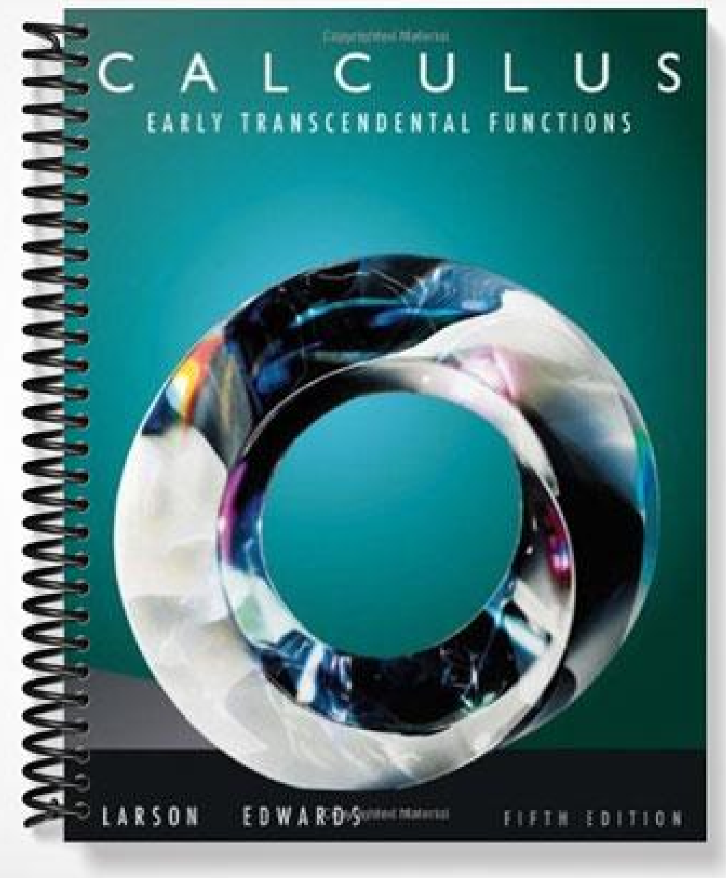
SOLUTIONS MANUAL



2 ☐ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

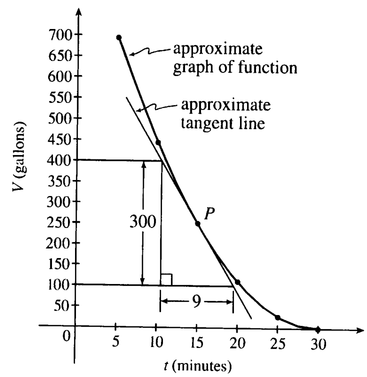
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948-2530}{42-36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948-2661}{42-38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948-2806}{42-40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080-2948}{44-42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. For the curve $y = x/(1+x)$ and the point $P(1, \frac{1}{2})$:

(a)

	x	Q	m_{PQ}
(i)	0.5	$(0.5, 0.333333)$	0.333333
(ii)	0.9	$(0.9, 0.473684)$	0.263158
(iii)	0.99	$(0.99, 0.497487)$	0.251256
(iv)	0.999	$(0.999, 0.499750)$	0.250125
(v)	1.5	$(1.5, 0.6)$	0.2
(vi)	1.1	$(1.1, 0.523810)$	0.238095
(vii)	1.01	$(1.01, 0.502488)$	0.248756
(viii)	1.001	$(1.001, 0.500250)$	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ or $y = \frac{1}{4}x + \frac{1}{4}$.

4. For the curve $y = \ln x$ and the point $P(2, \ln 2)$:

(a)

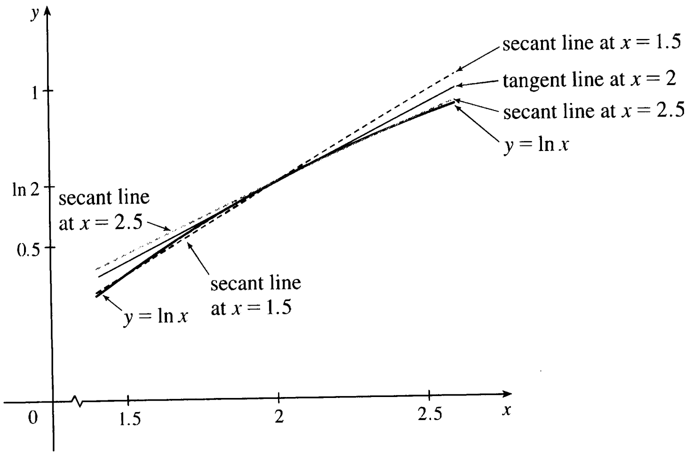
	x	Q	m_{PQ}
(i)	1.5	(1.5, 0.405465)	0.575364
(ii)	1.9	(1.9, 0.641854)	0.512933
(iii)	1.99	(1.99, 0.688135)	0.501254
(iv)	1.999	(1.999, 0.692647)	0.500125
(v)	2.5	(2.5, 0.916291)	0.446287
(vi)	2.1	(2.1, 0.741937)	0.487902
(vii)	2.01	(2.01, 0.698135)	0.498754
(viii)	2.001	(2.001, 0.693647)	0.499875

(b) The slope appears to be $\frac{1}{2}$.

(c) $y - \ln 2 = \frac{1}{2}(x - 2)$ or

$y = \frac{1}{2}x - 1 + \ln 2$

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is $v_{\text{ave}} = \dfrac{y(2+h) - y(2)}{(2+h) - 2} = \dfrac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \dfrac{-24h - 16h^2}{h} = -24 - 16h$, if $h \neq 0$.

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. The average velocity between t and $t + h$ seconds is

$$\frac{58(t+h) - 0.83(t+h)^2 - (58t - 0.83t^2)}{h} = \frac{58h - 1.66th - 0.83h^2}{h} = 58 - 1.66t - 0.83h \text{ if } h \neq 0.$$

(a) Here $t = 1$, so the average velocity is $58 - 1.66 - 0.83h = 56.34 - 0.83h$.

(i) $[1, 2]$: $h = 1$, 55.51 m/s

(ii) $[1, 1.5]$: $h = 0.5$, 55.925 m/s

(iii) $[1, 1.1]$: $h = 0.1$, 56.257 m/s

(iv) $[1, 1.01]$: $h = 0.01$, 56.3317 m/s

(v) $[1, 1.001]$: $h = 0.001$, 56.33917 m/s

(b) The instantaneous velocity after 1 second is 56.34 m/s.

7. $s = s(t) = t^3/6$. Average velocity between times 1 and $1 + h$ is

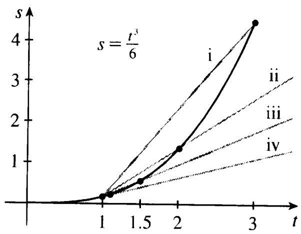
$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{(1+h)^3/6 - 1/6}{h} = \frac{h^3 + 3h^2 + 3h}{6h} = \frac{h^2 + 3h + 3}{6} \text{ if } h \neq 0.$$

(a) (i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = \frac{13}{6}$ ft/s (ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = \frac{7}{6}$ ft/s

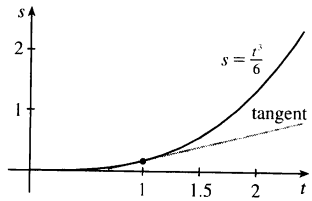
(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = \frac{19}{24}$ ft/s (iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = \frac{331}{600}$ ft/s

(b) As h approaches 0, the velocity approaches $\frac{3}{6} = \frac{1}{2}$ ft/s.

(c)



(d)



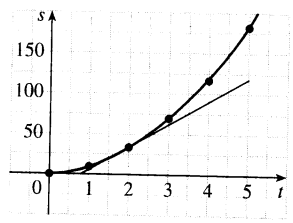
8. Average velocity between times $t = 2$ and $t = 2 + h$ is given by $\dfrac{s(2+h) - s(2)}{h}$.

(a) (i) $h = 3 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(5) - s(2)}{5 - 2} = \dfrac{178 - 32}{3} = \dfrac{146}{3} \approx 48.7$ ft/s

(ii) $h = 2 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{119 - 32}{2} = \dfrac{87}{2} = 43.5$ ft/s

(iii) $h = 1 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(3) - s(2)}{3 - 2} = \dfrac{70 - 32}{1} = 38$ ft/s

(b) Using the points $(0.8, 0)$ and $(5, 118)$ from the approximate tangent line, the instantaneous velocity at $t = 2$ is about $\frac{118 - 0}{5 - 0.8} \approx 28$ ft/s.



9. For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

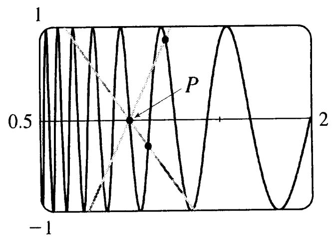
(a)

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) $\lim_{x \to 0} f(x) = 3$ (b) $\lim_{x \to 3^-} f(x) = 4$ (c) $\lim_{x \to 3^+} f(x) = 2$

(d) $\lim_{x \to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.

(e) $f(3) = 3$

5. (a) $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \to 1^-} f(x) = 2$.

(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \to 1^+} f(x) = 3$.

(c) $\lim_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \to 5} f(x) = 4$.

(e) $f(5)$ is not defined, so it doesn't exist.

6. (a) $\lim_{x \to -2^-} g(x) = -1$ (b) $\lim_{x \to -2^+} g(x) = 1$ (c) $\lim_{x \to -2} g(x)$ doesn't exist

(d) $g(-2) = 1$ (e) $\lim_{x \to 2^-} g(x) = 1$ (f) $\lim_{x \to 2^+} g(x) = 2$

(g) $\lim_{x \to 2} g(x)$ doesn't exist (h) $g(2) = 2$ (i) $\lim_{x \to 4^+} g(x)$ doesn't exist

(j) $\lim_{x \to 4^-} g(x) = 2$ (k) $g(0)$ doesn't exist (l) $\lim_{x \to 0} g(x) = 0$

7. (a) $\lim_{t \to 0^-} g(t) = -1$ (b) $\lim_{t \to 0^+} g(t) = -2$

(c) $\lim_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \to 2^-} g(t) = 2$ (e) $\lim_{t \to 2^+} g(t) = 0$

(f) $\lim_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$ (h) $\lim_{t \to 4} g(t) = 3$

8. (a) $\lim_{x \to 2} R(x) = -\infty$ (b) $\lim_{x \to 5} R(x) = \infty$ (c) $\lim_{x \to -3^-} R(x) = -\infty$

(d) $\lim_{x \to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

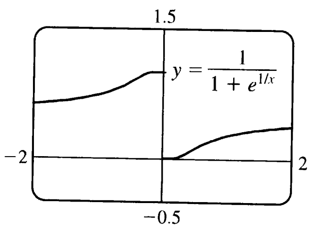
9. (a) $\lim_{x \to -7} f(x) = -\infty$ (b) $\lim_{x \to -3} f(x) = \infty$ (c) $\lim_{x \to 0} f(x) = \infty$

(d) $\lim_{x \to 6^-} f(x) = -\infty$ (e) $\lim_{x \to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t \to 12^-} f(t) = 150$ mg and $\lim_{t \to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11.

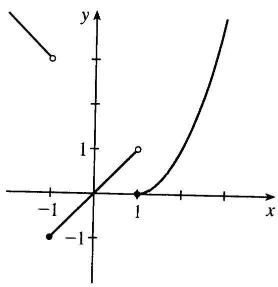


(a) $\lim_{x \to 0^-} f(x) = 1$

(b) $\lim_{x \to 0^+} f(x) = 0$

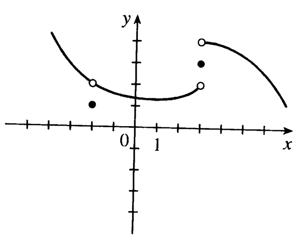
(c) $\lim_{x \to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

12. $\lim_{x \to a} f(x)$ exists for all a except $a = \pm 1$.



13. $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to 3^-} f(x) = 2$,

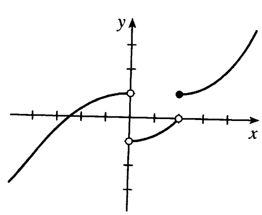
$\lim_{x \to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



14. $\lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$,

$\lim_{x \to 2^-} f(x) = 0$, $\lim_{x \to 2^+} f(x) = 1$, $f(2) = 1$,

$f(0)$ is undefined



15. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

16. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
0	0
-0.5	-1
-0.9	-9
-0.95	-19
-0.99	-99
-0.999	-999

x	$f(x)$
-2	2
-1.5	3
-1.1	11
-1.01	101
-1.001	1001

It appears that $\lim\limits_{x \to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$ *does not exist* since $f(x) \to -\infty$ as $x \to -1^-$ and $f(x) \to \infty$ as $x \to -1^+$.

17. For $f(x) = \dfrac{e^x - 1 - x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
-1	0.367879
-0.5	0.426123
-0.1	0.483742
-0.05	0.491770
-0.01	0.498337

It appears that $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2}$.

18. For $f(x) = x \ln(x + x^2)$:

x	$f(x)$
1	0.693147
0.5	-0.143841
0.1	-0.220727
0.05	-0.147347
0.01	-0.045952
0.005	-0.026467
0.001	-0.006907

It appears that $\lim\limits_{x \to 0^+} x \ln(x + x^2) = 0$.

19. For $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

20. For $f(x) = \dfrac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

21. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim_{x\to 1} \dfrac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

22. For $f(x) = \dfrac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
−0.5	0.227761
−0.1	0.485984
−0.05	0.534447
−0.01	0.576706
−0.001	0.586669

It appears that $\lim_{x\to 0} \dfrac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

23. $\lim_{x\to 5^+} \dfrac{6}{x-5} = \infty$ since $(x-5) \to 0$ as $x \to 5^+$ and $\dfrac{6}{x-5} > 0$ for $x > 5$.

24. $\lim_{x\to 5^-} \dfrac{6}{x-5} = -\infty$ since $(x-5) \to 0$ as $x \to 5^-$ and $\dfrac{6}{x-5} < 0$ for $x < 5$.

25. $\lim_{x\to 1} \dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

26. $\lim_{x\to 0} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $0 < x < 1$ and for $-2 < x < 0$.

27. $\lim_{x\to -2^+} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \to 0$ as $x \to -2^+$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $-2 < x < 0$.

28. $\lim_{x\to \pi^-} \csc x = \lim_{x\to \pi^-} (1/\sin x) = \infty$ since $\sin x \to 0$ as $x \to \pi^-$ and $\sin x > 0$ for $0 < x < \pi$.

29. $\lim_{x\to (-\pi/2)^-} \sec x = \lim_{x\to (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \to 0$ as $x \to (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

30. $\lim_{x\to 5^+} \ln(x-5) = -\infty$ since $x - 5 \to 0^+$ as $x \to 5^+$.

31. (a) $f(x) = 1/(x^3 - 1)$

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33,333.7

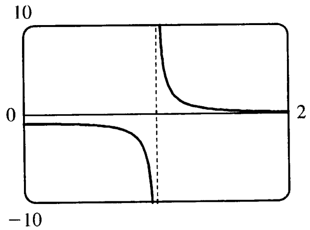
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

From these calculations, it seems that $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x\to 1^-} f(x) = -\infty$.

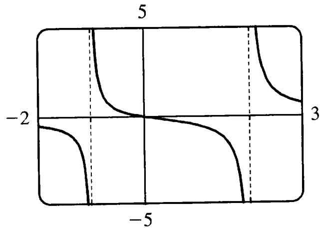
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.



32. (a) $y = \dfrac{x}{x^2 - x - 2} = \dfrac{x}{(x-2)(x+1)}$. Therefore, as $x \to -1^+$ or $x \to 2^+$, the denominator approaches 0, and $y > 0$ for $x < -1$ and for $x > 2$, so $\lim_{x\to -1^+} y = \lim_{x\to 2^+} y = \infty$. Also, as $x \to -1^-$ or $x \to 2^-$, the denominator approaches 0 and $y < 0$ for $-1 < x < 2$, so $\lim_{x\to -1^-} y = \lim_{x\to 2^-} y = -\infty$.

(b)



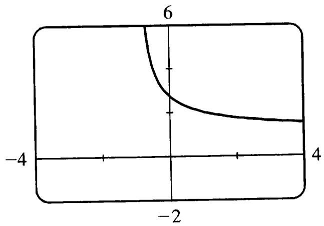
33. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
−0.001	2.71964
−0.0001	2.71842
−0.00001	2.71830
−0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

It appears that $\lim_{x\to 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 3.8 we will see that the value of the limit is exactly e.

(b)



34. For the curve $y = 2^x$ and the points $P(0,1)$ and $Q(x, 2^x)$:

x	Q	m_{PQ}
0.1	(0.1, 1.0717735)	0.71773
0.01	(0.01, 1.0069556)	0.69556
0.001	(0.001, 1.0006934)	0.69339
0.0001	(0.0001, 1.0000693)	0.69317

The slope appears to be about 0.693.

35. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	−0.000614
0.01	−0.000907
0.005	−0.000978
0.003	−0.000993
0.001	−0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

36. $h(x) = \dfrac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

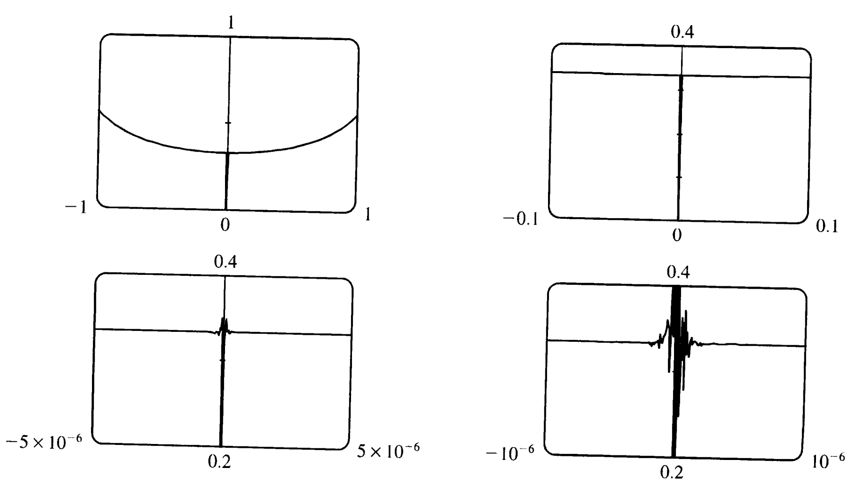
(b) It seems that $\lim_{x \to 0} h(x) = \frac{1}{3}$.

(c)

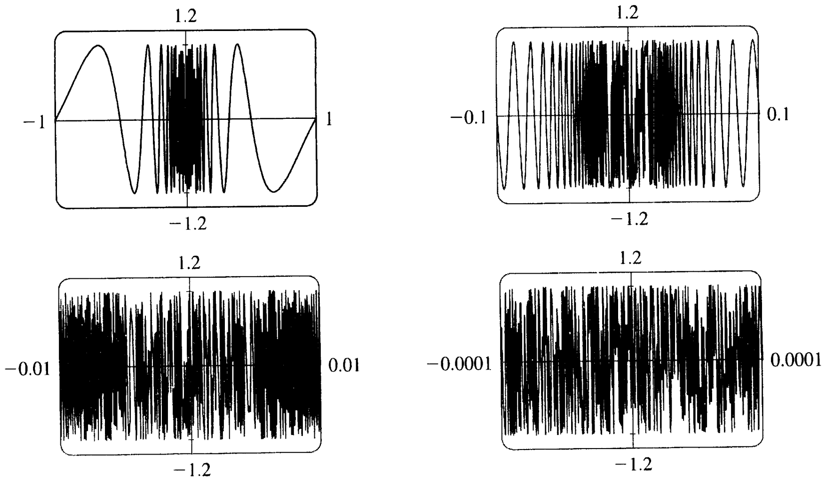
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values.*

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

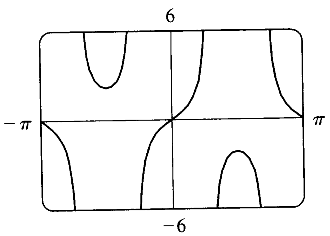


37. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.



38. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

39.

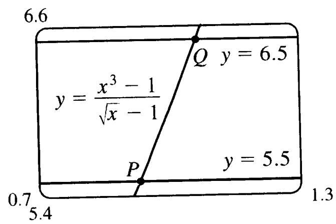


There appear to be vertical asymptotes of the curve $y = \tan(2\sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2} n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$).

Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of the vertical asymptotes (corresponding to $x \approx \pm 2.24$).

40. (a) Let $y = \left(x^3 - 1\right)/\left(\sqrt{x} - 1\right)$.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P(0.9313853, 5.5)$ and $Q(1.0649004, 6.5)$. Now $1 - 0.9313853 \approx 0.0686$ and $1.0649004 - 1 \approx 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to a}[f(x)+h(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} h(x) = -3+8=5$

(b) $\lim\limits_{x\to a}[f(x)]^2 = \left[\lim\limits_{x\to a} f(x)\right]^2 = (-3)^2 = 9$

(c) $\lim\limits_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{\lim\limits_{x\to a} h(x)} = \sqrt[3]{8} = 2$

(d) $\lim\limits_{x\to a}\dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x\to a} f(x)} = \dfrac{1}{-3} = -\dfrac{1}{3}$

(e) $\lim\limits_{x\to a}\dfrac{f(x)}{h(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} h(x)} = \dfrac{-3}{8} = -\dfrac{3}{8}$

(f) $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)} = \dfrac{\lim\limits_{x\to a} g(x)}{\lim\limits_{x\to a} f(x)} = \dfrac{0}{-3} = 0$

(g) The limit does not exist, since $\lim\limits_{x\to a} g(x) = 0$ but $\lim\limits_{x\to a} f(x) \neq 0$.

(h) $\lim\limits_{x\to a}\dfrac{2f(x)}{h(x)-f(x)} = \dfrac{2\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} h(x) - \lim\limits_{x\to a} f(x)} = \dfrac{2(-3)}{8-(-3)} = -\dfrac{6}{11}$

2. (a) $\lim\limits_{x\to 2}[f(x)+g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 2+0=2$

(b) $\lim\limits_{x\to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x\to 0}[f(x)g(x)] = \lim\limits_{x\to 0} f(x)\cdot \lim\limits_{x\to 0} g(x) = 0\cdot 1.3 = 0$

(d) Since $\lim\limits_{x\to -1} g(x) = 0$ and g is in the denominator, but $\lim\limits_{x\to -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim\limits_{x\to 2} x^3 f(x) = \left[\lim\limits_{x\to 2} x^3\right]\left[\lim\limits_{x\to 2} f(x)\right] = 2^3\cdot 2 = 16$

(f) $\lim\limits_{x\to 1}\sqrt{3+f(x)} = \sqrt{3+\lim\limits_{x\to 1} f(x)} = \sqrt{3+1} = 2$

3.
$$\begin{aligned}\lim_{x\to -2}(3x^4+2x^2-x+1) &= \lim_{x\to -2} 3x^4 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && \text{[Limit Laws 1 and 2]}\\ &= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && [3]\\ &= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && \text{[9, 8, and 7]}\\ &= 48+8+2+1 = 59\end{aligned}$$

4.
$$\begin{aligned}\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4} &= \frac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(x^2+6x-4)} && \text{[Limit Law 5]}\\ &= \frac{2\lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2 + 6\lim\limits_{x\to 2} x - \lim\limits_{x\to 2} 4} && \text{[2, 1, and 3]}\\ &= \frac{2(2)^2+1}{(2)^2+6(2)-4} = \frac{9}{12} = \frac{3}{4} && \text{[9, 7, and 8]}\end{aligned}$$

5.
$$\begin{aligned}\lim_{x\to 3}(x^2-4)(x^3+5x-1) &= \lim_{x\to 3}(x^2-4)\cdot\lim_{x\to 3}(x^3+5x-1) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to 3} x^2 - \lim_{x\to 3} 4\right)\cdot\left(\lim_{x\to 3} x^3 + 5\lim_{x\to 3} x - \lim_{x\to 3} 1\right) && \text{[2, 1, and 3]}\\ &= (3^2-4)\cdot(3^3+5\cdot 3-1) && \text{[7, 8, and 9]}\\ &= 5\cdot 41 = 205\end{aligned}$$

6. $\lim_{t\to-1}(t^2+1)^3(t+3)^5 = \lim_{t\to-1}(t^2+1)^3 \cdot \lim_{t\to-1}(t+3)^5$ [Limit Law 4]

$= \left[\lim_{t\to-1}(t^2+1)\right]^3 \cdot \left[\lim_{t\to-1}(t+3)\right]^5$ [6]

$= \left[\lim_{t\to-1}t^2 + \lim_{t\to-1}1\right]^3 \cdot \left[\lim_{t\to-1}t + \lim_{t\to-1}3\right]^5$ [1]

$= \left[(-1)^2+1\right]^3 \cdot \left[-1+3\right]^5 = 8\cdot 32 = 256$ [9, 7, and 8]

7. $\lim_{x\to1}\left(\dfrac{1+3x}{1+4x^2+3x^4}\right)^3 = \left(\lim_{x\to1}\dfrac{1+3x}{1+4x^2+3x^4}\right)^3$ [6]

$= \left[\dfrac{\lim_{x\to1}(1+3x)}{\lim_{x\to1}(1+4x^2+3x^4)}\right]^3$ [5]

$= \left[\dfrac{\lim_{x\to1}1 + 3\lim_{x\to1}x}{\lim_{x\to1}1 + 4\lim_{x\to1}x^2 + 3\lim_{x\to1}x^4}\right]^3$ [2, 1, and 3]

$= \left[\dfrac{1+3(1)}{1+4(1)^2+3(1)^4}\right]^3 = \left[\dfrac{4}{8}\right]^3 = \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$ [7, 8, and 9]

8. $\lim_{u\to-2}\sqrt{u^4+3u+6} = \sqrt{\lim_{u\to-2}(u^4+3u+6)}$ [11]

$= \sqrt{\lim_{u\to-2}u^4 + 3\lim_{u\to-2}u + \lim_{u\to-2}6}$ [1, 2, and 3]

$= \sqrt{(-2)^4+3(-2)+6}$ [9, 8, and 7]

$= \sqrt{16-6+6} = \sqrt{16} = 4$

9. $\lim_{x\to4^-}\sqrt{16-x^2} = \sqrt{\lim_{x\to4^-}(16-x^2)}$ [11]

$= \sqrt{\lim_{x\to4^-}16 - \lim_{x\to4^-}x^2}$ [2]

$= \sqrt{16-(4)^2} = 0$ [7 and 9]

10. (a) The left-hand side of the equation is not defined for $x=2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim_{x\to a} f(x)$, we never consider $x = a$.

