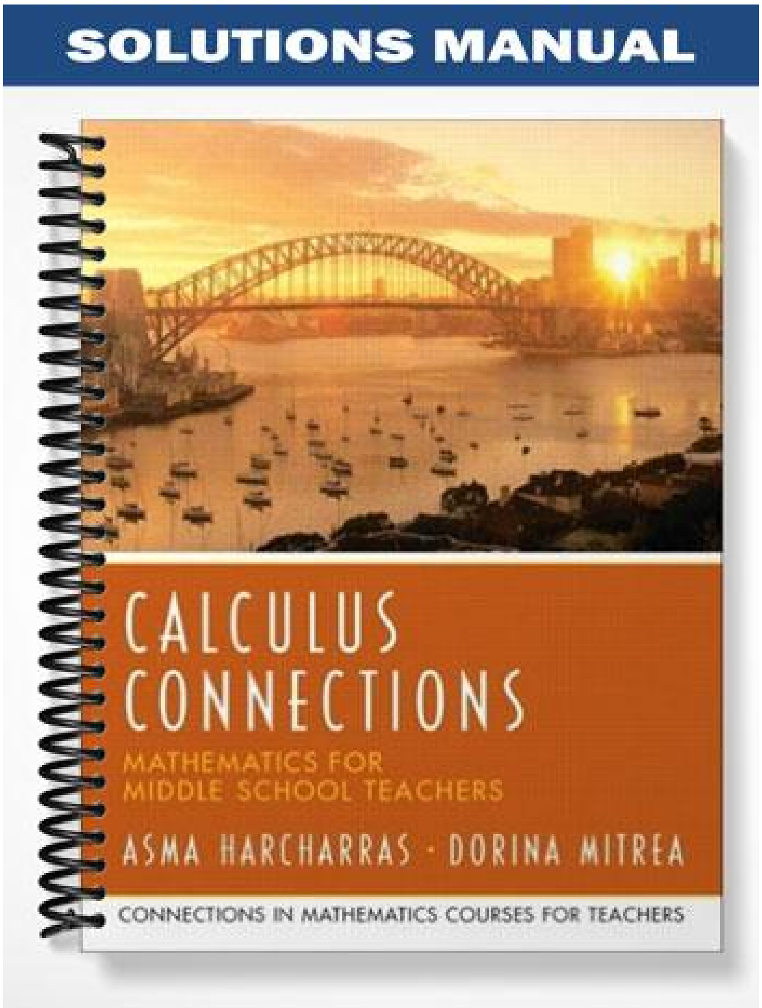

SOLUTIONS MANUAL
CALCULUS CONNECTIONS
MATHEMATICS FOR MIDDLE SCHOOL TEACHERS
ASMA HARCHARRAS · DORINA MITREA
CONNECTIONS IN MATHEMATICS COURSES FOR TEACHERS

Chapter 2: Functions, Limits, and Continuity

Section 2.1: Functions

1. function, Image= $\{0, 1, 4, 9, 16, 25, 36, \ldots\}$.

2. function, Image= The grades of the students.

3. not a function; for $x = 1$ there are two values of $y \in [-2, 2]$ such that $x^2 + y^2 = 4$.

4. function, Image= $\{x_1, x_2, x_3, \ldots\}$.

5. **a.** $[0, \frac{5\sqrt{10}}{2}]$

b. $h(0) = 1000$, $h(1) = 984$, $h(3) = 856$, $h(t+1) = -16(t+1)^2 + 1000$.

c. $h(t+1) - h(t)$ represents the distance covered by the falling object between t and $t+1$ seconds.

d. False. Corresponds to $h(t+1) = h(t) - h(1)$, while $h(t+1) = h(t) - 32t - 16 \neq h(t) - h(1)$.

6. A. Graphing a race.

1. a. B, about 2 hours 48 minutes;

b. A, the line has the greatest slope;

c. had a break;

d. about 25 km, all the riders slowed down;

e. 3 hours 30 minutes.

