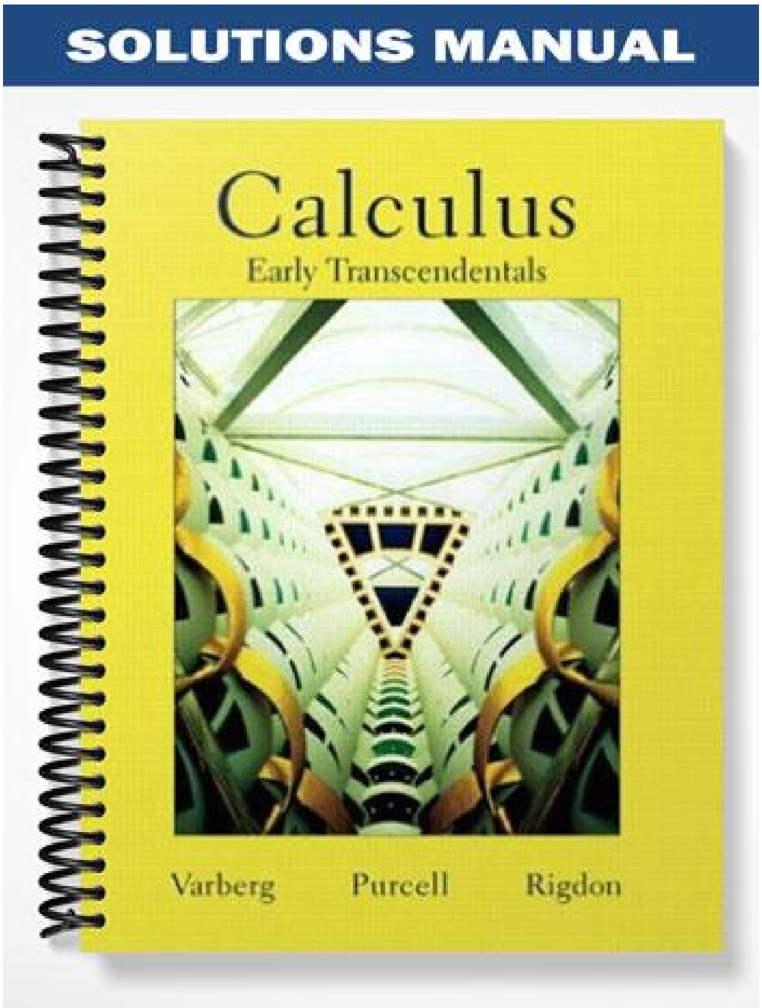

SOLUTIONS MANUAL
Calculus
Early Transcendentals
Varberg
Purcell
Rigdon

CHAPTER 2 Limits

2.1 Concepts Review

1. L; c

2. 6

3. L; right

4. $\lim_{x\to c} f(x) = M$

Problem Set 2.1

1. $\lim_{x\to 3}(x-5) = -2$

2. $\lim_{t\to -1}(1-2t) = 3$

3. $\lim_{x\to -2}(x^2+2x-1) = (-2)^2 + 2(-2) - 1 = -1$

4. $\lim_{x\to -2}(x^2+2t-1) = (-2)^2 + 2t - 1 = 3 + 2t$

5. $\lim_{t\to -1}\left(t^2-1\right) = \left((-1)^2-1\right) = 0$

6. $\lim_{t\to -1}\left(t^2-x^2\right) = \left((-1)^2-x^2\right) = 1-x^2$

7. $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$
$= \lim_{x\to 2}(x+2)$
$= 2 + 2 = 4$

8. $\lim_{t\to -7}\frac{t^2+4t-21}{t+7} = \lim_{t\to -7}\frac{(t+7)(t-3)}{t+7}$
$= \lim_{t\to -7}(t-3)$
$= -7-3$
$= -10$

9. $\lim_{x\to -1}\frac{x^3-4x^2+x+6}{x+1}$
$= \lim_{x\to -1}\frac{(x+1)(x^2-5x+6)}{x+1}$
$= \lim_{x\to -1}(x^2-5x+6)$
$= (-1)^2 - 5(-1) + 6$
$= 12$

10. $\lim_{x\to 0}\frac{x^4+2x^3-x^2}{x^2} = \lim_{x\to 0}(x^2+2x-1) = -1$

11. $\lim_{x\to -t}\frac{x^2-t^2}{x+t} = \lim_{x\to -t}\frac{(x+t)(x-t)}{x+t}$
$= \lim_{x\to -t}(x-t)$
$= -t-t$
$= -2t$

12. $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$
$= \lim_{x\to 3}(x+3)$
$= 3 + 3$
$= 6$

13. $\lim_{t\to 2}\frac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2} = \lim_{t\to 2}\frac{(t-2)^2\sqrt{t+4}}{9(t-2)^2}$
$= \lim_{t\to 2}\frac{\sqrt{t+4}}{9}$
$= \frac{\sqrt{2+4}}{9}$
$= \frac{\sqrt{6}}{9}$

14. $\lim_{t\to 7^+}\frac{\sqrt{(t-7)^3}}{t-7} = \lim_{t\to 7^+}\frac{(t-7)\sqrt{t-7}}{t-7}$
$= \lim_{t\to 7^+}\sqrt{t-7}$
$= \sqrt{7-7}$
$= 0$

15. $\lim_{x\to 3}\frac{x^4-18x^2+81}{(x-3)^2}=\lim_{x\to 3}\frac{(x^2-9)^2}{(x-3)^2}$

$=\lim_{x\to 3}\frac{(x-3)^2(x+3)^2}{(x-3)^2}=\lim_{x\to 3}(x+3)^2=(3+3)^2$

$=36$

16. $\lim_{u\to 1}\frac{(3u+4)(2u-2)^3}{(u-1)^2}=\lim_{u\to 1}\frac{8(3u+4)(u-1)^3}{(u-1)^2}$

$=\lim_{u\to 1}8(3u+4)(u-1)=8[3(1)+4](1-1)=0$

17. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$

$=\lim_{h\to 0}\frac{h^2+4h}{h}=\lim_{h\to 0}(h+4)=4$

18. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}\frac{h^2+2xh}{h}=\lim_{h\to 0}(h+2x)=2x$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.49999992
–1.	0.420735
–0.1	0.499167
–0.01	0.499992
–0.001	0.49999992

$\lim_{x\to 0}\frac{\sin x}{2x}=0.5$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.00024999998
–1.	–0.229849
–0.1	–0.0249792
–0.01	–0.00249998
–0.001	–0.00024999998

$\lim_{t\to 0}\frac{1-\cos t}{2t}=0$

21.

x	$(x-\sin x)^2/x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.77775×10^{-10}
0.001	2.77778×10^{-14}
–1.	0.0251314
–0.1	2.775×10^{-6}
–0.01	2.77775×10^{-10}
–0.001	2.77778×10^{-14}

$\lim_{x\to 0}\frac{(x-\sin x)^2}{x^2}=0$

22.

x	$(1-\cos x)^2/x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
–1.	0.211322
–0.1	0.00249584
–0.01	0.0000249996
–0.001	2.5×10^{-7}

$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2}=0$

23.

t	$(t^2-1)/(\sin(t-1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$\lim_{t\to 1}\frac{t^2-1}{\sin(t-1)}=2$

24.

x	$\frac{x-\sin(x-3)-3}{x-3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x\to3}\frac{x-\sin(x-3)-3}{x-3}=0$$

25.

x	$(1+\sin(x-3\pi/2))/(x-\pi)$
$1. + \pi$	0.4597
$0.1 + \pi$	0.0500
$0.01 + \pi$	0.0050
$0.001 + \pi$	0.0005
$-1. + \pi$	–0.4597
$-0.1 + \pi$	–0.0500
$-0.01 + \pi$	–0.0050
$-0.001 + \pi$	–0.0005

$$\lim_{x\to\pi}\frac{1+\sin\left(x-\frac{3\pi}{2}\right)}{x-\pi}=0$$

26.

t	$(1-\cot t)/(1/t)$
1.	0.357907
0.1	–0.896664
0.01	–0.989967
0.001	–0.999
–1.	–1.64209
–0.1	–1.09666
–0.01	–1.00997
–0.001	–1.001

$$\lim_{t\to0}\frac{1-\cot t}{\frac{1}{t}}=-1$$

27.

x	$(x-\pi/4)^2/(\tan x-1)^2$
$1.+\frac{\pi}{4}$	0.0320244
$0.1+\frac{\pi}{4}$	0.201002
$0.01+\frac{\pi}{4}$	0.245009
$0.001+\frac{\pi}{4}$	0.2495
$-1.+\frac{\pi}{4}$	0.674117
$-0.1+\frac{\pi}{4}$	0.300668
$-0.01+\frac{\pi}{4}$	0.255008
$-0.001+\frac{\pi}{4}$	0.2505

$$\lim_{x\to\frac{\pi}{4}}\frac{\left(x-\frac{\pi}{4}\right)^2}{(\tan x-1)^2}=0.25$$

28.

u	$(2-2\sin u)/3u$
$1.+\frac{\pi}{2}$	0.11921
$0.1+\frac{\pi}{2}$	0.00199339
$0.01+\frac{\pi}{2}$	0.0000210862
$0.001+\frac{\pi}{2}$	2.12072×10^{-7}
$-1.+\frac{\pi}{2}$	0.536908
$-0.1+\frac{\pi}{2}$	0.00226446
$-0.01+\frac{\pi}{2}$	0.0000213564
$-0.001+\frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u\to\frac{\pi}{2}}\frac{2-2\sin u}{3u}=0$$

29. **a.** $\lim_{x\to-3} f(x)=2$

b. $f(-3) = 1$

c. $f(-1)$ does not exist.

d. $\lim_{x\to-1} f(x)=\frac{5}{2}$

e. $f(1) = 2$

f. $\lim_{x\to1} f(x)$ does not exist.

g. $\lim_{x\to1^-} f(x)=2$

h. $\lim_{x\to1^+} f(x)=1$

i. $\lim_{x\to-1^+} f(x)=\frac{5}{2}$

30. **a.** $\lim_{x \to -3} f(x)$ does not exist.

b. $f(-3) = 1$

c. $f(-1) = 1$

d. $\lim_{x \to -1} f(x) = 2$

e. $f(1) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

g. $\lim_{x \to 1^-} f(x) = 1$

h. $\lim_{x \to 1^+} f(x)$ does not exist.

i. $\lim_{x \to -1^+} f(x) = 2$

31. **a.** $f(-3) = 2$

b. $f(3)$ is undefined.

c. $\lim_{x \to -3^-} f(x) = 2$

d. $\lim_{x \to -3^+} f(x) = 4$

e. $\lim_{x \to -3} f(x)$ does not exist.

f. $\lim_{x \to 3^+} f(x)$ does not exist.

32. **a.** $\lim_{x \to -1^-} f(x) = -2$

b. $\lim_{x \to -1^+} f(x) = -2$

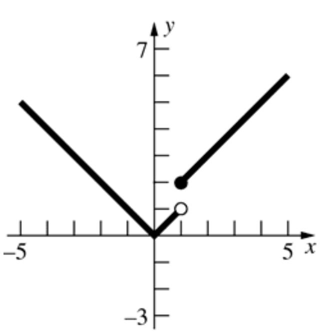
c. $\lim_{x \to -1} f(x) = -2$

d. $f(-1) = -2$

e. $\lim_{x \to 1} f(x) = 0$

f. $f(1) = 0$

33.



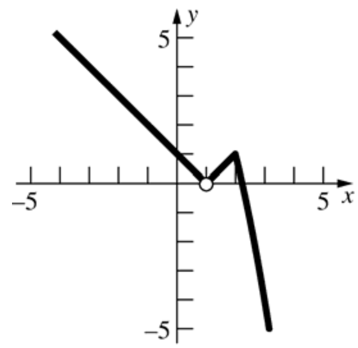
a. $\lim_{x \to 0} f(x) = 0$

b. $\lim_{x \to 1} f(x)$ does not exist.

c. $f(1) = 2$

d. $\lim_{x \to 1^+} f(x) = 2$

34.



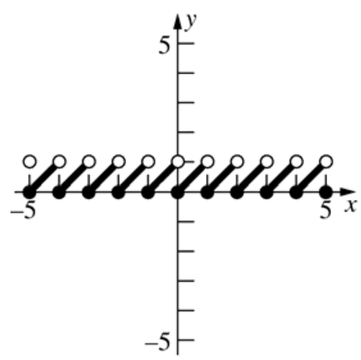
a. $\lim_{x \to 1} g(x) = 0$

b. $g(1)$ does not exist.

c. $\lim_{x \to 2} g(x) = 1$

d. $\lim_{x \to 2^+} g(x) = 1$

35. $f(x) = x - [\![x]\!]$



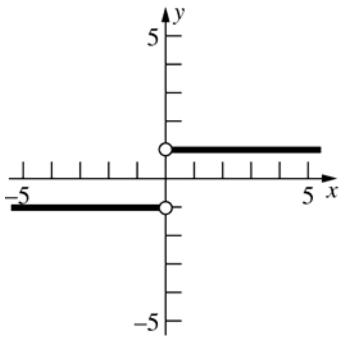
a. $f(0) = 0$

b. $\lim_{x \to 0} f(x)$ does not exist.

c. $\lim_{x \to 0^-} f(x) = 1$

d. $\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$

36. $f(x)=\dfrac{x}{|x|}$



a. $f(0)$ does not exist.

b. $\lim_{x\to 0} f(x)$ does not exist.

c. $\lim_{x\to 0^-} f(x)=-1$

d. $\lim_{x\to \frac{1}{2}} f(x)=1$

37. $\lim_{x\to 1}\dfrac{x^2-1}{|x-1|}$ does not exist.

$\lim_{x\to 1^-}\dfrac{x^2-1}{|x-1|}=-2$ and $\lim_{x\to 1^+}\dfrac{x^2-1}{|x-1|}=2$

38. $\lim_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

$=\lim_{x\to 0}\dfrac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})}$

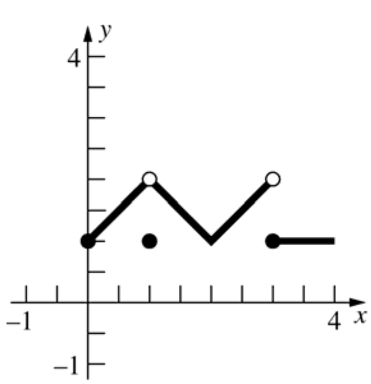
$=\lim_{x\to 0}\dfrac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}=\lim_{x\to 0}\dfrac{x}{x(\sqrt{x+2}+\sqrt{2})}$

$=\lim_{x\to 0}\dfrac{1}{\sqrt{x+2}+\sqrt{2}}=\dfrac{1}{\sqrt{0+2}+\sqrt{2}}=\dfrac{1}{2\sqrt{2}}=\dfrac{\sqrt{2}}{4}$

39. **a.** $\lim_{x\to 1} f(x)$ does not exist.

b. $\lim_{x\to 0} f(x)=0$

40.



41. $\lim_{x\to a} f(x)$ exists for $a=-1, 0, 1$.

42. The changed values will not change $\lim_{x\to a} f(x)$ at any a. As x approaches a, the limit is still a^2.

43. **a.** $\lim_{x\to 1}\dfrac{|x-1|}{x-1}$ does not exist.

$\lim_{x\to 1^-}\dfrac{|x-1|}{x-1}=-1$ and $\lim_{x\to 1^+}\dfrac{|x-1|}{x-1}=1$

b. $\lim_{x\to 1^-}\dfrac{|x-1|}{x-1}=-1$

c. $\lim_{x\to 1^-}\dfrac{x^2-|x-1|-1}{|x-1|}=-3$

d. $\lim_{x\to 1^-}\left[\dfrac{1}{x-1}-\dfrac{1}{|x-1|}\right]$ does not exist.

44. **a.** $\lim_{x\to 1^+}\sqrt{x-[\![x]\!]}=0$

b. $\lim_{x\to 0^+}\left[\!\left[\dfrac{1}{x}\right]\!\right]$ does not exist.

c. $\lim_{x\to 0^+} x(-1)^{[\![1/x]\!]}=0$

d. $\lim_{x\to 0^+} [\![x]\!](-1)^{[\![1/x]\!]}=0$

45. **a)** 1 **b)** 0

c) –1 **d)** –1

46. **a)** Does not exist **b)** 0

c) 1 **d)** 0.556

47. $\lim_{x\to 0}\sqrt{x}$ does not exist since $\sqrt{x}$ is not defined for $x<0$.

48. $\lim_{x\to 0^+} x^x=1$

49. $\lim_{x\to 0}\sqrt{|x|}=0$

50. $\lim_{x\to 0}|x|^x=1$

51. $\lim_{x\to 0}\dfrac{\sin 2x}{4x}=\dfrac{1}{2}$

52. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\frac{5}{3}$

53. $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

54. $\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)=0$

55. $\lim_{x\to 1}\frac{x^3-1}{\sqrt{2x+2}-2}=6$

56. $\lim_{x\to 0}\frac{x\sin 2x}{\sin(x^2)}=2$

57. $\lim_{x\to 2^-}\frac{x^2-x-2}{|x-2|}=-3$

58. $\lim_{x\to 1^+}\frac{2}{1+2^{1/(x-1)}}=0$

59. $\lim_{x\to 0}\sqrt{x}$; The computer gives a value of 0, but $\lim_{x\to 0^-}\sqrt{x}$ does not exist.

2.2 Concepts Review

1. $L-\varepsilon$; $L+\varepsilon$

2. $0<|x-a|<\delta$; $|f(x)-L|<\varepsilon$

3. $\frac{\varepsilon}{3}$

4. $ma+b$

Problem Set 2.2

1. $0<|t-a|<\delta\Rightarrow|f(t)-M|<\varepsilon$

2. $0<|u-b|<\delta\Rightarrow|g(u)-L|<\varepsilon$

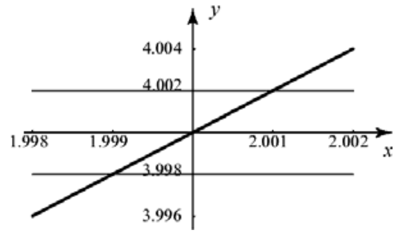
3. $0<|z-d|<\delta\Rightarrow|h(z)-P|<\varepsilon$

4. $0<|y-e|<\delta\Rightarrow|\phi(y)-B|<\varepsilon$

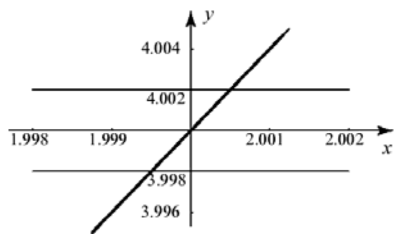
5. $0<c-x<\delta\Rightarrow|f(x)-L|<\varepsilon$

6. $0<t-a<\delta\Rightarrow|g(t)-D|<\varepsilon$

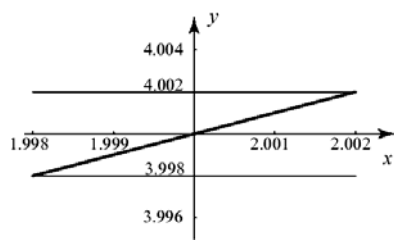
7. If x is within 0.001 of 2, then $2x$ *is* within 0.002 of 4.