11. $\lim_{x\to2}\dfrac{x^2+x-6}{x-2} = \lim_{x\to2}\dfrac{(x+3)(x-2)}{x-2} = \lim_{x\to2}(x+3) = 2+3 = 5$

12. $\lim_{x\to-4}\dfrac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to-4}\dfrac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x\to-4}\dfrac{x+1}{x-1} = \dfrac{-4+1}{-4-1} = \dfrac{-3}{-5} = \dfrac{3}{5}$

13. $\lim_{x\to2}\dfrac{x^2-x+6}{x-2}$ does not exist since $x-2\to0$ but $x^2-x+6\to8$ as $x\to2$.

14. $\lim_{x\to4}\dfrac{x^2-4x}{x^2-3x-4} = \lim_{x\to4}\dfrac{x(x-4)}{(x-4)(x+1)} = \lim_{x\to4}\dfrac{x}{x+1} = \dfrac{4}{4+1} = \dfrac{4}{5}$

15. $\lim_{t\to-3}\dfrac{t^2-9}{2t^2+7t+3} = \lim_{t\to-3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t\to-3}\dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

16. $\lim_{x\to-1}\dfrac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to0$ but $x^2-4x\to5$ as $x\to-1$.

17. $\displaystyle\lim_{h\to0}\frac{(4+h)^2-16}{h}=\lim_{h\to0}\frac{(16+8h+h^2)-16}{h}=\lim_{h\to0}\frac{8h+h^2}{h}=\lim_{h\to0}\frac{h(8+h)}{h}=\lim_{h\to0}(8+h)=8+0=8$

18. $\displaystyle\lim_{x\to1}\frac{x^3-1}{x^2-1}=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}=\lim_{x\to1}\frac{x^2+x+1}{x+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$

19. $\displaystyle\lim_{h\to0}\frac{(1+h)^4-1}{h}=\lim_{h\to0}\frac{(1+4h+6h^2+4h^3+h^4)-1}{h}=\lim_{h\to0}\frac{4h+6h^2+4h^3+h^4}{h}$

$\displaystyle=\lim_{h\to0}\frac{h(4+6h+4h^2+h^3)}{h}=\lim_{h\to0}(4+6h+4h^2+h^3)=4+0+0+0=4$

20. $\displaystyle\lim_{h\to0}\frac{(2+h)^3-8}{h}=\lim_{h\to0}\frac{(8+12h+6h^2+h^3)-8}{h}=\lim_{h\to0}\frac{12h+6h^2+h^3}{h}$

$\displaystyle=\lim_{h\to0}(12+6h+h^2)=12+0+0=12$

21. $\displaystyle\lim_{t\to9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to9}\frac{(3+\sqrt{t})(3-\sqrt{t})}{3-\sqrt{t}}=\lim_{t\to9}(3+\sqrt{t})=3+\sqrt{9}=6$

22. $\displaystyle\lim_{h\to0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim_{h\to0}\frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}=\lim_{h\to0}\frac{h}{h\left(\sqrt{1+h}+1\right)}$

$\displaystyle=\lim_{h\to0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$

23. $\displaystyle\lim_{x\to7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to7}\frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)}$

$\displaystyle=\lim_{x\to7}\frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}=\lim_{x\to7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$

24. $\displaystyle\lim_{x\to2}\frac{x^4-16}{x-2}=\lim_{x\to2}\frac{(x+2)(x-2)(x^2+4)}{x-2}=\lim_{x\to2}(x+2)(x^2+4)=\lim_{x\to2}(x+2)\lim_{x\to2}(x^2+4)$

$=(2+2)(2^2+4)=32$

25. $\displaystyle\lim_{x\to-4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim_{x\to-4}\frac{\frac{x+4}{4x}}{4+x}=\lim_{x\to-4}\frac{x+4}{4x(4+x)}=\lim_{x\to-4}\frac{1}{4x}=\frac{1}{4(-4)}=-\frac{1}{16}$

26. $\displaystyle\lim_{t\to0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to0}\frac{(t^2+t)-t}{t(t^2+t)}=\lim_{t\to0}\frac{t^2}{t\cdot t(t+1)}=\lim_{t\to0}\frac{1}{t+1}=\frac{1}{0+1}=1$

27. $\displaystyle\lim_{x\to9}\frac{x^2-81}{\sqrt{x}-3}=\lim_{x\to9}\frac{(x-9)(x+9)}{\sqrt{x}-3}=\lim_{x\to9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x}-3}$ [factor $x-9$ as a difference of squares]

$\displaystyle=\lim_{x\to9}[(\sqrt{x}+3)(x+9)]=(\sqrt{9}+3)(9+9)=6\cdot18=108$

28. $\displaystyle\lim_{h\to0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to0}\frac{3-(3+h)}{h(3+h)3}=\lim_{h\to0}\frac{-h}{h(3+h)3}$

$\displaystyle=\lim_{h\to0}\left[-\frac{1}{3(3+h)}\right]=-\frac{1}{\lim_{h\to0}[3(3+h)]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$

29. $\lim\limits_{t\to 0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)=\lim\limits_{t\to 0}\dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim\limits_{t\to 0}\dfrac{\left(1-\sqrt{1+t}\,\right)\left(1+\sqrt{1+t}\,\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\,\right)}=\lim\limits_{t\to 0}\dfrac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\,\right)}=-\frac{1}{2}$$

30. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim\limits_{x\to 1}\dfrac{\sqrt{x}\left(1-x^{3/2}\right)}{1-\sqrt{x}}=\lim\limits_{x\to 1}\dfrac{\sqrt{x}\,(1-\sqrt{x}\,)(1+\sqrt{x}+x)}{1-\sqrt{x}}$ [difference of cubes]

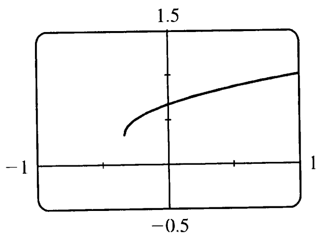
$$=\lim_{x\to 1}\left[\sqrt{x}\,(1+\sqrt{x}+x)\right]=\lim_{x\to 1}\left[1(1+1+1)\right]=3$$

Another method: We "add and subtract" 1 in the numerator, and then split up the fraction:

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{(\sqrt{x}-1)+(1-x^2)}{1-\sqrt{x}}=\lim_{x\to 1}\left[-1+\frac{(1-x)(1+x)}{1-\sqrt{x}}\right]$$

$$=\lim_{x\to 1}\left[-1+\frac{(1-\sqrt{x}\,)(1+\sqrt{x}\,)(1+x)}{1-\sqrt{x}}\right]=-1+\left(1+\sqrt{1}\,\right)(1+1)=3$$

31. (a)



$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt{1+3x}-1}\cdot\dfrac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim\limits_{x\to 0}\dfrac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim\limits_{x\to 0}\dfrac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$=\dfrac{1}{3}\lim\limits_{x\to 0}\left(\sqrt{1+3x}+1\right)$ [Limit Law 3]

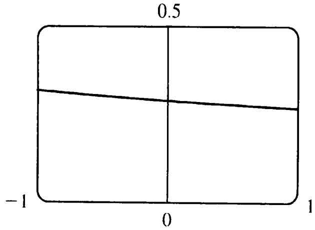
$=\dfrac{1}{3}\left[\sqrt{\lim\limits_{x\to 0}(1+3x)}+\lim\limits_{x\to 0}1\right]$ [1 and 11]

$=\dfrac{1}{3}\left(\sqrt{\lim\limits_{x\to 0}1+3\lim\limits_{x\to 0}x}+1\right)$ [1, 3, and 7]

$=\dfrac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)$ [7 and 8]

$=\dfrac{1}{3}(1+1)=\dfrac{2}{3}$

32. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

(b)

x	$f(x)$
−0.001	0.2886992
−0.0001	0.2886775
−0.00001	0.2886754
−0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

(c) $$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

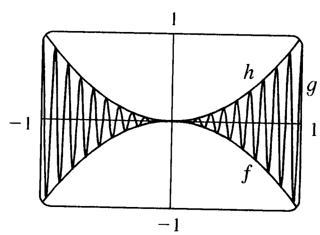
$$=\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\sqrt{3+x}+\lim\limits_{x\to 0}\sqrt{3}}$$ [Limit Laws 5 and 1]

$$=\frac{1}{\sqrt{\lim\limits_{x\to 0}(3+x)}+\sqrt{3}}$$ [7 and 11]

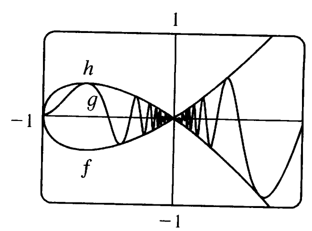
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}$$ [1, 7, and 8]

$$=\frac{1}{2\sqrt{3}}$$

33. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le\cos 20\pi x\le 1\;\Rightarrow\;-x^2\le x^2\cos 20\pi x\le x^2\;\Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



34. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le\sin(\pi/x)\le 1\;\Rightarrow$ $-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\;\Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



35. $1\le f(x)\le x^2+2x+2$ for all x. Now $\lim\limits_{x\to -1}1=1$ and $\lim\limits_{x\to -1}(x^2+2x+2)=\lim\limits_{x\to -1}x^2+2\lim\limits_{x\to -1}x+\lim\limits_{x\to -1}2=(-1)^2+2(-1)+2=1$. Therefore, by the Squeeze Theorem, $\lim\limits_{x\to -1}f(x)=1$.

36. $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$. Now $\lim_{x\to 1} 3x = 3$ and $\lim_{x\to 1} (x^3+2) = \lim_{x\to 1} x^3 + \lim_{x\to 1} 2 = 1^3 + 2 = 3$. Therefore, by the Squeeze Theorem, $\lim_{x\to 1} f(x) = 3$.

37. $-1 \le \cos(2/x) \le 1 \;\Rightarrow\; -x^4 \le x^4\cos(2/x) \le x^4$. Since $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0} x^4 = 0$, we have $\lim_{x\to 0}\left[x^4\cos(2/x)\right] = 0$ by the Squeeze Theorem.

38. $-1 \le \sin(\pi/x) \le 1 \;\Rightarrow\; e^{-1} \le e^{\sin(\pi/x)} \le e^1 \;\Rightarrow\; \sqrt{x}/e \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$. Since $\lim_{x\to 0^+}(\sqrt{x}/e) = 0$ and $\lim_{x\to 0^+}(\sqrt{x}\, e) = 0$, we have $\lim_{x\to 0^+}\left[\sqrt{x}\, e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

39. If $x > -4$, then $|x+4| = x+4$, so $\lim_{x\to -4^+}|x+4| = \lim_{x\to -4^+}(x+4) = -4+4 = 0$.
If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x\to -4^-}|x+4| = \lim_{x\to -4^-} -(x+4) = -(-4+4) = 0$.
Since the right and left limits are equal, $\lim_{x\to -4}|x+4| = 0$.

40. If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4} = \lim_{x\to -4^-}\dfrac{-(x+4)}{x+4} = \lim_{x\to -4^-}(-1) = -1$.

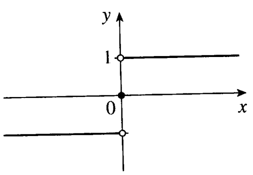
41. If $x > 2$, then $|x-2| = x-2$, so $\lim_{x\to 2^+}\dfrac{|x-2|}{x-2} = \lim_{x\to 2^+}\dfrac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$. If $x < 2$, then $|x-2| = -(x-2)$, so $\lim_{x\to 2^-}\dfrac{|x-2|}{x-2} = \lim_{x\to 2^-}\dfrac{-(x-2)}{x-2} = \lim_{x\to 2^-} -1 = -1$. The right and left limits are different, so $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

42. If $x > \frac{3}{2}$, then $|2x-3| = 2x-3$, so
$\lim_{x\to 1.5^+}\dfrac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5^+}\dfrac{2x^2-3x}{2x-3} = \lim_{x\to 1.5^+}\dfrac{x(2x-3)}{2x-3} = \lim_{x\to 1.5^+} x = 1.5$. If $x < \frac{3}{2}$, then
$|2x-3| = 3-2x$, so $\lim_{x\to 1.5^-}\dfrac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5^-}\dfrac{2x^2-3x}{-(2x-3)} = \lim_{x\to 1.5^-}\dfrac{x(2x-3)}{-(2x-3)} = \lim_{x\to 1.5^-} -x = -1.5$.
The right and left limits are different, so $\lim_{x\to 1.5}\dfrac{2x^2-3x}{|2x-3|}$ does not exist.

43. Since $|x| = -x$ for $x < 0$, we have $\lim_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim_{x\to 0^-}\dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x| = x$ for $x > 0$, we have $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim_{x\to 0^+} 0 = 0$.

45. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim_{x\to 0^+}\operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim_{x\to 0^-}\operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1$.

(iii) Since $\lim_{x\to 0^-}\operatorname{sgn} x \ne \lim_{x\to 0^+}\operatorname{sgn} x$, $\lim_{x\to 0}\operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim_{x\to 0}|\operatorname{sgn} x| = \lim_{x\to 0} 1 = 1$.

46. (a) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (4 - x^2) = \lim\limits_{x\to 2^-} 4 - \lim\limits_{x\to 2^-} x^2$

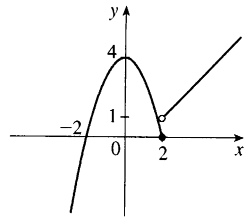
$= 4 - 4 = 0$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x - 1) = \lim\limits_{x\to 2^+} x - \lim\limits_{x\to 2^+} 1$

$= 2 - 1 = 1$

(b) No, $\lim\limits_{x\to 2} f(x)$ does not exist since $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$.

(c)

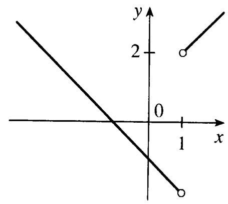


47. (a) (i) $\lim\limits_{x\to 1^+} \dfrac{x^2 - 1}{|x - 1|} = \lim\limits_{x\to 1^+} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x\to 1^+} (x + 1) = 2$

(ii) $\lim\limits_{x\to 1^-} \dfrac{x^2 - 1}{|x - 1|} = \lim\limits_{x\to 1^-} \dfrac{x^2 - 1}{-(x - 1)} = \lim\limits_{x\to 1^-} -(x + 1) = -2$

(b) No, $\lim\limits_{x\to 1} F(x)$ does not exist since $\lim\limits_{x\to 1^+} F(x) \neq \lim\limits_{x\to 1^-} F(x)$.

(c)



48. (a) (i) $\lim\limits_{x\to 0^+} h(x) = \lim\limits_{x\to 0^+} x^2 = 0^2 = 0$

(ii) $\lim\limits_{x\to 0^-} h(x) = \lim\limits_{x\to 0^-} x = 0$, so $\lim\limits_{x\to 0} h(x) = 0$.

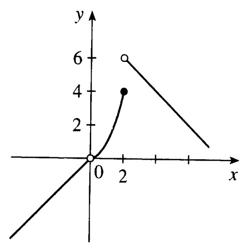
(iii) $\lim\limits_{x\to 1} h(x) = \lim\limits_{x\to 1} x^2 = 1^2 = 1$

(iv) $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^-} x^2 = 2^2 = 4$

(v) $\lim\limits_{x\to 2^+} h(x) = \lim\limits_{x\to 2^+} (8 - x) = 8 - 2 = 6$

(vi) Since $\lim\limits_{x\to 2^-} h(x) \neq \lim\limits_{x\to 2^+} h(x)$, $\lim\limits_{x\to 2} h(x)$ does not exist.

(b)



49. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x\to -2^+} [\![x]\!] = \lim\limits_{x\to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2^-} [\![x]\!] = \lim\limits_{x\to -2^-} (-3) = -3$. The right and left limits are different, so $\lim\limits_{x\to -2} [\![x]\!]$ does not exist.

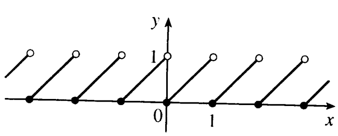
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2.4} [\![x]\!] = \lim\limits_{x\to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n - 1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

50. (a)



(b) (i) $\lim\limits_{x\to n^-} f(x) = \lim\limits_{x\to n^-} (x - [\![x]\!]) = \lim\limits_{x\to n^-} [x - (n - 1)] = n - (n - 1) = 1$

(ii) $\lim\limits_{x\to n^+} f(x) = \lim\limits_{x\to n^+} (x - [\![x]\!]) = \lim\limits_{x\to n^+} (x - n) = n - n = 0$

(c) $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow$ a is not an integer.

51. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim_{x\to 2^-} f(x) = -1$ and $\lim_{x\to 2^+} f(x) = -1$, so $\lim_{x\to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim_{x\to 2} f(x) \neq f(2)$.

52. $\lim_{v\to c^-}\left(L_0\sqrt{1-\dfrac{v^2}{c^2}}\right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\\ &= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n\\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim_{x\to a} p(x) = p(a)$.

54. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 53]} \quad = r(a).$$

55. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

56. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist (Example 10) but $\lim_{x\to 3}[f(x) + g(x)] = \lim_{x\to 3}([\![x]\!] - [\![x]\!]) = \lim_{x\to 3} 0 = 0$.

57. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59. Thus, either f or g is 0 for any value of x. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}[f(x)g(x)] = \lim_{x\to 0} 0 = 0$.

58.
$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)\\ &= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)\\ &= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}\end{aligned}$$

59. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim_{x\to -2}(3x^2 + ax + a + 3) = 0 \Leftrightarrow$ $3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

60. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \Leftrightarrow r^2 = 1 + 2x - 1 \Leftrightarrow x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \Leftrightarrow y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

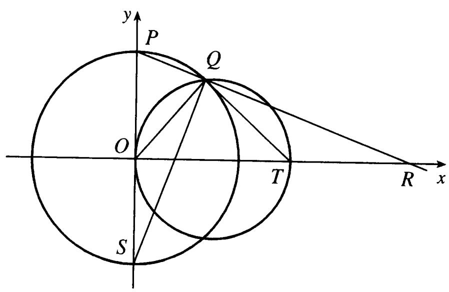
$y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right).$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r \to 0^+} x = \lim\limits_{r \to 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim\limits_{r \to 0^+} 2\left(\sqrt{1} + 1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

1. (a) To have $5x + 3$ within a distance of 0.1 of 13, we must have $12.9 \le 5x + 3 \le 13.1 \Rightarrow 9.9 \le 5x \le 10.1 \Rightarrow 1.98 \le x \le 2.02$. Thus, x must be within 0.02 units of 2 so that $5x + 3$ is within 0.1 of 13.

(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. (a) To have $6x - 1$ within a distance of 0.01 of 29, we must have $28.99 \le 6x - 1 \le 29.01 \Rightarrow 29.99 \le 6x \le 30.01 \Rightarrow 4.998\overline{3} \le x \le 5.001\overline{6}$. Thus, x must be within $0.001\overline{6}$ units of 5 so that $6x - 1$ is within 0.01 of 29.

(b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\overline{6}$.

(c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\overline{6}$.

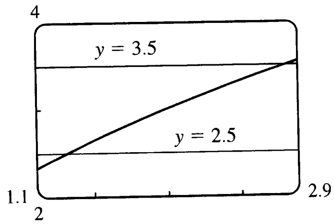
3. On the left side of $x = 2$, we need $|x - 2| < \left|\frac{10}{7} - 2\right| = \frac{4}{7}$. On the right side, we need $|x - 2| < \left|\frac{10}{3} - 2\right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x - 2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

4. On the left side, we need $|x - 5| < |4 - 5| = 1$. On the right side, we need $|x - 5| < |5.7 - 5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x - 5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.

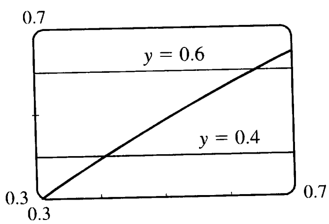
5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

6. The left-hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right-hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

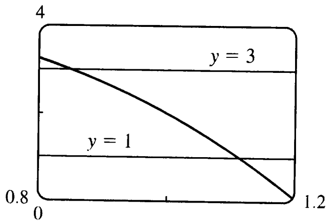
7. $\left|\sqrt{4x + 1} - 3\right| < 0.5 \quad \Leftrightarrow \quad 2.5 < \sqrt{4x + 1} < 3.5$. We plot the three parts of this inequality on the same screen and identify the x-coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125 \leq x \leq 2.8125$. Since $|2 - 1.3125| = 0.6875$ and $|2 - 2.8125| = 0.8125$, we choose $0 < \delta < \min\{0.6875, 0.8125\} = 0.6875$.

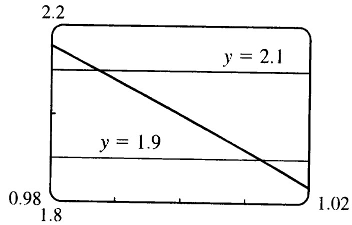


8. $\left|\sin x - \frac{1}{2}\right| < 0.1 \quad \Leftrightarrow \quad 0.4 < \sin x < 0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \leq x \leq 0.64$. So since $|0.5 - 0.42| = 0.08$ and $|0.5 - 0.64| = 0.14$, we choose $0 < \delta \leq \min\{0.08, 0.14\} = 0.08$.

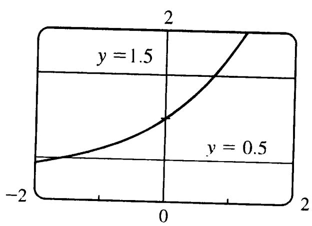


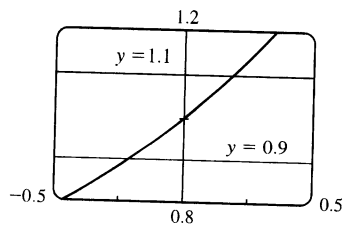
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $\left|(4 + x - 3x^3) - 2\right| < 1 \Leftrightarrow$ $1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left|(4 + x - 3x^3) - 2\right| < 0.1 \Leftrightarrow 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



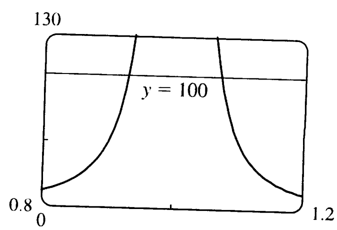


10. For $\varepsilon = 0.5$, the definition of a limit requires that we find δ such that $\left|\dfrac{e^x - 1}{x} - 1\right| < 0.5 \Leftrightarrow$ $0.5 < \dfrac{e^x - 1}{x} < 1.5$ whenever $0 < |x - 0| < \delta$. If we plot the graphs of $y = 0.5$, $y = \dfrac{e^x - 1}{x}$, and $y = 1.5$ on the same screen, we see that we need $-1.59 \leq x \leq 0.76$. So since $|0 - (-1.59)| = 1.59$ and $|0 - 0.76| = 0.76$, we choose $\delta = 0.76$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left|\dfrac{e^x - 1}{x} - 1\right| < 0.1 \Leftrightarrow$ $0.9 < \dfrac{e^x - 1}{x} < 1.1$ whenever $0 < |x - 0| < \delta$. From the graph, we see that we need $-0.21 \leq x \leq 0.18$. So since $|0 - (-0.21)| = 0.21$ and $|0 - 0.18| = 0.18$, we choose $\delta = 0.18$ (or any smaller positive number) for the inequality to hold.