There are three functions, f_A, f_B, f_C, the distances travelled by the riders A, B, C, respectively.
f_A: Domain=$[0, 3]$, Image=$[0, 50]$,
f_B: Domain=$[0, 3]$, Image=$[0, 50]$,
f_C: Domain=$[0, 3]$, Image=$[0, 32.5]$.

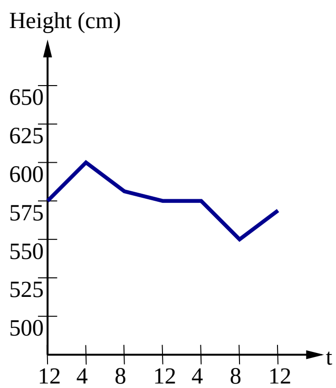
B. Filling in temperatures.

1. The function is the height of the river in centimeters

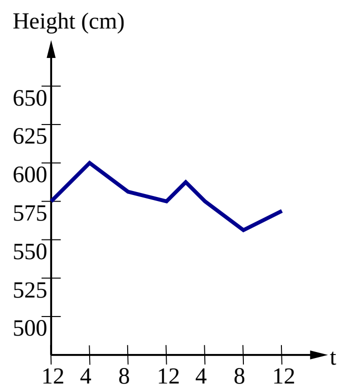
Day 1		Day 2		Day 3		Day 4	
Time	Height	Time	Height	Time	Height	Time	Height
4 a.m.	510 cm	4 a.m.	585 cm	4 a.m.	580 cm	4 a.m.	635 cm
8 a.m.	510 cm	8 a.m.	585 cm	8 a.m.	605 cm	8 a.m.	635 cm
12 p.m.	510 cm	12 p.m.	580 cm	12 p.m.	620 cm	12 p.m.	625 cm
4 p.m.	510 cm	4 p.m.	575 cm	4 p.m.	620 cm	4 p.m.	600 cm
8 p.m.	550 cm	8 p.m.	573 cm	8 p.m.	630 cm	8 p.m.	580 cm
12 a.m.	575 cm	12 a.m.	570 cm	12 a.m.	635 cm	12 a.m.	580 cm

Domain=The 4 days time period.
Image=$[510, 635]$.

2.a.



b.

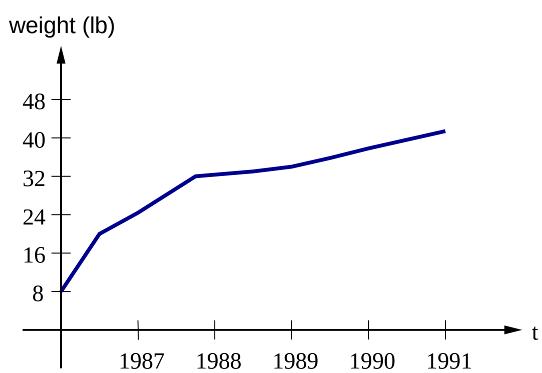


C. Graphs over a large time period.

1.a. 18 pounds;

b. April;

2. The function is Tameeka's weight in pounds. Domain=The 5 years time period (1987, Jan – 1991, Dec).
Image=$[8, 41]$.



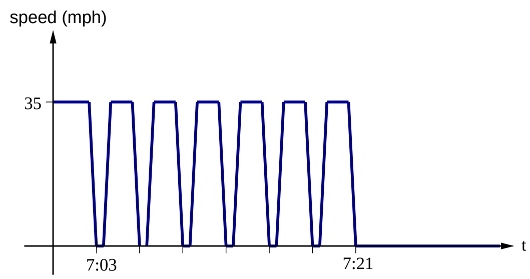
3.a. Tameeka's weight began to change slower and more uniform;

b. no;

c. the first year.

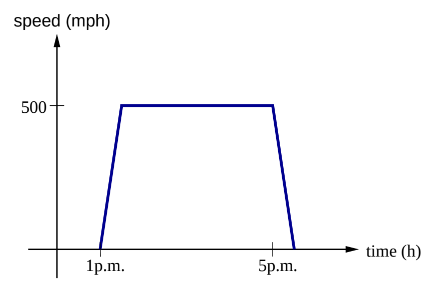
D. Graphs depicting speed.

