0 = 1.$$

2.M.17: First simplify:

$$\frac{\dfrac{1}{\sqrt{13+x}}-\dfrac{1}{3}}{x+4} = \frac{1}{x+4}\cdot\frac{3-\sqrt{13+x}}{3\sqrt{13+x}} = \frac{3-\sqrt{13+x}}{3(x+4)\sqrt{13+x}}.$$

Then we multiply both numerator and denominator by $3+\sqrt{13+x}$, the conjugate of the numerator, to obtain

$$\frac{9-(13+x)}{3(x+4)\sqrt{13+x}\left(3+\sqrt{13+x}\,\right)} = \frac{-(x+4)}{3(x+4)\sqrt{13+x}\left(3+\sqrt{13+x}\,\right)}$$

$$= -\frac{1}{3\sqrt{13+x}\left(3+\sqrt{13+x}\,\right)}.$$

Now let $x\to -4$ to obtain the limit $-\dfrac{1}{3\cdot 3\cdot(3+3)} = -\dfrac{1}{54}.$

2.M.18: Because $x \to 1^+$, $x > 1$, so that $1 - x < 0$. Therefore

$$\lim_{x \to 1^+} \frac{1-x}{|1-x|} = \lim_{x \to 1^+} \frac{1-x}{-(1-x)} = \lim_{x \to 1^+} (-1) = -1.$$

2.M.19: First, $4 - 4x + x^2 = (2-x)^2 = (x-2)^2$. Because $x \to 2^+$, $x > 2$, so that $x - 2 > 0$. Hence $\sqrt{4 - 4x + x^2} = \sqrt{(x-2)^2} = |x-2| = x - 2$. Therefore

$$\lim_{x \to 2^+} \frac{2-x}{\sqrt{4-4x+x^2}} = \lim_{x \to 2^+} \frac{2-x}{x-2} = \lim_{x \to 2^+} (-1) = -1.$$

2.M.20: As $x \to -2^-$, $x < -2$, so that $x + 2 < 0$. Hence $|x+2| = -(x+2)$. Thus

$$\lim_{x \to -2^-} \frac{x+2}{|x+2|} = \lim_{x \to -2^-} \frac{x+2}{-(x+2)} = \lim_{x \to -2^-} (-1) = -1.$$

2.M.21: As $x \to 4^+$, $x > 4$, so that $x - 4 > 0$. Therefore $|x-4| = x - 4$, and thus

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = \lim_{x \to 4^+} \frac{x-4}{x-4} = \lim_{x \to 4^+} 1 = 1.$$

2.M.22: As $x \to 3^-$, $x < 3$, so that $x^2 - 9 < 0$ for $-3 < x < 3$. Therefore $\sqrt{x^2-9}$ is undefined for such x, and consequently $\lim_{x \to 3^-} \sqrt{x^2-9}$ does not exist.

2.M.23: As $x \to 2^+$, $x > 2$, so that $4 - x^2 < 0$. Therefore $\sqrt{4-x^2}$ is undefined for all such x, and consequently $\lim_{x \to 2^+} \sqrt{4-x^2}$ does not exist.

2.M.24: As $x \to -3$, $(x+3)^2 \to 0$ whereas the numerator x approaches -3. Therefore this limit does not exist. Because $(x+3)^2$ is approaching zero through positive values, it is also correct to write

$$\lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty.$$

2.M.25: As $x \to 2$, the denominator $(x-2)^2$ is approaching zero, while the numerator $x+2$ is approaching 4. So this limit does not exist. Because the denominator is approaching zero through positive values, it is also correct (and more informative) to write

$$\lim_{x \to 2} \frac{x+2}{(x-2)^2} = +\infty.$$

2.M.26: As $x \to 1^-$, the denominator $x - 1$ is approaching zero, but the numerator x is not. Therefore this limit does not exist. Because the numerator is approaching 1 and the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x}{x-1} = -\infty.$$