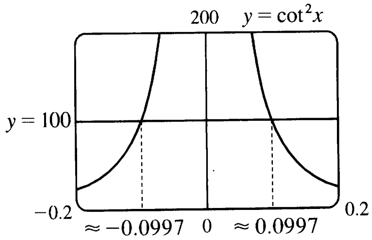




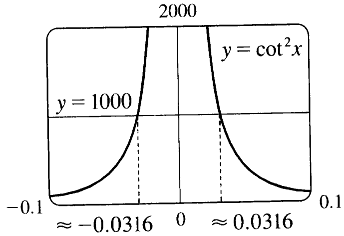
11. From the graph, we see that $\dfrac{x}{(x^2 + 1)(x - 1)^2} > 100$ whenever $0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M = 100$, we need $-0.0997 < x < 0$ or $0 < x < 0.0997$. Thus, we choose $\delta = 0.0997$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 100$.



For $M = 1000$, we need $-0.0316 < x < 0$ or $0 < x < 0.0316$. Thus, we choose $\delta = 0.0316$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 1000$.

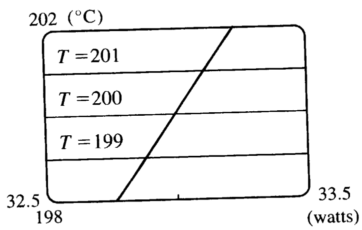


13. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow$
$r = \sqrt{\frac{1000}{\pi}}$ $[r > 0]$ ≈ 17.8412 cm.

(b) $|A - 1000| \le 5 \Rightarrow -5 \le \pi r^2 - 1000 \le 5 \Rightarrow 1000 - 5 \le \pi r^2 \le 1000 + 5 \Rightarrow$
$\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

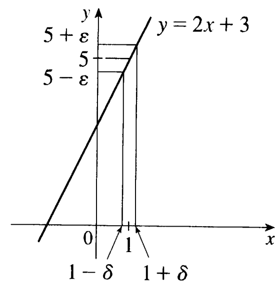
14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$
$0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts $(w > 0)$



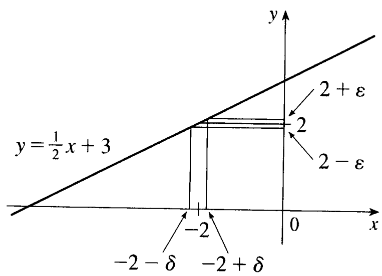
(b) From the graph, $199 \le T \le 201 \Rightarrow 32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

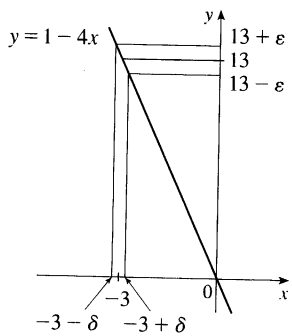
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x + 3) - 5| < \varepsilon$. But $|(2x + 3) - 5| < \varepsilon \iff |2x - 2| < \varepsilon \iff 2\,|x - 1| < \varepsilon \iff |x - 1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 1| < \delta \Rightarrow |(2x + 3) - 5| < \varepsilon$. Thus, $\lim\limits_{x\to 1}(2x + 3) = 5$ by the definition of a limit.



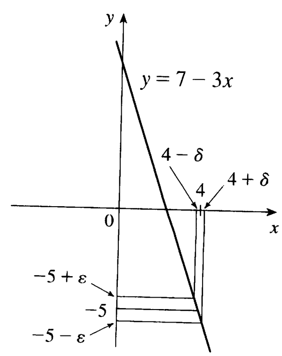
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon \iff \left|\frac{1}{2}x + 1\right| < \varepsilon \iff \frac{1}{2}\,|x + 2| < \varepsilon \iff |x - (-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \Rightarrow \left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon$. Thus, $\lim\limits_{x\to -2}\left(\frac{1}{2}x + 3\right) = 2$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \iff |-4x - 12| < \varepsilon \iff |-4|\,|x + 3| < \varepsilon \iff |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim\limits_{x\to -3}(1 - 4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(7 - 3x) - (-5)| < \varepsilon$. But $|(7 - 3x) - (-5)| < \varepsilon \iff |-3x + 12| < \varepsilon \iff |-3|\,|x - 4| < \varepsilon \iff |x - 4| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x - 4| < \delta \Rightarrow |(7 - 3x) - (-5)| < \varepsilon$. Thus, $\lim\limits_{x\to 4}(7 - 3x) = -5$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon \iff \frac{1}{5}\,|x - 3| < \varepsilon \iff |x - 3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x - 3| < \delta \Rightarrow |x - 3| < 5\varepsilon \Rightarrow \dfrac{|x - 3|}{5} < \varepsilon \Rightarrow \left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 3}\dfrac{x}{5} = \dfrac{3}{5}$.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 6| < \delta$, then $\left|\left(\frac{x}{4} + 3\right) - \frac{9}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{x}{4} - \frac{3}{2}\right| < \varepsilon \quad \Leftrightarrow$ $\frac{1}{4}|x - 6| < \varepsilon \quad \Leftrightarrow \quad |x - 6| < 4\varepsilon$. So choose $\delta = 4\varepsilon$. Then $0 < |x - 6| < \delta \quad \Rightarrow \quad |x - 6| < 4\varepsilon \quad \Rightarrow$ $\dfrac{|x - 6|}{4} < \varepsilon \quad \Rightarrow \quad \left|\frac{x}{4} - \frac{6}{4}\right| < \varepsilon \quad \Rightarrow \quad \left|\left(\frac{x}{4} + 3\right) - \frac{9}{2}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 6} \left(\frac{x}{4} + 3\right) = \frac{9}{2}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-5)| < \delta$, then $\left|\left(4 - \frac{3}{5}x\right) - 7\right| < \varepsilon \quad \Leftrightarrow$ $\left|-\frac{3}{5}x - 3\right| < \varepsilon \quad \Leftrightarrow \quad \frac{3}{5}|x + 5| < \varepsilon \quad \Leftrightarrow \quad |x - (-5)| < \frac{5}{3}\varepsilon$. So choose $\delta = \frac{5}{3}\varepsilon$. Then $|x - (-5)| < \delta \quad \Rightarrow$ $\left|\left(4 - \frac{3}{5}x\right) - 7\right| < \varepsilon$. Thus, $\lim\limits_{x \to -5} \left(4 - \frac{3}{5}x\right) = 7$ by the definition of a limit.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\dfrac{x^2 + x - 12}{x - 3} - 7\right| < \varepsilon$. Notice that if $0 < |x - 3|$, then $x \neq 3$, so $\dfrac{x^2 + x - 12}{x - 3} = \dfrac{(x + 4)(x - 3)}{x - 3} = x + 4$. Thus, when $0 < |x - 3|$, we have $\left|\dfrac{x^2 + x - 12}{x - 3} - 7\right| < \varepsilon \quad \Leftrightarrow \quad |(x + 4) - 7| < \varepsilon \quad \Leftrightarrow \quad |x - 3| < \varepsilon$. We take $\delta = \varepsilon$ and see that $0 < |x - 3| < \delta$ $\Rightarrow \quad \left|\dfrac{x^2 + x - 12}{x - 3} - 7\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 3} \dfrac{x^2 + x - 12}{x - 3} = 7$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \quad \Leftrightarrow \quad x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^2 - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \quad \Leftrightarrow \quad |x|^3 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x \to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\big||x| - 0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x \to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9 - \delta < x < 9$, then $\left|\sqrt[4]{9 - x} - 0\right| < \varepsilon \quad \Leftrightarrow \quad \sqrt[4]{9 - x} < \varepsilon \quad \Leftrightarrow$ $9 - x < \varepsilon^4 \quad \Leftrightarrow \quad 9 - \varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9 - \delta < x < 9 \quad \Rightarrow \quad \left|\sqrt[4]{9 - x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 9^-} \sqrt[4]{9 - x} = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\left(x^2 - 4x + 5\right) - 1\right| < \varepsilon \quad \Leftrightarrow$ $\left|x^2 - 4x + 4\right| < \varepsilon \quad \Leftrightarrow \quad \left|(x - 2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \quad \Leftrightarrow \quad |x - 2| < \sqrt{\varepsilon} \quad \Leftrightarrow$ $\left|(x - 2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 2} \left(x^2 - 4x + 5\right) = 1$ by the definition of a limit.

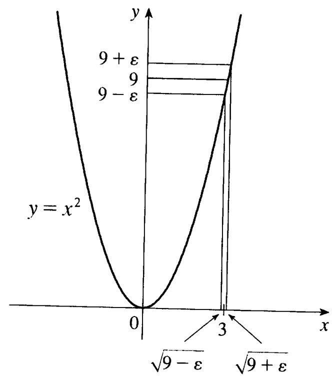
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\left(x^2 + x - 4\right) - 8\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2 + x - 12\right| < \varepsilon$ $\Leftrightarrow \quad |(x - 3)(x + 4)| < \varepsilon$. Notice that if $|x - 3| < 1$, then $-1 < x - 3 < 1 \quad \Rightarrow$ $6 < x + 4 < 8 \quad \Rightarrow \quad |x + 4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $0 < |x - 3| < \delta \quad \Leftrightarrow$ $|(x - 3)(x + 4)| \leq |8(x - 3)| = 8 \cdot |x - 3| < 8\delta \leq \varepsilon$. Thus, $\lim\limits_{x \to 3} \left(x^2 + x - 4\right) = 8$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|(x^2 - 1) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \Rightarrow$ $-5 < x - 2 < -3 \Rightarrow |x - 2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \Rightarrow |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $\left|(x^2 - 1) - 3\right| = |(x + 2)(x - 2)| = |x + 2|\,|x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to -2} (x^2 - 1) = 3$.

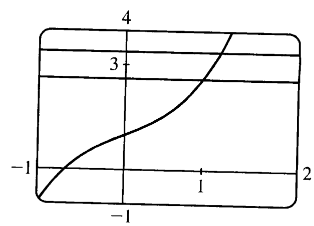
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x - 2)(x^2 + 2x + 4)\right|$. If $|x - 2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < 19\,|x - 2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x - 2| < \delta \Rightarrow \left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow$ $4 < x + 3 < 8 \Rightarrow |x + 3| < 8$. Also $|x - 3| < \frac{\varepsilon}{8}$, so $\left|x^2 - 9\right| = |x + 3|\,|x - 3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x \to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9 - \varepsilon}$ and $\delta_2 = \sqrt{9 + \varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9 + \varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x - 2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2 - x}{2x}\right| = \frac{|x - 2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x - 2|}{|2x|} < C\,|x - 2|$ and we can make $C\,|x - 2| < \varepsilon$ by taking $|x - 2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x - 2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x - 2| < \delta$, then $|x - 2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x - 2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x - 2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim_{x\to 2}(1/x) = \frac{1}{2}$.

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then $|x - a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$$|\sqrt{x} - \sqrt{a}\,| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{a/2} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2} + \sqrt{a}\right)} = \varepsilon.$$ Therefore, $\lim_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |H(t) - L| < \frac{1}{2} \Leftrightarrow L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \Rightarrow L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \Rightarrow L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \Rightarrow 0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Thus, $\lim_{x\to a^-} f(x) = L$. Also $a < x < a+\delta \Rightarrow 0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Hence, $\lim_{x\to a^+} f(x) = L$.

Now suppose $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \Rightarrow |f(x) - L| < \varepsilon$. Since $\lim_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \Rightarrow |f(x)-L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x\to a} f(x) = L$. So we have proved that $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \Leftrightarrow (x+3)^4 < \dfrac{1}{10{,}000} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \Leftrightarrow |x-(-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \Leftrightarrow (x+3)^4 < \dfrac{1}{M} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \Rightarrow \dfrac{1}{(x+3)^4} > M$, so $\lim_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim_{x\to 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow f(x) > M + 1 - c$. Since $\lim_{x\to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow |g(x) - c| < 1 \Rightarrow g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow f(x) + g(x) > (M+1-c) + (c-1) = M$. Thus, $\lim_{x\to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim_{x\to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < c/2 \Rightarrow g(x) > c/2$. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x)\,g(x) > \dfrac{2M}{c}\dfrac{c}{2} = M$, so $\lim_{x\to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim_{x\to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < -c/2 \Rightarrow g(x) < c/2$. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim_{x\to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim_{x\to4} f(x) = f(4)$.

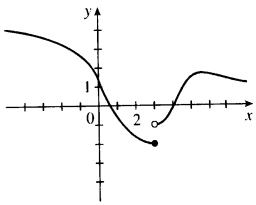
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).

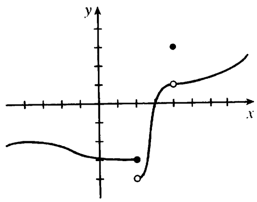
(b) f is continuous from the left at -2 since $\lim_{x\to-2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x\to2^+} f(x) = f(2)$ and $\lim_{x\to4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

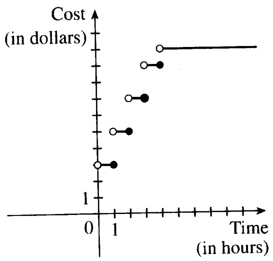
5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim_{x\to3^-} f(x) = f(3)$.



6.



7. (a)



(b) There are discontinuities at times $t = 1, 2, 3$, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x\to3}[2f(x) - g(x)] &= 2\lim_{x\to3} f(x) - \lim_{x\to3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3]\\ &= 2\cdot5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim_{x\to 3}[2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

10. $\lim_{x\to 4} f(x) = \lim_{x\to 4}\left(x^2 + \sqrt{7-x}\right) = \lim_{x\to 4} x^2 + \sqrt{\lim_{x\to 4} 7 - \lim_{x\to 4} x} = 4^2 + \sqrt{7-4} = 16 + \sqrt{3} = f(4)$.

By the definition of continuity, f is continuous at $a = 4$.

11. $\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(x + 2x^3\right)^4 = \left(\lim_{x\to -1} x + 2\lim_{x\to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim_{x\to 4} g(x) = \lim_{x\to 4}\dfrac{x+1}{2x^2-1} = \dfrac{\lim_{x\to 4} x + \lim_{x\to 4} 1}{2\lim_{x\to 4} x^2 - \lim_{x\to 4} 1} = \dfrac{4+1}{2(4)^2-1} = \dfrac{5}{31} = g(4)$. So g is continuous at 4.

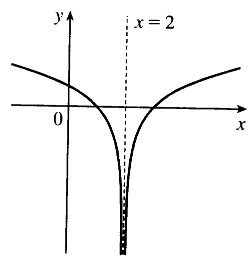
13. For $a > 2$, we have $\lim_{x\to a} f(x) = \lim_{x\to a}\dfrac{2x+3}{x-2} = \dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5] $= \dfrac{2\lim_{x\to a} x + \lim_{x\to a} 3}{\lim_{x\to a} x - \lim_{x\to a} 2}$

[1, 2, and 3] $= \dfrac{2a+3}{a-2}$ [7 and 8] $= f(a)$. Thus, f is continuous at $x = a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

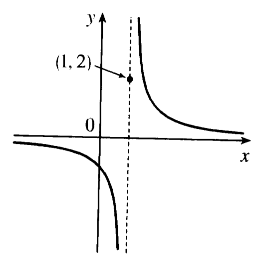
14. For $a < 3$, we have $\lim_{x\to a} g(x) = \lim_{x\to a} 2\sqrt{3-x} = 2\lim_{x\to a}\sqrt{3-x}$ [Limit Law 3] $= 2\sqrt{\lim_{x\to a}(3-x)}$ [11]

$= 2\sqrt{\lim_{x\to a} 3 - \lim_{x\to a} x}$ [2] $= 2\sqrt{3-a}$ [7 and 8] $= g(a)$, so g is continuous at $x = a$ for every a in $(-\infty, 3)$. Also, $\lim_{x\to 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3. Thus, g is continuous on $(-\infty, 3]$.

15. $f(x) = \ln|x-2|$ is discontinuous at 2 since $f(2) = \ln 0$ is not defined.

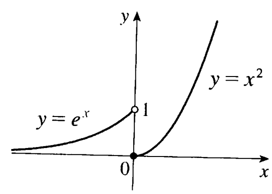


16. $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



17. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

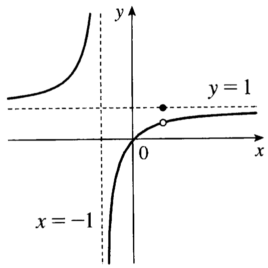
The left-hand limit of f at $a = 0$ is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = 1$. The right-hand limit of f at $a = 0$ is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$. Since these limits are not equal, $\lim_{x\to 0} f(x)$ does not exist and f is discontinuous at 0.



18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2},$$

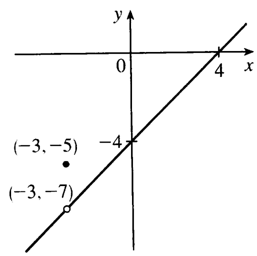
but $f(1) = 1$, so f is discontinous at 1.



19. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x+3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} = \begin{cases} x - 4 & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$

So $\lim_{x \to -3} f(x) = \lim_{x \to -3} (x - 4) = -7$ and $f(-3) = -5$.

Since $\lim_{x \to -3} f(x) \neq f(-3)$, f is discontinuous at -3.

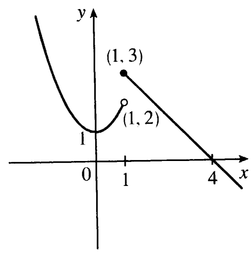


20. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 + x^2) = 1 + 1^2 = 2$ and

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4 - x) = 4 - 1 = 3.$

Thus, f is discontinuous at 1 because $\lim_{x \to 1} f(x)$ does not exist.



21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x+3)(x+2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}\,(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $[\frac{1}{2}, \infty)$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

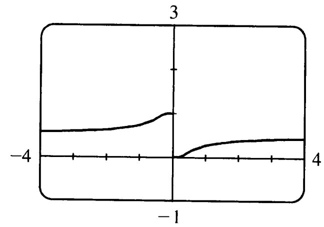
25. By Theorem 5, the polynomial $5x$ is continuous on $(-\infty, \infty)$. By Theorems 9 and 7, $\sin 5x$ is continuous on $(-\infty, \infty)$. By Theorem 7, e^x is continuous on $(-\infty, \infty)$. By part 4 of Theorem 4, the product of e^x and $\sin 5x$ is continuous at all numbers which are in both of their domains, that is, on $(-\infty, \infty)$.

26. By Theorem 5, the polynomial $x^2 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\sin^{-1}$ is continuous on its domain, $[-1, 1]$. By Theorem 9, $\sin^{-1}(x^2 - 1)$ is continuous on its domain, which is $\{x \mid -1 \leq x^2 - 1 \leq 1\} = \{x \mid 0 \leq x^2 \leq 2\} = \{x \mid |x| \leq \sqrt{2}\} = [-\sqrt{2}, \sqrt{2}]$.

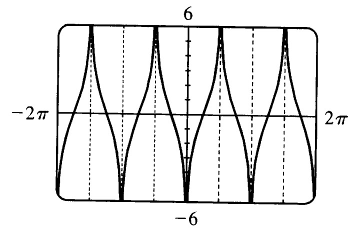
27. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$. By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is $\{t \mid t^4 - 1 > 0\} = \{t \mid t^4 > 1\} = \{t \mid |t| > 1\} = (-\infty, -1) \cup (1, \infty)$.

28. By Theorem 7, $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorems 7 and 9, $e^{\sqrt{x}}$ is continuous on $[0, \infty)$. Also by Theorems 7 and 9, $\cos\left(e^{\sqrt{x}}\right)$ is continuous on $[0, \infty)$.

29. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



30. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where k is any integer. The function $y = \ln(\tan^2 x)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln(\tan^2 x)$ is discontinuous at $x = \frac{\pi}{2} n$, n any integer.



31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x + 5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim\limits_{x \to 4} f(x) = f(4) = \frac{7}{3}$.

32. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to \pi} f(x) = f(\pi) = \sin(\pi + \sin \pi) = \sin \pi = 0$.

33. Because $x^2 - x$ is continuous on $\mathbb{R}$, the composite function $f(x) = e^{x^2 - x}$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to 1} f(x) = f(1) = e^{1 - 1} = e^0 = 1$.

34. Because arctan is a continuous function, we can apply Theorem 8.

$$\lim_{x \to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right) = \arctan\left(\lim_{x \to 2} \frac{(x + 2)(x - 2)}{3x(x - 2)}\right) = \arctan\left(\lim_{x \to 2} \frac{x + 2}{3x}\right) = \arctan \tfrac{2}{3} \approx 0.588$$

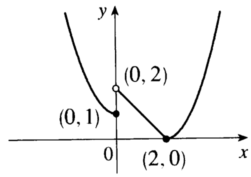
35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x \to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

36. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

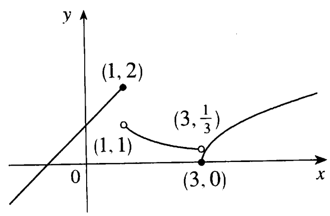
By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x \to (\pi/4)^-} f(x) = \lim\limits_{x \to (\pi/4)^-} \sin x = \sin \frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x \to (\pi/4)^+} f(x) = \lim\limits_{x \to (\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x \to (\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$



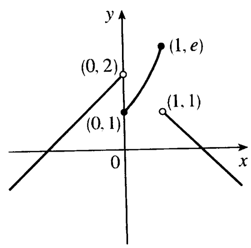
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (1 + x^2) = 1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2 - x) = 0$, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x - 2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

38. $f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geq 3 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x + 1) = 2$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (1/x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1. Also, $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (1/x) = 1/3$, and $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \sqrt{x - 3} = 0 = f(3)$, so f is discontinuous at 3, but it is continuous from the right at 3.

39. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential. Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} e^x = e$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

40. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$. $\lim_{r\to R^-} F(r) = \lim_{r\to R^-} \frac{GMr}{R^3} = \frac{GM}{R^2}$ and $\lim_{r\to R^+} F(r) = \lim_{r\to R^+} \frac{GM}{r^2} = \frac{GM}{R^2}$, so $\lim_{r\to R} F(r) = \frac{GM}{R^2}$. Since $F(R) = \frac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

41. f is continuous on $(-\infty, 3)$ and $(3, \infty)$. Now $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (cx+1) = 3c+1$ and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (cx^2-1) = 9c-1$. So f is continuous $\Leftrightarrow$ $3c+1 = 9c-1$ $\Leftrightarrow$ $6c = 2$ $\Leftrightarrow$ $c = \frac{1}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{1}{3}$.

42. The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c. So the only possible discontinuity is at $x = 4$. For the function to be continuous at $x = 4$, the left-hand and right-hand limits must be the same. Now $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x^2 - c^2) = 16 - c^2$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} (cx + 20) = 4c + 20 = g(4)$. Thus, $16 - c^2 = 4c + 20$ $\Leftrightarrow$ $c^2 + 4c + 4 = 0$ $\Leftrightarrow$ $c = -2$.

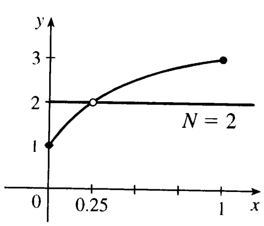
43. (a) $f(x) = \frac{x^2 - 2x - 8}{x+2} = \frac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x) = x - 4$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -2$. [The discontinuity is removed by defining $f(-2) = -6$.]

(b) $f(x) = \frac{x-7}{|x-7|}$ $\Rightarrow$ $\lim_{x\to 7^-} f(x) = -1$ and $\lim_{x\to 7^+} f(x) = 1$. Thus, $\lim_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

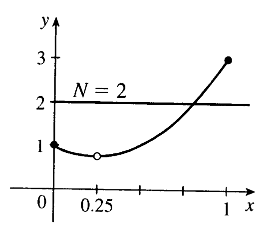
(c) $f(x) = \frac{x^3 + 64}{x+4} = \frac{(x+4)(x^2 - 4x + 16)}{x+4}$ has a removable discontinuity at -4 because $g(x) = x^2 - 4x + 16$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \frac{3 - \sqrt{x}}{9 - x} = \frac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = \frac{1}{3 + \sqrt{x}}$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

44.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

45. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.