1.



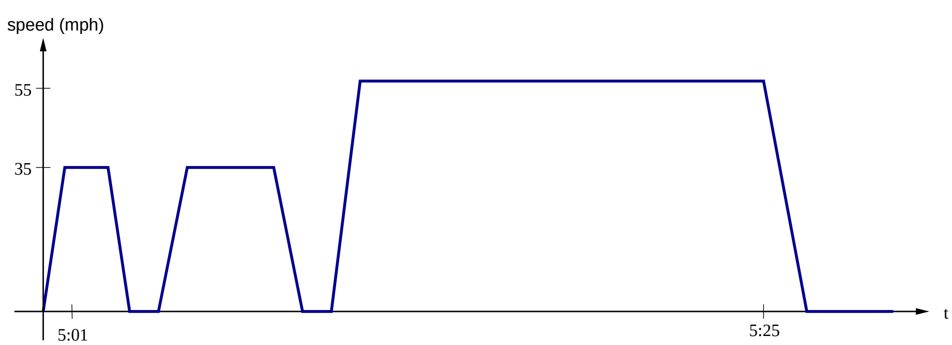
The function is the speed of the train in mph.
Domain=The time period from 7 a.m. to 12 : 21 a.m.
Image=$[0, 35]$.

2.



The function is the speed of the plain in mph.
Domain=The time period from 1 p.m. to 5 p.m.
Image=$[0, 500]$.

3. The part of the graph corresponding to time after 5 : 25 p.m. can be different depending on the assumptions made.



The function is the speed of the car in mph.
Domain=The time period from 5 p.m. to 5 : 25 p.m.
Image=$[0, 55]$.

E. Graphs of tides.

1. The best times are 6 a.m. and 6 p.m.;
the worst times are 12 a.m. and 12 p.m.

2. 6 a.m. and 6 p.m.

3. from 7 a.m. to 11 p.m. and from 7p.m. to 11 a.m.

4. 11 : 30 a.m. and 11 : 30 p.m.
The function is the depth of the harbor in meters.
Domain=The one-day time period.
Image=$[2,4]$.

7. $y = 4x - 2$ is a function of x.

8. $y = 2 + x^2$ is a function of x.

9. $y = \pm\sqrt{x^2+1}$ is not a function of x.

10. $y = \pm\sqrt{4x-1}$ is not a function of x.

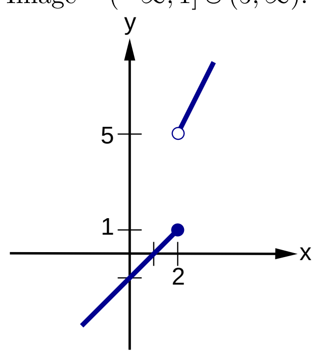
11. $y = \sqrt[5]{\frac{100+2x}{3}}$ is a function of x.

12. $y = \sqrt[3]{2+x}$ is a function of x.

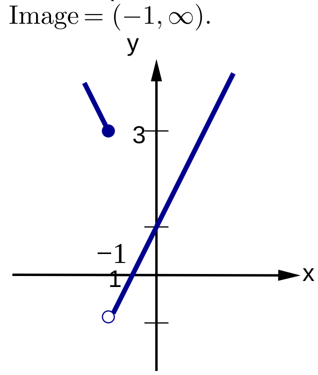
13. $y = -3$ is a function of x.

14. $y = \pm\frac{1}{2}\sqrt{1-3x^2}$ is not a function of x.

15. Image$= (-\infty, 1] \cup (5, \infty)$.



16. $f(x) = \begin{cases} 1 - 2x & \text{if } x \leq -1 \\ 2x + 1 & \text{if } x > -1 \end{cases}$
Image$= (-1, \infty)$.



17. $\mathbb{R}$, $[0, \infty)$.

18. $\mathbb{R}$, $[1, \infty)$.