2.M.27: Because $x \to 3^+$, the denominator $x-3$ is approaching zero, but the numerator x is not. Therefore this limit does not exist. Because the denominator is approaching zero through positive values while the numerator is approaching 3, it is also correct to write

$$\lim_{x \to 3^+} \frac{x}{x-3} = +\infty.$$

2.M.28: Because

$$\frac{x-2}{x^2-3x+2} = \frac{x-2}{(x-1)(x-2)} = \frac{1}{x-1}$$

if $x \neq 2$, the limit of this fraction as $x \to 1^-$ does not exist: The numerator is approaching 1 while the denominator is approaching zero. Because the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x-2}{x^2-3x+2} = -\infty.$$

2.M.29: As $x \to 1^-$, the numerator of the fraction is approaching 2, but the denominator is approaching zero. Therefore this limit does not exist. Because the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x+1}{(x-1)^3} = -\infty.$$

2.M.30: Note first that

$$\frac{25-x^2}{x^2-10x+25} = \frac{(5+x)(5-x)}{(x-5)^2} = \frac{(5+x)(5-x)}{(5-x)^2} = \frac{5+x}{5-x}.$$

Thus as $x \to 5^+$, the numerator approaches 10 while the denominator is approaching zero through negative values. Therefore this limit does not exist. It is also correct to write

$$\lim_{x \to 5^+} \frac{25-x^2}{x^2-10x+25} = -\infty.$$

2.M.31: Let $u = 3x$. Then $x = \frac{1}{3}u$; also, $x \to 0$ is equivalent to $u \to 0$. Thus

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{u \to 0} \frac{\sin u}{\frac{1}{3}u} = \lim_{u \to 0} \frac{3 \sin u}{u} = \left(\lim_{u \to 0} 3\right) \cdot \left(\lim_{u \to 0} \frac{\sin u}{u}\right) = 3 \cdot 1 = 3.$$

2.M.32: Let $u = 5x$; then $x = \frac{1}{5}u$; moreover, $x \to 0$ is equivalent to $u \to 0$. Therefore

$$\lim_{x \to 0} \frac{\tan 5x}{x} = \lim_{u \to 0} \frac{\tan u}{\frac{1}{5}u} = \lim_{u \to 0} \frac{5 \sin u}{u \cos u} = 5 \cdot \left(\lim_{u \to 0} \frac{\sin u}{u}\right) \cdot \left(\lim_{u \to 0} \frac{1}{\cos u}\right) = 5 \cdot 1 \cdot \frac{1}{1} = 5.$$

2.M.33: The substitution $u = kx$ shows that if $k \neq 0$, then

$$\lim_{x \to 0} \frac{\sin kx}{kx} = 1.$$

It also follows that $\lim_{x\to 0} \dfrac{kx}{\sin kx} = 1$. Therefore

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{2x}{\sin 2x} \cdot \frac{3x}{2x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{2x}{\sin 2x} \cdot \frac{3}{2} = 1 \cdot 1 \cdot \frac{3}{2} = \frac{3}{2}.$$

2.M.34: We saw in the solution of Problem 2.3.14 that if k is a nonzero constant, then

$$\lim_{x\to 0} \frac{\tan kx}{kx} = 1 = \lim_{x\to 0} \frac{kx}{\tan kx}.$$

Therefore

$$\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x\to 0} \frac{\tan 2x}{2x} \cdot \frac{3x}{\tan 3x} \cdot \frac{2x}{3x} = \lim_{x\to 0} \frac{\tan 2x}{2x} \cdot \frac{3x}{\tan 3x} \cdot \frac{2}{3} = 1 \cdot 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

2.M.35: Let $x = u^2$ where $u > 0$. Then $x \to 0^+$ is equivalent to $u \to 0^+$. Hence

$$\lim_{x\to 0^+} \frac{x}{\sin \sqrt{x}} = \lim_{u\to 0^+} \frac{u^2}{\sin u} = \lim_{u\to 0^+} u \cdot \frac{u}{\sin u} = 0 \cdot 1 = 0.$$

2.M.36: First multiply numerator and denominator by the conjugate $1 + \cos 3x$ of the numerator:

$$\begin{aligned} \frac{1-\cos 3x}{2x} &= \frac{(1-\cos 3x)(1+\cos 3x)}{2x(1+\cos 3x)} = \frac{1-\cos^2 3x}{2x(1+\cos 3x)} = \frac{\sin^2 3x}{2x(1+\cos 3x)} \\ &= \frac{\sin 3x}{2x} \cdot \frac{\sin 3x}{1+\cos 3x} = \frac{\sin 3x}{3x} \cdot \frac{3x}{2x} \cdot \frac{\sin 3x}{1+\cos 3x} = \frac{\sin 3x}{3x} \cdot \frac{3}{2} \cdot \frac{\sin 3x}{1+\cos 3x}. \end{aligned}$$

Now let $x \to 0$ to obtain the limit $1 \cdot \dfrac{3}{2} \cdot \dfrac{0}{1+1} = 0$.