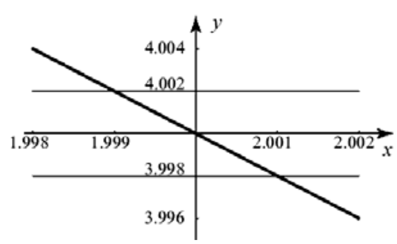
8. If x is within 0.0005 of 2, then x^2 is within 0.002 of 4.



9. If x is within 0.0019 of 2, then $\sqrt{8x}$ is within 0.002 of 4.



10. If x is within 0.001 of 2, then $\frac{8}{x}$ is within 0.002 of 4.



11. $0<|x-0|<\delta\Rightarrow|(2x-1)-(-1)|<\varepsilon$

$|2x-1+1|<\varepsilon\Leftrightarrow|2x|<\varepsilon$

$\Leftrightarrow 2|x|<\varepsilon$

$\Leftrightarrow |x|<\frac{\varepsilon}{2}$

$\delta=\frac{\varepsilon}{2}$; $0<|x-0|<\delta$

$|(2x-1)-(-1)|=|2x|=2|x|<2\delta=\varepsilon$

12. $0 < |x+21| < \delta \Rightarrow |(3x-1)-(-64)| < \varepsilon$

$|3x-1+64| < \varepsilon \Leftrightarrow |3x+63| < \varepsilon$

$\Leftrightarrow |3(x+21)| < \varepsilon$

$\Leftrightarrow 3|x+21| < \varepsilon$

$\Leftrightarrow |x+21| < \frac{\varepsilon}{3}$

$\delta = \frac{\varepsilon}{3};\ 0 < |x+21| < \delta$

$|(3x-1)-(-64)| = |3x+63| = 3|x+21| < 3\delta = \varepsilon$

13. $0 < |x-5| < \delta \Rightarrow \left|\frac{x^2-25}{x-5}-10\right| < \varepsilon$

$\left|\frac{x^2-25}{x-5}-10\right| < \varepsilon \Leftrightarrow \left|\frac{(x-5)(x+5)}{x-5}-10\right| < \varepsilon$

$\Leftrightarrow |x+5-10| < \varepsilon$

$\Leftrightarrow |x-5| < \varepsilon$

$\delta = \varepsilon;\ 0 < |x-5| < \delta$

$\left|\frac{x^2-25}{x-5}-10\right| = \left|\frac{(x-5)(x+5)}{x-5}-10\right| = |x+5-10|$

$= |x-5| < \delta = \varepsilon$

14. $0 < |x-0| < \delta \Rightarrow \left|\frac{2x^2-x}{x}-(-1)\right| < \varepsilon$

$\left|\frac{2x^2-x}{x}+1\right| < \varepsilon \Leftrightarrow \left|\frac{x(2x-1)}{x}+1\right| < \varepsilon$

$\Leftrightarrow |2x-1+1| < \varepsilon$

$\Leftrightarrow |2x| < \varepsilon$

$\Leftrightarrow 2|x| < \varepsilon$

$\Leftrightarrow |x| < \frac{\varepsilon}{2}$

$\delta = \frac{\varepsilon}{2};\ 0 < |x-0| < \delta$

$\left|\frac{2x^2-x}{x}-(-1)\right| = \left|\frac{x(2x-1)}{x}+1\right| = |2x-1+1|$

$= |2x| = 2|x| < 2\delta = \varepsilon$

15. $0 < |x-5| < \delta \Rightarrow \left|\frac{2x^2-11x+5}{x-5}-9\right| < \varepsilon$

$\left|\frac{2x^2-11x+5}{x-5}-9\right| < \varepsilon \Leftrightarrow \left|\frac{(2x-1)(x-5)}{x-5}-9\right| < \varepsilon$

$\Leftrightarrow |2x-1-9| < \varepsilon$

$\Leftrightarrow |2(x-5)| < \varepsilon$

$\Leftrightarrow |x-5| < \frac{\varepsilon}{2}$

$\delta = \frac{\varepsilon}{2};\ 0 < |x-5| < \delta$

$\left|\frac{2x^2-11x+5}{x-5}-9\right| = \left|\frac{(2x-1)(x-5)}{x-5}-9\right|$

$= |2x-1-9| = |2(x-5)| = 2|x-5| < 2\delta = \varepsilon$

16. $0 < |x-1| < \delta \Rightarrow \left|\sqrt{2x}-\sqrt{2}\right| < \varepsilon$

$\left|\sqrt{2x}-\sqrt{2}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{2x-2}{\sqrt{2x}+\sqrt{2}}\right| < \varepsilon$

$\Leftrightarrow 2\left|\frac{x-1}{\sqrt{2x}+\sqrt{2}}\right| < \varepsilon$

$\delta = \frac{\sqrt{2}\varepsilon}{2};\ 0 < |x-1| < \delta$

$\left|\sqrt{2x}-\sqrt{2}\right| = \left|\frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|$

$= \left|\frac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|$

$\frac{2|x-1|}{\sqrt{2x}+\sqrt{2}} \le \frac{2|x-1|}{\sqrt{2}} < \frac{2\delta}{\sqrt{2}} = \varepsilon$

17. $0 < |x-4| < \delta \Rightarrow \left|\frac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right| < \varepsilon$

$\left|\frac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right| < \varepsilon \Leftrightarrow \left|\frac{\sqrt{2x-1}-\sqrt{7(x-3)}}{\sqrt{x-3}}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{(\sqrt{2x-1}-\sqrt{7(x-3)})(\sqrt{2x-1}+\sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{2x-1-(7x-21)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right| < \varepsilon$

$\Leftrightarrow \left|\frac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right| < \varepsilon$

$$\Leftrightarrow |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$$

To bound $\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree that $\delta \le \frac{1}{2}$. If $\delta \le \frac{1}{2}$, then $\frac{7}{2} < x < \frac{9}{2}$, so $0.65 < \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < 1.65$ and hence $|x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$

$$\Leftrightarrow |x-4| < \frac{\varepsilon}{1.65}$$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{1.65}\right\}$, $0 < |x-4| < \delta$

$$\left|\frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7}\right| = |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$$

$< |x-4|(1.65) < 1.65\delta \le \varepsilon$

since $\delta = \frac{1}{2}$ only when $\frac{1}{2} \le \frac{\varepsilon}{1.65}$ so $1.65\delta \le \varepsilon$.

18. $0 < |x-1| < \delta \Rightarrow \left|\frac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon$

$$\left|\frac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon \Leftrightarrow \left|\frac{2(7x-3)(x-1)}{x-1} - 8\right| < \varepsilon$$

$\Leftrightarrow |2(7x-3)-8| < \varepsilon$

$\Leftrightarrow |14(x-1)| < \varepsilon$

$\Leftrightarrow 14|x-1| < \varepsilon$

$\Leftrightarrow |x-1| < \frac{\varepsilon}{14}$

$\delta = \frac{\varepsilon}{14}; 0 < |x-1| < \delta$

$$\left|\frac{14x^2-20x+6}{x-1} - 8\right| = \left|\frac{2(7x-3)(x-1)}{x-1} - 8\right|$$

$= |2(7x-3)-8|$

$= |14(x-1)| = 14|x-1| < 14\delta = \varepsilon$

19. $0 < |x-1| < \delta \Rightarrow \left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$

$$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right| < \varepsilon$$

$\Leftrightarrow |10x-6-4| < \varepsilon$

$\Leftrightarrow |10(x-1)| < \varepsilon$

$\Leftrightarrow 10|x-1| < \varepsilon$

$\Leftrightarrow |x-1| < \frac{\varepsilon}{10}$

$\delta = \frac{\varepsilon}{10}; 0 < |x-1| < \delta$

$$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| = \left|\frac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right|$$

$= |10x-6-4| = |10(x-1)|$

$= 10|x-1| < 10\delta = \varepsilon$

20. $0 < |x-1| < \delta \Rightarrow |(2x^2+1)-3| < \varepsilon$

$|2x^2+1-3| = |2x^2-2| = 2|x+1||x-1|$

To bound $|2x+2|$, agree that $\delta \le 1$.

$|x-1| < \delta$ implies

$|2x+2| = |2x-2+4|$

$\le |2x-2| + |4|$

$< 2 + 4 = 6$

$\delta \le \frac{\varepsilon}{6}; \delta = \min\left\{1, \frac{\varepsilon}{6}\right\}; 0 < |x-1| < \delta$

$|(2x^2+1)-3| = |2x^2-2|$

$= |2x+2||x-1| < 6 \cdot \left(\frac{\varepsilon}{6}\right) = \varepsilon$

21. $0<|x+1|<\delta \Rightarrow |(x^2-2x-1)-2|<\varepsilon$

$|x^2-2x-1-2|=|x^2-2x-3|=|x+1||x-3|$

To bound $|x-3|$, agree that $\delta \le 1$.

$|x+1|<\delta$ implies

$|x-3|=|x+1-4| \le |x+1|+|-4|<1+4=5$

$\delta \le \frac{\varepsilon}{5}; \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}; 0<|x+1|<\delta$

$|(x^2-2x-1)-2|=|x^2-2x-3|$

$=|x+1||x-3|<5\cdot\frac{\varepsilon}{5}=\varepsilon$

22. $0<|x|<\delta \Rightarrow |x^4-0|=|x^4|<\varepsilon$

$|x^4|=|x||x^3|$. To bound $|x^3|$, agree that $\delta \le 1$. $|x|<\delta \le 1$ implies $|x^3|=|x|^3 \le 1$ so $\delta \le \varepsilon$.

$\delta = \min\{1, \varepsilon\}; 0<|x|<\delta \Rightarrow |x^4|=|x||x^3|<\varepsilon\cdot 1$

$=\varepsilon$

23. Choose $\varepsilon>0$. Then since $\lim_{x\to c} f(x)=L$, there is some $\delta_1>0$ such that

$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$.

Since $\lim_{x\to c} f(x)=M$, there is some $\delta_2>0$ such that $0<|x-c|<\delta_2 \Rightarrow |f(x)-M|<\varepsilon$.

Let $\delta=\min\{\delta_1,\delta_2\}$ and choose x_0 such that $0<|x_0-c|<\delta$.

Thus, $|f(x_0)-L|<\varepsilon \Rightarrow -\varepsilon<f(x_0)-L<\varepsilon$

$\Rightarrow -f(x_0)-\varepsilon<-L<-f(x_0)+\varepsilon$

$\Rightarrow f(x_0)-\varepsilon<L<f(x_0)+\varepsilon$.

Similarly,

$f(x_0)-\varepsilon<M<f(x_0)+\varepsilon$.

Thus,

$-2\varepsilon<L-M<2\varepsilon$. As $\varepsilon \Rightarrow 0$, $L-M\to 0$, so $L=M$.

24. Since $\lim_{x\to c} G(x)=0$, then given any $\varepsilon>0$, we can find $\delta>0$ such that whenever $|x-c|<\delta, |G(x)|<\varepsilon$.

Take any $\varepsilon>0$ and the corresponding δ that works for $G(x)$, then $|x-c|<\delta$ implies $|F(x)-0|=|F(x)|\le|G(x)|<\varepsilon$ since $\lim_{x\to c} G(x)=0$.

Thus, $\lim_{x\to c} F(x)=0$.

25. For all $x\ne 0$, $0\le \sin^2\left(\frac{1}{x}\right)\le 1$ so

$x^4\sin^2\left(\frac{1}{x}\right)\le x^4$ for all $x\ne 0$.

$\lim_{x\to 0} x^4=0$, so,

$\lim_{x\to 0} x^4\sin^2\left(\frac{1}{x}\right)=0$.

26. $0<x<\delta \Rightarrow |\sqrt{x}-0|=|\sqrt{x}|=\sqrt{x}<\varepsilon$

For $x>0$, $(\sqrt{x})^2=x$.

$\sqrt{x}<\varepsilon \Leftrightarrow (\sqrt{x})^2=x<\varepsilon^2$

$\delta=\varepsilon^2; 0<x<\delta \Rightarrow \sqrt{x}<\sqrt{\delta}=\sqrt{\varepsilon^2}=\varepsilon$

27. $\lim_{x\to 0^+}|x|: 0<x<\delta \Rightarrow ||x|-0|<\varepsilon$

For $x\ge 0$, $|x|=x$.

$\delta=\varepsilon; 0<x<\delta \Rightarrow ||x|-0|=|x|=x<\delta=\varepsilon$

Thus, $\lim_{x\to 0^+}|x|=0$.

$\lim_{x\to 0^-}|x|: 0<0-x<\delta \Rightarrow ||x|-0|<\varepsilon$

For $x<0$, $|x|=-x$; note also that $||x||=|x|$ since $|x|\ge 0$.

$\delta=\varepsilon; 0<-x<\delta \Rightarrow ||x||=|x|=-x<\delta=\varepsilon$

Thus, $\lim_{x\to 0^-}|x|=0$,

since $\lim_{x\to 0^+}|x|=\lim_{x\to 0^-}|x|=0$, $\lim_{x\to 0}|x|=0$.

28. Choose $\varepsilon>0$. Since $\lim_{x\to a} g(x)=0$ there is some $\delta_1>0$ such that

$0<|x-a|<\delta_1 \Rightarrow |g(x)-0|<\frac{\varepsilon}{B}$.

Let $\delta=\min\{1,\delta_1\}$, then $|f(x)|<B$ for $|x-a|<\delta$ or $|x-a|<\delta \Rightarrow |f(x)|<B$. Thus,

$|x-a|<\delta \Rightarrow |f(x)g(x)-0|=|f(x)g(x)|$

$=|f(x)||g(x)|<B\cdot\frac{\varepsilon}{B}=\varepsilon$ so $\lim_{x\to a} f(x)g(x)=0$.

29. Choose $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
That is, for
$a - \delta < x < a$ or $a < x < a + \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$.
Let $f(a) = A$,
$M = \max\{|L - \varepsilon|, |L + \varepsilon|, |A|\}$, $c = a - \delta$,
$d = a + \delta$. Then for x in (c, d), $|f(x)| \le M$, since either $x = a$, in which case
$|f(x)| = |f(a)| = |A| \le M$ or $0 < |x - a| < \delta$ so
$L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

30. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where
$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and
$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.
Thus, for $0 < |x - a| < \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.
Combine the inequalities and use the fact that $f(x) \le g(x)$ to get
$L - \varepsilon < f(x) \le g(x) < M + \varepsilon$ which leads to
$L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.
However,
$L - M = \alpha > 2\varepsilon$
which is a contradiction.
Thus $L \le M$.

31. (b) and (c) are equivalent to the definition of limit.

32. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with $0 < |x - c| < \delta$ such that $|f(x) - L| > \varepsilon$.

33. **a.** $g(x) = \dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\dfrac{x + 6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has an asymptote at $x \approx 3.49$.

c. If $\delta \le \frac{1}{4}$, then $2.75 < x < 3$ or $3 < x < 3.25$ and by graphing
$$y = |g(x)| = \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right|$$
on the interval [2.75, 3.25], we see that
$$0 < \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right| < 3$$
so m must be at least three.

2.3 Concepts Review

1. 48

2. 4

3. – 8; – 4 + 5c

4. 0

Problem Set 2.3

1.
$\lim_{x \to 1}(2x + 1)$ 4
$= \lim_{x \to 1} 2x + \lim_{x \to 1} 1$ 3
$= 2\lim_{x \to 1} x + \lim_{x \to 1} 1$ 2,1
$= 2(1) + 1 = 3$

2.
$\lim_{x \to -1}(3x^2 - 1)$ 5
$= \lim_{x \to -1} 3x^2 - \lim_{x \to -1} 1$ 3
$= 3\lim_{x \to -1} x^2 - \lim_{x \to -1} 1$ 8
$= 3\left(\lim_{x \to -1} x\right)^2 - \lim_{x \to -1} 1$ 2, 1
$= 3(-1)^2 - 1 = 2$

3.
$\lim_{x \to 0}[(2x + 1)(x - 3)]$ 6
$= \lim_{x \to 0}(2x + 1) \cdot \lim_{x \to 0}(x - 3)$ 4, 5
$= \left(\lim_{x \to 0} 2x + \lim_{x \to 0} 1\right) \cdot \left(\lim_{x \to 0} x - \lim_{x \to 0} 3\right)$ 3
$= \left(2\lim_{x \to 0} x + \lim_{x \to 0} 1\right) \cdot \left(\lim_{x \to 0} x - \lim_{x \to 0} 3\right)$ 2, 1
$= [2(0) + 1](0 - 3) = -3$

4.
$\lim_{x \to \sqrt{2}}[(2x^2 + 1)(7x^2 + 13)]$ 6
$= \lim_{x \to \sqrt{2}}(2x^2 + 1) \cdot \lim_{x \to \sqrt{2}}(7x^2 + 13)$ 4, 3
$= \left(2\lim_{x \to \sqrt{2}} x^2 + \lim_{x \to \sqrt{2}} 1\right) \cdot \left(7\lim_{x \to \sqrt{2}} x^2 + \lim_{x \to \sqrt{2}} 13\right)$ 8,1
$= \left[2\left(\lim_{x \to \sqrt{2}} x\right)^2 + 1\right]\left[7\left(\lim_{x \to \sqrt{2}} x\right)^2 + 13\right]$ 2
$= [2(\sqrt{2})^2 + 1][7(\sqrt{2})^2 + 13] = 135$

5. $\lim_{x\to 2}\frac{2x+1}{5-3x}$ 7

$=\frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(5-3x)}$ 4, 5

$=\frac{\lim_{x\to 2}2x+\lim_{x\to 2}1}{\lim_{x\to 2}5-\lim_{x\to 2}3x}$ 3, 1

$=\frac{2\lim_{x\to 2}x+1}{5-3\lim_{x\to 2}x}$ 2

$=\frac{2(2)+1}{5-3(2)}=-5$

6. $\lim_{x\to -3}\frac{4x^3+1}{7-2x^2}$ 7

$=\frac{\lim_{x\to -3}(4x^3+1)}{\lim_{x\to -3}(7-2x^2)}$ 4, 5

$=\frac{\lim_{x\to -3}4x^3+\lim_{x\to -3}1}{\lim_{x\to -3}7-\lim_{x\to -3}2x^2}$ 3, 1