19. $f(x) = \frac{1}{\sqrt[3]{x}}$, so $x \neq 0$;
Domain$= \mathbb{R} \setminus \{0\}$, Image$= \mathbb{R} \setminus \{0\}$.

20. $f(x) = \frac{1}{\sqrt[5]{x-1}}$, so $x - 1 \neq 0$;
Domain$= \mathbb{R} \setminus \{1\}$, Image$= \mathbb{R} \setminus \{0\}$.

21. $\mathbb{R}$, $(0, 3]$.

22. $\mathbb{R}$, $(0, 4]$.

23. $\mathbb{R}\backslash\{-1\}$, $\mathbb{R}\backslash\{0\}$.

24. $\mathbb{R}\backslash\{2\}$, $\mathbb{R}$.

25. $\mathbb{R}$, $[0, \infty)$.

26. $\mathbb{R}$, $[1, \infty)$.

27. $[1, \infty)$, $[0, \infty)$.

28. $x^2 + 2x \geq 0$, $x(x+2) \geq 0$, so Domain$= (-\infty, -2] \cup [0, \infty)$, Image$= [0, \infty)$.

29. $\mathbb{R}$, $\mathbb{R}$.

30. $\mathbb{R}$, $\mathbb{R}$.

31. $x + 2 > 0$, so Domain$= (-2, \infty)$, Image$= (0, \infty)$.

32. $x - 4 > 0$, so Domain$= (4, \infty)$, Image$= (0, \infty)$.

33. $(f \circ g)(x) = 3 + 2(-2x + 1) = 5 - 4x$, Domain$\,=\mathbb{R}$.

34. $(f \circ g)(x) = (x - 4)^2 + 2(x - 4) = x^2 - 8x + 16 + 2x - 8 = x^2 - 6x + 8$, Domain$\,=\mathbb{R}$.

35. $(f \circ g)(x) = |-x| = |x|$, Domain$\,=\mathbb{R}$.

36. $(f \circ g)(x) = |x^3 + 2 + 1| = |x^3 + 3|$, Domain$\,=\mathbb{R}$.

37. $(g \circ f)(x) = \frac{1}{x+3}$, Domain$\,=\mathbb{R}\backslash\{-3\}$.

38. $(g \circ f)(x) = \sqrt{\frac{x}{x^2+1}}$, Domain$\,=[0, \infty)$.

39. $(g \circ f)(x) = (x^4 + 1)^{\frac{1}{3}}$, Domain$\,=\mathbb{R}$.

40. $(g \circ f)(x) = (2x + 5)^{\frac{1}{4}}$, Domain$\,=[-\frac{5}{2}, \infty)$.

41. $f(g(-1)) = f(1) = 2$

42. $(f \circ g)(0) = f(-1) = 3$.

43. $g(f(1)) = g(2) = 5$

44. $(g \circ f)(3) = g(0) = -1$.

45. $f(f(-1)) = f(3) = 0$

46. $(g \circ g)(0) = g(-1) = 1$.

47. $f(x) \geq 0$ for all $x \in$ Domain(f).

48. $f(x) \geq 2$ for all $x \in$ Domain(f).

49. The domain of f contains $[0, \infty)$.

50. The domain of f contains $[-2, \infty)$.

51. $f(x) > -1$ for all $x \in \text{Domain}(f)$.

52. $f(x) < 1$ for all $x \in \text{Domain}(f)$.

53. Let $y = x - 1$. Then, $x = y + 1$ so $f(y) = (y + 1)^2$. This implies $f(x + 1) = (x + 1 + 1)^2 = (x + 2)^2$.

54. Let $y = 2x + 1$. Then, $x = \frac{1}{2}(y - 1)$ so $f(y) = \frac{1}{8}(y - 1)^3 - 1$. This implies $f(x + 2) = \frac{1}{8}(x + 1)^3 - 1$.