2.M.37: First multiply numerator and denominator by the conjugate $1 + \cos 3x$ of the numerator:

$$\begin{aligned} \frac{1-\cos 3x}{2x^2} &= \frac{(1-\cos 3x)(1+\cos 3x)}{2x^2(1+\cos 3x)} = \frac{1-\cos^2 3x}{2x^2(1+\cos 3x)} = \frac{\sin^2 3x}{2x^2(1+\cos 3x)} \\ &= \frac{\sin 3x}{2x} \cdot \frac{\sin 3x}{x} \cdot \frac{1}{1+\cos 3x} = \frac{\sin 3x}{3x} \cdot \frac{3x}{2x} \cdot \frac{\sin 3x}{3x} \cdot \frac{3x}{x} \cdot \frac{1}{1+\cos 3x} \\ &= \frac{\sin 3x}{3x} \cdot \frac{3}{2} \cdot \frac{\sin 3x}{3x} \cdot \frac{3}{1} \cdot \frac{1}{1+\cos 3x}. \end{aligned}$$

Now let $x \to 0$ to obtain the limit $1 \cdot \dfrac{3}{2} \cdot 1 \cdot \dfrac{3}{1} \cdot \dfrac{1}{1+1} = \dfrac{9}{4}$.

2.M.38: Express the cotangent and cosecant functions in terms of the sine and cosine functions to obtain

$$\lim_{x\to 0} x^3 \cot x \csc x = \lim_{x\to 0} (x^3) \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = \lim_{x\to 0} x \cdot \frac{x}{\sin x} \cdot (\cos x) \cdot \frac{x}{\sin x} = 0 \cdot 1 \cdot 1 \cdot 1 = 0.$$

2.M.39: Let $u = 2x$; then $x = \frac{1}{2}u$, and $x \to 0$ is then equivalent to $u \to 0$. Also express the secant and tangent functions in terms of the sine and cosine functions. Result:

$$\lim_{x\to 0} \frac{\sec 2x \tan 2x}{x} = \lim_{u\to 0} \frac{\sec u \tan u}{\frac{1}{2}u} = \lim_{u\to 0} \frac{2\sin u}{u\cos^2 u} = \lim_{u\to 0} \frac{2}{\cos^2 u} \cdot \frac{\sin u}{u} = \frac{2}{1} \cdot 1 = 2.$$

2.M.40: Let $u = 3x$; then $x = \frac{1}{3}u$, and $x \to 0$ is then equivalent to $u \to 0$. Also express the cotangent function in terms of sines and cosines. Result:

$$\lim_{x\to 0} x^2 \cot^2 3x = \lim_{x\to 0} \frac{x^2 \cos^2 3x}{\sin^2 3x} = \lim_{u\to 0} \frac{\frac{1}{9}u^2 \cos^2 u}{\sin^2 u} = \lim_{u\to 0} \frac{\cos^2 u}{9} \cdot \frac{u}{\sin u} \cdot \frac{u}{\sin u} = \frac{1}{9} \cdot 1 \cdot 1 = \frac{1}{9}.$$

2.M.41: Given $f(x) = 2x^2 + 3$, a slope-predictor for f is $m(x) = 4x$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, 5)$ is therefore $m(1) = 4$. So an equation of that line is $y - 5 = 4(x - 1)$; that is, $y = 4x + 1$.

2.M.42: Given $f(x) = -5x^2 + x$, a slope-predictor for f is $m(x) = -10x + 1$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, -4)$ is therefore $m(1) = -9$. So an equation of that line is $y + 4 = -9(x - 1)$; that is, $y = -9x + 5$.