$=\frac{4\lim_{x\to -3}x^3+1}{7-2\lim_{x\to -3}x^2}$ 8

$=\frac{4\left(\lim_{x\to -3}x\right)^3+1}{7-2\left(\lim_{x\to -3}x\right)^2}$ 2

$=\frac{4(-3)^3+1}{7-2(-3)^2}=\frac{107}{11}$

7. $\lim_{x\to 3}\sqrt{3x-5}$ 9

$=\sqrt{\lim_{x\to 3}(3x-5)}$ 5, 3

$=\sqrt{3\lim_{x\to 3}x-\lim_{x\to 3}5}$ 2, 1

$=\sqrt{3(3)-5}=2$

8. $\lim_{x\to -3}\sqrt{5x^2+2x}$ 9

$=\sqrt{\lim_{x\to -3}(5x^2+2x)}$ 4, 3

$=\sqrt{5\lim_{x\to -3}x^2+2\lim_{x\to -3}x}$ 8

$=\sqrt{5\left(\lim_{x\to -3}x\right)^2+2\lim_{x\to -3}x}$ 2

$=\sqrt{5(-3)^2+2(-3)}=\sqrt{39}$

9. $\lim_{t\to -2}(2t^3+15)^{13}$ 8

$=\left[\lim_{t\to -2}(2t^3+15)\right]^{13}$ 4, 3

$=\left[2\lim_{t\to -2}t^3+\lim_{t\to -2}15\right]^{13}$ 8

$=\left[2\left(\lim_{t\to -2}t\right)^3+\lim_{t\to -2}15\right]^{13}$ 2, 1

$=[2(-2)^3+15]^{13}=-1$

10. $\lim_{w\to -2}\sqrt{-3w^3+7w^2}$ 9

$=\sqrt{\lim_{w\to -2}(-3w^3+7w^2)}$ 4, 3

$=\sqrt{-3\lim_{w\to -2}w^3+7\lim_{w\to -2}w^2}$ 8

$=\sqrt{-3\left(\lim_{w\to -2}w\right)^3+7\left(\lim_{w\to -2}w\right)^2}$ 2

$=\sqrt{-3(-2)^3+7(-2)^2}=2\sqrt{13}$

11. $\lim_{y\to 2}\left(\frac{4y^3+8y}{y+4}\right)^{1/3}$ 9

$=\left(\lim_{y\to 2}\frac{4y^3+8y}{y+4}\right)^{1/3}$ 7

$=\left[\frac{\lim_{y\to 2}(4y^3+8y)}{\lim_{y\to 2}(y+4)}\right]^{1/3}$ 4, 3

$=\left(\frac{4\lim_{y\to 2}y^3+8\lim_{y\to 2}y}{\lim_{y\to 2}y+\lim_{y\to 2}4}\right)^{1/3}$ 8, 1

$=\left[\frac{4\left(\lim_{y\to 2}y\right)^3+8\lim_{y\to 2}y}{\lim_{y\to 2}y+4}\right]^{1/3}$ 2

$=\left[\frac{4(2)^3+8(2)}{2+4}\right]^{1/3}=2$

12. $\lim_{w\to 5}(2w^4-9w^3+19)^{-1/2}$

$= \lim_{w\to 5}\frac{1}{\sqrt{2w^4-9w^3+19}}$ 7

$= \frac{\lim_{w\to 5} 1}{\lim_{w\to 5}\sqrt{2w^4-9w^3+19}}$ 1, 9

$= \frac{1}{\sqrt{\lim_{w\to 5}(2w^4-9w^3+19)}}$ 4,5

$= \frac{1}{\sqrt{\lim_{w\to 5} 2w^4 - \lim_{w\to 5} 9w^3 + \lim_{w\to 5} 19}}$ 1,3

$= \frac{1}{\sqrt{2\lim_{w\to 5} w^4 - 9\lim_{w\to 5} w^3 + 19}}$ 8

$= \frac{1}{\sqrt{2\left(\lim_{w\to 5} w\right)^4 - 9\left(\lim_{w\to 5} w\right)^3 + 19}}$ 2

$= \frac{1}{\sqrt{2(5)^4-9(5)^3+19}}$

$= \frac{1}{\sqrt{144}} = \frac{1}{12}$

13. $\lim_{x\to 2}\frac{x^2-4}{x^2+4} = \frac{\lim_{x\to 2}\left(x^2-4\right)}{\lim_{x\to 2}\left(x^2+4\right)} = \frac{4-4}{4+4} = 0$

14. $\lim_{x\to 2}\frac{x^2-5x+6}{x-2} = \lim_{x\to 2}\frac{(x-3)(x-2)}{(x-2)}$

$= \lim_{x\to 2}(x-3) = -1$

15. $\lim_{x\to -1}\frac{x^2-2x-3}{x+1} = \lim_{x\to -1}\frac{(x-3)(x+1)}{(x+1)}$

$= \lim_{x\to -1}(x-3) = -4$

16. $\lim_{x\to -1}\frac{x^2+x}{x^2+1} = \frac{\lim_{x\to -1}\left(x^2+x\right)}{\lim_{x\to -1}\left(x^2+1\right)} = \frac{0}{2} = 0$

17. $\lim_{x\to -1}\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x\to -1}\frac{x-3}{x+7}$

$= \frac{-1-3}{-1+7} = -\frac{2}{3}$

18. $\lim_{x\to 2}\frac{x^2+7x+10}{x+2} = \lim_{x\to 2}\frac{(x+2)(x+5)}{x+2}$

$= \lim_{x\to 2}(x+5) = 7$

19. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(x+1)(x-1)}$

$= \lim_{x\to 1}\frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$

20. $\lim_{x\to -3}\frac{x^2-14x-51}{x^2-4x-21} = \lim_{x\to -3}\frac{(x+3)(x-17)}{(x+3)(x-7)}$

$= \lim_{x\to -3}\frac{x-17}{x-7} = \frac{-3-17}{-3-7} = 2$

21. $\lim_{u\to -2}\frac{u^2-ux+2u-2x}{u^2-u-6} = \lim_{u\to -2}\frac{(u+2)(u-x)}{(u+2)(u-3)}$

$= \lim_{u\to -2}\frac{u-x}{u-3} = \frac{x+2}{5}$

22. $\lim_{x\to 1}\frac{x^2+ux-x-u}{x^2+2x-3} = \lim_{x\to 1}\frac{(x-1)(x+u)}{(x-1)(x+3)}$

$= \lim_{x\to 1}\frac{x+u}{x+3} = \frac{1+u}{1+3} = \frac{u+1}{4}$

23. $\lim_{x\to \pi}\frac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2} = \lim_{x\to \pi}\frac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)}$

$= \lim_{x\to \pi}\frac{2(x-2\pi)}{x+\pi} = \frac{2(\pi-2\pi)}{\pi+\pi} = -1$

24. $\lim_{w\to -2}\frac{(w+2)(w^2-w-6)}{w^2+4w+4}$

$= \lim_{w\to -2}\frac{(w+2)^2(w-3)}{(w+2)^2} = \lim_{w\to -2}(w-3)$

$= -2-3 = -5$

25. $\lim_{x\to a}\sqrt{f^2(x)+g^2(x)}$

$= \sqrt{\lim_{x\to a} f^2(x) + \lim_{x\to a} g^2(x)}$

$= \sqrt{\left(\lim_{x\to a} f(x)\right)^2 + \left(\lim_{x\to a} g(x)\right)^2}$

$= \sqrt{(3)^2+(-1)^2} = \sqrt{10}$

26. $\lim_{x\to a}\frac{2f(x)-3g(x)}{f(x)+g(x)} = \frac{\lim_{x\to a}[2f(x)-3g(x)]}{\lim_{x\to a}[f(x)+g(x)]}$

$= \frac{2\lim_{x\to a} f(x) - 3\lim_{x\to a} g(x)}{\lim_{x\to a} f(x) + \lim_{x\to a} g(x)} = \frac{2(3)-3(-1)}{3+(-1)} = \frac{9}{2}$

27. $\lim_{x\to a}\sqrt[3]{g(x)}[f(x)+3] = \lim_{x\to a}\sqrt[3]{g(x)}\cdot\lim_{x\to a}[f(x)+3]$

$= \sqrt[3]{\lim_{x\to a} g(x)}\cdot\left[\lim_{x\to a} f(x)+\lim_{x\to a} 3\right] = \sqrt[3]{-1}\cdot(3+3)$

$= -6$

28. $\lim_{x\to a}[f(x)-3]^4 = \left[\lim_{x\to a}(f(x)-3)\right]^4$

$= \left[\lim_{x\to a} f(x) - \lim_{x\to a} 3\right]^4 = (3-3)^4 = 0$

29. $\lim_{t\to a}\left[|f(t)|+|3g(t)|\right] = \lim_{t\to a}|f(t)| + 3\lim_{t\to a}|g(t)|$

$= \left|\lim_{t\to a} f(t)\right| + 3\left|\lim_{t\to a} g(t)\right|$

$= |3| + 3|-1| = 6$

30. $\lim_{u\to a}[f(u)+3g(u)]^3 = \left(\lim_{u\to a}[f(u)+3g(u)]\right)^3$

$= \left[\lim_{u\to a} f(u) + 3\lim_{u\to a} g(u)\right]^3 = [3+3(-1)]^3 = 0$

31. $\lim_{x\to 2}\frac{3x^2-12}{x-2} = \lim_{x\to 2}\frac{3(x-2)(x+2)}{x-2}$

$= 3\lim_{x\to 2}(x+2) = 3(2+2) = 12$

32. $\lim_{x\to 2}\frac{(3x^2+2x+1)-17}{x-2} = \lim_{x\to 2}\frac{3x^2+2x-16}{x-2}$

$= \lim_{x\to 2}\frac{(3x+8)(x-2)}{x-2} = \lim_{x\to 2}(3x+8)$

$= 3\lim_{x\to 2} x + 8 = 3(2)+8 = 14$

33. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2} = \lim_{x\to 2}\frac{-\frac{x-2}{2x}}{x-2}$

$= \lim_{x\to 2} -\frac{1}{2x} = \frac{-1}{2\lim_{x\to 2} x} = \frac{-1}{2(2)} = -\frac{1}{4}$

34. $\lim_{x\to 2}\frac{\frac{3}{x^2}-\frac{3}{4}}{x-2} = \lim_{x\to 2}\frac{\frac{3(4-x^2)}{4x^2}}{x-2} = \lim_{x\to 2}\frac{\frac{-3(x+2)(x-2)}{4x^2}}{x-2}$

$= \lim_{x\to 2}\frac{-3(x+2)}{4x^2} = \frac{-3\left(\lim_{x\to 2} x+2\right)}{4\left(\lim_{x\to 2} x\right)^2} = \frac{-3(2+2)}{4(2)^2}$

$= -\frac{3}{4}$

35. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.

$|f(x)g(x)-LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

as shown in the text. Choose $\varepsilon_1 = 1$. Since $\lim_{x\to c} g(x) = M$, there is some $\delta_1 > 0$ such that if $0 < |x-c| < \delta_1$, $|g(x)-M| < \varepsilon_1 = 1$ or $M-1 < g(x) < M+1$

$|M-1| \le |M|+1$ and $|M+1| \le |M|+1$ so for $0 < |x-c| < \delta_1, |g(x)| < |M|+1$. Choose $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist δ_2 and δ_3 such that $0 < |x-c| < \delta_2 \Rightarrow$ $|f(x)-L| < \frac{\varepsilon}{|L|+|M|+1}$ and $0 < |x-c| < \delta_3 \Rightarrow$ $|g(x)-M| < \frac{\varepsilon}{|L|+|M|+1}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $0 < |x-c| < \delta \Rightarrow$

$|f(x)g(x)-LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

$< (|M|+1)\frac{\varepsilon}{|L|+|M|+1} + |L|\frac{\varepsilon}{|L|+|M|+1} = \varepsilon$

Hence,

$\lim_{x\to c} f(x)g(x) = LM = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

36. Say $\lim_{x\to c} g(x) = M$, $M \ne 0$, and choose $\varepsilon_1 = \frac{1}{2}|M|$.

There is some $\delta_1 > 0$ such that

$0 < |x-c| < \delta_1 \Rightarrow |g(x)-M| < \varepsilon_1 = \frac{1}{2}|M|$ or

$M - \frac{1}{2}|M| < g(x) < M + \frac{1}{2}|M|$.

$\left|M - \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$ and $\left|M + \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$

so $|g(x)| > \frac{1}{2}|M|$ and $\frac{1}{|g(x)|} < \frac{2}{|M|}$

Choose $\varepsilon > 0$.

Since $\lim_{x\to c} g(x) = M$ there is $\delta_2 > 0$ such that

$0 < |x-c| < \delta_2 \Rightarrow |g(x)-M| < \frac{1}{2}M^2$.

Let $\delta = \min\{\delta_1, \delta_2\}$, then

$0 < |x-c| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{g(x)M}\right|$

$= \frac{1}{|M||g(x)|}|g(x)-M| < \frac{2}{M^2}|g(x)-M| = \frac{2}{M^2}\cdot\frac{1}{2}M^2\varepsilon$

$= \varepsilon$

Thus, $\lim_{x\to c}\frac{1}{g(x)}=\frac{1}{M}=\frac{1}{\lim_{x\to c} g(x)}$.

Using statement 6 and the above result,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c} f(x)\cdot\lim_{x\to c}\frac{1}{g(x)}$$

$$=\lim_{x\to c} f(x)\cdot\frac{1}{\lim_{x\to c} g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

37. $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c} f(x)=\lim_{x\to c} L$

$\Leftrightarrow \lim_{x\to c} f(x)-\lim_{x\to c} L=0$

$\Leftrightarrow \lim_{x\to c}[f(x)-L]=0$

38. $\lim_{x\to c} f(x)=0 \Leftrightarrow \left[\lim_{x\to c} f(x)\right]^2=0$

$\Leftrightarrow \lim_{x\to c} f^2(x)=0$

$\Leftrightarrow \sqrt{\lim_{x\to c} f^2(x)}=0$

$\Leftrightarrow \lim_{x\to c}\sqrt{f^2(x)}=0$

$\Leftrightarrow \lim_{x\to c}|f(x)|=0$

39. $\lim_{x\to c}|x|=\sqrt{\left(\lim_{x\to c}|x|\right)^2}=\sqrt{\lim_{x\to c}|x|^2}=\sqrt{\lim_{x\to c}x^2}$

$=\sqrt{\left(\lim_{x\to c}x\right)^2}=\sqrt{c^2}=|c|$

40. a. If $f(x)=\frac{x+1}{x-2}, g(x)=\frac{x-5}{x-2}$ and $c=2$, then $\lim_{x\to c}[f(x)+g(x)]$ exists, but neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists.

b. If $f(x)=\frac{2}{x}, g(x)=x$, and $c=0$, then $\lim_{x\to c}[f(x)\cdot g(x)]$ exists, but $\lim_{x\to c} f(x)$ does not exist.

41. $\lim_{x\to -3^+}\frac{\sqrt{3+x}}{x}=\frac{\sqrt{3-3}}{-3}=0$

42. $\lim_{x\to -\pi^+}\frac{\sqrt{\pi^3+x^3}}{x}=\frac{\sqrt{\pi^3+(-\pi)^3}}{-\pi}=0$

43. $\lim_{x\to 3^+}\frac{x-3}{\sqrt{x^2-9}}=\lim_{x\to 3^+}\frac{(x-3)\sqrt{x^2-9}}{x^2-9}$

$=\lim_{x\to 3^+}\frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)}=\lim_{x\to 3^+}\frac{\sqrt{x^2-9}}{x+3}$

$=\frac{\sqrt{3^2-9}}{3+3}=0$

44. $\lim_{x\to 1^-}\frac{\sqrt{1+x}}{4+4x}=\frac{\sqrt{1+1}}{4+4(1)}=\frac{\sqrt{2}}{8}$

45. $\lim_{x\to 2^+}\frac{(x^2+1)[\![x]\!]}{(3x-1)^2}=\frac{(2^2+1)[\![2]\!]}{(3\cdot 2-1)^2}=\frac{5\cdot 2}{5^2}=\frac{2}{5}$

46. $\lim_{x\to 3^-}(x-[\![x]\!])=\lim_{x\to 3^-}x-\lim_{x\to 3^-}[\![x]\!]=3-2=1$

47. $\lim_{x\to 0^-}\frac{x}{|x|}=-1$

48. $\lim_{x\to 3^+}[\![x^2+2x]\!]=[\![3^2+2\cdot 3]\!]=15$

49. $f(x)g(x)=1; g(x)=\frac{1}{f(x)}$

$\lim_{x\to a} g(x)=0 \Leftrightarrow \lim_{x\to a}\frac{1}{f(x)}=0$

$\Leftrightarrow \frac{1}{\lim_{x\to a} f(x)}=0$

No value satisfies this equation, so $\lim_{x\to a} f(x)$ must not exist.

50. R has the vertices $\left(\pm\frac{x}{2}, \pm\frac{1}{2}\right)$

Each side of Q has length $\sqrt{x^2+1}$ so the perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of length 1 and two sides of length $\sqrt{x^2}$ so the perimeter of R is $2+2\sqrt{x^2}$.