55. Suppose such functions exist. If we set $x = 0$ we obtain $f(0) + g(y) = 0$ for all $y \in \mathbb{R}$ and if we set $y = 0$ we obtain $f(x) + g(0) = 0$ for all $x \in \mathbb{R}$. Hence,

$$f(x) = -g(0) = -(-f(0)) = f(0) \quad \text{for all } x \in \mathbb{R},$$

while

$$g(y) = -f(0) = -(-g(0)) = g(0) \quad \text{for all } y \in \mathbb{R}.$$

So far we have obtained that f and g are constant functions. On the other hand, this information combined with the identity $f(x) + g(y) = xy$ gives for $y = 1$

$$f(x) = x - g(1) = x - g(0) \quad \text{for all } x \in \mathbb{R},$$

which contradicts the fact that f was found to be constant. Therefore, there do not exist functions f and g satisfying $f(x) + g(y) = xy$ for all $x, y \in \mathbb{R}$.

56. Suppose such functions exist. If we set $x = 0$, we obtain $f(0)g(y) = y$ for all $y \in \mathbb{R}$ and if we set $y = 0$, we obtain $f(x)g(0) = x$ for all $x \in \mathbb{R}$. If we set $x = y = 0$, then we obtain $f(0)g(0) = 0$ and hence either $f(0) = 0$ or $g(0) = 0$. On the other hand $g(0) \neq 0$ since $f(x)g(0) = x$ for all $x \in \mathbb{R}$, and $f(0) \neq 0$ since $f(0)g(y) = y$ for all $y \in \mathbb{R}$. Therefore, there do not exist functions f and g satisfying $f(x)g(y) = x + y$ for all $x, y \in \mathbb{R}$.

57. The condition $f \circ g_1 = h$ implies $2g_1(x) - 3 = x + 6$ for all $x \in \mathbb{R}$. Hence, $g_1(x) = \frac{1}{2}x + \frac{9}{2}$ for all $x \in \mathbb{R}$. From $g_2 \circ f = h$, it follows that $g_2(2x - 3) = x + 6$ for all $x \in \mathbb{R}$, thus $g_2(x) = \frac{x+3}{2} + 6 = \frac{1}{2}x + \frac{15}{2}$ for all $x \in \mathbb{R}$.

Section 2.2: Limits of Functions

1. 0, 1.5, $\lim_{x \to 2} f(x)$ does not exist.

2. 0.5, 0.5, 0.5.

3. 2, 2, 2.

4. 0, 0, 0.

5. 0, 0, 0, 0.

6. $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{3}$, $\lim_{x\to 4} f(x)$ does not exist.

7. 2, $+\infty$, 2, does not exist.

8. $\frac{1}{3}$, 1, $-\infty$, $\lim_{x\to 2} f(x)$ does not exist.

9. 0, 0, 0, 0.

10. $-\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{2}{3}$.

11. $1-6-4=-9$.

12. $3(-1)^4+5(-1)^3+2(-1)-1=-5$.

13. $\sqrt{0+3}=\sqrt{3}$.

14. $\sqrt{4^2-3}=\sqrt{13}$.

15. $\frac{2\cdot(-1)+1}{-1-3}=\frac{1}{4}$.

16. $\frac{0-6}{3\cdot 0+1}=-6$.

17. $4-2\cdot 2=0$.

18. $5-3^2=-4$.

19. $\frac{-1}{0+0+1}=-1$.

20. $\frac{0+1}{0-0-1}=-1$.

21. $\sqrt[3]{1-5+1}=\sqrt[3]{-3}$.

22. $\sqrt[3]{2(-2)^3-(-2)^2+3(-2)-4}=\sqrt[3]{2}$.

23. $\frac{1}{2}-\frac{1}{2}=0$.

24. $\frac{2}{4+1}=\frac{2}{5}$.

25. $\sqrt[4]{6^2-4\cdot 6+4}=2$.

26. $\sqrt[4]{2^3+2-1}=\sqrt{3}$.

27. $\frac{(-2)^3-2\cdot(-2)-3}{-2+3}=-7$.

28. $\frac{0-1}{0-1}=0$.