46. $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$, and $f(2) = 4$. Since $1 < 2 < 4$, there is a number c in $(1, 2)$ such that $f(c) = c^2 = 2$ by the Intermediate Value Theorem.

47. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

48. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

49. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

50. $f(x) = \ln x - e^{-x}$ is continuous on the interval $[1, 2]$, $f(1) = -e^{-1} \approx -0.37$, and $f(2) = \ln 2 - e^{-2} \approx 0.56$. Since $-0.37 < 0 < 0.56$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - e^{-x} = 0$, or $\ln x = e^{-x}$, in the interval $(1, 2)$.

51. (a) $f(x) = e^x + x - 2$ is continuous on the interval $[0, 1]$, $f(0) = -1 < 0$, and $f(1) = e - 1 \approx 1.72 > 0$. Since $-1 < 0 < 1.72$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + x - 2 = 0$, or $e^x = 2 - x$, in the interval $(0, 1)$.

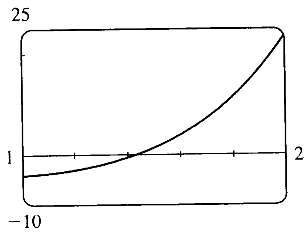
(b) $f(0.44) \approx -0.007 < 0$ and $f(0.45) \approx 0.018 > 0$, so there is a root between 0.44 and 0.45.

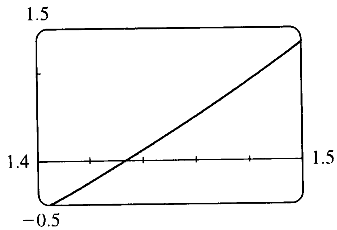
52. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between -0.88 and -0.87.

53. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.

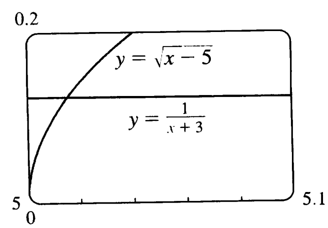
(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.





54. (a) Let $f(x) = \sqrt{x - 5} - \dfrac{1}{x + 3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c + 3} = \sqrt{c - 5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



55. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\Big(\lim_{h\to 0}(a+h)\Big) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \;\Rightarrow$

$|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x-a| < \delta$, then $|f(x) - f(a)| = |f(a + (x-a)) - f(a)| < \varepsilon$.
Thus, $\lim_{x\to a} f(x) = f(a)$ and so f is continuous at a.

56. $\lim_{h\to 0} \sin(a+h) = \lim_{h\to 0}(\sin a\cos h + \cos a \sin h) = \lim_{h\to 0}(\sin a \cos h) + \lim_{h\to 0}(\cos a \sin h)$

$$= \Big(\lim_{h\to 0}\sin a\Big)\Big(\lim_{h\to 0}\cos h\Big) + \Big(\lim_{h\to 0}\cos a\Big)\Big(\lim_{h\to 0}\sin h\Big)$$
$$= (\sin a)(1) + (\cos a)(0) = \sin a$$

57. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h - \sin a \sin h)\\ &= \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)\\ &= \Big(\lim_{h\to 0}\cos a\Big)\Big(\lim_{h\to 0}\cos h\Big) - \Big(\lim_{h\to 0}\sin a\Big)\Big(\lim_{h\to 0}\sin h\Big)\\ &= (\cos a)(1) - (\sin a)(0) = \cos a\end{aligned}$$

58. (a) Since f is continuous at a, $\lim_{x\to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have $\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law of Limits: $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

59. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x\to a} f(x) \neq f(a)$. [In fact, $\lim_{x\to a} f(x)$ does not even exist.]

60. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0. To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze Theorem $\lim_{x\to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x\to a} g(x) \neq g(a)$.

61. If there is such a number, it satisfies the equation $x^3 + 1 = x \;\Leftrightarrow\; x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

62. (a) $\lim_{x\to 0^+} F(x) = 0$ and $\lim_{x\to 0^-} F(x) = 0$, so $\lim_{x\to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x\to a} F(x) = \lim_{x\to a} x = a = F(a)$. For $a < 0$, $\lim_{x\to a} F(x) = \lim_{x\to a} (-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

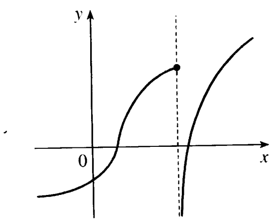
63. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow \quad u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

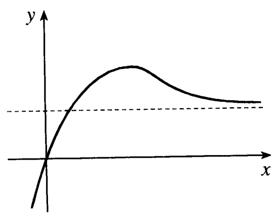
1. (a) As x becomes large, the values of $f(x)$ approach 5.

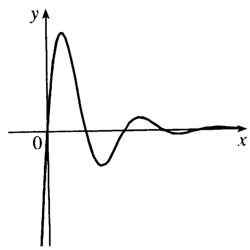
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

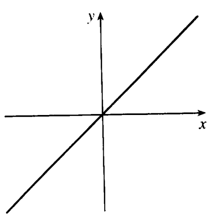


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

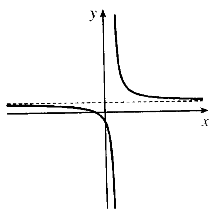




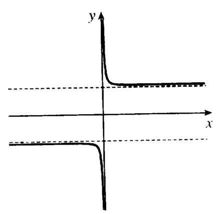
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. (a) $\lim_{x\to 2} f(x) = \infty$ (b) $\lim_{x\to -1^-} f(x) = \infty$

(c) $\lim_{x\to -1^+} f(x) = -\infty$ (d) $\lim_{x\to \infty} f(x) = 1$

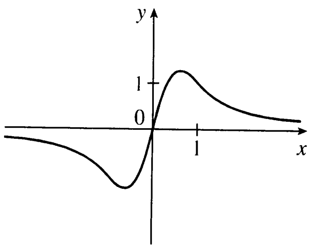
(e) $\lim_{x\to -\infty} f(x) = 2$ (f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

4. (a) $\lim_{x\to \infty} g(x) = 2$ (b) $\lim_{x\to -\infty} g(x) = -2$

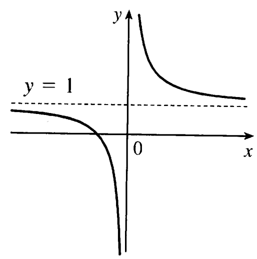
(c) $\lim_{x\to 3} g(x) = \infty$ (d) $\lim_{x\to 0} g(x) = -\infty$

(e) $\lim_{x\to -2^+} g(x) = -\infty$ (f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

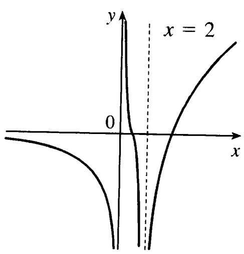
5. $f(0) = 0$, $f(1) = 1$, $\lim_{x\to\infty} f(x) = 0$, f is odd



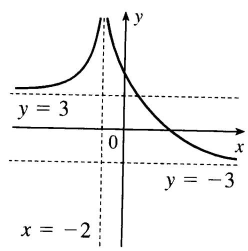
6. $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to -\infty} f(x) = 1$



7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$



8. $\lim_{x\to -2} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = -3$



9. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim_{x\to\infty} (x^2/2^x) = 0$.

10. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10.000]$ by $[0, 0.2]$, we estimate that $\lim_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

11. $$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x\to\infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2}$$ [divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]

$$= \frac{\lim_{x\to\infty}(3 - 1/x + 4/x^2)}{\lim_{x\to\infty}(2 + 5/x - 8/x^2)}$$ [Limit Law 5]

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}(1/x) + \lim_{x\to\infty}(4/x^2)}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}(5/x) - \lim_{x\to\infty}(8/x^2)}$$ [Limit Laws 1 and 2]

$$= \frac{3 - \lim_{x\to\infty}(1/x) + 4\lim_{x\to\infty}(1/x^2)}{2 + 5\lim_{x\to\infty}(1/x) - 8\lim_{x\to\infty}(1/x^2)}$$ [Limit Laws 7 and 3]

$$= \frac{3 - 0 + 4(0)}{2 + 5(0) - 8(0)}$$ [Theorem 5 of Section 2.5]

$$= \frac{3}{2}$$

12. $$\lim_{x\to\infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}} = \sqrt{\lim_{x\to\infty} \frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$ [Limit Law 11]

$$= \sqrt{\lim_{x\to\infty} \frac{12 - 5/x^2 + 2/x^3}{1/x^3 + 4/x + 3}}$$ [divide by x^3]

$$= \sqrt{\frac{\lim_{x\to\infty}(12 - 5/x^2 + 2/x^3)}{\lim_{x\to\infty}(1/x^3 + 4/x + 3)}}$$ [Limit Law 5]

$$= \sqrt{\frac{\lim_{x\to\infty} 12 - \lim_{x\to\infty}(5/x^2) + \lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1/x^3) + \lim_{x\to\infty}(4/x) + \lim_{x\to\infty} 3}}$$ [Limit Laws 1 and 2]

$$= \sqrt{\frac{12 - 5\lim_{x\to\infty}(1/x^2) + 2\lim_{x\to\infty}(1/x^3)}{\lim_{x\to\infty}(1/x^3) + 4\lim_{x\to\infty}(1/x) + 3}}$$ [Limit Laws 7 and 3]

$$= \sqrt{\frac{12 - 5(0) + 2(0)}{0 + 4(0) + 3}}$$ [Theorem 5 of Section 2.5]

$$= \sqrt{\frac{12}{3}} = \sqrt{4} = 2$$

13. $$\lim_{x\to\infty} \frac{1}{2x+3} = \lim_{x\to\infty} \frac{1/x}{(2x+3)/x} = \frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(2 + 3/x)} = \frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty} 2 + 3\lim_{x\to\infty}(1/x)} = \frac{0}{2 + 3(0)} = \frac{0}{2} = 0$$

14. $\displaystyle \lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{(3x+5)/x}{(x-4)/x} = \lim_{x\to\infty} \frac{3+5/x}{1-4/x} = \frac{\lim\limits_{x\to\infty} 3 + 5 \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 1 - 4 \lim\limits_{x\to\infty} \frac{1}{x}} = \frac{3+5(0)}{1-4(0)} = 3$

15. $\displaystyle \lim_{x\to-\infty} \frac{1-x-x^2}{2x^2-7} = \lim_{x\to-\infty} \frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2} = \frac{\lim\limits_{x\to-\infty} (1/x^2 - 1/x - 1)}{\lim\limits_{x\to-\infty} (2-7/x^2)}$

$\displaystyle = \frac{\lim\limits_{x\to-\infty} (1/x^2) - \lim\limits_{x\to-\infty} (1/x) - \lim\limits_{x\to-\infty} 1}{\lim\limits_{x\to-\infty} 2 - 7 \lim\limits_{x\to-\infty} (1/x^2)} = \frac{0-0-1}{2-7(0)} = -\frac{1}{2}$

16. $\displaystyle \lim_{y\to\infty} \frac{2-3y^2}{5y^2+4y} = \lim_{y\to\infty} \frac{(2-3y^2)/y^2}{(5y^2+4y)/y^2} = \frac{\lim\limits_{y\to\infty} (2/y^2-3)}{\lim\limits_{y\to\infty} (5+4/y)} = \frac{2\lim\limits_{y\to\infty} (1/y^2) - \lim\limits_{y\to\infty} 3}{\lim\limits_{y\to\infty} 5 + 4 \lim\limits_{y\to\infty} (1/y)} = \frac{2(0)-3}{5+4(0)} = -\frac{3}{5}$

17. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$\displaystyle \lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4} = \lim_{x\to\infty} \frac{\frac{x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{5}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2-\frac{1}{x}+\frac{4}{x^3}\right)}$

$\displaystyle = \frac{\lim\limits_{x\to\infty} 1 + 5\lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{1}{x} + 4\lim\limits_{x\to\infty} \frac{1}{x^3}} = \frac{1+5(0)}{2-0+4(0)} = \frac{1}{2}$

18. $\displaystyle \lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1} = \lim_{t\to-\infty} \frac{(t^2+2)/t^3}{(t^3+t^2-1)/t^3} = \lim_{t\to-\infty} \frac{1/t+2/t^3}{1+1/t-1/t^3} = \frac{0+0}{1+0-0} = 0$

19. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$\displaystyle \lim_{u\to\infty} \frac{4u^4+5}{(u^2-2)(2u^2-1)} = \lim_{u\to\infty} \frac{4u^4+5}{2u^4-5u^2+2} = \lim_{u\to\infty} \frac{\frac{4u^4+5}{u^4}}{\frac{2u^4-5u^2+2}{u^4}} = \lim_{u\to\infty} \frac{4+\frac{5}{u^4}}{2-\frac{5}{u^2}+\frac{2}{u^4}}$

$\displaystyle = \frac{\lim\limits_{u\to\infty}\left(4+\frac{5}{u^4}\right)}{\lim\limits_{u\to\infty}\left(2-\frac{5}{u^2}+\frac{2}{u^4}\right)} = \frac{\lim\limits_{u\to\infty} 4 + 5\lim\limits_{u\to\infty}\frac{1}{u^4}}{\lim\limits_{u\to\infty} 2 - 5\lim\limits_{u\to\infty}\frac{1}{u^2} + 2\lim\limits_{u\to\infty}\frac{1}{u^4}} = \frac{4+5(0)}{2-5(0)+2(0)}$

$\displaystyle = \frac{4}{2} = 2$

20. $\displaystyle \lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1+2/x}{\sqrt{9+1/x^2}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$

21. $\displaystyle \lim_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+1} = \lim_{x\to\infty} \frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3} = \frac{\lim\limits_{x\to\infty}\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to\infty}(1+1/x^3)}$ [since $x^3 = \sqrt{x^6}$ for $x > 0$]

$\displaystyle = \frac{\lim\limits_{x\to\infty}\sqrt{9-1/x^5}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}(1/x^3)} = \frac{\sqrt{\lim\limits_{x\to\infty} 9 - \lim\limits_{x\to\infty}(1/x^5)}}{1+0}$

$= \sqrt{9-0} = 3$

22. $$\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}\,/x^3}{(x^3+1)/x^3}=\frac{\lim\limits_{x\to-\infty}-\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to-\infty}(1+1/x^3)}$$ [since $x^3=-\sqrt{x^6}$ for $x<0$]

$$=\frac{\lim\limits_{x\to-\infty}-\sqrt{9-1/x^5}}{\lim\limits_{x\to-\infty}1+\lim\limits_{x\to-\infty}(1/x^3)}=\frac{-\sqrt{\lim\limits_{x\to-\infty}9-\lim\limits_{x\to-\infty}(1/x^5)}}{1+0}$$

$$=-\sqrt{9-0}=-3$$

23. $$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$$

$$=\lim_{x\to\infty}\frac{\left(9x^2+x\right)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

24. $$\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)=\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)\left[\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right]=\lim_{x\to-\infty}\frac{x^2-\left(x^2+2x\right)}{x-\sqrt{x^2+2x}}$$

$$=\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}}=\frac{-2}{1+\sqrt{1+2\,(0)}}=-1$$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

25. $$\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

26. $\lim\limits_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1.

27. $\sqrt{x}$ is large when x is large, so $\lim\limits_{x\to\infty}\sqrt{x}=\infty$.

28. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim\limits_{x\to-\infty}\sqrt[3]{x}=-\infty$.

29. $\lim\limits_{x\to\infty}\left(x-\sqrt{x}\right)=\lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x}-1\right)=\infty$ since $\sqrt{x}\to\infty$ and $\sqrt{x}-1\to\infty$ as $x\to\infty$.

30. $$\lim_{x\to\infty}\frac{x^3-2x+3}{5-2x^2}=\lim_{x\to\infty}\frac{(x^3-2x+3)/x^2}{(5-2x^2)/x^2}$$ [divide by the highest power of x in the denominator]

$$=\lim_{x\to\infty}\frac{x-2/x+3/x^2}{5/x^2-2}=-\infty$$ because $x-2/x+3/x^2\to\infty$ and $5/x^2-2\to-2$ as $x\to\infty$.

31. $\lim\limits_{x\to-\infty}\left(x^4+x^5\right)=\lim\limits_{x\to-\infty}x^5\left(\frac{1}{x}+1\right)$ [factor out the largest power of x] $=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

32. $\lim_{x\to\infty} \tan^{-1}(x^2 - x^4) = \lim_{x\to\infty} \tan^{-1}(x^2(1-x^2))$. If we let $t = x^2(1-x^2)$, we know that $t \to -\infty$ as $x \to \infty$, since $x^2 \to \infty$ and $1 - x^2 \to -\infty$. So $\lim_{x\to\infty} \tan^{-1}(x^2(1-x^2)) = \lim_{t\to-\infty} \tan^{-1} t = -\frac{\pi}{2}$.

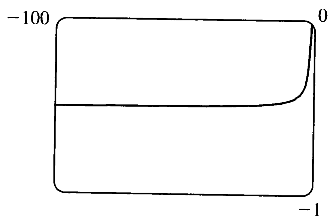
33. $$\lim_{x\to\infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim_{x\to\infty} \frac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4}$$ [divide by the highest power of x in the denominator]

$$= \lim_{x\to\infty} \frac{1/x^3 + 1/x + x}{1/x^4 - 1/x^2 + 1} = \infty$$

because $(1/x^3 + 1/x + x) \to \infty$ and $(1/x^4 - 1/x^2 + 1) \to 1$ as $x \to \infty$.

34. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\lim_{x\to(\pi/2)^+} e^{\tan x} = \lim_{t\to-\infty} e^t = 0$.

35. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x\to-\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	-0.4999625
$-100{,}000$	-0.4999962
$-1{,}000{,}000$	-0.4999996

From the table, we estimate the limit to be -0.5.

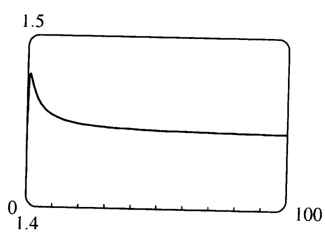
(c) $$\lim_{x\to-\infty} \left(\sqrt{x^2+x+1} + x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2+x+1} + x\right)\left[\frac{\sqrt{x^2+x+1} - x}{\sqrt{x^2+x+1} - x}\right] = \lim_{x\to-\infty} \frac{(x^2+x+1) - x^2}{\sqrt{x^2+x+1} - x}$$

$$= \lim_{x\to-\infty} \frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1} - x\right)(1/x)} = \lim_{x\to-\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$

$$= \frac{1+0}{-\sqrt{1+0+0} - 1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\frac{1}{x}\sqrt{x^2+x+1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1 + (1/x) + (1/x^2)}$.

36. (a)



From the graph of $f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$, we estimate (to one decimal place) the value of $\lim_{x\to\infty} f(x)$ to be 1.4.

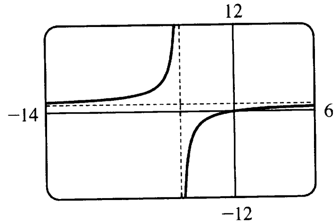
(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

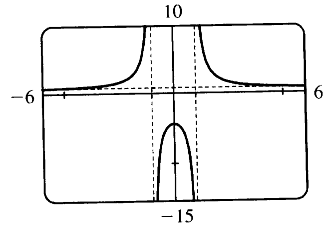
From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c) $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$
$$= \lim_{x\to\infty} \frac{(3x^2+8x+6)-(3x^2+3x+1)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$
$$= \lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$
$$= \lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

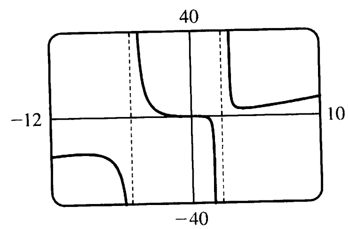
37. $\displaystyle\lim_{x\to\pm\infty} \frac{x}{x+4} = \lim_{x\to\pm\infty} \frac{1}{1+4/x} = \frac{1}{1+0} = 1$, so $y=1$ is a horizontal asymptote. $\displaystyle\lim_{x\to-4^-} \frac{x}{x+4} = \infty$ and $\displaystyle\lim_{x\to-4^+} \frac{x}{x+4} = -\infty$, so $x=-4$ is a vertical asymptote. The graph confirms these calculations.



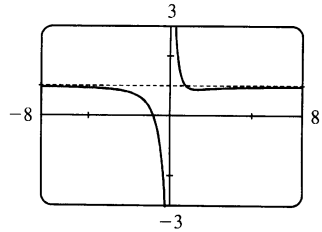
38. Since $x^2-1\to 0$ as $x\to\pm 1$ and $y<0$ for $-1<x<1$ and $y>0$ for $x<-1$ and $x>1$, we have $\displaystyle\lim_{x\to 1^-} \frac{x^2+4}{x^2-1} = -\infty$, $\displaystyle\lim_{x\to 1^+} \frac{x^2+4}{x^2-1} = \infty$, $\displaystyle\lim_{x\to -1^-} \frac{x^2+4}{x^2-1} = \infty$, and $\displaystyle\lim_{x\to -1^+} \frac{x^2+4}{x^2-1} = -\infty$, so $x=1$ and $x=-1$ are vertical asymptotes. Also
$\displaystyle\lim_{x\to\pm\infty} \frac{x^2+4}{x^2-1} = \lim_{x\to\pm\infty} \frac{1+4/x^2}{1-1/x^2} = \frac{1+0}{1-0} = 1$, so $y=1$ is a horizontal asymptote. The graph confirms these calculations.



39. $\displaystyle\lim_{x\to\pm\infty} \frac{x^3}{x^2+3x-10} = \lim_{x\to\pm\infty} \frac{x}{1+(3/x)-(10/x^2)} = \pm\infty$, so there is no horizontal asymptote.
$\displaystyle\lim_{x\to 2^+} \frac{x^3}{x^2+3x-10} = \lim_{x\to 2^+} \frac{x^3}{(x+5)(x-2)} = \infty$, since $\displaystyle\frac{x^3}{(x+5)(x-2)} > 0$ for $x>2$. Similarly, $\displaystyle\lim_{x\to 2^-} \frac{x^3}{x^2+3x-10} = -\infty$, $\displaystyle\lim_{x\to -5^-} \frac{x^3}{x^2+3x-10} = -\infty$, and $\displaystyle\lim_{x\to -5^+} \frac{x^3}{x^2+3x-10} = \infty$, so $x=2$ and $x=-5$ are vertical asymptotes. The graph confirms these calculations.



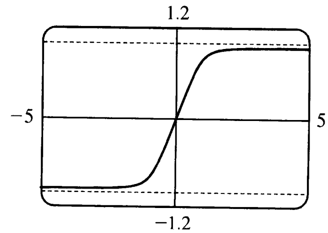
40. $\displaystyle\lim_{x\to\pm\infty}\frac{x^3+1}{x^3+x}=\lim_{x\to\pm\infty}\frac{1+1/x^3}{1+1/x^2}=1$, so $y=1$ is a horizontal asymptote. Since $y=\dfrac{x^3+1}{x^3+x}=\dfrac{x^3+1}{x(x^2+1)}>0$ for $x>0$ and $y<0$ for $-1<x<0$, $\displaystyle\lim_{x\to0^+}\frac{x^3+1}{x^3+x}=\infty$ and $\displaystyle\lim_{x\to0^-}\frac{x^3+1}{x^3+x}=-\infty$, so $x=0$ is a vertical asymptote.



41. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt[4]{x^4+1}}\cdot\frac{1/x}{1/\sqrt[4]{x^4}}=\lim_{x\to\infty}\frac{1}{\sqrt[4]{1+\dfrac{1}{x^4}}}=\frac{1}{\sqrt[4]{1+0}}=1$ and

$$\lim_{x\to-\infty}\frac{x}{\sqrt[4]{x^4+1}}\cdot\frac{1/x}{-1/\sqrt[4]{x^4}}=\lim_{x\to-\infty}\frac{1}{-\sqrt[4]{1+\dfrac{1}{x^4}}}=\frac{1}{-\sqrt[4]{1+0}}=-1,$$

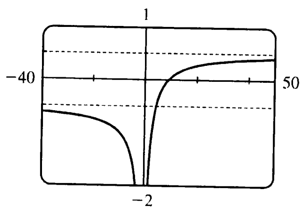
so $y=\pm1$ are horizontal asymptotes. There is no vertical asymptote.