$\lim_{x\to 0^+}\frac{\text{perimeter of } R}{\text{perimeter of } Q}=\lim_{x\to 0^+}\frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}}$

$=\frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}}=\frac{2}{4}=\frac{1}{2}$

51. a. $NO = \sqrt{(0-0)^2 + (1-0)^2} = 1$

$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

$= \sqrt{x^2 + x}$

$NP = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + y^2 - 2y + 1}$

$= \sqrt{x^2 + x - 2\sqrt{x} + 1}$

$MO = \sqrt{(1-0)^2 + (0-0)^2} = 1$

$MP = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{y^2 + x^2 - 2x + 1}$

$= \sqrt{x^2 - x + 1}$

$$\lim_{x\to 0^+} \frac{\text{perimeter of } \Delta NOP}{\text{perimeter of } \Delta MOP}$$

$$= \lim_{x\to 0^+} \frac{1 + \sqrt{x^2 + x} + \sqrt{x^2 + x - 2\sqrt{x} + 1}}{1 + \sqrt{x^2 + x} + \sqrt{x^2 - x + 1}}$$

$$= \frac{1 + \sqrt{1}}{1 + \sqrt{1}} = 1$$

b. Area of $\Delta NOP = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area of $\Delta MOP = \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2}$

$$\lim_{x\to 0^+} \frac{\text{area of } \Delta NOP}{\text{area of } \Delta MOP} = \lim_{x\to 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \sqrt{x} = 0$$

2.4 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound

2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left

3. $y = 6$; horizontal

4. $x = 6$; vertical

Problem Set 2.4

1. $$\lim_{x\to\infty} \frac{x}{x-5} = \lim_{x\to\infty} \frac{1}{1-\frac{5}{x}} = 1$$

2. $$\lim_{x\to\infty} \frac{x^2}{5-x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{5}{x^3} - 1} = 0$$

3. $$\lim_{t\to-\infty} \frac{t^2}{7-t^2} = \lim_{t\to-\infty} \frac{1}{\frac{7}{t^2} - 1} = -1$$

4. $$\lim_{t\to-\infty} \frac{t}{t-5} = \lim_{t\to-\infty} \frac{1}{1-\frac{5}{t}} = 1$$

5. $$\lim_{x\to\infty} \frac{x^2}{(x-5)(3-x)} = \lim_{x\to\infty} \frac{x^2}{-x^2 + 8x - 15}$$

$$= \lim_{x\to\infty} \frac{1}{-1 + \frac{8}{x} - \frac{15}{x^2}} = -1$$

6. $$\lim_{x\to\infty} \frac{x^2}{x^2 - 8x + 15} = \lim_{x\to\infty} \frac{1}{1 - \frac{8}{x} + \frac{15}{x^2}} = 1$$

7. $$\lim_{x\to\infty} \frac{x^3}{2x^3 - 100x^2} = \lim_{x\to\infty} \frac{1}{2 - \frac{100}{x}} = \frac{1}{2}$$

8. $$\lim_{\theta\to-\infty} \frac{\pi\theta^5}{\theta^5 - 5\theta^4} = \lim_{\theta\to-\infty} \frac{\pi}{1 - \frac{5}{\theta}} = \pi$$

9. $$\lim_{x\to\infty} \frac{3x^3 - x^2}{\pi x^3 - 5x^2} = \lim_{x\to\infty} \frac{3 - \frac{1}{x}}{\pi - \frac{5}{x}} = \frac{3}{\pi}$$

10. $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2 - 5}$; $0 \le \sin^2\theta \le 1$ for all θ and

$$\lim_{\theta\to\infty} \frac{1}{\theta^2 - 5} = \lim_{\theta\to\infty} \frac{\frac{1}{\theta^2}}{1 - \frac{5}{\theta^2}} = 0 \text{ so } \lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2 - 5} = 0$$

11. $$\lim_{x\to\infty} \frac{3\sqrt{x^3} + 3x}{\sqrt{2x^3}} = \lim_{x\to\infty} \frac{3x^{3/2} + 3x}{\sqrt{2}x^{3/2}}$$

$$= \lim_{x\to\infty} \frac{3 + \frac{3}{\sqrt{x}}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

12. $$\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}} = \sqrt[3]{\lim_{x\to\infty} \frac{\pi x^3 + 3x}{\sqrt{2}x^3 + 7x}}$$

$$= \sqrt[3]{\lim_{x\to\infty} \frac{\pi + \frac{3}{x^2}}{\sqrt{2} + \frac{7}{x^2}}} = \sqrt[3]{\frac{\pi}{\sqrt{2}}}$$

13. $\lim_{x\to\infty}\sqrt[3]{\frac{1+8x^2}{x^2+4}}=\sqrt[3]{\lim_{x\to\infty}\frac{1+8x^2}{x^2+4}}$

$=\sqrt[3]{\lim_{x\to\infty}\frac{\frac{1}{x^2}+8}{1+\frac{4}{x^2}}}=\sqrt[3]{8}=2$

14. $\lim_{x\to\infty}\sqrt{\frac{x^2+x+3}{(x-1)(x+1)}}=\sqrt{\lim_{x\to\infty}\frac{x^2+x+3}{x^2-1}}$

$=\sqrt{\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{3}{x^2}}{1-\frac{1}{x^2}}}=\sqrt{1}=1$

15. $\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2}$

16. $\lim_{n\to\infty}\frac{n^2}{n^2+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=\frac{1}{1+0}=1$

17. $\lim_{n\to\infty}\frac{n^2}{n+1}=\lim_{n\to\infty}\frac{n}{1+\frac{1}{n}}=\frac{\lim_{n\to\infty}n}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}=\frac{\infty}{1+0}=\infty$

18. $\lim_{n\to\infty}\frac{n}{n^2+1}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}}=\frac{0}{1+0}=0$

19. For $x>0$, $x=\sqrt{x^2}$.

$\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\frac{\sqrt{x^2+3}}{\sqrt{x^2}}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{3}{x^2}}}$

$=\frac{2}{\sqrt{1}}=2$

20. $\lim_{x\to\infty}\frac{\sqrt{2x+1}}{x+4}=\lim_{x\to\infty}\frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1+\frac{4}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{2}{x}+\frac{1}{x^2}}}{1+\frac{4}{x}}=0$

21. $\lim_{x\to\infty}\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)$

$=\lim_{x\to\infty}\frac{\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)\left(\sqrt{2x^2+3}+\sqrt{2x^2-5}\right)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$=\lim_{x\to\infty}\frac{2x^2+3-(2x^2-5)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$=\lim_{x\to\infty}\frac{8}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}=\lim_{x\to\infty}\frac{\frac{8}{x}}{\frac{\sqrt{2x^2+3}+\sqrt{2x^2-5}}{\sqrt{x^2}}}$

$=\lim_{x\to\infty}\frac{\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}+\sqrt{2-\frac{5}{x^2}}}=0$

22. $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}$

$=\lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$

$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\frac{2}{2}=1$

23. $\lim_{y\to-\infty}\frac{9y^3+1}{y^2-2y+2}=\lim_{y\to-\infty}\frac{9y+\frac{1}{y^2}}{1-\frac{2}{y}+\frac{2}{y^2}}=-\infty$

24. $\lim_{x\to\infty}\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$

$=\lim_{x\to\infty}\frac{a_0+\frac{a_1}{x}+\ldots+\frac{a_{n-1}}{x^{n-1}}+\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\ldots+\frac{b_{n-1}}{x^{n-1}}+\frac{b_n}{x^n}}=\frac{a_0}{b_0}$

25. $\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=\frac{1}{\sqrt{1+0}}=1$

26. $\lim_{n\to\infty}\frac{n^2}{\sqrt{n^3+2n+1}}=\lim_{n\to\infty}\frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+\frac{2}{n^2}+\frac{1}{n^3}}}=\frac{\infty}{1}=\infty$

27. As $x\to4^+, x\to4$ while $x-4\to0^+$.

$\lim_{x\to4^+}\frac{x}{x-4}=\infty$

28. $\lim_{t\to -3^+} \frac{t^2-9}{t+3} = \lim_{t\to -3^+} \frac{(t+3)(t-3)}{t+3}$
$= \lim_{t\to -3^+} (t-3) = -6$

29. As $t \to 3^-, t^2 \to 9$ while $9-t^2 \to 0^+$.
$\lim_{t\to 3^-} \frac{t^2}{9-t^2} = \infty$

30. As $x \to \sqrt[3]{5}^+, x^2 \to 5^{2/3}$ while $5-x^3 \to 0^-$.
$\lim_{x\to \sqrt[3]{5}^+} \frac{x^2}{5-x^3} = -\infty$

31. As $x \to 5^-, x^2 \to 25, x-5 \to 0^-$, and $3-x \to -2$.
$\lim_{x\to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty$

32. As $\theta \to \pi^+, \theta^2 \to \pi^2$ while $\sin\theta \to 0^-$.
$\lim_{\theta\to \pi^+} \frac{\theta^2}{\sin\theta} = -\infty$

33. As $x \to 3^-, x^3 \to 27$, while $x-3 \to 0^-$.
$\lim_{x\to 3^-} \frac{x^3}{x-3} = -\infty$

34. As $\theta \to \frac{\pi}{2}^+, \pi\theta \to \frac{\pi^2}{2}$ while $\cos\theta \to 0^-$.
$\lim_{\theta\to \frac{\pi}{2}^+} \frac{\pi\theta}{\cos\theta} = -\infty$

35. $\lim_{x\to 3^-} \frac{x^2-x-6}{x-3} = \lim_{x\to 3^-} \frac{(x+2)(x-3)}{x-3}$
$= \lim_{x\to 3^-} (x+2) = 5$

36. $\lim_{x\to 2^+} \frac{x^2+2x-8}{x^2-4} = \lim_{x\to 2^+} \frac{(x+4)(x-2)}{(x+2)(x-2)}$
$= \lim_{x\to 2^+} \frac{x+4}{x+2} = \frac{6}{4} = \frac{3}{2}$

37. For $0 \le x < 1$, $[\![x]\!] = 0$, so for $0 < x < 1$, $\frac{[\![x]\!]}{x} = 0$
thus $\lim_{x\to 0^+} \frac{[\![x]\!]}{x} = 0$

38. For $-1 \le x < 0$, $[\![x]\!] = -1$, so for $-1 < x < 0$,
$\frac{[\![x]\!]}{x} = -\frac{1}{x}$ thus $\lim_{x\to 0^-} \frac{[\![x]\!]}{x} = \infty$.
(Since $x < 0$, $-\frac{1}{x} > 0$.)

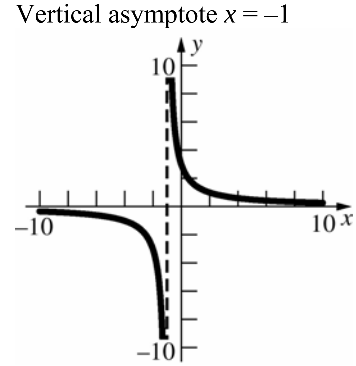
39. For $x < 0$, $|x| = -x$, thus
$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$

40. For $x > 0$, $|x| = x$, thus $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$

41. As $x \to 0^-, 1+\cos x \to 2$ while $\sin x \to 0^-$.
$\lim_{x\to 0^-} \frac{1+\cos x}{\sin x} = -\infty$

42. $-1 \le \sin x \le 1$ for all x, and
$\lim_{x\to\infty} \frac{1}{x} = 0$, so $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

43. $\lim_{x\to\infty} \frac{3}{x+1} = 0$, $\lim_{x\to -\infty} \frac{3}{x+1} = 0$;
Horizontal asymptote $y = 0$.
$\lim_{x\to -1^+} \frac{3}{x+1} = \infty$, $\lim_{x\to -1^-} \frac{3}{x+1} = -\infty$;
Vertical asymptote $x = -1$

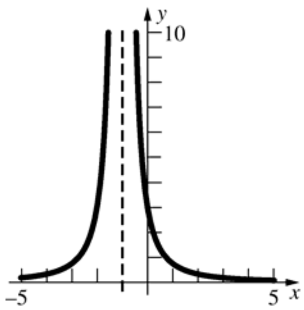


44. $\lim_{x\to\infty} \frac{3}{(x+1)^2} = 0,\ \lim_{x\to-\infty} \frac{3}{(x+1)^2} = 0;$

Horizontal asymptote $y = 0$.

$\lim_{x\to-1^+} \frac{3}{(x+1)^2} = \infty,\ \lim_{x\to-1^-} \frac{3}{(x+1)^2} = \infty;$

Vertical asymptote $x = -1$



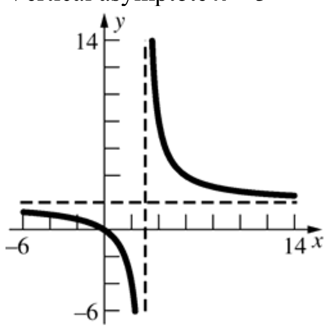
45. $\lim_{x\to\infty} \frac{2x}{x-3} = \lim_{x\to\infty} \frac{2}{1-\frac{3}{x}} = 2,$

$\lim_{x\to-\infty} \frac{2x}{x-3} = \lim_{x\to-\infty} \frac{2}{1-\frac{3}{x}} = 2,$

Horizontal asymptote $y = 2$

$\lim_{x\to3^+} \frac{2x}{x-3} = \infty,\ \lim_{x\to3^-} \frac{2x}{x-3} = -\infty;$

Vertical asymptote $x = 3$



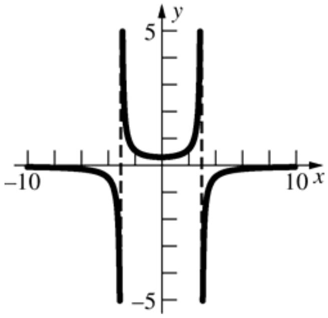
46. $\lim_{x\to\infty} \frac{3}{9-x^2} = 0,\ \lim_{x\to-\infty} \frac{3}{9-x^2} = 0;$

Horizontal asymptote $y = 0$

$\lim_{x\to3^+} \frac{3}{9-x^2} = -\infty,\ \lim_{x\to3^-} \frac{3}{9-x^2} = \infty,$

$\lim_{x\to-3^+} \frac{3}{9-x^2} = \infty,\ \lim_{x\to-3^-} \frac{3}{9-x^2} = -\infty;$

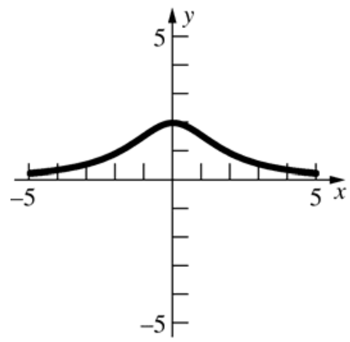
Vertical asymptotes $x = -3$, $x = 3$



47. $\lim_{x\to\infty} \frac{14}{2x^2+7} = 0,\ \lim_{x\to-\infty} \frac{14}{2x^2+7} = 0;$

Horizontal asymptote $y = 0$

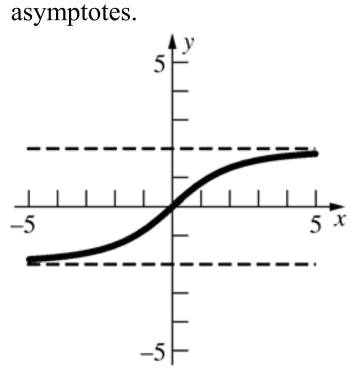
Since $2x^2 + 7 > 0$ for all x, $g(x)$ has no vertical asymptotes.



48. $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{5}{x^2}}} = \frac{2}{\sqrt{1}} = 2,$

$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to-\infty} \frac{2}{-\sqrt{1+\frac{5}{x^2}}} = \frac{2}{-\sqrt{1}} = -2$

Since $\sqrt{x^2+5} > 0$ for all x, $g(x)$ has no vertical asymptotes.



49. $f(x) = 2x + 3 - \frac{1}{x^3-1}$, thus

$\lim_{x\to\infty} [f(x) - (2x+3)] = \lim_{x\to\infty} \left[-\frac{1}{x^3-1}\right] = 0$

The oblique asymptote is $y = 2x + 3$.

50. $f(x) = 3x + 4 - \frac{4x+3}{x^2+1}$, thus

$\lim_{x\to\infty} [f(x) - (3x+4)] = \lim_{x\to\infty} \left[-\frac{4x+3}{x^2+1}\right]$

$= \lim_{x\to\infty} \left[-\frac{\frac{4}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}}\right] = 0$.

The oblique asymptote is $y = 3x + 4$.

51. a. We say that $\lim_{x \to c^+} f(x) = -\infty$ if to each negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

b. We say that $\lim_{x \to c^-} f(x) = \infty$ if to each positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

52. a. We say that $\lim_{x \to \infty} f(x) = \infty$ if to each positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

b. We say that $\lim_{x \to -\infty} f(x) = \infty$ if to each positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

53. Let $\varepsilon > 0$ be given. Since $\lim_{x \to \infty} f(x) = A$, there is a corresponding number M_1 such that

$x > M_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$. Similarly, there is a

number M_2 such that $x > M_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$.

Let $M = \max\{M_1, M_2\}$, then

$$x > M \Rightarrow |f(x) + g(x) - (A + B)|$$

$$= |f(x) - A + g(x) - B| \le |f(x) - A| + |g(x) - B|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\lim_{x \to \infty} [f(x) + g(x)] = A + B$

54. Written response

55. a. $\lim_{x \to \infty} \sin x$ does not exist as $\sin x$ oscillates between –1 and 1 as x increases.

b. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{u \to 0^+} \sin u = 0$$

c. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{u \to 0^+} \frac{1}{u} \sin u = \lim_{u \to 0^+} \frac{\sin u}{u} = 1$$

d. Let $u = \frac{1}{x}$, then

$$\lim_{x \to \infty} x^{3/2} \sin \frac{1}{x} = \lim_{u \to 0^+} \left(\frac{1}{u}\right)^{3/2} \sin u$$

$$= \lim_{u \to 0^+} \left[\left(\frac{1}{\sqrt{u}}\right)\left(\frac{\sin u}{u}\right)\right] = \infty$$

e. As $x \to \infty$, $\sin x$ oscillates between –1 and 1, while $x^{-1/2} = \frac{1}{\sqrt{x}} \to 0$.

$$\lim_{x \to \infty} x^{-1/2} \sin x = 0$$

f. Let $u = \frac{1}{x}$, then

$$\lim_{x \to \infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right) = \lim_{u \to 0^+} \sin\left(\frac{\pi}{6} + u\right)$$

$$= \sin \frac{\pi}{6} = \frac{1}{2}$$

g. As $x \to \infty, x + \frac{1}{x} \to \infty$, so $\lim_{x \to \infty} \sin\left(x + \frac{1}{x}\right)$ does not exist. (See part a.)

h. $\sin\left(x + \frac{1}{x}\right) = \sin x \cos \frac{1}{x} + \cos x \sin \frac{1}{x}$

$$\lim_{x \to \infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$$

$$= \lim_{x \to \infty} \left[\sin x\left(\cos \frac{1}{x} - 1\right) + \cos x \sin \frac{1}{x}\right]$$

As $x \to \infty, \cos \frac{1}{x} \to 1$ so $\cos \frac{1}{x} - 1 \to 0$.

From part **b.**, $\lim_{x \to \infty} \sin \frac{1}{x} = 0$.