29. **a.** $\mathbb{R}\setminus\{1\}$. We can not compute $\lim\limits_{x\to 1} f(x)$ using **P4** since the denominator of $f(x)$ is zero at $x=1$.

b. $1, x, \dots, x^n$ are terms of a geometric sequence with $a=1$, $r=x$.
$S=x^n+x^{n-1}+\dots+x+1$
$xS+1=x^{n+1}+x^n+\dots+x^2+x+1$
$xS+1=x^{n+1}+S$
$S=\frac{x^{n+1}-1}{x-1}$

c. $\lim\limits_{x\to 1}\frac{x^{n+1}-1}{x-1}=\lim\limits_{x\to 1}(x^n+x^{n-1}+\dots+x+1)=n+1$

30. Let $f(x)=x^3+2$. Fix an open interval I centered at 3. We want to find an open interval J containing 1, with $J\setminus\{1\}$ contained in the domain of f, such that for every $x\in J\setminus\{1\}$ we have $f(x)\in I$.

Let $2l$ be the length of the interval I, that is, $I=(3-l,3+l)$. The condition $f(x)\in I$ is equivalent to $|f(x)-3|<l$, which is in turn equivalent to $3-l<x^3+2<3+l$, $1-l<x^3<1+l$. This suggests that a good candidate for the interval J is $J=(\sqrt[3]{1-l},\sqrt[3]{1+l})$. Indeed, $1\in J$, $J\setminus\{1\}$ is contained in the domain of f, and

$$x\in J\setminus\{1\}\ \Rightarrow\ \sqrt[3]{1-l}<x<\sqrt[3]{1+l}\ \Rightarrow\ 1-l<x^3<1+l$$
$$\Rightarrow\ -l<x^3-1<l\ \Rightarrow\ |f(x)-3|<l\ \Rightarrow\ f(x)\in I.$$

31. Let $f(x)=3x^3-1$. Fix an open interval I centered at 2. We want to find an open interval J containing 1, with $J\setminus\{1\}$ contained in the domain of f, such that for every $x\in J\setminus\{1\}$ we have $f(x)\in I$.

Let $2l$ be the length of the interval I, that is, $I=(2-l,2+l)$. The condition $f(x)\in I$ is equivalent to $|f(x)-2|<l$, which is in turn equivalent to $1-\frac{l}{3}<x^3<1+\frac{l}{3}$. This suggests that a good candidate for the interval J is $J=(\sqrt[3]{1-\frac{l}{3}},\sqrt[3]{1+\frac{l}{3}})$. Indeed, $1\in J$, $J\setminus\{1\}$ is contained in the domain of f, and

$$x\in J\setminus\{1\}\ \Rightarrow\ \sqrt[3]{1-\frac{l}{3}}<x<\sqrt[3]{1+\frac{l}{3}}\ \Rightarrow\ -\frac{l}{3}<x^3-1<\frac{l}{3}$$
$$\Rightarrow\ -l<3x^3-3<l\ \Rightarrow\ |f(x)-2|<l\ \Rightarrow\ f(x)\in I.$$

Section 2.3: Continuity

1. Continuous on $\mathbb{R} \setminus \{-1\}$, discontinuous at $x = 1$.

2. Continuous on $\mathbb{R} \setminus \{2\}$, discontinuous at $x = 2$.

3. Continuous on $\mathbb{R}$.

4. Continuous on $\mathbb{R}$.

5. $2 \cdot 1^2 - 3 = 3 \cdot 1 - a$, $a = 4$.

6. $a = -1$.

7. $a = 2$.

8. $-2 - a^2 = 1 - 4a,\ \ a^2 - 4a + 3 = 0,\ \ a = 1$ or $a = 3$.

9. $f(x) = x^2 - x$.

10. $f(x) = x^3 + x^2$.

11. $f(x) = \begin{cases} 1 & \text{if } x \neq 4 \\ 0 & \text{if } x = 4 \end{cases}$

12. $f(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

13. $f(x) = \begin{cases} 0 & \text{if } x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } x \leq 2 \\ x+1 & \text{if } x > 2 \end{cases}$

15. Solving for $f(x)$ we obtain that for each $x \in \mathbb{R}$, $f(x)$ is either x or $-x$. Hence the graph of f is contained in the lines $y = x$ and $y = -x$. Since f is continuous, we have four possibilities: $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = |x|$, and $f_4(x) = -|x|$.