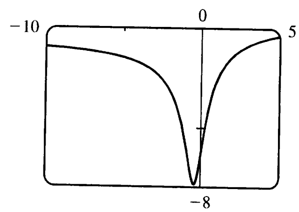


42. $\displaystyle\lim_{x\to\infty}\frac{x-9}{\sqrt{4x^2+3x+2}}=\lim_{x\to\infty}\frac{1-9/x}{\sqrt{4+(3/x)+(2/x^2)}}=\frac{1-0}{\sqrt{4+0+0}}=\frac{1}{2}.$

Using the fact that $\sqrt{x^2}=|x|=-x$ for $x<0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$. Thus, $\displaystyle\lim_{x\to-\infty}\frac{x-9}{\sqrt{4x^2+3x+2}}=\lim_{x\to-\infty}\frac{-1+9/x}{\sqrt{4+(3/x)+(2/x^2)}}=\frac{-1+0}{\sqrt{4+0+0}}=-\frac{1}{2}.$

The horizontal asymptotes are $y=\pm\frac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x, so the denominator never approaches zero, and thus there is no vertical asymptote.





43. Let's look for a rational function.

(1) $\displaystyle\lim_{x\to\pm\infty}f(x)=0\ \Rightarrow$ degree of numerator < degree of denominator

(2) $\displaystyle\lim_{x\to0}f(x)=-\infty\ \Rightarrow$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x=0$), and the function is negative near $x=0$.

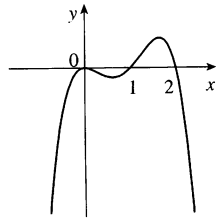
(3) $\displaystyle\lim_{x\to3^-}f(x)=\infty$ and $\displaystyle\lim_{x\to3^+}f(x)=-\infty\ \Rightarrow$ vertical asymptote at $x=3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2)=0\ \Rightarrow$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

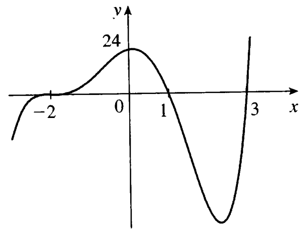
Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x)=\dfrac{2-x}{x^2(x-3)}$ as one possibility.

44. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x - 1)$ and $(x - 3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

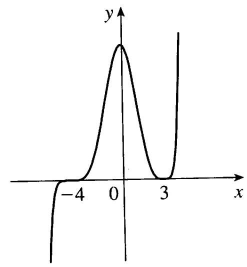
45. $y = f(x) = x^2(x-2)(1-x)$. The y-intercept is $f(0) = 0$, and the x-intercepts occur when $y = 0 \Rightarrow x = 0, 1,$ and 2. Notice that, since x^2 is always positive, the graph does not cross the x-axis at 0, but does cross the x-axis at 1 and 2. $\lim\limits_{x\to\infty} x^2(x-2)(1-x) = -\infty$, since the first two factors are large positive and the third large negative when x is large positive. $\lim\limits_{x\to-\infty} x^2(x-2)(1-x) = -\infty$ because the first and third factors are large positive and the second large negative as $x \to -\infty$.



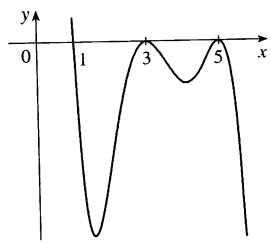
46. $y = (2+x)^3(1-x)(3-x)$. As $x \to \infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$. As $x \to -\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim\limits_{x\to-\infty} f(x) = -\infty$. Now the y-intercept is $f(0) = (2)^3(1)(3) = 24$ and the x-intercepts are the solutions to $f(x) = 0 \Rightarrow x = -2, 1$ and 3, and the graph crosses the x-axis at all of these points.



47. $y = f(x) = (x+4)^5(x-3)^4$. The y-intercept is $f(0) = 4^5(-3)^4 = 82{,}944$. The x-intercepts occur when $y = 0 \Rightarrow x = -4, 3$. Notice that the graph does not cross the x-axis at 3 because $(x-3)^4$ is always positive, but does cross the x-axis at -4. $\lim\limits_{x\to\infty} (x+4)^5(x-3)^4 = \infty$ since both factors are large positive when x is large positive. $\lim\limits_{x\to-\infty} (x+4)^5(x-3)^4 = -\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.

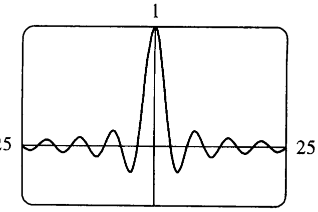


48. $y = (1-x)(x-3)^2(x-5)^2$. As $x \to \infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞. Therefore, $\lim\limits_{x\to\infty} (x) = -\infty$. As $x \to -\infty$, the factors all approach ∞. Therefore, $\lim\limits_{x\to-\infty} (x) = \infty$. Now the y-intercept is $f(0) = (1)(-3)^2(-5)^2 = 225$ and the x-intercepts are the solutions to $f(x) = 0 \Rightarrow x = 1, 3,$ and 5. Notice that $f(x)$ does not change sign at $x = 3$ or $x = 5$ because the factors $(x-3)^2$ and $(x-5)^2$ are always positive, so the graph does not cross the x-axis at $x = 3$ or $x = 5$, but does cross the x-axis at $x = 1$.

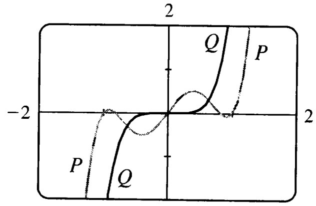


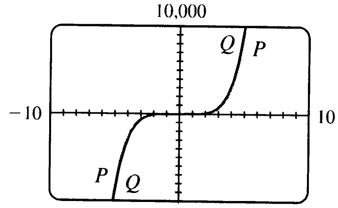
49. (a) Since $-1 \le \sin x \le 1$ for all x, $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*



50. (a) In both viewing rectangles, $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} Q(x) = \infty$ and $\lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.





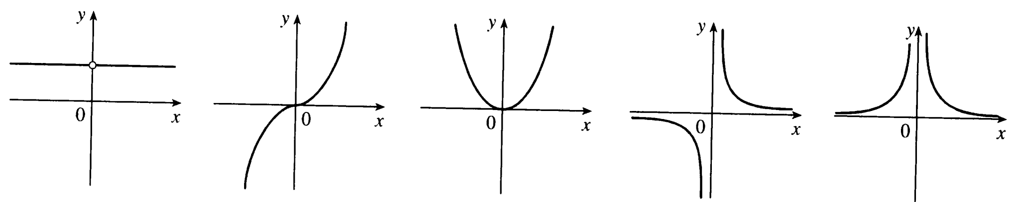
(b) $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x\to\infty}\left(1 - \frac{5}{3}\cdot\frac{1}{x^2} + \frac{2}{3}\cdot\frac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \Rightarrow$
P and Q have the same end behavior.

51. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

52.



(i) $n = 0$ (ii) $n > 0$ (n odd) (iii) $n > 0$ (n even) (iv) $n < 0$ (n odd) (v) $n < 0$ (n even)

From these sketches we see that

(a) $\lim_{x\to0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$

(b) $\lim_{x\to0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

(d) $\lim_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

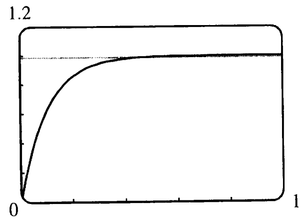
53. $\lim_{x\to\infty} \frac{4x-1}{x} = \lim_{x\to\infty}\left(4-\frac{1}{x}\right) = 4$, and $\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = \lim_{x\to\infty}\left(4+\frac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 4$.

54. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \frac{750t}{5000+25t} = \frac{30t}{200+t}\ \frac{\text{g}}{\text{L}}$.

(b) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{30t/t}{200/t + t/t} = \frac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

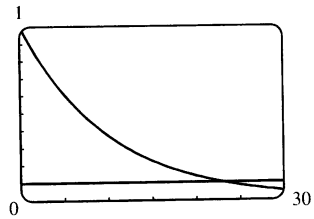
55. (a) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

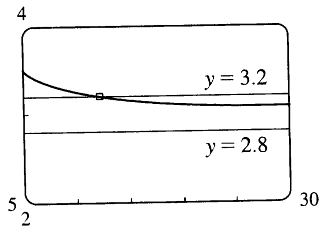


56. (a) $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$. If $x > x_1$, then $e^{-x/10} < 0.1$.

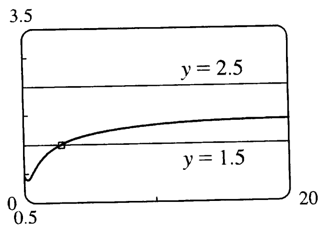
(b) $e^{-x/10} < 0.1 \;\Rightarrow\; -x/10 < \ln 0.1 \;\Rightarrow\; x > -10\ln\frac{1}{10} = -10\ln 10^{-1} = 10\ln 10$

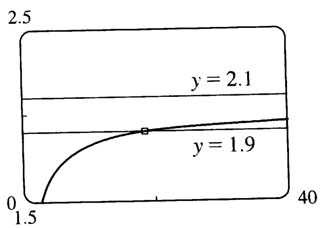


57. $\left|\frac{6x^2+5x-3}{2x^2-1} - 3\right| < 0.2 \;\Leftrightarrow\; 2.8 < \frac{6x^2+5x-3}{2x^2-1} < 3.2$. So we graph the three parts of this inequality on the same screen, and find that the curve $y = \frac{6x^2+5x-3}{2x^2-1}$ seems to lie between the lines $y = 2.8$ and $y = 3.2$ whenever $x > 12.8$. So we can choose $N = 13$ (or any larger number) so that the inequality holds whenever $x \geq N$.

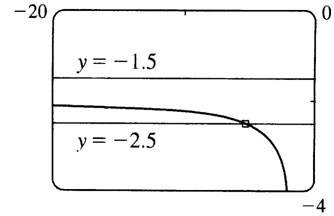


58. For $\varepsilon = 0.5$, we must find N such that whenever $x \geq N$, we have $\left|\frac{\sqrt{4x^2+1}}{x+1} - 2\right| < 0.5 \;\Leftrightarrow\; 1.5 < \frac{\sqrt{4x^2+1}}{x+1} < 2.5$. We graph the three parts of this inequality on the same screen, and find that it holds whenever $x \geq 3$. So we choose $N = 3$ (or any larger number). For $\varepsilon = 0.1$, we must have $1.9 < \frac{\sqrt{4x^2+1}}{x+1} < 2.1$, and the graphs show that this holds whenever $x \geq 19$. So we choose $N = 19$ (or any larger number).

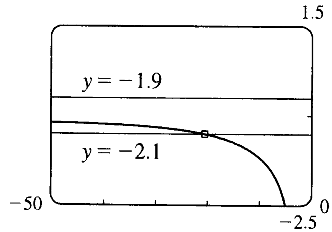




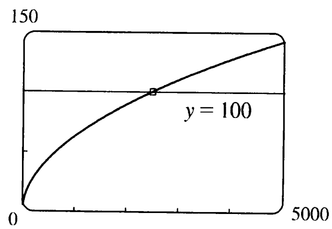
59. For $\varepsilon = 0.5$, we need to find N such that $\left|\dfrac{\sqrt{4x^2+1}}{x+1} - (-2)\right| < 0.5$ $\Leftrightarrow$ $-2.5 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \leq N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).



For $\varepsilon = 0.1$, we need $-2.1 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose $N = -22$ (or any smaller number).



60. We need N such that $\dfrac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \geq N$. From the graph, we see that this inequality holds for $x \geq 2500$. So we choose $N = 2500$ (or any larger number).



61. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10{,}000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\displaystyle\lim_{x\to\infty} \frac{1}{x^2} = 0$.

62. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$.

63. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\displaystyle\lim_{x\to-\infty} (1/x) = 0$.

64. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\displaystyle\lim_{x\to\infty} x^3 = \infty$.

65. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad e^x > M$. Now $e^x > M \quad \Leftrightarrow \quad x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \quad \Rightarrow \quad e^x > \max(e, M) \geq M$, so $\displaystyle\lim_{x\to\infty} e^x = \infty$.

66. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim\limits_{x\to-\infty}(1 + x^3) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\lim\limits_{x\to-\infty}(1 + x^3) = -\infty$.

67. Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim\limits_{t\to0^+} f(1/t) = L = \lim\limits_{x\to\infty} f(x)$.

Now suppose that $\lim\limits_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t\to0^-} f(1/t) = L = \lim\limits_{x\to-\infty} f(x)$.

2.7 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

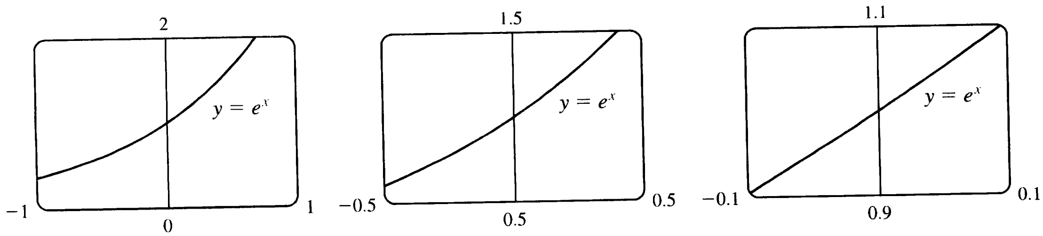
(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x\to3} \dfrac{f(x) - f(3)}{x - 3}$.

2. (a) Average velocity $= \dfrac{\Delta s}{\Delta t} = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$

(b) Instantaneous velocity $= \lim\limits_{h\to0} \dfrac{f(a+h) - f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E. The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A, and B.

4. The curve looks more like a line as the viewing rectangle gets smaller.



5. (a) (i) Using Definition 1,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)} = \lim_{x \to -3} \frac{(x^2 + 2x) - (3)}{x - (-3)} = \lim_{x \to -3} \frac{(x+3)(x-1)}{x+3}$$

$$= \lim_{x \to -3} (x - 1) = -4$$

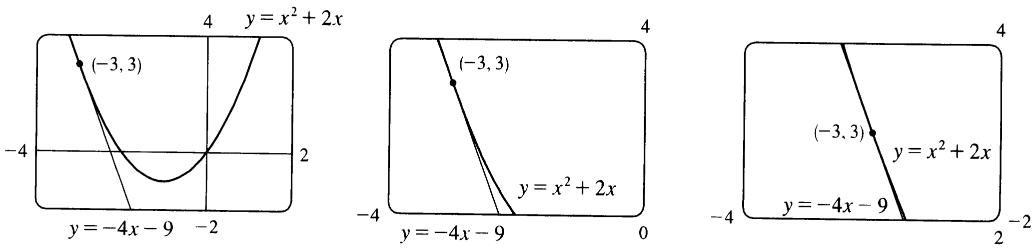
(ii) Using Equation 2,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \frac{\left[(-3+h)^2 + 2(-3+h)\right] - (3)}{h}$$

$$= \lim_{h \to 0} \frac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4$$

(b) Using the point-slope form of the equation of a line, an equation of the tangent line is $y - 3 = -4(x + 3)$. Solving for y gives us $y = -4x - 9$, which is the slope-intercept form of the equation of the tangent line.

(c)



6. (a) (i) $m = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{x^3 - (-1)}{x + 1} = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{x + 1}$

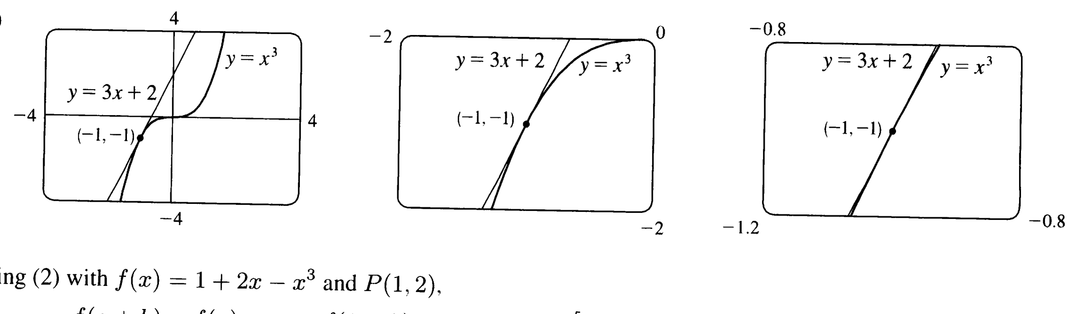
$$= \lim_{x \to -1} \left(x^2 - x + 1\right) = 3$$

(ii) $m = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{(-1+h)^3 - (-1)}{h} = \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h - 1 + 1}{h}$

$$= \lim_{h \to 0} \left(h^2 - 3h + 3\right) = 3$$

(b) $y - (-1) = 3\,[x - (-1)] \quad \Leftrightarrow \quad y + 1 = 3x + 3 \quad \Leftrightarrow \quad y = 3x + 2$

(c)



7. Using (2) with $f(x) = 1 + 2x - x^3$ and $P(1, 2)$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[1 + 2(1+h) - (1+h)^3\right] - 2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2 + 2h - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(-h^2 - 3h - 1)}{h} = \lim_{h \to 0} (-h^2 - 3h - 1) = -1$$

Tangent line: $y - 2 = -1(x - 1) \quad \Leftrightarrow \quad y - 2 = -x + 1 \quad \Leftrightarrow \quad y = -x + 3$

8. Using (1),

$$m = \lim_{x\to 4} \frac{\sqrt{2x+1} - \sqrt{2(4)+1}}{x-4} = \lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}$$

$$= \lim_{x\to 4} \frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)}$$

$$= \lim_{x\to 4} \frac{2}{(\sqrt{2x+1}+3)} = \frac{2}{3+3} = \frac{1}{3}.$$

Tangent line: $y - 3 = \frac{1}{3}(x-4) \quad \Leftrightarrow \quad y - 3 = \frac{1}{3}x - \frac{4}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{5}{3}$

9. Using (1) with $f(x) = \dfrac{x-1}{x-2}$ and $P(3,2)$,

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 3} \frac{\dfrac{x-1}{x-2} - 2}{x-3} = \lim_{x\to 3} \frac{\dfrac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x\to 3} \frac{3-x}{(x-2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-1}{x-2} = \frac{-1}{1} = -1.$$

Tangent line: $y - 2 = -1(x-3) \quad \Leftrightarrow \quad y - 2 = -x + 3 \quad \Leftrightarrow \quad y = -x + 5$

10. Using (1), $m = \lim\limits_{x\to 0} \dfrac{\dfrac{2x}{(x+1)^2} - 0}{x-0} = \lim\limits_{x\to 0} \dfrac{2x}{x(x+1)^2} = \lim\limits_{x\to 0} \dfrac{2}{(x+1)^2} = \dfrac{2}{1^2} = 2.$

Tangent line: $y - 0 = 2(x-0) \quad \Leftrightarrow \quad y = 2x$

11. (a) $m = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{2/(x+3) - 2/(a+3)}{x-a} = \lim\limits_{x\to a} \dfrac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$

$$= \lim_{x\to a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$$

(b) (i) $a = -1 \quad \Rightarrow \quad m = \dfrac{-2}{(-1+3)^2} = -\dfrac{1}{2}$

(ii) $a = 0 \quad \Rightarrow \quad m = \dfrac{-2}{(0+3)^2} = -\dfrac{2}{9}$

(iii) $a = 1 \quad \Rightarrow \quad m = \dfrac{-2}{(1+3)^2} = -\dfrac{1}{8}$

12. (a) Using (1),

$$m = \lim_{x\to a} \frac{(1+x+x^2)-(1+a+a^2)}{x-a} = \lim_{x\to a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x\to a} \frac{x-a+(x-a)(x+a)}{x-a}$$

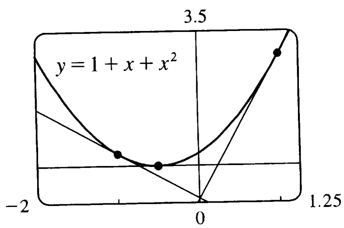
$$= \lim_{x\to a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x\to a} (1+x+a) = 1+2a$$

(b) (i) $x = -1 \quad \Rightarrow \quad m = 1 + 2(-1) = -1$

(ii) $x = -\frac{1}{2} \quad \Rightarrow \quad m = 1 + 2(-\frac{1}{2}) = 0$

(iii) $x = 1 \quad \Rightarrow \quad m = 1 + 2(1) = 3$

(c)



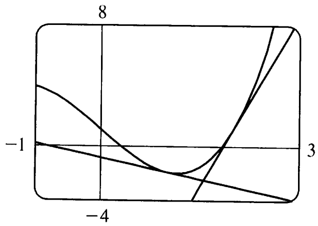
13. (a) Using (1),

$$m = \lim_{x \to a} \frac{(x^3 - 4x + 1) - (a^3 - 4a + 1)}{x - a} = \lim_{x \to a} \frac{(x^3 - a^3) - 4(x - a)}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2) - 4(x - a)}{x - a} = \lim_{x \to a} (x^2 + ax + a^2 - 4) = 3a^2 - 4$$

(b) At $(1, -2)$: $m = 3(1)^2 - 4 = -1$, so an equation of the tangent line is $y - (-2) = -1(x - 1) \Leftrightarrow y = -x - 1$. At $(2, 1)$: $m = 3(2)^2 - 4 = 8$, so an equation of the tangent line is $y - 1 = 8(x - 2) \Leftrightarrow y = 8x - 15$.

(c)



14. (a) Using (1),

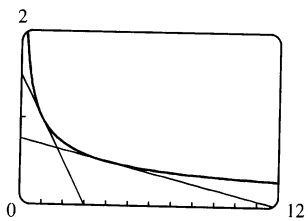
$$m = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x \to a} \frac{a - x}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2}$$

(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x - 4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



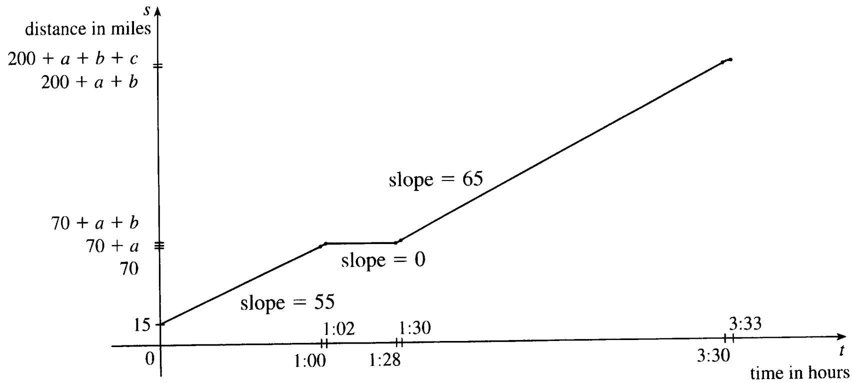
15. (a) Since the slope of the tangent at $t = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B, so the car was going faster at C.

(c) Near A, the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B, the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C, so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E, the slope of the tangent is 0, so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t = 0$ at the beginning of the trip.



17. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t\to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t\to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t\to 2} \frac{-8(2t^2 - 5t + 2)}{t - 2}$$

$$= \lim_{t\to 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t\to 2}(2t - 1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

18. (a) $v(1) = \lim_{h\to 0} \dfrac{H(1+h) - H(1)}{h}$

$$= \lim_{h\to 0} \frac{(58 + 58h - 0.83 - 1.66h - 0.83h^2) - 57.17}{h} = \lim_{h\to 0}(56.34 - 0.83h) = 56.34 \text{ m/s}$$

(b) $v(a) = \lim_{h\to 0} \dfrac{H(a+h) - H(a)}{h}$

$$= \lim_{h\to 0} \frac{(58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2) - (58a - 0.83a^2)}{h}$$

$$= \lim_{h\to 0} (58 - 1.66a - 0.83h) = 58 - 1.66a \text{ m/s}$$

(c) The arrow strikes the moon when the height is 0, that is, $58t - 0.83t^2 = 0 \Leftrightarrow t(58 - 0.83t) = 0 \Leftrightarrow$ $t = \dfrac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\dfrac{58}{0.83}\right) = 58 - 1.66\left(\dfrac{58}{0.83}\right) = -58$ m/s. Thus, the arrow will have a velocity of -58 m/s.

19. $v(a) = \lim_{h\to 0} \dfrac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \dfrac{4(a+h)^3 + 6(a+h) + 2 - (4a^3 + 6a + 2)}{h}$

$$= \lim_{h\to 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h}$$

$$= \lim_{h\to 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h\to 0} (12a^2 + 12ah + 4h^2 + 6) = (12a^2 + 6) \text{ m/s}$$

So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

20. (a) The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t}=\frac{(t+h)^2-8(t+h)+18-(t^2-8t+18)}{h}$$

$$=\frac{t^2+2th+h^2-8t-8h+18-t^2+8t-18}{h}=\frac{2th+h^2-8h}{h}$$

$$=(2t+h-8)\ \text{m/s}$$

(i) $[3,4]$: $t=3$, $h=4-3=1$, so the average velocity is $2(3)+1-8=-1$ m/s.