2.M.43: Given $f(x) = 3x^2 + 4x - 5$, a slope-predictor for f is $m(x) = 6x + 4$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, 2)$ is therefore $m(1) = 10$. So an equation of that line is $y - 2 = 10(x - 1)$; that is, $y = 10x - 8$.

2.M.44: Given $f(x) = -3x^2 - 2x + 1$, a slope-predictor for f is $m(x) = -6x - 2$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, -4)$ is therefore $m(1) = -8$. So an equation of that line is $y + 4 = -8(x - 1)$; that is, $y = -8x + 4$.

2.M.45: Given $f(x) = (x - 1)(2x - 1) = 2x^2 - 3x + 1$, a slope-predictor for f is $m(x) = 4x - 3$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, 0)$ is therefore $m(1) = 1$. So an equation of that line is $y = x - 1$.

2.M.46: Given $f(x) = \frac{1}{3}x - \left(\frac{1}{4}x\right)^2 = -\frac{1}{16}x^2 + \frac{1}{3}x$, a slope-predictor for f is $m(x) = -\frac{1}{8}x + \frac{1}{3}$. The slope of the line tangent to the graph of f at $(1, f(1)) = \left(1, \frac{13}{48}\right)$ is therefore $m(1) = -\frac{1}{8} + \frac{1}{3} = \frac{5}{24}$. So an equation of that line is $y - \frac{13}{48} = \frac{5}{24}(x - 1)$; that is, $48y = 10x + 3$.

2.M.47: If $f(x) = 2x^2 + 3x$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 + 3(x+h) - (2x^2 + 3x)}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 3x + 3h - 2x^2 - 3x}{h} = \lim_{h\to 0} \frac{4xh + 2h^2 + 3h}{h} = \lim_{h\to 0} \frac{h(4x + 2h + 3)}{h}$$

$$= \lim_{h\to 0} (4x + 2h + 3) = 4x + 3.$$

2.M.48: If $f(x) = x - x^3$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h) - (x+h)^3 - (x - x^3)}{h}$$

$$= \lim_{h\to 0} \frac{x + h - (x^3 + 3x^2h + 3xh^2 + h^3) - x + x^3}{h}$$

$$= \lim_{h\to 0} \frac{x + h - x^3 - 3x^2h - 3xh^2 - h^3 - x + x^3}{h}$$

$$= \lim_{h\to 0} \frac{h - 3x^2h - 3xh^2 - h^3}{h} = \lim_{h\to 0} \frac{h(1 - 3x^2 - 3xh - h^2)}{h}$$

$$= \lim_{h\to 0} (1 - 3x^2 - 3xh - h^2) = 1 - 3x^2.$$

2.4.49: If $f(x) = \dfrac{1}{3-x}$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{3-(x+h)} - \dfrac{1}{3-x}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(3-x) - (3-x-h)}{(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{3 - x - 3 + x + h}{(3-x-h)(3-x)} = \lim_{h\to 0} \frac{h}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{(3-x-h)(3-x)} = \frac{1}{(3-x)^2}.$$

2.M.50: If $f(x) = \dfrac{1}{2x+1}$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)+1} - \dfrac{1}{2x+1}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(2x+1) - (2x+2h+1)}{(2x+2h+1)(2x+1)}$$

$$= \lim_{h\to 0} \frac{2x + 1 - 2x - 2h - 1}{h(2x+2h+1)(2x+1)} = \lim_{h\to 0} \frac{-2h}{h(2x+2h+1)(2x+1)}$$

$$= \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)} = -\frac{2}{(2x+1)^2}.$$

2.M.51: If $f(x) = x - \dfrac{1}{x}$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h) - \dfrac{1}{x+h} - \left(x - \dfrac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \left(x + h - \frac{1}{x+h} - x + \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h} \cdot \left(h + \frac{1}{x} - \frac{1}{x+h}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \left(h + \frac{x + h - x}{x(x+h)}\right) = \lim_{h\to 0} \left(1 + \frac{h}{hx(x+h)}\right)$$

$$= \lim_{h\to 0} \left(1 + \frac{1}{x(x+h)}\right) = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}.$$