Chapter 2 Review

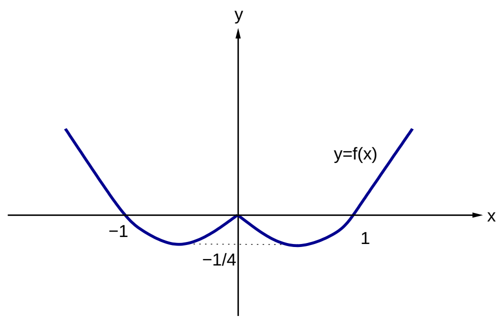
1. $y = \pm\sqrt{x^3 - 2}$ is not a function of x.

2. $y = \sqrt[3]{2 + x^2}$ is a function of x.

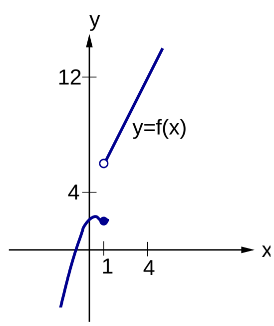
3. $y = \frac{5}{2}\,x + \frac{1}{2}$ is a function of x.

4. $y = -\frac{2}{x}$ is a function of x.

5. $f(x) = \begin{cases} x^2 + x & \text{if } x \le 0 \\ x^2 - x & \text{if } x > 0 \end{cases}$, Image $(f) = [-\frac{1}{4}, \infty)$.



6. $f(x) = \begin{cases} 2 + x - x^2 & \text{if } x \le 1 \\ 2x + 4 & \text{if } x > 1 \end{cases}$, Image $(f) = (-\infty, \frac{9}{4}] \cup (6, \infty)$.



7. Domain $(f) = \mathbb{R}$, Image $(f) = [\,\frac{7}{4}, \infty)$.

8. Domain $(f) = \mathbb{R}$, Image $(f) = (-\infty, 4]$.

9. Domain $(f) = [0, \infty)$, Image $(f) = [0, \infty)$.

10. $x^2 - 2x - 3 = (x+1)(x-3) \ge 0$.
Domain $(f) = (-\infty, -1] \cup [3, \infty)$, Image $(f) = [0, \infty)$.

11. Domain $(f) = \mathbb{R}$, Image $(f) = \mathbb{R}$.

12. Domain $(f) = \mathbb{R}$, Image $(f) = \mathbb{R}$.

13. Domain $(f) = \mathbb{R} \setminus \{-2\}$, Image $(f) = \{-1, 1\}$.

14. $x - 1 \ge 0$, so Domain $(f) = [1, \infty)$, Image $(f) = \{1\}$.

15. Domain $(f) = \mathbb{R} \setminus \{-3, 3\}$, Image $(f) = \mathbb{R} \setminus \{0, -\frac{1}{6}\}$.

16. $x - 2 \ne 0$, so Domain $(f) = \mathbb{R} \setminus \{2\}$, Image $(f) = \mathbb{R}$.

17. $f(g(x)) = \frac{x^2+2x}{x^2+2x-3}$, Domain $(f \circ g) = \mathbb{R} \setminus \{-3, 1\}$.

18. $f(g(x)) = \frac{\sqrt{x^2-4x}}{x^2-4x-1}$, $x^2 - 4x = x(x-4) \geq 0$ and $x^2 - 4x - 1 = (x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) \neq 0$, so Domain $(f \circ g) = (-\infty, 2 - \sqrt{5}) \cup (2 - \sqrt{5}, 0] \cup [4, 2 + \sqrt{5}) \cup (2 + \sqrt{5}, \infty)$.

19. $g(f(x)) = \sqrt{x-3}$, Domain $(g \circ f) = [3, \infty)$.

20. $g(f(x)) = \sqrt{x^2 + x}$, $x^2 + x = x(x+1) \geq 0$, so Domain $(g \circ f) = (-\infty, -1] \cup [0, \infty)$.