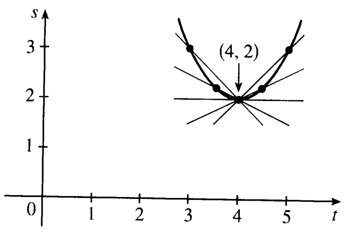
(ii) $[3.5,4]$: $t=3.5$, $h=0.5$, so the average velocity is $2(3.5)+0.5-8=-0.5$ m/s.

(iii) $[4,5]$: $t=4$, $h=1$, so the average velocity is $2(4)+1-8=1$ m/s.

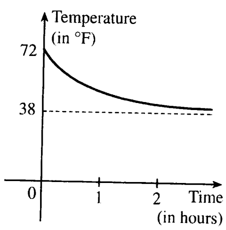
(iv) $[4,4.5]$: $t=4$, $h=0.5$, so the average velocity is $2(4)+0.5-8=0.5$ m/s.

(b) $v(t)=\lim\limits_{h\to 0}\dfrac{s(t+h)-s(t)}{h}=\lim\limits_{h\to 0}(2t+h-8)=2t-8$, so $v(4)=0$.

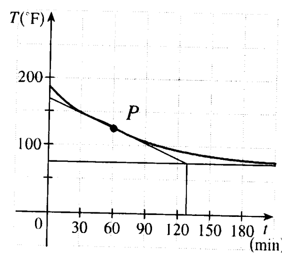
(c)



21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t=1$ h seems to be about $\dfrac{75-168}{132-0}\approx -0.7$ °F/min.

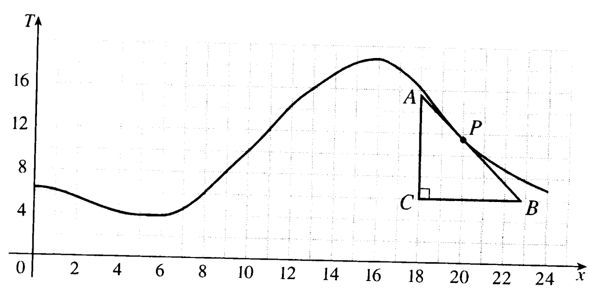


23. (a) (i) $[20,23]$: $\dfrac{7.9-11.5}{23-20}=-1.2$ °C/h

(ii) $[20,22]$: $\dfrac{9.0-11.5}{22-20}=-1.25$ °C/h

(iii) $[20,21]$: $\dfrac{10.2-11.5}{21-20}=-1.3$ °C/h

(b) In the figure, we estimate A to be $(18,15.5)$ and B as $(23,6)$. So the slope is $\dfrac{6-15.5}{23-18}=-1.9$ °C/h at 8:00 P.M.



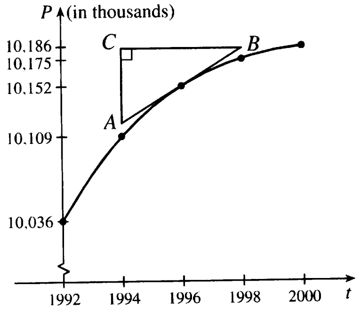
24. (a) (i) $[1992, 1996]$: $\dfrac{P(1996) - P(1992)}{1996 - 1992} = \dfrac{10{,}152 - 10{,}036}{4} = \dfrac{116}{4} = 29$ thousand/year

(ii) $[1994, 1996]$: $\dfrac{P(1996) - P(1994)}{1996 - 1994} = \dfrac{10{,}152 - 10{,}109}{2} = \dfrac{43}{2} = 21.5$ thousand/year

(iii) $[1996, 1998]$: $\dfrac{P(1998) - P(1996)}{1998 - 1996} = \dfrac{10{,}175 - 10{,}152}{2} = \dfrac{23}{2} = 11.5$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{21.5 + 11.5}{2} = 16.5$ thousand/year.

(c) Estimating A as $(1994, 10{,}125)$ and B as $(1998, 10{,}182)$, the slope at 1996 is $\dfrac{10{,}182 - 10{,}125}{1998 - 1994} = \dfrac{57}{4} = 14.25$ thousand/year.



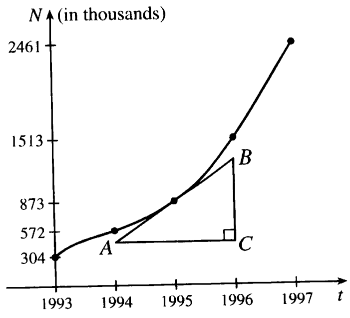
25. (a) (i) $[1995, 1997]$: $\dfrac{N(1997) - N(1995)}{1997 - 1995} = \dfrac{2461 - 873}{2} = \dfrac{1588}{2} = 794$ thousand/year

(ii) $[1995, 1996]$: $\dfrac{N(1996) - N(1995)}{1996 - 1995} = \dfrac{1513 - 873}{1} = 640$ thousand/year

(iii) $[1994, 1995]$: $\dfrac{N(1995) - N(1994)}{1995 - 1994} = \dfrac{873 - 572}{1} = 301$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{640 + 301}{2} = \dfrac{941}{2} = 470.5$ thousand/year.

(c) Estimating A as $(1994, 420)$ and B as $(1996, 1275)$, the slope at 1995 is $\dfrac{1275 - 420}{1996 - 1994} = \dfrac{855}{2} = 427.5$ thousand/year



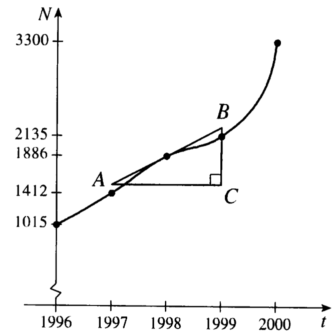
26. (a) (i) $[1996, 1998]$: $\dfrac{N(1998) - N(1996)}{1998 - 1996} = \dfrac{1886 - 1015}{2} = \dfrac{871}{2} = 435.5$ locations/year

(ii) $[1997, 1998]$: $\dfrac{N(1998) - N(1997)}{1998 - 1997} = \dfrac{1886 - 1412}{1} = 474$ locations/year

(iii) $[1998, 1999]$: $\dfrac{N(1999) - N(1998)}{1999 - 1998} = \dfrac{2135 - 1886}{1} = 249$ locations/year

(b) Using the values from (ii) and (iii), we have $\dfrac{474 + 249}{2} = \dfrac{723}{2} = 361.5 \approx 362$ locations/year.

(c) Estimating A as $(1997, 1525)$ and B as $(1999, 2250)$, the slope at 1998 is $\dfrac{2250 - 1525}{1999 - 1997} = \dfrac{725}{2} = 362.5$ locations/year.



27. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $\dfrac{C(100 + h) - C(100)}{h} = \dfrac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \dfrac{20h + 0.05h^2}{h}$

$= 20 + 0.05h,\ h \neq 0$

So the instantaneous rate of change is $\lim\limits_{h \to 0} \dfrac{C(100 + h) - C(100)}{h} = \lim\limits_{h \to 0} (20 + 0.05h) = \$20/\text{unit}.$

28. $\Delta V = V(t + h) - V(t) = 100{,}000\left(1 - \dfrac{t + h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$= 100{,}000\left[\left(1 - \dfrac{t + h}{30} + \dfrac{(t + h)^2}{3600}\right) - \left(1 - \dfrac{t}{30} + \dfrac{t^2}{3600}\right)\right] = 100{,}000\left(-\dfrac{h}{30} + \dfrac{2th}{3600} + \dfrac{h^2}{3600}\right)$

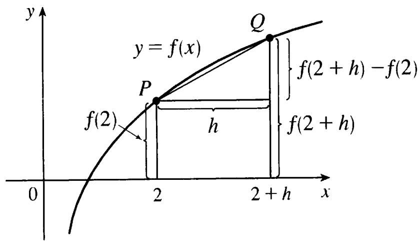
$= \dfrac{100{,}000}{3600} h\,(-120 + 2t + h) = \dfrac{250}{9} h\,(-120 + 2t + h)$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

2.8 Derivatives

1. 

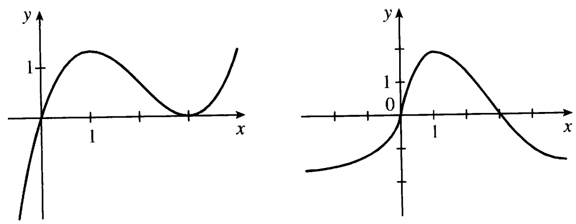
The line from $P(2, f(2))$ to $Q(2+h, f(2+h))$ is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$.

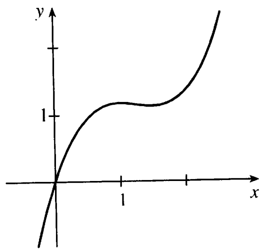
2. As h decreases, the line PQ becomes steeper, so its slope increases. So
$0 < \dfrac{f(4)-f(2)}{4-2} < \dfrac{f(3)-f(2)}{3-2} < \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$. Thus, $0 < \frac{1}{2}\,[f(4)-f(2)] < f(3)-f(2) < f'(2)$.

3. $g'(0)$ is the only negative value. The slope at $x=4$ is smaller than the slope at $x=2$ and both are smaller than the slope at $x=-2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

4. Since $(4,3)$ is on $y=f(x)$, $f(4)=3$. The slope of the tangent line between $(0,2)$ and $(4,3)$ is $\frac{1}{4}$, so $f'(4)=\frac{1}{4}$.

5. We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0)=0$ and $f'(0)=3$. Since $f'(1)=0$, we will round off our figure so that there is a horizontal tangent directly over $x=1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x=2$. Two of the many possibilities are shown.



6. 

7. Using Definition 2 with $f(x)=3x^2-5x$ and the point $(2,2)$, we have

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{[3(2+h)^2-5(2+h)]-2}{h}$$
$$= \lim_{h\to 0}\frac{(12+12h+3h^2-10-5h)-2}{h} = \lim_{h\to 0}\frac{3h^2+7h}{h} = \lim_{h\to 0}(3h+7) = 7.$$

So an equation of the tangent line at $(2,2)$ is $y-2=7(x-2)$ or $y=7x-12$.

8. Using Definition 2 with $g(x)=1-x^3$ and the point $(0,1)$, we have

$$g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{[1-(0+h)^3]-1}{h} = \lim_{h\to 0}\frac{(1-h^3)-1}{h} = \lim_{h\to 0}(-h^2) = 0.$$

So an equation of the tangent line is $y-1=0(x-0)$ or $y=1$.

9. (a) Using Definition 2 with $F(x) = x^3 - 5x + 1$ and the point $(1, -3)$, we have

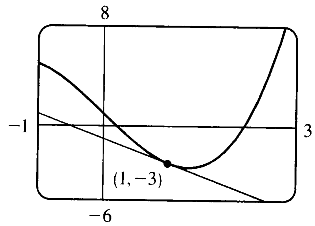
$$F'(1) = \lim_{h\to 0} \frac{F(1+h) - F(1)}{h} = \lim_{h\to 0} \frac{\left[(1+h)^3 - 5(1+h) + 1\right] - (-3)}{h}$$

$$= \lim_{h\to 0} \frac{(1 + 3h + 3h^2 + h^3 - 5 - 5h + 1) + 3}{h} = \lim_{h\to 0} \frac{h^3 + 3h^2 - 2h}{h}$$

$$= \lim_{h\to 0} \frac{h(h^2 + 3h - 2)}{h} = \lim_{h\to 0}(h^2 + 3h - 2) = -2$$

So an equation of the tangent line at $(1, -3)$ is $y - (-3) = -2(x - 1) \Leftrightarrow y = -2x - 1$.

(b)

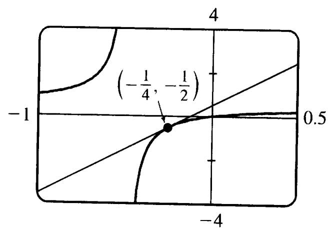


10. (a) $$G'(a) = \lim_{h\to 0} \frac{G(a+h) - G(a)}{h} = \lim_{h\to 0} \frac{\dfrac{a+h}{1+2(a+h)} - \dfrac{a}{1+2a}}{h}$$

$$= \lim_{h\to 0} \frac{a + 2a^2 + h + 2ah - a - 2a^2 - 2ah}{h(1 + 2a + 2h)(1 + 2a)} = \lim_{h\to 0} \frac{1}{(1 + 2a + 2h)(1 + 2a)} = (1 + 2a)^{-2}$$

So the slope of the tangent at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ is $m = \left[1 + 2\left(-\frac{1}{4}\right)\right]^{-2} = 4$, and thus an equation is $y + \frac{1}{2} = 4\left(x + \frac{1}{4}\right)$ or $y = 4x + \frac{1}{2}$.

(b)



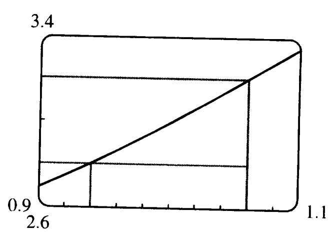
11. (a) $f'(1) = \lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \dfrac{3^{1+h} - 3^1}{h}$.

So let $F(h) = \dfrac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$	h	$F(h)$
0.1	3.484	−0.1	3.121
0.01	3.314	−0.01	3.278
0.001	3.298	−0.001	3.294
0.0001	3.296	−0.0001	3.296

We estimate that $f'(1) \approx 3.296$.

(b)



From the graph, we estimate that the slope of the tangent is about

$$\frac{3.2 - 2.8}{1.06 - 0.94} = \frac{0.4}{0.12} \approx 3.3.$$

12. (a) $g'\left(\frac{\pi}{4}\right) = \lim\limits_{h\to 0} \dfrac{g\left(\frac{\pi}{4}+h\right) - g\left(\frac{\pi}{4}\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\tan\left(\frac{\pi}{4}+h\right) - \tan\left(\frac{\pi}{4}\right)}{h}.$

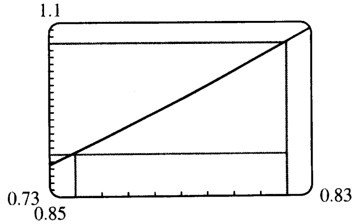
So let $G(h) = \dfrac{\tan\left(\frac{\pi}{4}+h\right) - 1}{h}.$

We calculate:

h	$G(h)$	h	$G(h)$
0.1	2.2305	−0.1	1.8237
0.01	2.0203	−0.01	1.9803
0.001	2.0020	−0.001	1.9980
0.0001	2.0002	−0.0001	1.9998

We estimate that $g'\left(\frac{\pi}{4}\right) = 2$.

(b)



From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07 - 0.91}{0.82 - 0.74} = \dfrac{0.16}{0.08} = 2.$

13. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\left[3 - 2(a+h) + 4(a+h)^2\right] - \left(3 - 2a + 4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(3 - 2a - 2h + 4a^2 + 8ah + 4h^2\right) - \left(3 - 2a + 4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{-2h + 8ah + 4h^2}{h} = \lim_{h\to 0} \frac{h(-2 + 8a + 4h)}{h} = \lim_{h\to 0}(-2 + 8a + 4h) = -2 + 8a$$

14. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(a+h)^4 - 5(a+h)\right] - \left(a^4 - 5a\right)}{h}$

$$= \lim_{h\to 0} \frac{\left(a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h\right) - \left(a^4 - 5a\right)}{h}$$

$$= \lim_{h\to 0} \frac{4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h}{h} = \lim_{h\to 0} \frac{h\left(4a^3 + 6a^2h + 4ah^2 + h^3 - 5\right)}{h}$$

$$= \lim_{h\to 0} \left(4a^3 + 6a^2h + 4ah^2 + h^3 - 5\right) = 4a^3 - 5$$

15. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h}$$

$$= \lim_{h\to 0} \frac{(2a + 2h + 1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{\left(2a^2 + 6a + 2ah + 6h + a + 3\right) - \left(2a^2 + 2ah + 6a + a + h + 3\right)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

16. $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{(a+h)^2+1}{(a+h)-2}-\dfrac{a^2+1}{a-2}}{h}$

$=\lim\limits_{h\to 0}\dfrac{(a^2+2ah+h^2+1)(a-2)-(a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$

$=\lim\limits_{h\to 0}\dfrac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2)-(a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$

$=\lim\limits_{h\to 0}\dfrac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)}=\lim\limits_{h\to 0}\dfrac{h\left(a^2-4a+ah-2h-1\right)}{h(a+h-2)(a-2)}$

$=\lim\limits_{h\to 0}\dfrac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)}=\dfrac{a^2-4a-1}{(a-2)^2}$

17. Use Definition 2 with $f(x)=1/\sqrt{x+2}$.

$f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{1}{\sqrt{(a+h)+2}}-\dfrac{1}{\sqrt{a+2}}}{h}$

$=\lim\limits_{h\to 0}\dfrac{\dfrac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\sqrt{a+2}}}{h}=\lim\limits_{h\to 0}\left[\dfrac{\sqrt{a+2}-\sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}}\cdot\dfrac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right]$

$=\lim\limits_{h\to 0}\dfrac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$

$=\lim\limits_{h\to 0}\dfrac{-h}{h\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$

$=\lim\limits_{h\to 0}\dfrac{-1}{\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$

$=\dfrac{-1}{\left(\sqrt{a+2}\right)^2\left(2\sqrt{a+2}\right)}=-\dfrac{1}{2\,(a+2)^{3/2}}$

18. $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{3\,(a+h)+1}-\sqrt{3a+1}}{h}$

$=\lim\limits_{h\to 0}\dfrac{\left(\sqrt{3a+3h+1}-\sqrt{3a+1}\right)\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$

$=\lim\limits_{h\to 0}\dfrac{(3a+3h+1)-(3a+1)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}=\lim\limits_{h\to 0}\dfrac{3h}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$

$=\lim\limits_{h\to 0}\dfrac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}}=\dfrac{3}{2\sqrt{3a+1}}$

Note that the answers to Exercises 19–24 are not unique.

19. By Definition 2, $\lim\limits_{h\to 0}\dfrac{(1+h)^{10}-1}{h}=f'(1)$, where $f(x)=x^{10}$ and $a=1$.

Or: By Definition 2, $\lim\limits_{h\to 0}\dfrac{(1+h)^{10}-1}{h}=f'(0)$, where $f(x)=(1+x)^{10}$ and $a=0$.

20. By Definition 2, $\lim\limits_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(16)$, where $f(x) = \sqrt[4]{x}$ and $a = 16$.

Or: By Definition 2, $\lim\limits_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(0)$, where $f(x) = \sqrt[4]{16+x}$ and $a = 0$.

21. By Equation 3, $\lim\limits_{x\to 5} \dfrac{2^x - 32}{x-5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

22. By Equation 3, $\lim\limits_{x\to \pi/4} \dfrac{\tan x - 1}{x - \pi/4} = f'(\pi/4)$, where $f(x) = \tan x$ and $a = \pi/4$.

23. By Definition 2, $\lim\limits_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim\limits_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

24. By Equation 3, $\lim\limits_{t\to 1} \dfrac{t^4+t-2}{t-1} = f'(1)$, where $f(t) = t^4 + t$ and $a = 1$.

25. $$v(2) = f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left[(2+h)^2 - 6(2+h) - 5\right] - \left[2^2 - 6(2) - 5\right]}{h}$$
$$= \lim_{h\to 0} \frac{\left(4 + 4h + h^2 - 12 - 6h - 5\right) - (-13)}{h} = \lim_{h\to 0} \frac{h^2 - 2h}{h} = \lim_{h\to 0}(h-2) = -2 \text{ m/s}$$

26. $$v(2) = f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\left[2(2+h)^3 - (2+h) + 1\right] - \left[2(2)^3 - 2 + 1\right]}{h}$$
$$= \lim_{h\to 0} \frac{\left(2h^3 + 12h^2 + 24h + 16 - 2 - h + 1\right) - 15}{h}$$
$$= \lim_{h\to 0} \frac{2h^3 + 12h^2 + 23h}{h} = \lim_{h\to 0} \left(2h^2 + 12h + 23\right) = 23 \text{ m/s}$$

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

28. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are $(\text{gal/h})/(\text{mi/h})$.

(b) The fuel consumption is decreasing by $0.05\ (\text{gal/h})/(\text{mi/h})$ as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about $0.05\ (\text{gal/h})/(\text{mi/h})$.

30. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let

$A = \dfrac{T(8) - T(10)}{8 - 10} = \dfrac{72 - 81}{-2} = 4.5$ and $B = \dfrac{T(12) - T(10)}{12 - 10} = \dfrac{88 - 81}{2} = 3.5$. Then

$T'(10) = \lim\limits_{t \to 10} \dfrac{T(t) - T(10)}{t - 10} \approx \dfrac{A + B}{2} = \dfrac{4.5 + 3.5}{2} = 4^\circ\text{F/h}.$

32. **For 1910:** We will average the difference quotients obtained using the years 1900 and 1920.

Let $A = \dfrac{E(1900) - E(1910)}{1900 - 1910} = \dfrac{48.3 - 51.1}{-10} = 0.28$ and

$B = \dfrac{E(1920) - E(1910)}{1920 - 1910} = \dfrac{55.2 - 51.1}{10} = 0.41$. Then

$E'(1910) = \lim\limits_{t \to 1910} \dfrac{E(t) - E(1910)}{t - 1910} \approx \dfrac{A + B}{2} = 0.345$. This means that life expectancy at birth was increasing at about 0.345 year/year in 1910.

For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain $E'(1950) \approx [0.31 + 0.10]/2 = 0.205$. So life expectancy at birth was increasing at about 0.205 year/year in 1950.

33. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of $0.25\ (\text{mg/L})/^\circ\text{C}$.

34. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/^\circ\text{C}$.

(b) For $T = 15\ ^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7\ (\text{cm/s})/^\circ\text{C}$. This tells us that at $T = 15\ ^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of $0.7\ (\text{cm/s})/^\circ\text{C}$. In a similar fashion for $T = 25\ ^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2\ (\text{cm/s})/^\circ\text{C}$. As it gets warmer than $20\ ^\circ\text{C}$, the maximum sustainable speed decreases rapidly.

35. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h \sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

36. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin \dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h| \sin \dfrac{1}{h} \leq |h| \quad \Rightarrow \quad -|h| \leq h \sin \dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0} (-|h|) = 0$ and $\lim\limits_{h\to 0} |h| = 0$, we know that

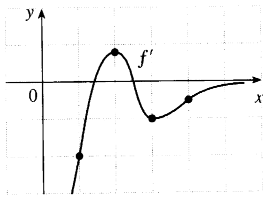
$\lim\limits_{h\to 0} \left(h \sin \dfrac{1}{h} \right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

2.9 The Derivative as a Function

1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(1) \approx -2$ (b) $f'(2) \approx 0.8$

(c) $f'(3) \approx -1$ (d) $f'(4) \approx -0.5$

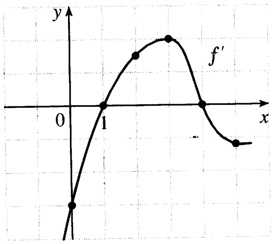


2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(0) \approx -3$ (b) $f'(1) \approx 0$

(c) $f'(2) \approx 1.5$ (d) $f'(3) \approx 2$

(e) $f'(4) \approx 0$ (f) $f'(5) \approx -1.2$



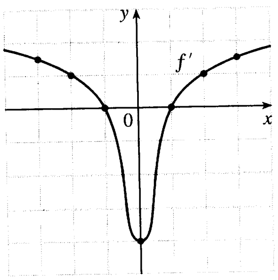
3. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$

(c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$

(e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$

(g) $f'(3) \approx 1.5$



4. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

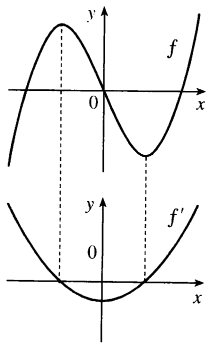
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 5–13: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f — there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

5.