2.M.52: If $f(x) = \dfrac{x}{x+1}$, then the slope-predicting function for f is

$$\begin{aligned} m(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{x+h}{x+h+1} - \frac{x}{x+1}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x+h)(x+1)-(x+h+1)(x)}{(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x^2+x+hx+h)-(x^2+xh+x)}{(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{x^2+x+hx+h-x^2-xh-x}{h(x+h+1)(x+1)} = \lim_{h\to 0} \frac{h}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}. \end{aligned}$$

2.M.53: If $f(x) = \dfrac{x+1}{x-1}$, then the slope-predicting function for f is

$$\begin{aligned} m(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x+h+1)(x-1)-(x+h-1)(x+1)}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{h(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{x^2+hx-h-1-x^2-hx-h+1}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2h}{h(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = -\frac{2}{(x-1)^2}. \end{aligned}$$

2.M.54: We must deal with $|2x+3|$, and to do so we need to know when $2x+3$ changes sign: When $2x+3=0$; that is, when $x = -\frac{3}{2}$. If $x > -\frac{3}{2}$, then $2x+3>0$, so that

$$f(x) = 3x - x^2 + (2x+3) = -x^2+5x+3 \qquad \text{if} \qquad x > -\tfrac{3}{2}.$$

By the theorem on page 58 (Section 2.1), the slope-predicting function for f will be $m_1(x) = -2x+5$ if $x > -\frac{3}{2}$. But if $x < -\frac{3}{2}$, then $2x+3<0$, so that

$$f(x) = 3x - x^2 - (2x+3) = -x^2+x-3 \qquad \text{if} \qquad x < -\tfrac{3}{2}.$$

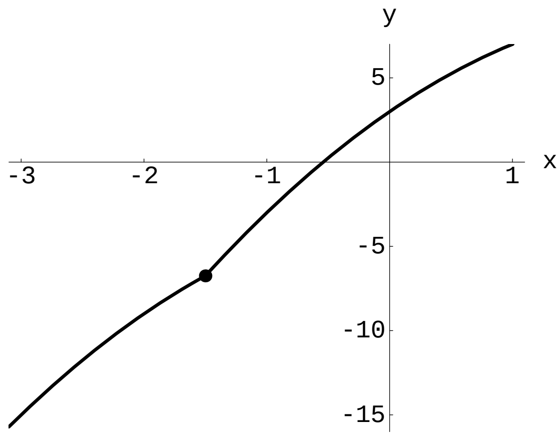
By the theorem just cited, the slope-predicting function for f will be $m_2(x) = -2x+1$ if $x < -\frac{3}{2}$. Therefore the general slope-predicting function for f will be

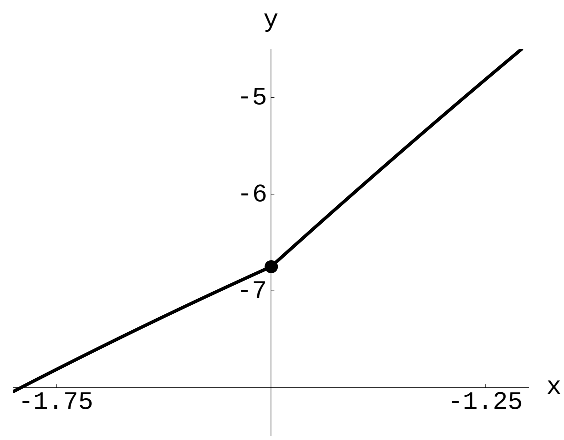
$$m(x) = \begin{cases} -2x+5 & \text{if } x > -\frac{3}{2}, \\ -2x+1 & \text{if } x < -\frac{3}{2}. \end{cases}$$

There will be no tangent line at $x = -\frac{3}{2}$. The reason is that

$$\lim_{x\to -1.5^-} \frac{f(x+h)-f(x)}{h} = 4 \quad \text{whereas} \quad \lim_{x\to -1.5^+} \frac{f(x+h)-f(x)}{h} = 8.$$

Therefore there is no tangent line at the point $\left(-\frac{3}{2},\, -\frac{27}{4}\right)$. But f is continuous at that point; indeed, f is continuous on the set $\boldsymbol{R}$ of all real numbers. The graph of f is shown next on the left; the part of the graph near the corner point at $\left(-\frac{3}{2},\, -\frac{27}{4}\right)$ is shown magnified on the right.