As $x \to \infty$ both $\sin x$ and $\cos x$ oscillate between –1 and 1.

$$\lim_{x \to \infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right] = 0.$$

56. $\lim_{v \to c^-} m(v) = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$

57. $\lim_{x \to \infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$

58. $\lim_{x \to -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$

59. $\lim_{x \to -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5}\right) = -\frac{3}{2\sqrt{2}}$

60. $\lim_{x \to \infty} \frac{2x + 1}{\sqrt{3x^2 + 1}} = \frac{2}{\sqrt{3}}$

61. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{10} = 1$

62. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \approx 2.718$

63. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2} = \infty$

64. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sin x} = 1$

65. $\lim_{x\to 3^-}\frac{\sin|x-3|}{x-3} = -1$

66. $\lim_{x\to 3^-}\frac{\sin|x-3|}{\tan(x-3)} = -1$

67. $\lim_{x\to 3^-}\frac{\cos(x-3)}{x-3} = -\infty$

68. $\lim_{x\to\frac{\pi}{2}^+}\frac{\cos x}{x-\frac{\pi}{2}} = -1$

69. $\lim_{x\to 0^+}(1+\sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$

70. $\lim_{x\to 0^+}(1+\sqrt{x})^{1/x} = \infty$

71. $\lim_{x\to 0^+}(1+\sqrt{x})^{x} = 1$

2.5 Concepts Review

1. 0

2. 1

3. the denominator is 0 when $t=0$.

4. 1

Problem Set 2.5

1. $\lim_{x\to 0}\frac{\cos x}{x+1} = \frac{1}{1} = 1$

2. $\lim_{\theta\to\pi/2}\theta\cos\theta = \frac{\pi}{2}\cdot 0 = 0$

3. $\lim_{t\to 0}\frac{\cos^2 t}{1+\sin t} = \frac{\cos^2 0}{1+\sin 0} = \frac{1}{1+0} = 1$

4. $\lim_{x\to 0}\frac{3x\tan x}{\sin x} = \lim_{x\to 0}\frac{3x(\sin x/\cos x)}{\sin x} = \lim_{x\to 0}\frac{3x}{\cos x}$
$= \frac{0}{1} = 0$

5. $\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$

6. $\lim_{\theta\to 0}\frac{\sin 3\theta}{2\theta} = \lim_{\theta\to 0}\frac{3}{2}\cdot\frac{\sin 3\theta}{3\theta} = \frac{3}{2}\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}$
$= \frac{3}{2}\cdot 1 = \frac{3}{2}$

7. $\lim_{\theta\to 0}\frac{\sin 3\theta}{\tan\theta} = \lim_{\theta\to 0}\frac{\sin 3\theta}{\frac{\sin\theta}{\cos\theta}} = \lim_{\theta\to 0}\frac{\cos\theta\sin 3\theta}{\sin\theta}$

$= \lim_{\theta\to 0}\left[\cos\theta\cdot 3\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right]$

$= 3\lim_{\theta\to 0}\left[\cos\theta\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right] = 3\cdot 1\cdot 1\cdot 1 = 3$

8. $\lim_{\theta\to 0}\frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta\to 0}\frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta\to 0}\frac{\sin 5\theta}{\cos 5\theta\sin 2\theta}$

$= \lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot 5\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{1}{2}\cdot\frac{2\theta}{\sin 2\theta}\right]$

$= \frac{5}{2}\lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{2\theta}{\sin 2\theta}\right]$

$= \frac{5}{2}\cdot 1\cdot 1\cdot 1 = \frac{5}{2}$

9. $\lim_{\theta\to 0}\frac{\cot\pi\theta\sin\theta}{2\sec\theta} = \lim_{\theta\to 0}\frac{\frac{\cos\pi\theta}{\sin\pi\theta}\sin\theta}{\frac{2}{\cos\theta}}$

$= \lim_{\theta\to 0}\frac{\cos\pi\theta\sin\theta\cos\theta}{2\sin\pi\theta}$

$= \lim_{\theta\to 0}\left[\frac{\cos\pi\theta\cos\theta}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\pi}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$

$= \frac{1}{2\pi}\lim_{\theta\to 0}\left[\cos\pi\theta\cos\theta\cdot\frac{\sin\theta}{\theta}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$

$= \frac{1}{2\pi}\cdot 1\cdot 1\cdot 1\cdot 1 = \frac{1}{2\pi}$

10. $\lim_{t\to 0}\frac{\sin^2 3t}{2t} = \lim_{t\to 0}\frac{9t}{2}\cdot\frac{\sin 3t}{3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1\cdot 1 = 0$

11. $\lim_{t\to 0}\frac{\tan^2 3t}{2t} = \lim_{t\to 0}\frac{\sin^2 3t}{(2t)(\cos^2 3t)}$

$= \lim_{t\to 0}\frac{3(\sin 3t)}{2\cos^2 3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1 = 0$

12. $\lim_{t\to 0}\frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$

13. $\lim_{t\to 0}\frac{\sin(3t)+4t}{t\sec t}=\lim_{t\to 0}\left(\frac{\sin 3t}{t\sec t}+\frac{4t}{t\sec t}\right)$

$=\lim_{t\to 0}\frac{\sin 3t}{t\sec t}+\lim_{t\to 0}\frac{4t}{t\sec t}$

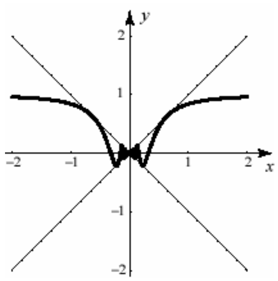
$=\lim_{t\to 0}3\cos t\cdot\frac{\sin 3t}{3t}+\lim_{t\to 0}4\cos t$

$=3\cdot 1+4=7$

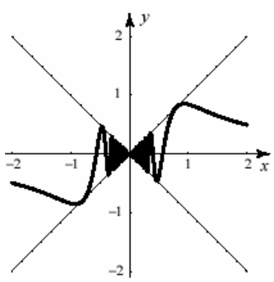
14. $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\times 1=1$

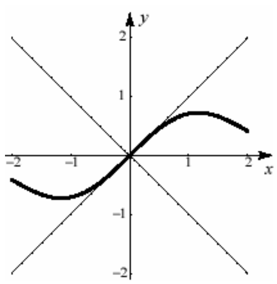
15. $\lim_{x\to 0}x\sin\left(1/x\right)=0$



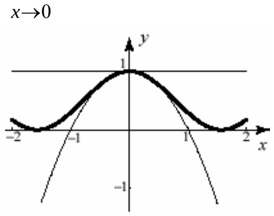
16. $\lim_{x\to 0}x\sin\left(1/x^2\right)=0$



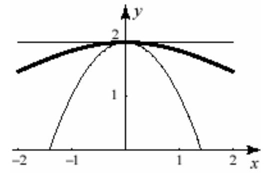
17. $\lim_{x\to 0}\left(1-\cos^2 x\right)/x=0$



18. $\lim_{x\to 0}\cos^2 x=1$



19. $\lim_{x\to 0}1+\frac{\sin x}{x}=2$



20. The result that $\lim_{t\to 0}\cos t=1$ was established in the proof of the theorem. Then

$\lim_{t\to c}\cos t=\lim_{h\to 0}\cos(c+h)$

$=\lim_{h\to 0}(\cos c\cos h-\sin c\sin h)$

$=\lim_{h\to 0}\cos c\lim_{h\to 0}\cos h-\sin c\lim_{h\to 0}\sin h$

$=\cos c$

21. $\lim_{t\to c}\tan t=\lim_{t\to c}\frac{\sin t}{\cos t}=\frac{\lim_{t\to c}\sin t}{\lim_{t\to c}\cos t}=\frac{\sin c}{\cos c}=\tan c$

$\lim_{t\to c}\cot t=\lim_{t\to c}\frac{\cos t}{\sin t}=\frac{\lim_{t\to c}\cos t}{\lim_{t\to c}\sin t}=\frac{\cos c}{\sin c}=\cot c$

22. $\lim_{t\to c}\sec t=\lim_{t\to c}\frac{1}{\cos t}=\frac{1}{\cos c}=\sec c$

$\lim_{t\to c}\csc t=\lim_{t\to c}\frac{1}{\sin t}=\frac{1}{\sin c}=\csc c$

23. $\overline{BP}=\sin t,\overline{OB}=\cos t$

$\text{area}(\Delta OBP)\le\text{area (sector }OAP)$

$\le\text{area }(\Delta OBP)+\text{area}(ABPQ)$

$\frac{1}{2}\overline{OB}\cdot\overline{BP}\le\frac{1}{2}t(1)^2\le\frac{1}{2}\overline{OB}\cdot\overline{BP}+(1-\overline{OB})\overline{BP}$

$\frac{1}{2}\sin t\cos t\le\frac{1}{2}t\le\frac{1}{2}\sin t\cos t+(1-\cos t)\sin t$

$\cos t\le\frac{t}{\sin t}\le 2-\cos t$

$\frac{1}{2-\cos t}\le\frac{\sin t}{t}\le\frac{1}{\cos t}$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$.

$\lim_{t\to 0}\frac{1}{2-\cos t}\le\lim_{t\to 0}\frac{\sin t}{t}\le\lim_{t\to 0}\frac{1}{\cos t}$

$1\le\lim_{t\to 0}\frac{\sin t}{t}\le 1$

Thus, $\lim_{t\to 0}\frac{\sin t}{t}=1$.

24. a. Written response

b. $D = \frac{1}{2}\overline{AB}\cdot\overline{BP} = \frac{1}{2}(1-\cos t)\sin t$

$= \frac{\sin t(1-\cos t)}{2}$

$E = \frac{1}{2}t(1)^2 - \frac{1}{2}\overline{OB}\cdot\overline{BP} = \frac{t}{2} - \frac{\sin t\cos t}{2}$

$\frac{D}{E} = \frac{\sin t(1-\cos t)}{t-\sin t\cos t}$

c. $\lim_{t\to 0^+}\left(\frac{D}{E}\right) = 0.75$

2.6 Concepts Review

1. $\lim_{n\to\infty} a^{r_n}$

2. natural logarithm; $\ln x$

3. $1000e^{(0.06)3} = \$1197.22$

4. even; odd

Problem Set 2.6

1. $10^{2\log_{10}5} = 10^{\log_{10}5^2} = 5^2 = 25$

2. $2^{2\log_2 x} = 2^{\log_2 x^2} = x^2$

3. $e^{3\ln x} = e^{\ln x^3} = x^3$

4. $e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

5. $\ln e^{\cos x} = \cos x$

6. $\ln e^{-2x-3} = -2x-3$

7. $\ln(x^3e^{-3x}) = \ln x^3 + \ln e^{-3x} = 3\ln x - 3x$

8. $e^{x-\ln x} = \frac{e^x}{e^{\ln x}} = \frac{e^x}{x}$

9. $e^{\ln 3+2\ln x} = e^{\ln 3}\cdot e^{2\ln x} = 3\cdot e^{\ln x^2} = 3x^2$

10. $e^{\ln x^2 - y\ln x} = \frac{e^{\ln x^2}}{e^{y\ln x}} = \frac{x^2}{e^{\ln x^y}} = \frac{x^2}{x^y} = x^{2-y}$

11. a. Graph D; the graph of $f(x) = e^{x-1}$ will be the graph of $y = e^x$, but shifted 1 unit to the right.

b. Graph B

c. Graph C; the graph of $y = e^x$ is strictly increasing and passes through the point $(0,1)$.

d. Graph A; the graph of $f(x) = e^{-x/4}$ will be the graph of $y = e^x$ reflected about the *y*-axis and stretched horizontally by a factor of 4. The graph will be strictly decreasing.

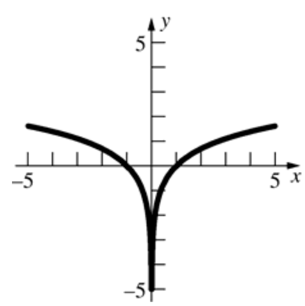
12. a. Graph A; The graph of $f(x) = \ln x$ will be strictly increasing, will rise slowly, and will have a vertical asymptote at $x = 0$.

b. Graph D; The graph of $f(x) = \ln(x-1)$ will be the graph of $y = \ln x$, but shifted 1 unit to the right.

c. Graph C; The graph of $f(x) = \ln\frac{1}{x} = \ln x^{-1} = -\ln x$ will be the graph of $y = \ln x$, but reflected about the *x*-axis. The graph will be strictly decreasing.

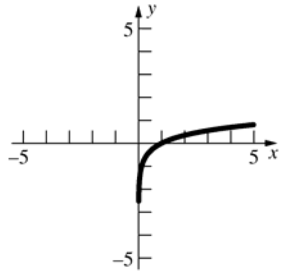
d. Graph B; The graph of $f(x) = \ln x^4 = 4\ln x\ \ (x > 0)$ will be the graph of $y = \ln x$, but stretched vertically by a factor of 4. The graph will be strictly increasing, but will rise at a faster rate than $y = \ln x$.

13.



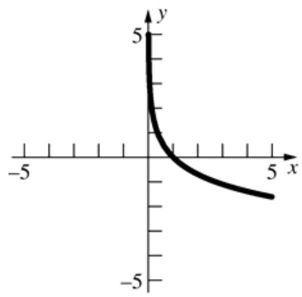
$y = \ln x$ is reflected across the *y*-axis.

14.



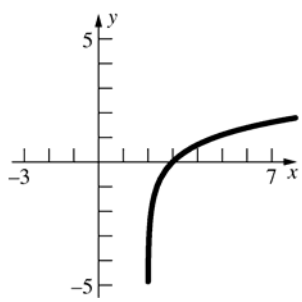
The y-values of $y = \ln x$ are multiplied by $\frac{1}{2}$, since $\ln\sqrt{x} = \frac{1}{2}\ln x$.

15.



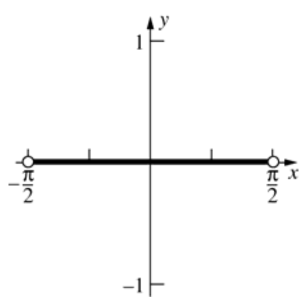
$y = \ln x$ is reflected across the x-axis since $\ln\left(\frac{1}{x}\right) = -\ln x$.

16.



$y = \ln x$ is shifted two units to the right.

17.



$$y = \ln\cos x + \ln\sec x$$
$$= \ln\cos x + \ln\frac{1}{\cos x}$$
$$= \ln\cos x - \ln\cos x = 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

18. **a.** $\lim_{x\to 0}(1+x)^{1000} = 1^{1000} = 1$

b. $\lim_{x\to 0} 1^{1/x} = \lim_{x\to 0} 1 = 1$

c. $\lim_{x\to 0^+}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}(1+\varepsilon)^n = \infty$

d. $\lim_{x\to 0^-}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}\frac{1}{(1+\varepsilon)^n} = 0$

19. **a.** $\lim_{x\to 0}(1-x)^{1/x} = \lim_{x\to 0}\frac{1}{[1+(-x)]^{1/(-x)}} = \frac{1}{e}$

b. $\lim_{x\to 0}(1+3x)^{1/x} = \lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^3 = e^3$

c. $\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$= \lim_{x\to 0^+}(1+2x)^{1/x}$

$= \lim_{x\to 0^+}\left[(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$

d. $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{2n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$

$= \lim_{x\to 0^+}(1-x)^{2/x}$

$= \lim_{x\to 0^+}\left[(1-x)^{\frac{1}{-x}}\right]^{-2} = \frac{1}{e^2}$

20. **a.** $\lim_{n\to 0}\left(1+\frac{2}{n}\right)^{100} = \infty$

(note that the exponent, while large, is still fixed)

b. $\lim_{n\to 0}(1.001)^n = 1.001^0 = 1$

c. $\lim_{n\to\infty}\left(\frac{n+3}{n}\right)^{n+1} = \lim_{n\to\infty}\left(\left(1+\frac{3}{n}\right)^{\frac{n+1}{3}}\right)^3 = e^3$

d. $\lim_{x\to 0}(1+x)^{1/x} = e$

21. **a.** $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$
$= 0.693 + 1.099 = 1.792$

b. $\ln 1.5 = \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2 = 1.099 - 0.693$
$= 0.406$

c. $\ln 81 = \ln 3^4 = 4\ln 3 = 4(1.099) = 4.396$

d. $\ln\sqrt{2} = \ln 2^{1/2} = \frac{1}{2}\ln 2 = \frac{1}{2}(0.693) = 0.3465$

e. $\ln\left(\frac{1}{36}\right) = -\ln 36 = -\ln(2^2 \cdot 3^2)$
$= -2\ln 2 - 2\ln 3 = -2(0.693) - 2(1.099)$
$= -3.584$

f. $\ln 48 = \ln(2^4 \cdot 3) = 4\ln 2 + \ln 3$
$= 4(0.693) + 1.099 = 3.871$

22. **a.** 1.792 **b.** 0.405

c. 4.394 **d.** 0.3466

e. –3.584 **f.** 3.871

23. $2\ln(x+1) - \ln x = \ln(x+1)^2 - \ln x = \ln\frac{(x+1)^2}{x}$

24. $\frac{1}{2}\ln(x-9) + \frac{1}{2}\ln x = \ln\sqrt{x-9} - \ln\sqrt{x}$
$= \ln\frac{\sqrt{x-9}}{\sqrt{x}} = \ln\sqrt{\frac{x-9}{x}}$

25. $\ln(x-2) - \ln(x+2) + 2\ln x$
$= \ln(x-2) - \ln(x+2) + \ln x^2$
$= \ln\frac{x^2(x-2)}{x+2}$

26. $\ln(x^2-9) - 2\ln(x-3) - \ln(x+3)$
$= \ln(x^2-9) - \ln(x-3)^2 - \ln(x+3)$
$= \ln\frac{x^2-9}{(x-3)^2(x+3)} = \ln\frac{1}{x-3}$

27. **a.** $(\$375)(1.035)^2 \approx \$401.71$

b. $(\$375)\left(1+\frac{0.035}{12}\right)^{24} \approx \$402.15$

c. $(\$375)\left(1+\frac{0.035}{365}\right)^{730} \approx \$402.19$

d. $(\$375)e^{0.035\cdot 2} \approx \$402.19$

28. **a.** $(\$375)(1.046)^2 = \$410.29$

b. $(\$375)\left(1+\frac{0.046}{12}\right)^{24} \approx \$411.06$

c. $(\$375)\left(1+\frac{0.046}{365}\right)^{730} \approx \$411.13$

d. $(\$375)e^{0.046\cdot 2} \approx \$411.14$

29. **a.** $\left(1+\frac{0.06}{12}\right)^{12t} = 2$
$1.005^{12t} = 2$
$12t = \frac{\ln 2}{\ln 1.005}$ so $t = \frac{\ln 2}{12\ln 1.005} \approx 11.58$
It will take about 11.58 years or
11 years, 6 months, 29 days.

b. $e^{0.06t} = 2 \Rightarrow t = \frac{\ln 2}{0.06} \approx 11.55$
It will take about 11.55 years
or 11 years, 6 months, and 18 days.