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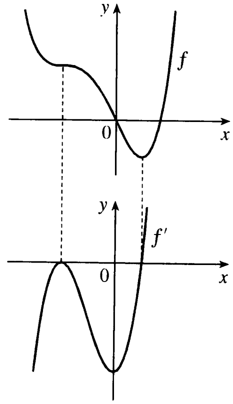

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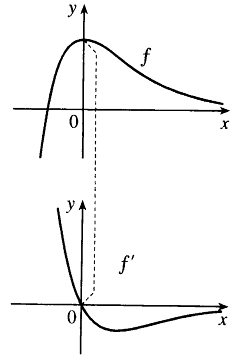

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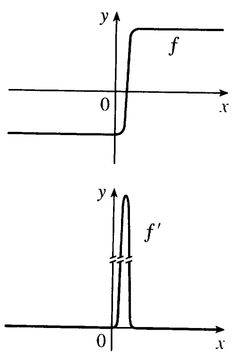

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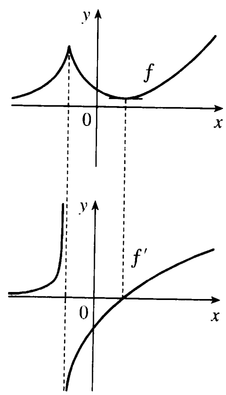

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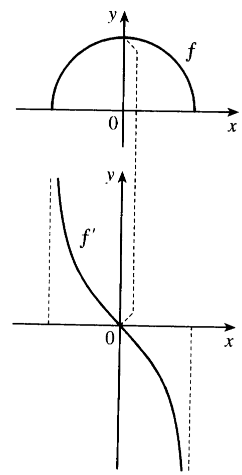

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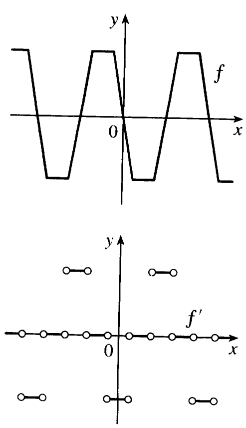

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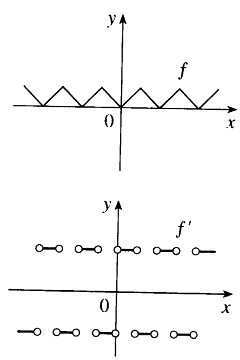
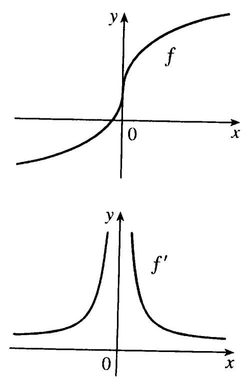
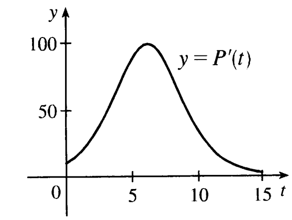


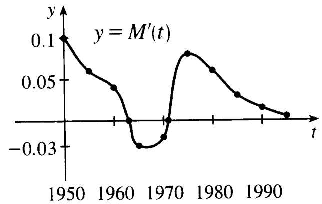
13.



14. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.

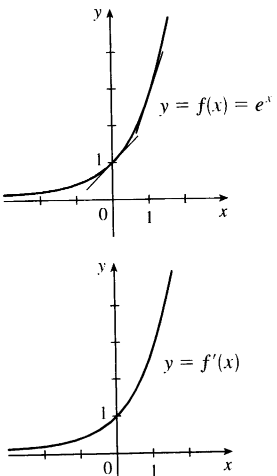


15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



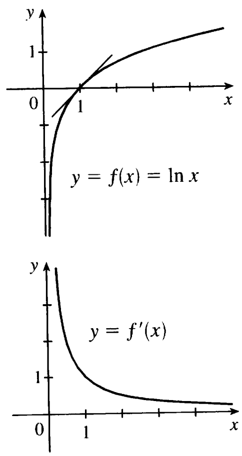
16. See Figure 1 in Section 3.4.

17.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

18.

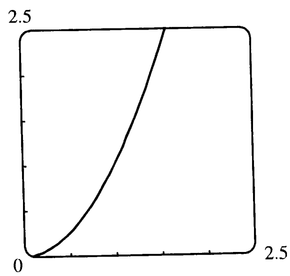


As x increases toward 1, $f'(x)$ decreases from very large numbers to 1. As x becomes large, $f'(x)$ gets closer to 0. As a guess, $f'(x) = 1/x^2$ or $f'(x) = 1/x$ make sense.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.



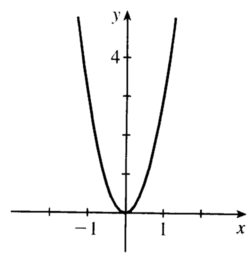
(d) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$

20. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$

21. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{37-37}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[12+7(x+h)]-(12+7x)}{h}$

$= \lim_{h\to 0} \frac{12+7x+7h-12-7x}{h} = \lim_{h\to 0} \frac{7h}{h} = \lim_{h\to 0} 7 = 7$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[1-3(x+h)^2]-(1-3x^2)}{h}$

$= \lim_{h\to 0} \frac{[1-3(x^2+2xh+h^2)]-(1-3x^2)}{h} = \lim_{h\to 0} \frac{1-3x^2-6xh-3h^2-1+3x^2}{h}$

$= \lim_{h\to 0} \frac{-6xh-3h^2}{h} = \lim_{h\to 0} \frac{h(-6x-3h)}{h} = \lim_{h\to 0}(-6x-3h) = -6x$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[5(x+h)^2+3(x+h)-2]-(5x^2+3x-2)}{h}$

$= \lim_{h\to 0} \frac{5x^2+10xh+5h^2+3x+3h-2-5x^2-3x+2}{h} = \lim_{h\to 0} \frac{10xh+5h^2+3h}{h}$

$= \lim_{h\to 0} \frac{h(10x+5h+3)}{h} = \lim_{h\to 0}(10x+5h+3) = 10x+3$

Domain of f = domain of $f' = \mathbb{R}$.

25. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(x+h)^3 - 3(x+h) + 5\right] - (x^3 - 3x + 5)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5) - (x^3 - 3x + 5)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim\limits_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2 - 3)}{h}$

$= \lim\limits_{h\to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$

Domain of f = domain of $f' = \mathbb{R}$.

26. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h}$

$= \lim\limits_{h\to 0} \left(\dfrac{h}{h} + \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim\limits_{h\to 0} \left[1 + \dfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\right]$

$= \lim\limits_{h\to 0} \left(1 + \dfrac{1}{\sqrt{x+h}+\sqrt{x}}\right) = 1 + \dfrac{1}{\sqrt{x}+\sqrt{x}} = 1 + \dfrac{1}{2\sqrt{x}}$

Domain of $f = [0, \infty)$, domain of $f' = (0, \infty)$.

27. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \left[\dfrac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}\right]$

$= \lim\limits_{h\to 0} \dfrac{(1+2x+2h) - (1+2x)}{h\left[\sqrt{1+2(x+h)} + \sqrt{1+2x}\right]} = \lim\limits_{h\to 0} \dfrac{2}{\sqrt{1+2x+2h} + \sqrt{1+2x}} = \dfrac{2}{2\sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}$

Domain of $g = \left[-\frac{1}{2}, \infty\right)$, domain of $g' = \left(-\frac{1}{2}, \infty\right)$.

28. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{3+(x+h)}{1-3(x+h)} - \dfrac{3+x}{1-3x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(3+x+h)(1-3x) - (3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}$

$= \lim\limits_{h\to 0} \dfrac{(3 - 9x + x - 3x^2 + h - 3hx) - (3 - 9x - 9h + x - 3x^2 - 3hx)}{h(1-3x-3h)(1-3x)}$

$= \lim\limits_{h\to 0} \dfrac{10h}{h(1-3x-3h)(1-3x)} = \lim\limits_{h\to 0} \dfrac{10}{(1-3x-3h)(1-3x)} = \dfrac{10}{(1-3x)^2}$

Domain of f = domain of $f' = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

29. $G'(t) = \lim\limits_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4(t+h)}{(t+h)+1} - \dfrac{4t}{t+1}}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4(t+h)(t+1) - 4t(t+h+1)}{(t+h+1)(t+1)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(4t^2 + 4ht + 4t + 4h) - (4t^2 + 4ht + 4t)}{h(t+h+1)(t+1)}$

$= \lim\limits_{h\to 0} \dfrac{4h}{h(t+h+1)(t+1)} = \lim\limits_{h\to 0} \dfrac{4}{(t+h+1)(t+1)} = \dfrac{4}{(t+1)^2}$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

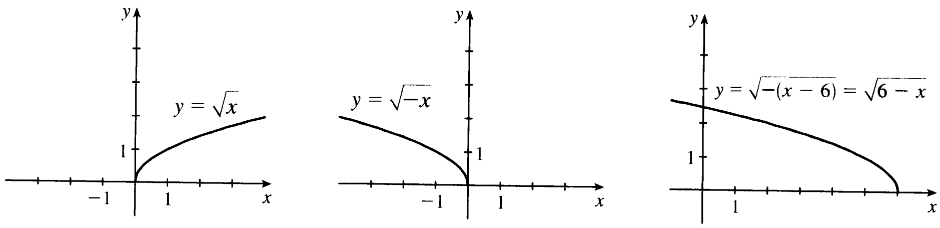
30. $$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{h(x+h)^2x^2}$$
$$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{h} = \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^4}$$
$$= -2x^{-3}$$

Domain of g = domain of $g' = \{x \mid x \neq 0\}$.

31. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$
$$= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0}\left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

32. (a)

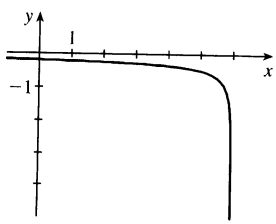


(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$$
$$= \lim_{h\to 0} \frac{[6-(x+h)]-(6-x)}{h\left[\sqrt{6-(x+h)}+\sqrt{6-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{6-x-h}+\sqrt{6-x}\right)}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}$$

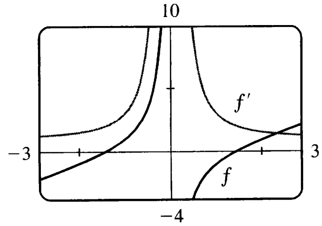
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



33. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[x+h-\left(\dfrac{2}{x+h}\right)\right]-\left[x-\left(\dfrac{2}{x}\right)\right]}{h}$$
$$= \lim_{h\to 0}\left[\frac{h-\dfrac{2}{(x+h)}+\dfrac{2}{x}}{h}\right] = \lim_{h\to 0}\left[1+\frac{-2x+2(x+h)}{hx(x+h)}\right] = \lim_{h\to 0}\left[1+\frac{2h}{hx(x+h)}\right]$$
$$= \lim_{h\to 0}\left[1+\frac{2}{x(x+h)}\right] = 1+\frac{2}{x^2}$$

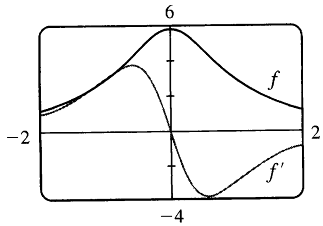
(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.



34. (a) $f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{\frac{6}{1+(t+h)^2} - \frac{6}{1+t^2}}{h} = \lim_{h\to 0} \frac{6+6t^2-6-6(t+h)^2}{h\left[1+(t+h)^2\right](1+t^2)}$

$= \lim_{h\to 0} \frac{-12th-6h^2}{h\left[1+(t+h)^2\right](1+t^2)} = \lim_{h\to 0} \frac{-12t-6h}{\left[1+(t+h)^2\right](1+t^2)} = \frac{-12t}{(1+t^2)^2}$

(b) Notice that f has a horizontal tangent when $t=0$. This corresponds to $f'(0)=0$. f' is positive when f is increasing and negative when f is decreasing.



35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h\to 0} \frac{U(t+h)-U(t)}{h} \approx \frac{U(t+h)-U(t)}{h}$ for small values of h.

For 1991: $U'(1991) = \frac{U(1992)-U(1991)}{1992-1991} = \frac{7.5-6.8}{1} = 0.70$

For 1992: We estimate $U'(1992)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \Rightarrow U'(1992) \approx \frac{U(1991)-U(1992)}{1991-1992} = \frac{6.8-7.5}{-1} = 0.70;$

$h=1 \Rightarrow U'(1992) \approx \frac{U(1993)-U(1992)}{1993-1992} = \frac{6.9-7.5}{1} = -0.60.$

So we estimate that $U'(1992) \approx \frac{1}{2}[0.70+(-0.60)] = 0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	−0.70	−0.65	−0.35	−0.35	−0.45	−0.35	−0.25	−0.20

36. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \frac{P(t+h)-P(t)}{h} \approx \frac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) = \frac{P(1960)-P(1950)}{1960-1950} = \frac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

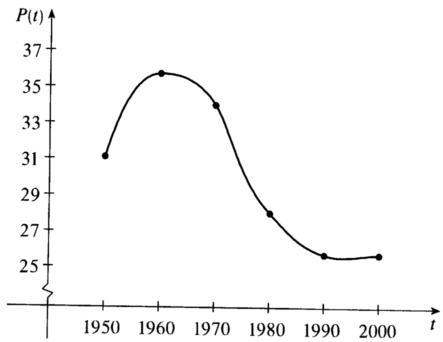
$h=-10 \Rightarrow P'(1960) \approx \frac{P(1950)-P(1960)}{1950-1960} = \frac{31.1-35.7}{-10} = 0.46$

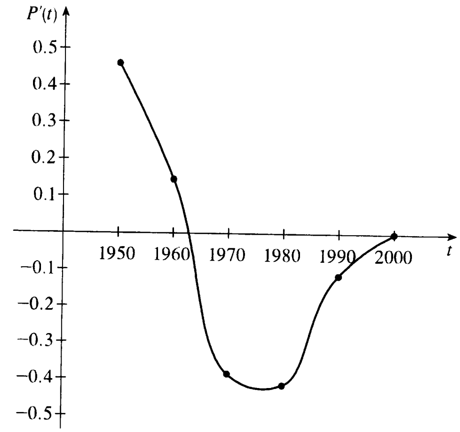
$$h = 10 \quad \Rightarrow \quad P'(1960) \approx \frac{P(1970) - P(1960)}{1970 - 1960} = \frac{34.0 - 35.7}{10} = -0.17$$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	0.000

(c)





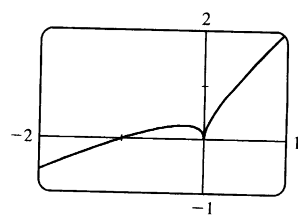
(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, and 1995.

37. f is not differentiable at $x = -1$ or at $x = 11$ because the graph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.

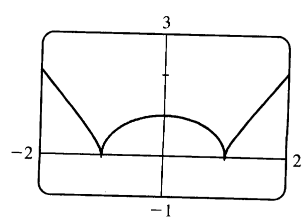
38. (a) g is discontinuous at $x = -2$ (a removable discontinuity), at $x = 0$ (g is not defined there), and at $x = 5$ (a jump discontinuity).

(b) g is not differentiable at the points mentioned in part (a) (by Theorem 4), nor is it differentiable at $x = -1$ (corner), $x = 2$ (vertical tangent), or $x = 4$ (vertical tangent).

39. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



40. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



41. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim_{h \to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim_{x \to 0} |f'(x)| = \lim_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

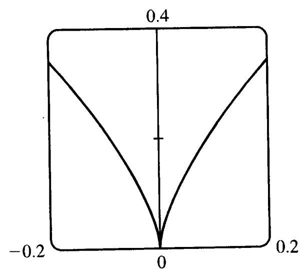
42. (a) $g'(0) = \lim_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \dfrac{x^{2/3} - 0}{x} = \lim_{x \to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

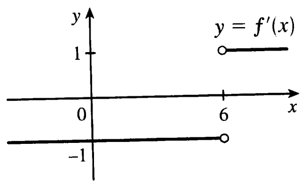
$$= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim_{x \to 0} |g'(x)| = \lim_{x \to 0} \dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)



43. $f(x) = |x - 6| = \begin{cases} -(x - 6) & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases} = \begin{cases} 6 - x & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases}$



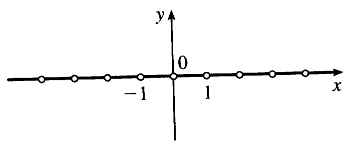
$$\lim_{x \to 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^+} \frac{x - 6}{x - 6} = \lim_{x \to 6^+} 1 = 1.$$

But $$\lim_{x \to 6^-} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^-} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^-} \frac{6 - x}{x - 6} = \lim_{x \to 6^-} (-1) = -1$$

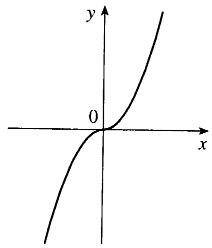
So $f'(6) = \lim_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist. However, $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

Another way of writing the answer is $f'(x) = \dfrac{x - 6}{|x - 6|}$.

44. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



45. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



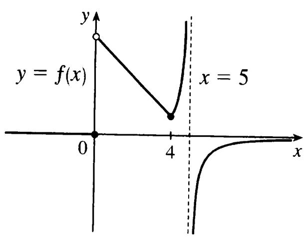
(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 2.9.19(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{x\,|x|}{x} = \lim_{x\to 0} |x| = 0$. So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{rl} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2\,|x|$.

46. (a) $f'_-(4) = \lim_{h\to 0^-} \dfrac{f(4+h) - f(4)}{h} = \lim_{h\to 0^-} \dfrac{5 - (4+h) - 1}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1$ and

$$f'_+(4) = \lim_{h\to 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0^+} \frac{\dfrac{1}{5-(4+h)} - 1}{h} = \lim_{h\to 0^+} \frac{1 - (1-h)}{h(1-h)} = \lim_{h\to 0^+} \frac{1}{1-h} = 1.$$

(b)



(c) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \geq 4 \end{cases}$

These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5 - x) = 5 \neq 0 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

At 4 we have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (5 - x) = 1$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \dfrac{1}{5-x} = 1$, so $\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5.

(d) From (a), f is not differentiable at 4 since $f'_-(4) \neq f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

47. (a) If f is even, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$

$$= -\lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = -\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = -f'(x)$$

Therefore, f' is odd.

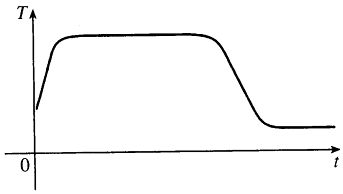
(b) If f is odd, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)] - f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h}$$

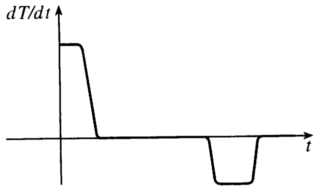
$$= \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

Therefore, f' is even.

48. (a)

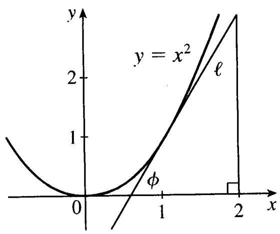


(c)



(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

49.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.

2 Review

CONCEPT CHECK

1. (a) $\lim\limits_{x \to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

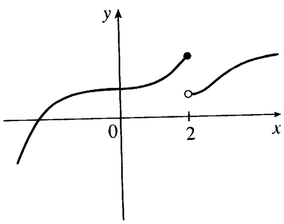
(b) $\lim\limits_{x \to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim\limits_{x \to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

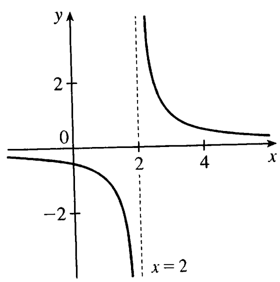
(d) $\lim\limits_{x \to a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.

(e) $\lim\limits_{x \to \infty} f(x) = L$: See Definition 2.6.1 and Figure 2 in Section 2.6.

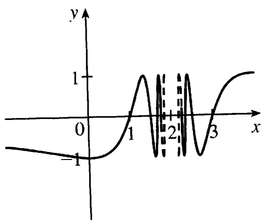
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

3. (a) – (g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

4. See Theorem 3 in Section 2.3.

5. (a) See Definition 2.2.6 and Figures 12–14 in Section 2.2.

(b) See Definition 2.6.3 and Figures 3 and 4 in Section 2.6.

6. (a) $y = x^4$: No asymptote

(b) $y = \sin x$: No asymptote

(c) $y = \tan x$: Vertical asymptotes $x = \frac{\pi}{2} + \pi n$, n an integer

(d) $y = \tan^{-1} x$: Horizontal asymptotes $y = \pm\frac{\pi}{2}$

(e) $y = e^x$: Horizontal asymptote $y = 0$ $\left(\lim_{x \to -\infty} e^x = 0\right)$

(f) $y = \ln x$: Vertical asymptote $x = 0$ $\left(\lim_{x \to 0^+} \ln x = -\infty\right)$

(g) $y = 1/x$: Vertical asymptote $x = 0$, horizontal asymptote $y = 0$

(h) $y = \sqrt{x}$: No asymptote

7. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

8. See Theorem 2.5.10.

9. See Definition 2.7.1.

10. See the paragraph containing Formula 3 in Section 2.7.

11. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

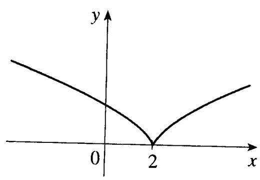
(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

12. See Definition 2.8.2. The pages following the definition discuss interpretations of $f'(a)$ as the slope of a tangent line to the graph of f at $x = a$ and as an instantaneous rate of change of $f(x)$ with respect to x when $x = a$.

13. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.

(b) See Theorem 2.9.4. This theorem also tells us that if f is *not* continuous at a, then f is *not* differentiable at a.

(c)



14. See the discussion and Figure 8 on page 172.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

5. False. Consider $\lim_{x \to 5} \frac{x(x-5)}{x-5}$ or $\lim_{x \to 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim_{x \to 6} [f(x)g(x)] = \lim_{x \to 6} \left[(x-6)\frac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim_{x \to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim_{x \to 0} [f(x) - g(x)] = \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. True. See Figure 4 in Section 2.6.

10. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x \to \infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

11. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

12. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let

$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

13. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

14. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

15. True, by the definition of a limit with $\varepsilon = 1$.

16. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 1) = 1$.

17. False. See the note after Theorem 4 in Section 2.9.

18. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x \to r} f(x) = f(r)$.

EXERCISES

1. (a) (i) $\lim_{x \to 2^+} f(x) = 3$

(ii) $\lim_{x \to -3^+} f(x) = 0$

(iii) $\lim_{x \to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x \to 4} f(x) = 2$

(v) $\lim_{x \to 0} f(x) = \infty$

(vi) $\lim_{x \to 2^-} f(x) = -\infty$

(vii) $\lim_{x \to \infty} f(x) = 4$

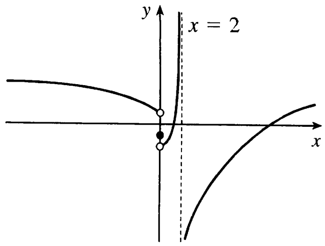
(viii) $\lim_{x \to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3, 0, 2$, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2.



3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

4. Since rational functions are continuous, $\lim_{x\to 3} \frac{x^2-9}{x^2+2x-3} = \frac{3^2-9}{3^2+2(3)-3} = \frac{0}{12} = 0$.

5. $\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$

6. $\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0$ as $x \to 1^+$ and $\frac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{(h^3-3h^2+3h-1)+1}{h} = \lim_{h\to 0} \frac{h^3-3h^2+3h}{h} = \lim_{h\to 0} (h^2-3h+3) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}$$
$$= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3$$

8. $\lim_{t\to 2} \frac{t^2-4}{t^3-8} = \lim_{t\to 2} \frac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \frac{t+2}{t^2+2t+4} = \frac{2+2}{4+4+4} = \frac{4}{12} = \frac{1}{3}$

9. $\lim_{r\to 9} \frac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0$ as $r \to 9$ and $\frac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\lim_{v\to 4^+} \frac{4-v}{|4-v|} = \lim_{v\to 4^+} \frac{4-v}{-(4-v)} = \lim_{v\to 4^+} \frac{1}{-1} = -1$

11. $\lim_{s\to 16} \frac{4-\sqrt{s}}{s-16} = \lim_{s\to 16} \frac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)} = \lim_{s\to 16} \frac{-1}{\sqrt{s}+4} = \frac{-1}{\sqrt{16}+4} = -\frac{1}{8}$

12. $\lim_{v\to 2} \frac{v^2+2v-8}{v^4-16} = \lim_{v\to 2} \frac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)} = \lim_{v\to 2} \frac{v+4}{(v+2)(v^2+4)} = \frac{2+4}{(2+2)(2^2+4)} = \frac{3}{16}$

13. $$\frac{|x-8|}{x-8} = \begin{cases} \dfrac{x-8}{x-8} & \text{if } x-8>0 \\ \dfrac{-(x-8)}{x-8} & \text{if } x-8<0 \end{cases} = \begin{cases} 1 & \text{if } x>8 \\ -1 & \text{if } x<8 \end{cases}$$

Thus, $\lim_{x\to 8^-} \frac{|x-8|}{x-8} = \lim_{x\to 8^-} (-1) = -1$.