2.M.55: Following the suggestion, the line tangent to the graph of $y = x^2$ at $(a,\, a^2)$ has slope $2a$ (because the slope-predicting function for $f(x) = x^2$ is $m(x) = 2x$). But using the two-point formula for slope, this line also has slope

$$\frac{a^2-4}{a-3} = 2a,$$

so that $a^2 - 4 = 2a^2 - 6a$; that is, $a^2 - 6a + 4 = 0$. The quadratic formula yields the two solutions $a = 3 \pm \sqrt{5}$, so one of the two lines in question has slope $2(3+\sqrt{5})$ and the other has slope $2(3-\sqrt{5})$. Both lines pass through (3, 4), so their equations are

$$y - 4 = 2\left(3+\sqrt{5}\right)(x-3) \quad \text{and} \quad y - 4 = 2\left(3-\sqrt{5}\right)(x-3).$$

2.M.56: The given line has equation $y = -x - 3$, so its slope is -1. The radius of the circle from its center $(2,3)$ to the point $(a,\, b)$ of tangency is perpendicular to that line, so has slope 1. So the radius lies on the line $y - 3 = x - 2$; that is, $y = x + 1$. We solve $y = x + 1$ and $y = -x - 3$ simultaneously to find the point of tangency $(a,\, b)$ to be $(-2,\, -1)$. The distance from the center of the circle to this point is $4\sqrt{2}$. Therefore an equation of the circle is $(x-2)^2 + (y-3)^2 = 32$.

2.M.57: First simplify $f(x)$:

$$f(x) = \frac{1-x}{1-x^2} = \frac{1-x}{(1+x)(1-x)} = \frac{1}{1+x} \tag{1}$$

if $x \neq 1$. Every rational function is continuous wherever it is defined, so f is continuous except at ± 1. The computations in (1) show that $f(x)$ has no limit as $x \to -1$, so f cannot be made continuous at $x = -1$. But the discontinuity at $x = 1$ is removable; if we redefine f at $x = 1$ to be its limit $\frac{1}{2}$ there, then f will be continuous there as well.

2.M.58: Every rational function is continuous where it is defined; that is, where its denominator is nonzero. So

$$f(x) = \frac{1-x}{(2-x)^2}$$

is continuous except at $x = 2$. This discontinuity is not removable because $f(x)$ has no limit at $x = 2$.

2.M.59: First simplify $f(x)$:

$$f(x) = \frac{x^2+x-2}{x^2+2x-3} = \frac{(x-1)(x+2)}{(x-1)(x+3)} = \frac{x+2}{x+3} \tag{1}$$

provided that $x \neq 1$. Note that f is a rational function, so f is continuous wherever it is defined: at every number other than 1 and -3. The computations in (1) show that $f(x)$ has no limit at $x = -3$, so it cannot be redefined in such a way to be continuous there. But the discontinuity at $x = 1$ is removable; if we redefine f at $x = 1$ to be its limit $\frac{3}{4}$ there, then f will be continuous everywhere except at $x = -3$.

2.M.60: Note that $f(x) = 1$ if $x^2 > 1$; that is, if $x > 1$ or $x < -1$. But if $x^2 < 1$, so that $-1 < x < 1$, then $f(x) = -1$. Hence f is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$. But f cannot be made continuous at either $x = 1$ or $x = -1$, because its left-hand and right-hand limits are unequal at each of these points. In any case, f is continuous wherever it is defined.

2.M.61: Let $f(x) = x^5 + x - 1$. Then $f(0) = -1 < 0 < 1 = f(1)$. Because $f(x)$ is a polynomial, it is continuous on $[0, 1]$, so f has the intermediate value property there. Hence there exists a number c in $(0, 1)$ such that $f(c) = 0$. Thus $c^5 + c - 1 = 0$, and so the equation $x^5 + x - 1 = 0$ has a solution. (The value of c is approximately 0.754877666.)