30. $\$20{,}000(1.025)^5 \approx \$22{,}628.16$

31. 1626 to 2000 is 374 years.
$y = 24e^{0.06\cdot 374} \approx \$133.6$ billion

32. $\$100(1.04)^{969} \approx \$3.201\times 10^{18}$

33. $\log_5 12 = \frac{\ln 12}{\ln 5} \approx 1.544$

34. $\log_7 0.11 = \frac{\ln 0.11}{\ln 7} \approx -1.1343$

35. $\log_{11}(8.12)^{1/5} = \frac{1}{5}\frac{\ln 8.12}{\ln 11} \approx 0.1747$

36. $\log_{10}(8.57)^7 = 7\frac{\ln 8.57}{\ln 10} \approx 6.5309$

37. $x\ln 2 = \ln 17$
$x = \frac{\ln 17}{\ln 2} \approx 4.08746$

38. $x\ln 5 = \ln 13$
$x = \frac{\ln 13}{\ln 5} \approx 1.5937$

39. $(2s-3)\ln 5 = \ln 4$
$2s - 3 = \frac{\ln 4}{\ln 5}$
$s = \frac{1}{2}\left(3+\frac{\ln 4}{\ln 5}\right) \approx 1.9307$

40. $\frac{1}{\theta-1}\ln 12=\ln 4$

$\frac{\ln 12}{\ln 4}=\theta-1$

$\theta=1+\frac{\ln 12}{\ln 4}\approx 2.7925$

41. $\cosh x+\sinh x=\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}$

$=\frac{2e^x}{2}=e^x$

42. $\cosh 2x+\sinh 2x=\frac{e^{2x}+e^{-2x}}{2}+\frac{e^{2x}-e^{-2x}}{2}$

$=\frac{2e^{2x}}{2}=e^{2x}$

43. $\cosh x-\sinh x=\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}$

$=\frac{2e^{-x}}{2}=e^{-x}$

44. $\cosh 2x-\sinh 2x=\frac{e^{2x}+e^{-2x}}{2}-\frac{e^{2x}-e^{-2x}}{2}$

$=\frac{2e^{-2x}}{2}=e^{-2x}$

45. $\sinh x\cosh y+\cosh x\sinh y=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}}{4}+\frac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}$

$=\frac{2e^{x+y}-2e^{-(x+y)}}{4}=\frac{e^{x+y}-e^{-(x+y)}}{2}=\sinh(x+y)$

46. $\sinh x\cosh y-\cosh x\sinh y=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}}{4}-\frac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}$

$=\frac{2e^{x-y}-2e^{-x+y}}{4}=\frac{e^{x-y}-e^{-(x-y)}}{2}=\sinh(x-y)$

47. $\cosh x\cosh y+\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}+\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x+y}+2e^{-x-y}}{4}=\frac{e^{x+y}+e^{-(x+y)}}{2}=\cosh(x+y)$

48. $\cosh x\cosh y-\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}-\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x-y}+2e^{-x+y}}{4}=\frac{e^{x-y}+e^{-(x-y)}}{2}=\cosh(x-y)$

49. $\dfrac{\tanh x+\tanh y}{1+\tanh x\tanh y}=\dfrac{\frac{\sinh x}{\cosh x}+\frac{\sinh y}{\cosh y}}{1+\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\dfrac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}=\dfrac{\sinh(x+y)}{\cosh(x+y)}$

$=\tanh(x+y)$

50. $\dfrac{\tanh x-\tanh y}{1-\tanh x\tanh y}=\dfrac{\frac{\sinh x}{\cosh x}-\frac{\sinh y}{\cosh y}}{1-\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\dfrac{\sinh x\cosh y-\cosh x\sinh y}{\cosh x\cosh y-\sinh x\sinh y}=\dfrac{\sinh(x-y)}{\cosh(x-y)}$

$=\tanh(x-y)$

51. $2\sinh x\cosh x=\sinh x\cosh x+\cosh x\sinh x$
$=\sinh(x+x)=\sinh 2x$

52. $\cosh^2 x+\sinh^2 x=\cosh x\cosh x+\sinh x\sinh x$
$=\cosh(x+x)=\cosh 2x$

53. (1) Let x be irrational and let y be any number. Further, let $\{r_n\}$ and $\{s_n\}$ be sequences of rational numbers such that $\lim_{n\to\infty} r_n=x$ and $\lim_{n\to\infty} s_n=y$.

(2) Since $\lim_{n\to\infty}(r_n-s_n)=\lim_{n\to\infty} r_n-\lim_{n\to\infty} s_n=x-y$ we have

$\dfrac{a^x}{a^y}=\dfrac{\lim_{n\to\infty} a^{r_n}}{\lim_{n\to\infty} a^{s_n}}=\lim_{n\to\infty}\left(\dfrac{a^{r_n}}{a^{s_n}}\right)$

$=\lim_{n\to\infty} a^{(r_n-s_n)}=a^{x-y}$

(3) (a) We first prove: if $\{u_n\}$ is *any* sequence of numbers (need not be rational) such that $\lim_{n\to\infty} u_n=z$, then $\lim_{n\to\infty} a^{u_n}=a^z$.

<u>Proof:</u> Let p_n be the truncation of the decimal expansion of u_n at the nth decimal place and let $q_n=p_n+\frac{1}{n}$; then

$0\le u_n-p_n<\frac{1}{n}$ and $0\le q_n-u_n<\frac{1}{n}$ so that

$u_n-\frac{1}{n}<p_n<q_n<u_n+\frac{1}{n}$. Thus

$\lim_{n\to\infty}\left(u_n-\frac{1}{n}\right)\le\lim_{n\to\infty} p_n\le\lim_{n\to\infty} q_n\le\lim_{n\to\infty}\left(u_n+\frac{1}{n}\right)$

or

$z\le\lim_{n\to\infty} p_n\le\lim_{n\to\infty}\le q_n\le z$,and we conclude that $\lim_{n\to\infty} p_n=\lim_{n\to\infty} q_n=x$. Since $p_n\le u_n<q_n$, it follows that a^{u_n} will be between a^{p_n} and a^{q_n} (the order will depend on whether $a<1$ or $a>1$). Hence, because $\lim_{n\to\infty} a^{p_n}=\lim_{n\to\infty} a^{q_n}=a^x$, $\lim_{n\to\infty} a^{u_n}=a^x$ by the Squeeze Theorem.

(b) Now choose any s_k; then , since $\lim_{n\to\infty} r_n s_k=s_k\lim_{n\to\infty} r_n=s_k x$,

$(a^x)^{s_k}=\left[\lim_{n\to\infty} a^{r_n}\right]^{s_k}=\lim_{n\to\infty}\left[\left(a^{r_n}\right)^{s_k}\right]$

$=\lim_{n\to\infty}\left[a^{r_n s_k}\right]=a^{xs_k}$

Define $u_n=xs_n$; then $\lim_{n\to\infty} u_n=x\lim_{n\to\infty} s_n=xy$.

Hence, from part (a),

$(a^x)^y=\lim_{n\to\infty}\left(a^x\right)^{s_n}=\lim_{n\to\infty}\left(a^{xs_n}\right)=a^{xy}$.

(4) Follows from (2) by letting $x=0$.

(5) $(ab)^x=\lim_{n\to\infty}(ab)^{r_n}=\lim_{n\to\infty}(a^{r_n}b^{r_n})$

$=\lim_{n\to\infty} a^{r_n}\cdot\lim_{n\to\infty} b^{r_n}=a^x b^x$

(6) $\left(\dfrac{a}{b}\right)^x=\left(a\cdot\dfrac{1}{b}\right)^x\overset{(5)}{=}a^x\left(\dfrac{1}{b}\right)^x\overset{(4)}{=}a^x\cdot\dfrac{1}{b^x}=\dfrac{a^x}{b^x}$

54. (1) $\lim_{t\to c}\sin^{-1}(t)=\sin^{-1}(c)$

$\lim_{t\to c}\cos^{-1}(t)=\cos^{-1}(c)$

$\lim_{t\to c}\tan^{-1}(t)=\tan^{-1}(c)$

$\lim_{t\to c}\cot^{-1}(t)=\cot^{-1}(c)$

$\lim_{t\to c}\sec^{-1}(t)=\sec^{-1}(c)$

$\lim_{t\to c}\csc^{-1}(t)=\csc^{-1}(c)$

(2) the validity of these statements follows directly from Theorem 2.5A and Theorem 2.6C. For example, let c be in the domain of $\sin^{-1}(t)$ and let a be such that $\sin(a)=c.$ By Theorem 2.5A, $\lim_{x\to a}\sin(x)=\sin(a)=c$; so by Theorem 2.6C, $\lim_{t\to c}\sin^{-1}(t)=\sin^{-1}(c)$.

55. A. (Hyperbolic)

(1) Statement:

$\lim_{x\to a} \sinh(x) = \sinh(a)$

$\lim_{x\to a} \cosh(x) = \cosh(a)$

$\lim_{x\to a} \tanh(x) = \tanh(a)$

$\lim_{x\to a} \coth(x) = \coth(a)$

$\lim_{x\to a} \operatorname{sech}(x) = \operatorname{sech}(a)$

$\lim_{x\to a} \operatorname{csch}(x) = \operatorname{csch}(a)$

(2) Proof

(a) By Theorem 2.3A and Theorem 2.6B(1),

$$\lim_{x\to a}\sinh(x) = \lim_{x\to a}\left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{\lim_{x\to a}(e^x - e^{-x})}{\lim_{x\to a} 2}\right) = \left(\frac{(\lim_{x\to a} e^x) - (\lim_{x\to a} e^{-x})}{2}\right) = \left(\frac{e^a - e^{-a}}{2}\right) = \sinh(a)$$

(b) The proof for $\cosh(x)$ is the same with "-" replaced by "+".

(c) The remaining proofs follow from Theorem 2.3A(7).

B. (Inverse Hyperbolic)

(1) Statement:

$\lim_{t\to c} \sinh^{-1}(t) = \sinh^{-1}(c)$

$\lim_{t\to c} \cosh^{-1}(t) = \cosh^{-1}(c)$

$\lim_{t\to c} \tanh^{-1}(t) = \tanh^{-1}(c)$

$\lim_{t\to c} \coth^{-1}(t) = \coth^{-1}(c)$

$\lim_{t\to c} \operatorname{sech}^{-1}(t) = \operatorname{sech}^{-1}(c)$

$\lim_{t\to c} \operatorname{csch}^{-1}(t) = \operatorname{csch}^{-1}(c)$

(2) Proof Follows directly from part A. and Theorem 2.6C.

56. Suppose the result is not true. Then there is an $\varepsilon_0 > 0$ such that: for every $\delta > 0$ there exists a value x_δ where $0 < |x_\delta - c| < \delta$ but $|a^{x_\delta} - a^c| > \varepsilon_0$. In particular, for every positive integer m there is an x_m such that $0 < |x_m - c| < \frac{1}{m}$ but $|a^{x_m} - a^c| > \varepsilon_0$. For each m, let $r_{m,1}, r_{m,2}, \ldots$ be a sequence of rational numbers that converges to x_m. This means $\lim_{n\to\infty} r_{m,n} = x_m$ and, by definition, $a^{x_m} = \lim_{n\to\infty} a^{r_{m,n}}$. Thus it is possible (by going far enough out in the sequence $r_{m,1}, r_{m,2}, \ldots$) to find a rational number t_m such that both $0 < |t_m - c| < \frac{1}{m}$ and $|a^{t_m} - a^c| > \varepsilon_0$

This yields $c - \frac{1}{m} < t_m < c + \frac{1}{m}$ and $a^{t_m} \notin (a^c - \varepsilon_0, a^c + \varepsilon_0)$ which means $\lim_{m\to\infty} t_m = c$ but $\lim_{m\to\infty} a^{t_m} \neq a^c$.

This contradicts the definition of a^c and so our original assumption is wrong, and, in fact, $\lim_{x\to c} a^x = a^c$.

57. $$\cosh(-x) = \frac{e^{(-x)} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

so cosh (x) is an even function.

58. $$\left(1 - \frac{1}{n}\right)^{-n} = \left(\frac{1}{\left(1 - \frac{1}{n}\right)}\right)^n = \left(\frac{n}{n-1}\right)^n = \left(\frac{(n-1)+1}{n-1}\right)^n = \left(1 + \frac{1}{n-1}\right)^n = \left(1 + \frac{1}{n-1}\right)^{n-1}\left(1 + \frac{1}{n-1}\right).$$

Thus:

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left[\left(1 + \frac{1}{n-1}\right)^{n-1}\left(1 + \frac{1}{n-1}\right)\right]_{m=n-1} = \lim_{m\to\infty}\left[\left(1 + \frac{1}{m}\right)^m\left(1 + \frac{1}{m}\right)\right] = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m \cdot \lim_{m\to\infty}\left(1 + \frac{1}{m}\right) = e \cdot (1 + 0) = e$$

59. (1) We first note that if $0 < a < 1$ and $0 < y < x$ then $a^{x-y} < 1$ so that $\frac{a^x}{a^y} < 1$ or $a^x < a^y$.

Hence, taking reciprocals, $a^{-y} < a^{-x}$.

Also: if $0 < a < b$ and $x > 0$ then $1 < \frac{b}{a}$ so that $1 < \left(\frac{b}{a}\right)^x = \left(\frac{b^x}{a^x}\right)$. Hence $a^x < b^x$.

(2) For any $-\frac{1}{2} < h < 0$ there is an integer m_h such that $m_h < -\frac{1}{h} < m_h + 1$. Thus

$-(m_h + 1) < \frac{1}{h} < -m_h$ and

$1 - \frac{1}{m_h} < 1 + h < 1 - \frac{1}{m_h + 1}$.

Since $0 < 1 - \frac{1}{m_h}, 1 + h, 1 - \frac{1}{m_h + 1} < 1$ it follows from (1) that

$$\left(1 - \frac{1}{m_h}\right)^{-m_h} < (1+h)^{\frac{1}{h}} < \left(1 - \frac{1}{m_h + 1}\right)^{-(m_h+1)}.$$

Using the fact that $m_h, (m_h + 1) \to \infty$ as $h \to 0^-$, and the result from Problem 58, we have

$$\lim_{h \to 0^-}\left(1 - \frac{1}{m_h}\right)^{-m_h} = \lim_{m_h \to \infty}\left(1 - \frac{1}{m_h}\right)^{-m_h} = e$$

and

$$\lim_{h \to 0^-}\left(1 - \frac{1}{m_h + 1}\right)^{-(m_h+1)}.$$

$$= \lim_{m_h + 1 \to \infty}\left(1 - \frac{1}{m_h + 1}\right)^{-(m_h+1)} = e$$

Thus, by the Squeeze Theorem,

$$\lim_{h \to 0^-}(1+h)^{\frac{1}{h}} = e.$$

2.7 Concepts Review

1. $\lim_{x \to c} f(x)$

2. every integer

3. $\lim_{x \to a^+} f(x) = f(a); \lim_{x \to b^-} f(x) = f(b)$

4. $a; b; f(c) = W$

Problem Set 2.7

1. $\lim_{x \to 3}[(x-3)(x-4)] = 0 = f(3)$; continuous

2. $\lim_{x \to 3}(x^2 - 9) = 0 = g(3)$; continuous

3. $\lim_{x \to 3} \frac{3}{x-3}$ and $h(3)$ do not exist, so $h(x)$ is not continuous at 3.

4. $\lim_{t \to 3} \sqrt{t-4}$ and $g(3)$ do not exist, so $g(t)$ is not continuous at 3.

5. $\lim_{t \to 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist, so $h(t)$ is not continuous at 3.

6. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

7. $\lim_{t \to 3} |t| = 3 = f(3)$; continuous

8. $\lim_{t \to 3} |t-2| = 1 = g(3)$; continuous

9. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

10. $f(3)$ does not exist, so $f(x)$ is not continuous at 3.

11. $\lim_{t \to 3} \frac{t^3 - 27}{t-3} = \lim_{t \to 3} \frac{(t-3)(t^2 + 3t + 9)}{t-3}$

$= \lim_{t \to 3}(t^2 + 3t + 9) = (3)^2 + 3(3) + 9 = 27 = r(3)$

continuous

12. From Problem 11, $\lim_{t \to 3} r(t) = 27$, so $r(t)$ is not continuous at 3 because $\lim_{t \to 3} r(t) \neq r(3)$.

13. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+}(3 - t) = 0$

$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-}(t - 3) = 0$

$\lim_{t \to 3} f(t) = f(3)$; continuous

14. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+}(3 - t)^2 = 0$

$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-}(t^2 - 9) = 0$

$\lim_{t \to 3} f(t) = f(3)$; continuous

15. $\lim_{t \to 3} f(x) = -2 = f(3)$; continuous

16. g is discontinuous at $x = -3, 4, 6, 8$; g is left continuous at $x = 4, 8$; g is right continuous at $x = -3, 6$

17. h is continuous on the intervals $(-\infty, -5), \ [-5, 4], \ (4, 6), \ [6, 8], \ (8, \infty)$

18. $\lim_{x\to 7} \frac{x^2 - 49}{x - 7} = \lim_{x\to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x\to 7}(x+7)$
$= 7 + 7 = 14$
Define $f(7) = 14$.

19. $\lim_{x\to 3} \frac{2x^2 - 18}{3 - x} = \lim_{x\to 3} \frac{2(x+3)(x-3)}{3-x}$
$= \lim_{x\to 3}[-2(x+3)] = -2(3+3) = -12$
Define $f(3) = -12$.

20. $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$
Define $g(0) = 1$

21. $\lim_{t\to 1} \frac{\sqrt{t} - 1}{t - 1} = \lim_{t\to 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}$
$= \lim_{t\to 1} \frac{t-1}{(t-1)(\sqrt{t}+1)} = \lim_{t\to 1} \frac{1}{\sqrt{t}+1} = \frac{1}{2}$
Define $H(1) = \frac{1}{2}$.

22. $\lim_{x\to -1} \frac{x^4 + 2x^2 - 3}{x + 1} = \lim_{x\to -1} \frac{(x^2-1)(x^2+3)}{x+1}$
$= \lim_{x\to -1} \frac{(x+1)(x-1)(x^2+3)}{x+1}$
$= \lim_{x\to -1} [(x-1)(x^2+3)]$
$= (-1-1)[(-1)^2 + 3] = -8$
Define $\phi(-1) = -8$.

23. $\lim_{x\to -1} \sin\left(\frac{x^2 - 1}{x + 1}\right) = \lim_{x\to -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$
$= \lim_{x\to -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$
Define $F(-1) = -\sin 2$.