14. $\lim_{x\to 9^+} \left(\sqrt{x-9} + [\![x+1]\!]\right) = \lim_{x\to 9^+} \sqrt{x-9} + \lim_{x\to 9^+} [\![x+1]\!] = \sqrt{9-9}+10 = 10$

15. $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x\to 0} \frac{1-(1-x^2)}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{x^2}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = 0$

16. $\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)} \cdot \frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2} \frac{-(x-2)}{x(x-2)\left(\sqrt{x+2}+\sqrt{2x}\right)} = \lim_{x\to 2} \frac{-1}{x\left(\sqrt{x+2}+\sqrt{2x}\right)} = -\frac{1}{8}$

17. $\lim\limits_{x\to\infty} \dfrac{1+2x-x^2}{1-x+2x^2} = \lim\limits_{x\to\infty} \dfrac{(1+2x-x^2)/x^2}{(1-x+2x^2)/x^2} = \lim\limits_{x\to\infty} \dfrac{1/x^2+2/x-1}{1/x^2-1/x+2} = \dfrac{0+0-1}{0-0+2} = -\dfrac{1}{2}$

18. $\lim\limits_{x\to-\infty} \dfrac{5x^3-x^2+2}{2x^3+x-3} = \lim\limits_{x\to-\infty} \dfrac{(5x^3-x^2+2)/x^3}{(2x^3+x-3)/x^3} = \lim\limits_{x\to-\infty} \dfrac{5-1/x+2/x^3}{2+1/x^2-3/x^3} = \dfrac{5-0+0}{2+0-0} = \dfrac{5}{2}$

19. Since x is positive, $\sqrt{x^2} = |x| = x$.

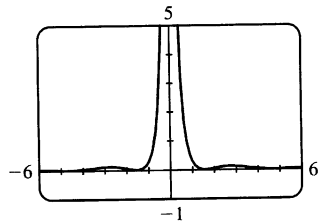
$$\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to\infty} \frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x} = \lim_{x\to\infty} \frac{\sqrt{1-9/x^2}}{2-6/x} = \frac{\sqrt{1-0}}{2-0} = \frac{1}{2}$$

20. $\lim\limits_{x\to10^-} \ln(100-x^2) = -\infty$ since as $x\to10^-$, $(100-x^2)\to0^+$.

21. $\lim\limits_{x\to\infty} e^{-3x} = 0$ since $-3x\to-\infty$ as $x\to\infty$ and $\lim\limits_{t\to-\infty} e^t = 0$.

22. If $y = x^3 - x = x(x^2-1)$, then as $x\to\infty$, $y\to\infty$. $\lim\limits_{x\to\infty} \arctan(x^3-x) = \lim\limits_{y\to\infty} \arctan y = \frac{\pi}{2}$ by (2.6.4).

23. From the graph of $y = (\cos^2 x)/x^2$, it appears that $y=0$ is the horizontal asymptote and $x=0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1$ $\Rightarrow$ $\dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$ $\Rightarrow$ $0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x\to\pm\infty} 0 = 0$ and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem,



$\lim\limits_{x\to\pm\infty} \dfrac{\cos^2 x}{x^2} = 0$. Thus, $y=0$ is the horizontal asymptote. $\lim\limits_{x\to0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2\to0$ as $x\to0$, so $x=0$ is the vertical asymptote.

24. From the graph of $y = f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$f_1(x) = \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$= \frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

Now

$$\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{2+(1/x)}{\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}} \qquad \text{(since } \sqrt{x^2} = x \text{ for } x>0\text{)}$$
$$= \frac{2}{1+1} = 1,$$

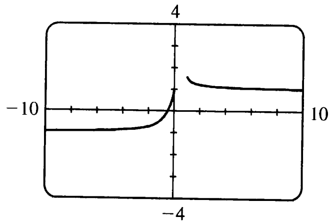
so $y=1$ is a horizontal asymptote. For $x<0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the denominator by x, with $x<0$, we get

$$\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{x} = -\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2}} = -\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}\right]$$

Therefore,

$$\lim_{x\to-\infty} f_1(x) = \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{2+(1/x)}{-\left[\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}\right]} = \frac{2}{-(1+1)} = -1,$$

so $y=-1$ is a horizontal asymptote. As $x \to 0^-$, $f(x) \to 1$, so $x=0$ is *not* a vertical asymptote. As $x \to 1^+$, $f(x) \to \sqrt{3}$, so $x=1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



25. Since $2x-1 \le f(x) \le x^2$ for $0<x<3$ and $\lim_{x\to1}(2x-1) = 1 = \lim_{x\to1} x^2$, we have $\lim_{x\to1} f(x) = 1$ by the Squeeze Theorem.

26. Let $f(x) = -x^2$, $g(x) = x^2\cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have $f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim_{x\to0} f(x) = \lim_{x\to0} h(x) = 0 \Rightarrow \lim_{x\to0} g(x) = 0$ by the Squeeze Theorem.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-5| < \delta$, then $|(7x-27)-8| < \varepsilon \Leftrightarrow |7x-35| < \varepsilon \Leftrightarrow |x-5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0 < |x-5| < \delta \Rightarrow |(7x-27)-8| < \varepsilon$. Thus, $\lim_{x\to5}(7x-27) = 8$ by the definition of a limit.

28. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x-0| < \delta$, then $|\sqrt[3]{x}-0| < \varepsilon$. Now $|\sqrt[3]{x}-0| = |\sqrt[3]{x}| < \varepsilon \Rightarrow |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x-0| = |x| < \varepsilon^3 \Rightarrow |\sqrt[3]{x}-0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x\to0}\sqrt[3]{x} = 0$.

29. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-2| < \delta$, then $\left|x^2-3x-(-2)\right| < \varepsilon$. First, note that if $|x-2| < 1$, then $-1 < x-2 < 1$, so $0 < x-1 < 2 \Rightarrow |x-1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x-2| < \delta \Rightarrow \left|x^2-3x-(-2)\right| = |(x-2)(x-1)| = |x-2|\,|x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim_{x\to2}\left(x^2-3x\right) = -2$ by the definition of a limit.

30. Given $M > 0$, we need $\delta > 0$ such that if $0 < x-4 < \delta$, then $2/\sqrt{x-4} > M$. This is true $\Leftrightarrow \sqrt{x-4} < 2/M \Leftrightarrow x-4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x-4 < \delta \Rightarrow 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim_{x\to4^+}\left(2/\sqrt{x-4}\right) = \infty$.

31. (a) $f(x) = \sqrt{-x}$ if $x<0$, $f(x) = 3-x$ if $0 \le x < 3$, $f(x) = (x-3)^2$ if $x>3$.

(i) $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}(3-x) = 3$

(ii) $\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x\to0} f(x)$ does not exist.

(iv) $\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(3-x) = 0$

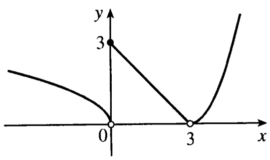
(v) $\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x\to3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x \to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

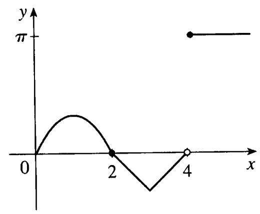
(c)



32. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$. Therefore, $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2x - x^2) = 0$ and $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (2 - x) = 0$. Thus, $\lim_{x \to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} (2 - x) = -1$ and

$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} (x - 4) = -1$. Thus, $\lim_{x \to 3} g(x) = -1 = g(3)$, so g is continuous at 3.
$\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x - 4) = 0$ and $\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} \pi = \pi$. Thus, $\lim_{x \to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x \to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



33. $\sin x$ is continuous on $\mathbb{R}$ by Theorem 7 in Section 2.5. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 9 in Section 2.5. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 4 in Section 2.5.

34. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \ge 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \ne 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.

35. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

36. $f(x) = e^{-x^2} - x$ is continuous on $\mathbb{R}$ so it is continuous on $[0, 1]$. $f(0) = 1 > 0 > 1/e - 1 = f(1)$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. Thus, $e^{-x^2} - x = 0$, or $e^{-x^2} = x$, has a root in $(0, 1)$.

37. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x \to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x \to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{-2(x - 2)(x + 2)}{x - 2}$$
$$= \lim_{x \to 2} [-2(x + 2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

38. For a general point with x-coordinate a, we have

$$m = \lim_{x \to a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x \to a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)}$$
$$= \lim_{x \to a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x \to a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x - 0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x + 1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

39. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}.$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim\limits_{h \to 0} \frac{s(1+h) - s(1)}{h} = \lim\limits_{h \to 0} \frac{10 + h}{4} = \frac{10}{4} = 2.5$ m/s.

40. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \frac{800}{250} - \frac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change is

$\frac{\Delta P}{\Delta V} = \frac{-0.8}{50} = -0.016 \, \frac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

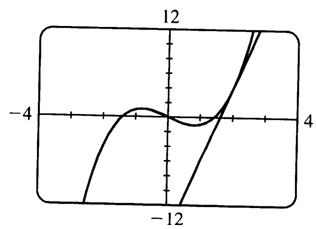
$$\lim_{h \to 0} \frac{\Delta V}{\Delta P} = \lim_{h \to 0} \frac{V(P+h) - V(P)}{h} = \lim_{h \to 0} \frac{800/(P+h) - 800/P}{h}$$
$$= \lim_{h \to 0} \frac{800\,[P - (P+h)]}{h(P+h)P} = \lim_{h \to 0} \frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

which is inversely proportional to the square of P.

41. (a)
$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2}$$
$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 2) = 10$$

(b) $y - 4 = 10(x - 2)$ or $y = 10x - 16$

(c)



42. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

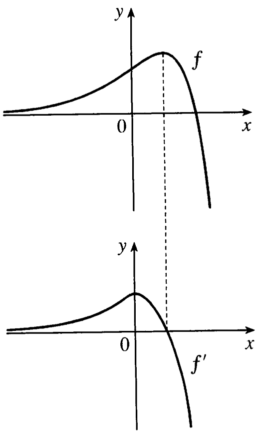
43. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

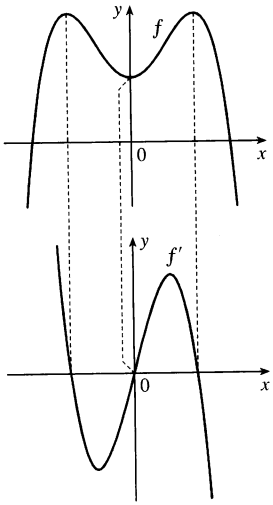
(c) As r increases, C increases. So $f'(r)$ will always be positive.

For Exercises 44–46, see the hints before Exercise 5 in Section 2.9.

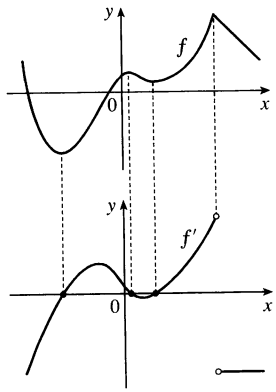
44.



45.



46.



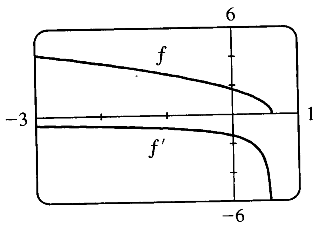
47. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

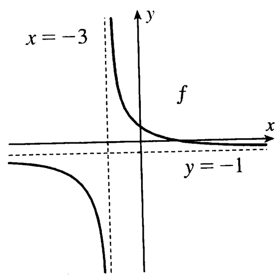
(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \;\Rightarrow\; 5x \leq 3 \;\Rightarrow\; x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in \left(-\infty, \frac{3}{5}\right)$

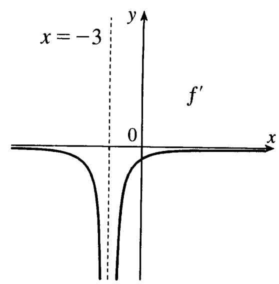
(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



48. (a) As $x \to \pm\infty$, $f(x) = (4-x)/(3+x) \to -1$, so there is a horizontal asymptote at $y=-1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x=-3$.



(b) Note that f is decreasing on $(-\infty, -3)$ and $(-3, \infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h}$$

$$= \lim_{h \to 0} \frac{(3+x)\,[4-(x+h)] - (4-x)\,[3+(x+h)]}{h\,[3+(x+h)]\,(3+x)}$$

$$= \lim_{h \to 0} \frac{(12 - 3x - 3h + 4x - x^2 - hx) - (12 + 4x + 4h - 3x - x^2 - hx)}{h[3+(x+h)](3+x)}$$

$$= \lim_{h \to 0} \frac{-7h}{h\,[3+(x+h)]\,(3+x)} = \lim_{h \to 0} \frac{-7}{[3+(x+h)]\,(3+x)} = -\frac{7}{(3+x)^2}$$

(d) The graphing device confirms our graph in part (b).

49. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

50. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

51. $B'(1990)$ is the rate at which the total value of U.S. banknotes in circulation is changing in billions of dollars per year. To estimate the value of $B'(1990)$, we will average the difference quotients obtained using the times $t = 1985$ and $t = 1995$. Let $A = \dfrac{B(1985) - B(1990)}{1985 - 1990} = \dfrac{182.0 - 268.2}{-5} = 17.24$ and

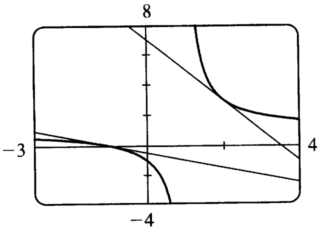
$C = \dfrac{B(1995) - B(1990)}{1995 - 1990} = \dfrac{401.5 - 268.2}{5} = 26.66$. Then

$$B'(1990) = \lim_{t \to 1990} \frac{B(t) - B(1990)}{t - 1990} \approx \frac{A + C}{2} = \frac{17.24 + 26.66}{2} = 21.95 \text{ billions of dollars/year.}$$

52. The slope of the tangent to $y = \dfrac{x+1}{x-1}$ is

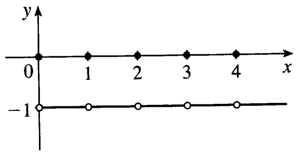
$$\lim_{h\to 0}\frac{\dfrac{(x+h)+1}{(x+h)-1}-\dfrac{x+1}{x-1}}{h} = \lim_{h\to 0}\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)}$$
$$= \lim_{h\to 0}\frac{-2h}{h(x-1)(x+h-1)} = -\frac{2}{(x-1)^2}$$

So at $(2,3)$, $m = -\dfrac{2}{(2-1)^2} = -2 \;\Rightarrow\; y-3 = -2(x-2) \;\Rightarrow$ $y = -2x+7$. At $(-1,0)$, $m = -\dfrac{2}{(-1-1)^2} = -\dfrac{1}{2} \;\Rightarrow$ $y = -\frac{1}{2}(x+1) \;\Rightarrow\; y = -\frac{1}{2}x - \frac{1}{2}$.



53. $|f(x)| \le g(x) \;\Leftrightarrow\; -g(x) \le f(x) \le g(x)$ and $\lim\limits_{x\to a} g(x) = 0 = \lim\limits_{x\to a} -g(x)$. Thus, by the Squeeze Theorem, $\lim\limits_{x\to a} f(x) = 0$.

54. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer,
$[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim\limits_{x\to a} f(x)$ exists (and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.

□ PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\lim\limits_{x\to 0}\dfrac{\sqrt{ax+b}-2}{x}\cdot\dfrac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim\limits_{x\to 0}\dfrac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes

$$\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \frac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4.$$ Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\lim\limits_{x\to 0}\dfrac{|2x-1|-|2x+1|}{x} = \lim\limits_{x\to 0}\dfrac{-(2x-1)-(2x+1)}{x} = \lim\limits_{x\to 0}\dfrac{-4x}{x} = \lim\limits_{x\to 0}(-4) = -4.$

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$. Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But

$m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \frac{1}{2}x^2 + \frac{1}{2}$.

As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. Since $\llbracket x \rrbracket \le x < \llbracket x \rrbracket + 1$, we have $\dfrac{\llbracket x \rrbracket}{\llbracket x \rrbracket} \le \dfrac{x}{\llbracket x \rrbracket} < \dfrac{\llbracket x \rrbracket + 1}{\llbracket x \rrbracket} \;\Rightarrow\; 1 \le \dfrac{x}{\llbracket x \rrbracket} < 1 + \dfrac{1}{\llbracket x \rrbracket}$ for $x \ge 1$. As $x \to \infty$, $\llbracket x \rrbracket \to \infty$, so $\dfrac{1}{\llbracket x \rrbracket} \to 0$ and $1 + \dfrac{1}{\llbracket x \rrbracket} \to 1$. Thus, $\lim\limits_{x\to\infty}\dfrac{x}{\llbracket x \rrbracket} = 1$ by the Squeeze Theorem.

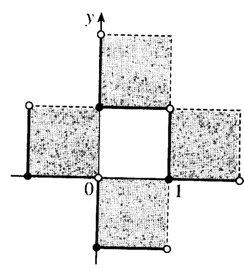
6. (a) $\llbracket x \rrbracket^2 + \llbracket y \rrbracket^2 = 1$. Since $\llbracket x \rrbracket^2$ and $\llbracket y \rrbracket^2$ are positive integers or 0, there are only 4 cases:

Case (i): $\llbracket x \rrbracket = 1, \llbracket y \rrbracket = 0 \;\Rightarrow\; 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $\llbracket x \rrbracket = -1, \llbracket y \rrbracket = 0 \;\Rightarrow\; -1 \le x < 0$ and $0 \le y < 1$

Case (iii): $\llbracket x \rrbracket = 0, \llbracket y \rrbracket = 1 \;\Rightarrow\; 0 \le x < 1$ and $1 \le y < 2$

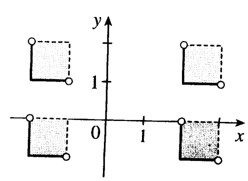
Case (iv): $\llbracket x \rrbracket = 0, \llbracket y \rrbracket = -1 \;\Rightarrow\; 0 \le x < 1$ and $-1 \le y < 0$



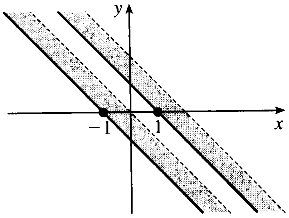
(b) $\llbracket x \rrbracket^2 - \llbracket y \rrbracket^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

$\{(x,y) \mid \llbracket x \rrbracket = \pm 2, \llbracket y \rrbracket = \pm 1\} =$

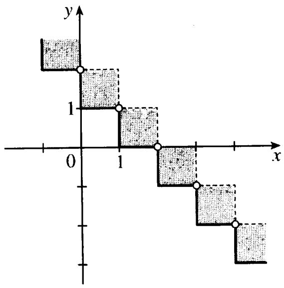
$$\left\{(x,y) \;\middle|\; \begin{array}{l} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{array}\right\}.$$



(c) $[\![x+y]\!]^2 = 1 \;\Rightarrow\; [\![x+y]\!] = \pm 1$
$\Rightarrow\; 1 \le x+y < 2$ or
$-1 \le x+y < 0$

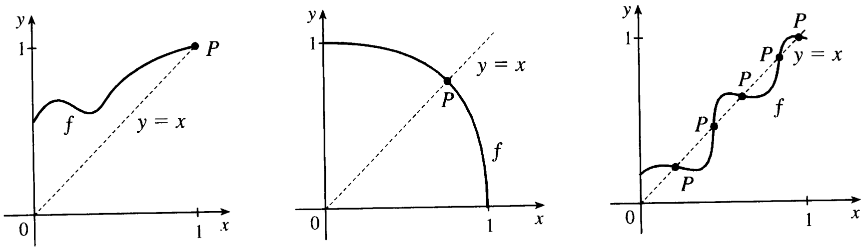


(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \;\Rightarrow$
$[\![y]\!] = 1-n \;\Rightarrow\; 1-n \le y < 2-n$. Choosing integer values for n produces the graph.



7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,
$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a^+} x^2 = \lim_{x\to a^-}(x+1) \;\Rightarrow\; a^2 = a+1 \;\Rightarrow\; a^2 - a - 1 = 0 \;\Rightarrow$
[by the quadratic formula] $a = \left(1 \pm \sqrt{5}\,\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point.
Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. $\lim_{x\to a} f(x) = \lim_{x\to a}\left(\frac{1}{2}[f(x)+g(x)] + \frac{1}{2}[f(x)-g(x)]\right)$
$= \frac{1}{2}\lim_{x\to a}[f(x)+g(x)] + \frac{1}{2}\lim_{x\to a}[f(x)-g(x)]$
$= \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 1 = \frac{3}{2}$, and
$\lim_{x\to a} g(x) = \lim_{x\to a}\Big([f(x)+g(x)] - f(x)\Big) = \lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}$.
So $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

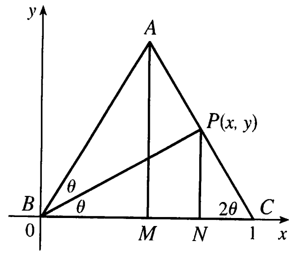
Another solution: Since $\lim_{x\to a}[f(x)+g(x)]$ and $\lim_{x\to a}[f(x)-g(x)]$ exist, we must have

$\lim_{x\to a}[f(x)+g(x)]^2 = \left(\lim_{x\to a}[f(x)+g(x)]\right)^2$ and $\lim_{x\to a}[f(x)-g(x)]^2 = \left(\lim_{x\to a}[f(x)-g(x)]\right)^2$, so

$$\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2 - [f(x)-g(x)]^2\right) \quad \text{[because all of the } f^2 \text{ and } g^2 \text{ cancel]}$$

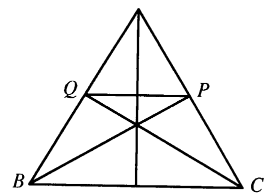
$$= \tfrac{1}{4}\left(\lim_{x\to a}[f(x)+g(x)]^2 - \lim_{x\to a}[f(x)-g(x)]^2\right) = \tfrac{1}{4}\left(2^2-1^2\right) = \tfrac{3}{4}.$$

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$.

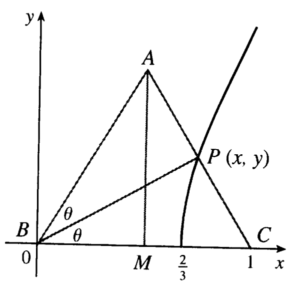


After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \Leftrightarrow$ $y^2 = x(3x-2)$. As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2)=0$. Obviously it is not $x=0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x-2=0$, that is, $x=\frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. (a) Consider $G(x) = T(x+180^\circ) - T(x)$. Fix any number a. If $G(a)=0$, we are done: Temperature at a = Temperature at $a+180^\circ$. If $G(a)>0$, then $G(a+180^\circ) = T(a+360^\circ) - T(a+180^\circ)$ $= T(a) - T(a+180^\circ) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a+180^\circ]$. If $G(a)<0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

12. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)f(x+h) - xf(x)}{h}$

$= \lim_{h\to 0} \left[\dfrac{xf(x+h) - xf(x)}{h} + \dfrac{hf(x+h)}{h} \right] = x \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x+h)$

$= xf'(x) + f(x)$ because f is differentiable and therefore continuous.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad \Rightarrow \quad f(0) = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h}$

$= \lim_{h\to 0} \dfrac{f(h)}{h} = \lim_{x\to 0} \dfrac{f(x)}{x} = 1$

(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{f(h) + x^2 h + xh^2}{h} = \lim_{h\to 0} \left[\dfrac{f(h)}{h} + x^2 + xh \right] = 1 + x^2$

14. We are given that $|f(x)| \le x^2$ for all x. In particular, $|f(0)| \le 0$, but $|a| \ge 0$ for all a. The only conclusion is that $f(0) = 0$. Now $\left| \dfrac{f(x) - f(0)}{x - 0} \right| = \left| \dfrac{f(x)}{x} \right| = \dfrac{|f(x)|}{|x|} \le \dfrac{x^2}{|x|} = \dfrac{|x^2|}{|x|} = |x| \quad \Rightarrow \quad -|x| \le \dfrac{f(x) - f(0)}{x - 0} \le |x|$. But $\lim_{x\to 0} (-|x|) = 0 = \lim_{x\to 0} |x|$, so by the Squeeze Theorem, $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$. So by the definition of a derivative, f is differentiable at 0 and, furthermore, $f'(0) = 0$.