2M.62: Let $f(x) = x^5 - 4x^2 + 1$. Here are some values of $f(x)$:

x	-1	0	1	2
$f(x)$	-4	1	-2	17

Because $f(x)$ is a polynomial, it is continuous everywhere. Therefore $f(x_1) = 0$ for some number x_1 in $(-1, 0)$, $f(x_2) = 0$ for some number x_2 in $(0, 1)$, and $f(x_3) = 0$ for some number x_3 in $(1, 2)$. The numbers x_1, x_2, and x_3 are distinct because they lie in nonoverlapping intervals. Therefore the equation $x^5 - 4x^2 + 1 = 0$ has at least three real solutions. (The actual values are $x_1 \approx 0.50842209$, $x_2 \approx 1.52864292$, and $x_3 \approx -0.49268877$.)

2.M.63: Let $g(x) = x - \cos x$. Then $g(0) = -1 < 0 < \pi/2 = g(\pi/2)$. Because g is continuous, $g(c) = 0$ for some number c in $(0, \pi/2)$. That is, $c - \cos c = 0$, so that $c = \cos c$.

2.M.64: Let $h(x) = x + \tan x$. Then $h(\pi) = \pi > 0$ and $h(x) \to -\infty$ as x approaches $\pi/2$ from above (from the right). This implies that $h(r) < 0$ for some number r slightly larger than $\pi/2$. Because h is continuous on the interval $[r, \pi]$, h has the intermediate value property there, so $h(c) = 0$ for some number c between r and π, and thus between $\pi/2$ and π. That is, $c + \tan c = 0$, so that $\tan c = -c$, and c does lie in the required interval $\left(\frac{1}{2}\pi, \pi\right)$.

2.M.65: Suppose that L is a straight line through $\left(12, \frac{15}{2}\right)$ that is normal to the graph of $y = x^2$ at the point (a, a^2). The line tangent to the graph of $y = x^2$ at that point has slope $2a$, and the slope of L is then $-1/(2a)$. We can equate this to the slope of L found by using the two-point formula:

$$\frac{a^2 - \frac{15}{2}}{a - 12} = -\frac{1}{2a};$$

$$2a\left(a^2 - \tfrac{15}{2}\right) = -(a - 12);$$

$$2a^3 - 15a = -a + 12;$$

$$2a^3 - 14a - 12 = 0;$$

$$a^3 - 7a - 6 = 0.$$

By inspection, one solution of the last equation is $a = -1$. By the factor theorem of algebra, we know that $a - (-1) = a + 1$ is a factor of the polynomial $a^3 - 7a - 6$, and division of the former into the latter yields

$$a^3 - 7a - 6 = (a + 1)(a^2 - a - 6) = (a + 1)(a - 3)(a + 2).$$

So the equation $a^3 - 7a - 6 = 0$ has the three solutions $a = -1$, $a = 3$, and $a = -2$. Therefore there are *three* lines through $\left(12, \frac{15}{2}\right)$ that are normal to the graph of $y = x^2$, and their slopes are $\frac{1}{4}$, $\frac{1}{2}$, and $-\frac{1}{6}$.

2.M.66: Let $(0, c)$ be the center of a too-big circle, r its radius, and (a, a^2) the point in the first quadrant where the too-big circle and the parabola are tangent. The idea is to solve for a in terms of c and (possibly) r, then to impose the condition that there is *exactly one* solution for a! This means that the circle just reaches to the bottom of the parabola and not beyond.

Consider the radius of the circle connecting $(0,\, c)$ with $(a,\, a^2)$. The circle and the parabola are mutually tangent at $(a,\, a^2)$, so this radius must be normal not only to the circle, but also to the parabola at the point $(a,\, a^2)$. We compute the slope of this radius in two ways to find that

$$\frac{a^2 - c}{a - 0} = -\frac{1}{2a};$$

$$a^2 - c = -\frac{1}{2};$$

$$a^2 = c - \frac{1}{2}.$$

Now we impose the condition that there is only one point at which the circle and the parabola meet. The last equation will have exactly one solution when $c = \frac{1}{2}$, and in this case the radius of the circle—because it touches the parabola only at $(0, 0)$—will also be $r = \frac{1}{2}$. Answer: $\frac{1}{2}$.