24. Discontinuous at $x = \pi, 30$

25. $f(x) = \frac{33 - x^2}{(\pi - x)(x - 3)}$
Discontinuous at $x = 3, \pi$

26. Continuous at all points

27. Discontinuous at all $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

28. Discontinuous at all $u \le -5$

29. Discontinuous at $u = -1$

30. Continuous at all points

31. $G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$
Discontinuous on $(-\infty, -2] \cup [2, \infty)$

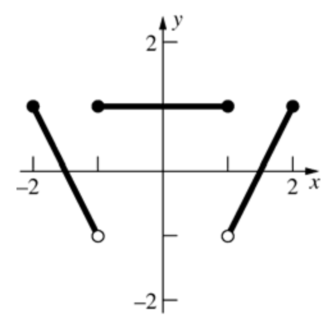
32. Continuous at all points since $\lim_{x\to 0} f(x) = 0 = f(0)$ and $\lim_{x\to 1} f(x) = 1 = f(1)$.

33. $\lim_{x\to 0} g(x) = 0 = g(0)$
$\lim_{x\to 1^+} g(x) = 1, \ \lim_{x\to 1^-} g(x) = -1$
$\lim_{x\to 1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = 1$.

34. Discontinuous at every integer

35. Discontinuous at $t = n + \frac{1}{2}$ where n is any integer

36.



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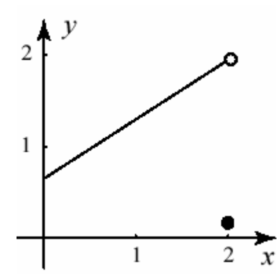

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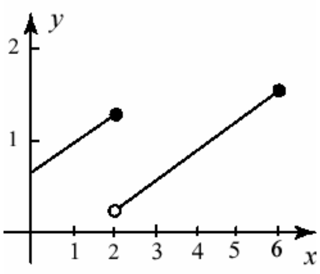

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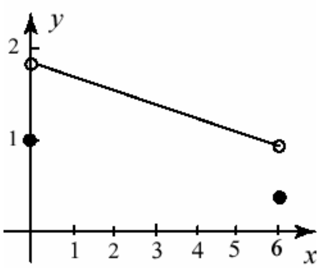

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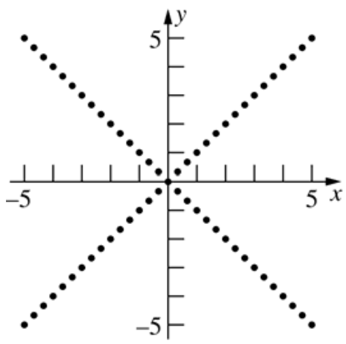


Discontinuous at all points except $x = 0$, because $\lim_{x\to c} f(x) \neq f(c)$ for $c \neq 0$. $\lim_{x\to c} f(x)$ exists only at $c = 0$ and $\lim_{x\to 0} f(x) = 0 = f(0)$.

41. Continuous.

42. Discontinuous: removable, define $f(10) = 20$

43. Discontinuous: removable, define $f(0) = 1$

44. Discontinuous: nonremovable.

45. Discontinuous, removable, redefine $g(0) = 1$

46. Discontinuous: removable, define $F(0) = 0$

47. Discontinuous: nonremovable.

48. Discontinuous: removable, define $f(4) = 4$

In problems 49-54 we will show that the function is continuous on its entire domain, which therefore is the largest interval.

49. $f(x) = \sqrt{25 - x^2}$

Let $g(x) = 25 - x^2$; then $f(x) = \sqrt{g(x)}$. To find the domain of $f(x)$ we require

$0 \le 25 - x^2$ or $-5 \le x \le 5$.

Now $g(x)$ is continuous on $[-5,5]$ by Thm. 2.7A; hence $f(x)$ is continuous on $[-5,5]$ by Thm 2.7C.

50. $f(x) = \dfrac{1}{\sqrt{25 - x^2}}$

Refer to problem 49. Let $g(x) = 25 - x^2$ and $h(x) = 1$; then $f(x) = \dfrac{h(x)}{\sqrt{g(x)}}$. The domain of $f(x)$ is $(-5,5)$; $g(x)$ is continuous on (-5,5) by prob. 49 and $h(x)$ by Thm. 2.7A.

Thus $f(x)$ is continuous on $(-5,5)$ by Thm 2.7C.

51. $f(x) = \sin^{-1}(x)$

The domain of $\sin^{-1}(x)$ is $[-1,1]$, so by Thm 2.7D, $\sin^{-1}(x)$ is continuous on $(-1,1)$. Now since (2.7D again) $\lim_{t\to -\frac{\pi}{2}^+} \sin(t) = \sin(-\frac{\pi}{2}) = -1$ and $\lim_{t\to \frac{\pi}{2}^-} \sin(t) = \sin(\frac{\pi}{2}) = 1$, we have (Thm. 2.6C)

$\lim_{x\to -1^+} \sin^{-1}(x) = \sin^{-1}(-1)$ and

$\lim_{x\to 1^-} \sin^{-1}(x) = \sin^{-1}(1)$.

Thus $\sin^{-1}(x)$ is continuous on $[-1,1]$.

52. $f(x) = \text{sech}(x)$

Since $\cosh(x)$ is positive for all x, the domain of $\text{sech}(x)$ is $(-\infty,\infty)$. Now $\cosh(x)$ is continuous on $(-\infty,\infty)$ (Thm. 2.7D) as is $g(x) = 1$ (Thm. 2.7A). Therefore $\text{sech}(x) = \dfrac{1}{\cosh(x)}$ is continuous on $(-\infty,\infty)$ by Thm 2.7C.

53. $f(x) = \sec^{-1}(x),\ x \ge 0$

The domain of this function is $[1,\infty)$ and by Thm. 2.7D, $\sec^{-1}(x)$ is continuous on $(1,\infty)$. Further, since $\lim_{t\to 0^+} \sec(t) = \sec(0) = 1$ (Thm. 2.7D), we have (Thm.. 2.6C) $\lim_{x\to 1^+} \sec^{-1}(x) = \sec^{-1}(1)$

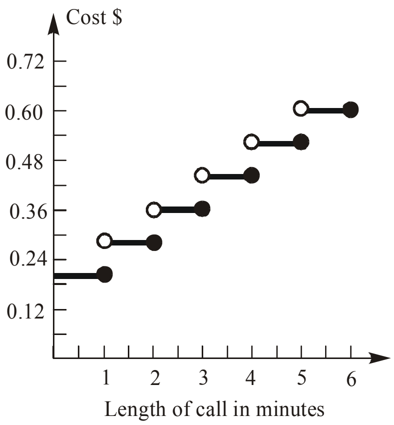
Thus $\sec^{-1}(x)$ is continuous on $[1,\infty)$.

54. $f(x) = \operatorname{sech}^{-1}(x)$

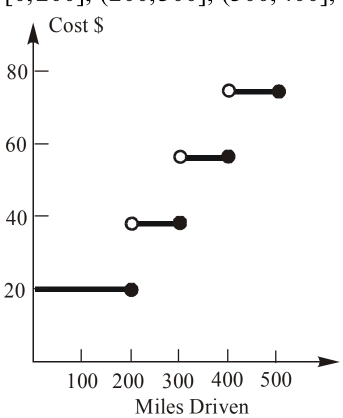
The domain of $\operatorname{sech}^{-1}(x)$ is $(0,1]$ and by Thm. 2.7D, $\operatorname{sech}^{-1}(x)$ is continuous on $(0,1)$. Further, since $\lim_{t \to 0^-} \operatorname{sech}(t) = \operatorname{sech}(0) = 1$ (Thm. 2.7D), we have (Thm. 2.6C)
$\lim_{x \to 1^-} \operatorname{sech}^{-1}(x) = \operatorname{sech}^{-1}(1)$.

Thus $\sec^{-1}(x)$ is continuous on $(0,1]$.

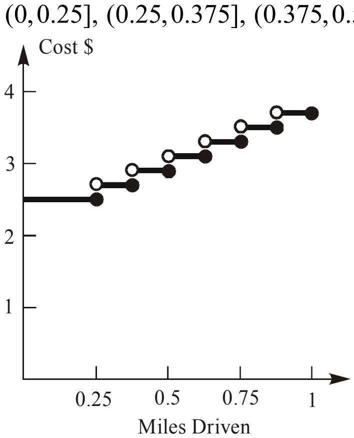
55. The function is continuous on the intervals $(0,1], (1,2], (2,3], \ldots$



56. The function is continuous on the intervals $[0, 200], (200, 300], (300, 400], \ldots$



57. The function is continuous on the intervals $(0, 0.25], (0.25, 0.375], (0.375, 0.5], \ldots$



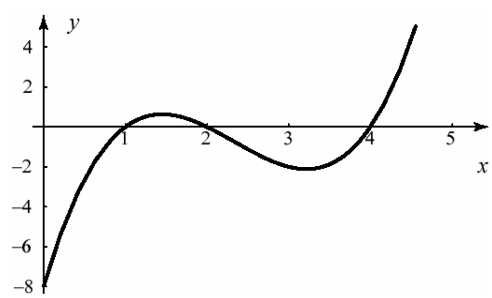
58. Let $f(x) = x^3 + 3x - 2$. f is continuous on [0, 1]. $f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

59. Because the function is continuous on $[0, 2\pi]$ and

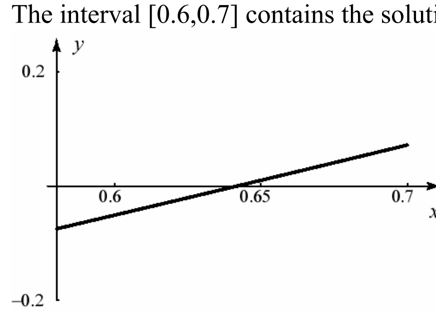
$(\cos 0)0^3 + 6\sin^5 0 - 3 = -3 < 0$,

$(\cos 2\pi)(2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0$, there is at least one number c between 0 and 2π such that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$.

60. Let $f(x) = x^3 - 7x^2 + 14x - 8$. $f(x)$ is continuous at all values of x.
$f(0) = -8$, $f(5) = 12$
Because 0 is between –8 and 12, there is at least one number c between 0 and 5 such that
$f(x) = x^3 - 7x^2 + 14x - 8 = 0$.
This equation has three solutions ($x = 1,2,4$)



61. Let $f(x) = \sqrt{x} - \cos x$.. $f(x)$ is continuous at all values of $x \geq 0$. $f(0) = -1$, $f(\pi/2) = \sqrt{\pi/2}$
Because 0 is between –1 and $\sqrt{\pi/2}$, there is at least one number c between 0 and $\pi/2$ such that
$f(x) = \sqrt{x} - \cos x = 0$.
The interval [0.6,0.7] contains the solution.



62. Let $f(x) = x^5 + 4x^3 - 7x + 14$
$f(x)$ is continuous at all values of x.
$f(-2) = -36$, $f(0) = 14$
Because 0 is between –36 and 14, there is at least one number c between –2 and 0 such that
$f(x) = x^5 + 4x^3 - 7x + 14 = 0$.

63. Suppose that f is continuous at c, so $\lim_{x\to c} f(x) = f(c)$. Let $x = c + t$, so $t = x - c$, then as $x \to c$, $t \to 0$ and the statement $\lim_{x\to c} f(x) = f(c)$ becomes $\lim_{t\to 0} f(t+c) = f(c)$.
Suppose that $\lim_{t\to 0} f(t+c) = f(c)$ and let $x = t + c$, so $t = x - c$. Since c is fixed, $t \to 0$ means that $x \to c$ and the statement $\lim_{t\to 0} f(t+c) = f(c)$ becomes $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c.

64. Since $f(x)$ is continuous at c, $\lim_{x\to c} f(x) = f(c) > 0$. Choose $\varepsilon = f(c)$, then there exists a $\delta > 0$ such that
$0 < |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
Thus, $f(x) - f(c) > -\varepsilon = -f(c)$, or $f(x) > 0$.
Since also $f(c) > 0$, $f(x) > 0$ for all x in $(c-\delta, c+\delta)$.

65. Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) = -f(0) \le 0$ and $g(1) = 1 - f(1) \ge 0$ since $0 \le f(x) \le 1$ on $[0, 1]$. If $g(0) = 0$, then $f(0) = 0$ and $c = 0$ is a fixed point of f. If $g(1) = 0$, then $f(1) = 1$ and $c = 1$ is a fixed point of f. If neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and $g(1) > 0$ so there is some c in $[0, 1]$ such that $g(c) = 0$. If $g(c) = 0$ then $c - f(c) = 0$ or $f(c) = c$ and c is a fixed point of f.

66. For $f(x)$ to be continuous everywhere, $f(1) = a(1) + b = 2$ and $f(2) = 6 = a(2) + b$

$$\begin{aligned} a + b &= 2 \\ \underline{2a + b} &\underline{= 6} \\ -a &= -4 \\ a = 4,\ \ b &= -2 \end{aligned}$$

67. For x in $[0, 1]$, let $f(x)$ indicate where the string originally at x ends up. Thus $f(0) = a$, $f(1) = b$. $f(x)$ is continuous since the string is unbroken. Since $0 \le a,\ b \le 1$, $f(x)$ satisfies the conditions of Problem 65, so there is some c in $[0, 1]$ with $f(c) = c$, i.e., the point of string originally at c ends up at c.

68. The Intermediate Value Theorem does not imply the existence of a number c between –2 and 2 such that $f(c) = 0$. The reason is that the function $f(x)$ is not continuous on $[-2, 2]$.

69. Let $f(x)$ be the difference in times on the hiker's watch where x is a point on the path, and suppose $x = 0$ at the bottom and $x = 1$ at the top of the mountain.
So $f(x)$ = (time on watch on the way up) – (time on watch on the way down).
$f(0) = 4 - 11 = -7$, $f(1) = 12 - 5 = 7$. Since time is continuous, $f(x)$ is continuous, hence there is some c between 0 and 1 where $f(c) = 0$. This c is the point where the hiker's watch showed the same time on both days.

70. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is the length of the side of the rectangle which makes angle θ with the x-axis minus the length of the sides perpendicular to it. f is continuous on $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is circumscribed by a square. If $f(0) \ne 0$, then observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the Intermediate Value Theorem, there is an angle θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$.
Hence, D can be circumscribed by a square.

71. Yes, g is continuous at R.

$$\lim_{r\to R^-} g(r) = \frac{GMm}{R^2} = \lim_{r\to R^+} g(r)$$

72. No. By the Intermediate Value Theorem, if f were to change signs on $[a,b]$, then f must be 0 at some c in $[a,b]$. Therefore, f cannot change sign.

73. **a.** $f(x) = f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. We want to prove that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, $\lim_{x\to c} [f(x) - f(c)] = 0$. But $f(x) - f(c) = f(x - c)$, so $\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} f(x-c)$. Let $h = x - c$ then as $x \to c$, $h \to 0$ and $\lim_{x\to c} f(x-c) = \lim_{h\to 0} f(h) = f(0) = 0$. Hence $\lim_{x\to c} f(x) = f(c)$ and f is continuous at c. Thus, f is continuous everywhere, since c was arbitrary.

b. By Problem 43 of Section 1.5, $f(t) = mt$ for all t in Q. Since $g(t) = mt$ is a polynomial function, it is continuous for all real numbers. $f(t) = g(t)$ for all t in Q, thus $f(t) = g(t)$ for all t in R, i.e. $f(t) = mt$.

74. If $f(x)$ is continuous on an interval then $\lim_{x\to c} f(x) = f(c)$ for all points in the interval:

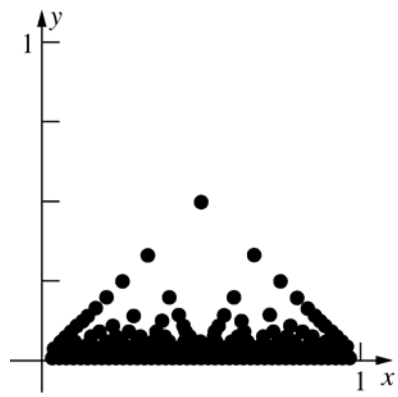
$$\lim_{x\to c} f(x) = f(c) \Rightarrow \lim_{x\to c} |f(x)|$$

$$= \lim_{x\to c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x\to c} f(x)\right)^2}$$

$$= \sqrt{(f(c))^2} = |f(c)|$$

75. Suppose $f(x) = \begin{cases} 1 \text{ if } x \ge 0 \\ -1 \text{ if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

76. **a.**



b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q}$. Hence, $\lim_{x\to r} f(x)$ does not exist.

If c is any irrational number in (0, 1), then as $x = \frac{p}{q} \to c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \to \infty$, so $f(x) \to 0$ as $x \to c$. Thus, $\lim_{x\to c} f(x) = 0 = f(c)$ for any irrational number c.

77. **a.** Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

Domain: $\left[-\frac{3}{4}, \frac{3}{4}\right]$;

Range: $\left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$

b. At $x = 0$

c. If $x = 0$, $f(x) = 0$, if $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, 0, \frac{3}{4}$ are fixed points of f.

2.8 Chapter Review

Concepts Test

1. False. Consider $f(x) = [\![x]\!]$ at $x = 2$.

2. False: c may not be in the domain of $f(x)$, or it may be defined separately.

3. False: c may not be in the domain of $f(x)$, or it may be defined separately.

4. True. By definition, where $c = 0, L = 0$.

5. False: If $f(c)$ is not defined, $\lim_{x\to c} f(x)$ might exist; e.g., $f(x) = \frac{x^2 - 4}{x + 2}$.
   $f(-2)$ does not exist, but $\lim_{x\to -2} \frac{x^2 - 4}{x + 2} = -4$.

6. True: $\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{x - 5}$
   $= \lim_{x\to 5} (x + 5) = 5 + 5 = 10$

7. True: Substitution Theorem

8. False: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

9. False: The tangent function is not defined for all values of c.

10. True: If x is in the domain of $\tan x = \frac{\sin x}{\cos x}$, then $\cos x \ne 0$, and Theorem A.7 applies.

11. True: Since both $\sinh x$ and $\cosh x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x) = 2\sinh^2 x - \cosh x$ is also continuous for all real numbers.

12. True. By definition, $\lim_{x \to c} f(x) = f(c)$.

13. False: $\lim_{x \to 0^-}$ may not exist

14. True: See Example 8 of section 2.7.

15. True: By Theorem C of section 2.6.

16. True: By definition, if r_n is a sequence of rational numbers that converges to the irrational number π, then π^π is defined to be $\pi^\pi = \lim_{n \to \infty} \pi^{r_n}$.

17. False: Consider $f(x) = \sin x$.

18. True. By the definition of continuity on an interval.

19. False: By Theorem C of section 2.6 and the subsequent discussion.

20. False. It could be the case where $\lim_{x \to -\infty} f(x) = 2$

21. False: The graph has many vertical asymptotes; e.g., $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

22. True: $x = 2$; $x = -2$

23. True: As $x \to 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.

24. False: $\lim_{x \to c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.

25. True: $\lim_{x \to c} f(x) = f\left(\lim_{x \to c} x\right) = f(c)$, so f is continuous at $x = c$.

26. True: $\lim_{x \to 2.3} \left[\!\left[\frac{x}{2} \right]\!\right] = 1 = f(2.3)$

27. True: Choose $\varepsilon = 0.001 f(2)$ then since $\lim_{x \to 2} f(x) = f(2)$, there is some δ such that $0 < |x - 2| < \delta \Rightarrow |f(x) - f(2)| < 0.001 f(2)$, or $-0.001 f(2) < f(x) - f(2) < 0.001 f(2)$
Thus, $0.999 f(2) < f(x) < 1.001 f(2)$ and $f(x) < 1.001 f(2)$ for $0 < |x - 2| < \delta$.
Since $f(2) < 1.001 f(2)$, as $f(2) > 0$, $f(x) < 1.001 f(2)$ on $(2 - \delta,\ 2 + \delta)$.

28. False: That $\lim_{x \to c} [f(x) + g(x)]$ exists does not imply that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and $g(x) = \frac{x+7}{x+2}$ for $c = -2$.

29. True: Squeeze Theorem

30. True: A function has only one limit at a point, so if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, $L = M$

31. False: That $f(x) \neq g(x)$ for all x does not imply that $\lim_{x \to c} f(x) \neq \lim_{x \to c} g(x)$. For example, if $f(x) = \frac{x^2 + x - 6}{x - 2}$ and $g(x) = \frac{5}{2} x$, then $f(x) \neq g(x)$ for all x, but $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 5$.

32. False: If $f(x) < 10$, $\lim_{x \to 2} f(x)$ could equal 10 if there is a discontinuity point (2, 10). For example,
$f(x) = \frac{-x^3 + 6x^2 - 2x - 12}{x - 2} < 10$ for all x, but $\lim_{x \to 2} f(x) = 10$.

33. True:
$$\lim_{x \to a} |f(x)| = \lim_{x \to a} \sqrt{f^2(x)} = \sqrt{\left[\lim_{x \to a} f(x)\right]^2} = \sqrt{(b)^2} = |b|$$

34. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

1. $\lim_{x\to 2}\frac{x-2}{x+2}=\frac{2-2}{2+2}=\frac{0}{4}=0$

2. $\lim_{u\to 1}\frac{u^2-1}{u+1}=\frac{1^2-1}{1+1}=0$

3. $\lim_{u\to 1}\frac{u^2-1}{u-1}=\lim_{u\to 1}\frac{(u-1)(u+1)}{u-1}=\lim_{u\to 1}(u+1)$
$= 1 + 1 = 2$

4. $\lim_{u\to 1}\frac{u+1}{u^2-1}=\lim_{u\to 1}\frac{u+1}{(u+1)(u-1)}=\lim_{u\to 1}\frac{1}{u-1}$;
does not exist

5. $\lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}=\lim_{x\to 2}\frac{\frac{x-2}{x}}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x(x+2)}$
$=\frac{1}{2(2+2)}=\frac{1}{8}$

6. $\lim_{z\to 2}\frac{z^2-4}{z^2+z-6}=\lim_{z\to 2}\frac{(z+2)(z-2)}{(z+3)(z-2)}$
$=\lim_{z\to 2}\frac{z+2}{z+3}=\frac{2+2}{2+3}=\frac{4}{5}$

7. $\lim_{x\to 0}\frac{\tan x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}=\lim_{x\to 0}\frac{1}{2\cos^2 x}$
$=\frac{1}{2\cos^2 0}=\frac{1}{2}$

8. $\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$
$=\lim_{y\to 1}\frac{y^2+y+1}{y+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$

9. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}$
$=\lim_{x\to 4}(\sqrt{x}+2)=\sqrt{4}+2=4$

10. $\lim_{x\to 0}\frac{\cos x}{x}$ does not exist.

11. $\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$

12. $\lim_{x\to (1/2)^+}[\![4x]\!]=2$

13. $\lim_{t\to 2^-}\left([\![t]\!]-t\right)=\lim_{t\to 2^-}[\![t]\!]-\lim_{t\to 2^-}t=1-2=-1$

14. $\lim_{x\to 1^-}\frac{|x-1|}{x-1}=\lim_{x\to 1^-}\frac{1-x}{x-1}=-1$ since $x-1<0$ as
$x\to 1^-$

15. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{5}{3}\frac{\sin 5x}{5x}$
$=\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{3}\times 1=\frac{5}{3}$

16. $\lim_{x\to 0}\frac{1-\cos 2x}{3x}=\lim_{x\to 0}\frac{2}{3}\frac{1-\cos 2x}{2x}$
$=\frac{2}{3}\lim_{x\to 0}\frac{1-\cos 2x}{2x}=\frac{2}{3}\times 0=0$

17. $\lim_{x\to\infty}\frac{x-1}{x+2}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{2}{x}}=\frac{1+0}{1+0}=1$

18. Since $-1\le \sin t\le 1$ for all t and $\lim_{t\to\infty}\frac{1}{t}=0$, we get $\lim_{t\to\infty}\frac{\sin t}{t}=0$.

19. $\lim_{x\to\infty}e^{x^2}=\infty$

20. $\lim_{x\to 0^+}\frac{\cos x}{x}=\infty$**,** because as $x\to 0^+$, $\cos x\to 1$ while the denominator goes to 0 from the right.

21. $\lim_{x\to \pi/4^-}\tan 2x=\infty$ because as $x\to(\pi/4)^-$, $2x\to(\pi/2)^-$, so $\tan 2x\to\infty$.

22. $\lim_{x\to 0^+}\ln x^2=-\infty$**,** because as $x\to 0^+$, $x^2\to 0^+$, and $\ln x^2$ decreases without bound.

23. Preliminary analysis: Let ε > 0. We need to find a δ > 0 such that
$0<|x-3|<\delta\Rightarrow|(2x+1)-7|<\varepsilon.$
$|2x-6|<\varepsilon\Leftrightarrow 2|x-3|<\varepsilon$
$\Leftrightarrow|x-3|<\frac{\varepsilon}{2}.$ Choose $\delta=\frac{\varepsilon}{2}.$

Let ε > 0. Choose $\delta=\varepsilon/2$. Thus,

$\left|(2x+1)-7\right|=\left|2x-6\right|=2\left|x-3\right|<2(\varepsilon/2)=\varepsilon.$

24. **a.** $f(1) = 0$

b. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x) = 0$

c. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

d. $\lim_{x\to -1} f(x) = -1$ because

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^3 = -1$ and

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$

25. **a.** f is discontinuous at $x = 1$ because $f(1) = 0$, but $\lim_{x\to 1} f(x)$ does not exist. f is discontinuous at $x = -1$ because $f(-1)$ does not exist.

b. Define $f(-1) = -1$

26. **a.** $0 < |u-a| < \delta \Rightarrow |g(u) - M| < \varepsilon$

b. $0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon$

27. **a.** $\lim_{x\to 3}[2f(x) - 4g(x)]$

$= 2\lim_{x\to 3} f(x) - 4\lim_{x\to 3} g(x)$

$= 2(3) - 4(-2) = 14$

b. $\lim_{x\to 3} g(x)\frac{x^2-9}{x-3} = \lim_{x\to 3} g(x)(x+3)$

$= \lim_{x\to 3} g(x)\cdot \lim_{x\to 3}(x+3) = -2\cdot(3+3) = -12$

c. $g(3) = -2$

d. $\lim_{x\to 3} g(f(x)) = g\left(\lim_{x\to 3} f(x)\right) = g(3) = -2$

e. $\lim_{x\to 3}\sqrt{f^2(x) - 8g(x)}$

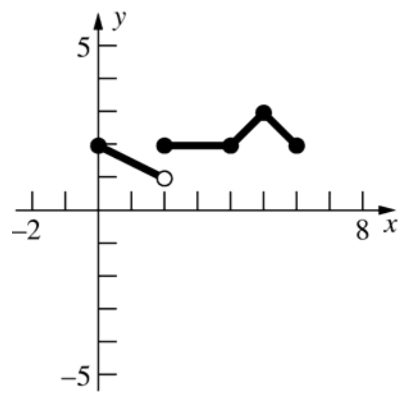
$= \sqrt{\left[\lim_{x\to 3} f(x)\right]^2 - 8\lim_{x\to 3} g(x)}$

$= \sqrt{(3)^2 - 8(-2)} = 5$

f. $\lim_{x\to 3}\frac{|g(x) - g(3)|}{f(x)} = \frac{|-2 - g(3)|}{3} = \frac{|-2-(-2)|}{3}$

$= 0$

28.



29. $a(0) + b = -1$ and $a(1) + b = 1$

$b = -1;\quad a + b = 1$

$a - 1 = 1$

$a = 2$

30. Let $f(x) = x^5 - 4x^3 - 3x + 1$

$f(2) = -5$, $f(3) = 127$

Because $f(x)$ is continuous on $[2, 3]$ and $f(2) < 0 < f(3)$, there exists some number c between 2 and 3 such that $f(c) = 0$.

31. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\frac{x}{x^2+1} = \lim_{x\to -\infty}\frac{x}{x^2+1} = 0$, so $y = 0$ is a horizontal asymptote.

32. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\frac{x^2}{x^2+1} = \lim_{x\to -\infty}\frac{x^2}{x^2+1} = 1$, so $y = 1$ is a horizontal asymptote.

33. Vertical: $x = 1, x = -1$ because $\lim_{x\to 1^+}\frac{x^2}{x^2-1} = \infty$

and $\lim_{x\to -1^-}\frac{x^2}{x^2-1} = \infty$

Horizontal: $\lim_{x\to\infty}\frac{x^2}{x^2-1} = \lim_{x\to -\infty}\frac{x^2}{x^2-1} = 1$, so $y = 1$ is a horizontal asymptote.

34. Vertical: $x = 2, x = -2$ because

$\lim_{x\to 2^+}\frac{x^3}{x^2-4} = \infty$ and $\lim_{x\to -2^-}\frac{x^3}{x^2-4} = \infty$

Horizontal: $\lim_{x\to\infty}\frac{x^3}{x^2-4} = \infty$ and

$\lim_{x\to -\infty}\frac{x^3}{x^2-4} = -\infty$, so there are no horizontal asymptotes.

35. Vertical: $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \ldots$ because $\lim_{x\to\pi/4^-} \tan 2x = \infty$ and similarly for other odd multiples of $\pi/4$.

Horizontal: None, because $\lim_{x\to\infty} \tan 2x$ and $\lim_{x\to-\infty} \tan 2x$ do not exist.

36. Vertical:
None since the domain of $\tan^{-1} x$ is all real numbers.

Horizontal:
$y = \pi$ because $\lim_{x\to\infty} 2\tan^{-1} x = 2 \lim_{x\to\infty} \tan^{-1} x$.

$$= 2\cdot\frac{\pi}{2} = \pi$$

$y = -\pi$ because $\lim_{x\to-\infty} 2\tan^{-1} x = 2 \lim_{x\to-\infty} \tan^{-1} x$

$$= 2\cdot\left(-\frac{\pi}{2}\right) = -\pi$$

37. $f(x) = \cos^{-1}\left(\frac{x}{2}\right)$

The domain of $\cos^{-1}(x)$ is $[-1,1]$, so by Thm 2.7D, $\cos^{-1}(x)$ is continuous on $(-1,1)$. Now since (2.7D again) $\lim_{t\to 0^+} \cos(t) = \cos(0) = 1$ and $\lim_{t\to\pi^-} \cos(t) = \cos(\pi) = -1$ we have (Thm. 2.6C) $\lim_{x\to -1^+} \cos^{-1}(x) = \cos^{-1}(-1)$ and $\lim_{x\to 1^-} \cos^{-1}(x) = \cos^{-1}(1)$.
Thus $\cos^{-1}(x)$ is continuous on $[-1,1]$.

Let $g(x) = \frac{x}{2}$, $\frac{x}{2} \in [-1,1]$; then $x \in [-2,2]$ and by Thm. 2.7A, $g(x)$ is continuous on $[-2,2]$.
Therefore $x \in [-2,2] \Rightarrow \frac{x}{2} \in [-1,1] \Rightarrow \cos^{-1}\left(\frac{x}{2}\right)$ is continuous at x by Thm. 2.7E. Thus the largest interval is $[-2,2]$.

38. $f(x) = \ln(25 - x^2)$
The domain of $f(x)$ is $\{x \mid 25 - x^2 > 0\} = (-5,5)$.
Now $25 - x^2$ is continuous on $(-5,5)$ by Thm. 2.7A and $\ln(x)$ is continuous on $(0,\infty)$ by Thm. 2.7D; thus $f(x)$ is continuous on $(-5,5)$ by Thm. 2.7E. This is the largest interval.

Review and Preview Problems

1. a. $f(2) = 2^2 = 4$

b. $f(2.1) = 2.1^2 = 4.41$

c. $f(2.1) - f(2) = 4.41 - 4 = 0.41$

d. $\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{0.41}{0.1} = 4.1$

e. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$

f. $f(a+h) - f(a) = a^2 + 2ah + h^2 - a^2$
$= 2ah + h^2$

g. $\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{2ah + h^2}{h} = 2a + h$

h. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h\to 0} (2a + h) = 2a$

2. a. $f(2) = 1/2$

b. $f(2.1) = 1/2.1 \approx 0.476$

c. $f(2.1) - f(2) = 0.476 - 0.5 = -0.024$

d. $\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{-0.024}{0.1} = -0.24$

e. $f(a+h) = 1/(a+h)$

f. $f(a+h) - f(a) = 1/(a+h) - 1/a$
$= \frac{-h}{a(a+h)}$

g. $\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{\frac{-h}{a(a+h)}}{h} = \frac{-1}{a(a+h)}$

h. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{-1}{a^2}$

3. a. $f(2) = \sqrt{2} \approx 1.414$

b. $f(2.1) = \sqrt{2.1} \approx 1.449$

c. $f(2.1) - f(2) = 1.449 - 1.414 = 0.035$

d. $\dfrac{f(2.1)-f(2)}{2.1-2}=\dfrac{0.035}{0.1}=0.35$

e. $f(a+h)=\sqrt{a+h}$

f. $f(a+h)-f(a)=\sqrt{a+h}-\sqrt{a}$

g. $\dfrac{f(a+h)-f(a)}{(a+h)-a}=\dfrac{\sqrt{a+h}-\sqrt{a}}{h}$

h. $\lim_{h\to0}\dfrac{f(a+h)-f(a)}{(a+h)-a}=\lim_{h\to0}\dfrac{\sqrt{a+h}-\sqrt{a}}{h}$

$=\lim_{h\to0}\dfrac{\left(\sqrt{a+h}-\sqrt{a}\right)\left(\sqrt{a+h}+\sqrt{a}\right)}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$

$=\lim_{h\to0}\dfrac{a+h-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$

$=\lim_{h\to0}\dfrac{h}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$

$=\lim_{h\to0}\dfrac{1}{\sqrt{a+h}+\sqrt{a}}=\dfrac{1}{2\sqrt{a}}=\dfrac{\sqrt{a}}{2a}$

4. a. $f(2)=(2)^3+1=8+1=9$

b. $f(2.1)=(2.1)^3+1=9.261+1=10.261$

c. $f(2.1)-f(2)=10.261-9=1.261$

d. $\dfrac{f(2.1)-f(2)}{2.1-2}=\dfrac{1.261}{0.1}=12.61$

e. $f(a+h)=(a+h)^3+1$

$=a^3+3a^2h+3ah^2+h^3+1$

f. $f(a+h)-f(a)=\left[(a+h)^3+1\right]-\left[a^3+1\right]$

$=\left(a^3+3a^2h+3ah^2+h^3+1\right)-\left(a^3+1\right)$

$=3a^2h+3ah^2+h^3$

g. $\dfrac{f(a+h)-f(a)}{(a+h)-a}=\dfrac{3a^2h+3ah^2+h^3}{h}$

$=3a^2+3ah+h^2$

h. $\lim_{h\to0}\dfrac{f(a+h)-f(a)}{(a+h)-a}=\lim_{h\to0}3a^2+3ah+h^2$

$=3a^2$

5. a. $(a+b)^3=a^3+3a^2b+\cdots$

b. $(a+b)^4=a^4+4a^3b+\cdots$

c. $(a+b)^5=a^5+5a^4b+\cdots$

6. $(a+b)^n=a^n+na^{n-1}b+\cdots$

7. $\sin(x+h)=\sin x\cos h+\cos x\sin h$

8. $\cos(x+h)=\cos x\cos h-\sin x\sin h$

9. a. The point will be at position $(10,0)$ in all three cases ($t=1,2,3$) because it will have made 4, 8, and 12 revolutions respectively.

b. Since the point is rotating at a rate of 4 revolutions per second, it will complete 1 revolution after $\frac{1}{4}$ second. Therefore, the point will first return to its starting position at time $t=\frac{1}{4}$.

10. $V_0=\dfrac{4}{3}\pi(2)^3=\dfrac{32\pi}{3}\text{cm}^3$

$V_1=\dfrac{4}{3}\pi(2.5)^3=\dfrac{62.5\pi}{3}=\dfrac{125\pi}{6}\text{cm}^3$

$\Delta V=V_1-V_0=\dfrac{125\pi}{6}\text{cm}^3-\dfrac{32\pi}{3}\text{cm}^3$

$=\dfrac{61}{6}\pi\,\text{cm}^3\approx31.940\,\text{cm}^3$

11. a. North plane has traveled 600miles. East plane has traveled 400 miles.

b. $d=\sqrt{600^2+400^2}$

$=721$ miles

c. $d=\sqrt{675^2+500^2}$

$=840$ miles

12. $\ln x + 2\ln\left(x^2+4\right) - 3\ln(x+1)$

$$= \ln x + \ln\left(x^2+4\right)^2 - \ln(x+1)^3$$

$$= \ln\left[x\left(x^2+4\right)^2\right] - \ln(x+1)^3$$

$$= \ln\left[\frac{x\left(x^2+4\right)^2}{(x+1)^3}\right]$$

13. From section 2.6 we have $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$.

Therefore, $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n = e^{-2}$.

14. Let $x = \frac{2}{h}$.

$$\lim_{h\to 0}\left(1+\frac{h}{2}\right)^{2/h} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

15. Let $x = \frac{3}{h}$.

$$\lim_{h\to 0}\left(1+\frac{h}{3}\right)^{1/h} = \lim_{h\to 0}\left[\left(1+\frac{h}{3}\right)^{3/h}\right]^{1/3}$$

$$= \left[\lim_{h\to 0}\left(1+\frac{h}{3}\right)^{3/h}\right]^{1/3}$$

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{1/3} = e^{1/3}$$

16. Let $n = \frac{x}{h}$.

$$\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h} = \lim_{h\to 0}\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}$$

$$= \left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{1/x} = e^{1/x}$$