21. $f(g(x)) = \begin{cases} 2x+5 & \text{if } x < -1 \\ x^2 + 2x + 4 & \text{if } x \geq -1 \end{cases}, \qquad g(f(x)) = \begin{cases} 2x+4 & \text{if } x < 0 \\ x^2 + 4 & \text{if } x \geq 0 \end{cases}.$

22. $f(g(x)) = \begin{cases} -8 & \text{if } x \leq -2 \\ 3x - 2 & \text{if } -2 < x \leq 0 \\ -2 & \text{if } x > 0 \end{cases}, \qquad g(f(x)) = -3.$

23. f is not defined at -2, $\frac{5}{6}$, $\frac{5}{6}$, $\frac{5}{6}$.

24. $-\frac{1}{2}$, $-\frac{1}{2}$, 1, $\lim_{x \to \frac{1}{2}} f(x)$ does not exist.

25. 1, $-\infty$, 1, does not exist.

26. 0, -1, ∞, $\lim_{x \to 3} f(x)$ does not exist.

27. 0, 0, 0, 0.

28. 2, 1, 1, 1.

29. 56.

30. $100 - 3(-1)^2 + (-1)^3 - 7(-1)^5 = 101$.

31. -1

32. $\sqrt{3 \cdot 1^5 - 2 \cdot 1 + 2} = \sqrt{3}$.

33. 0

34. $\frac{3+2}{3^2-6} = \frac{5}{3}$.

35. -3.

36. $\frac{3 \cdot 0 - 2}{0^3 - 1} + \sqrt[3]{2 - 0^2 - 0} = 2 + \sqrt[3]{2}$.

37. $f(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ -x & \text{if } x > 0 \end{cases}$

38. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x+1 & \text{if } x \geq -1 \end{cases}.$

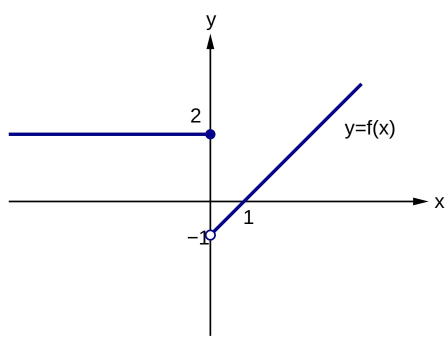
39. $f(x) = \begin{cases} 2x-1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}.$

40. $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}.$

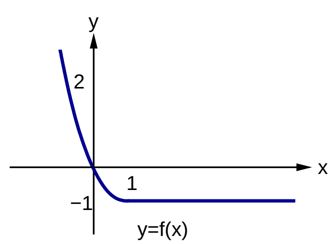
41. $f(x) = x^2 + 3x - 1.$

42. $f(x) = \frac{x}{x^2+1}.$

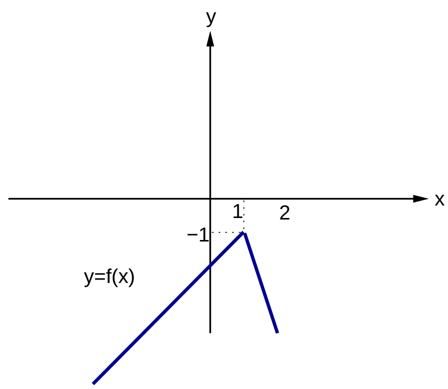
43. f is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0.



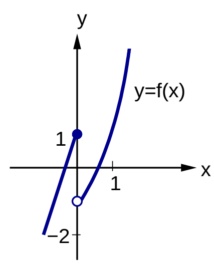
44. f is continuous on $\mathbb{R}$.



45. f is continuous on $\mathbb{R}$.



46. f is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at $\{0\}$.



47. $a = -1$.

48. $a^2 + 2 = 1 - 1 + 6$, $a^2 = 4$, $a = \pm 2$.

49. If $f(x) = ax+b$, $(f \circ f)(x) = x$ for all $x \in \mathbb{R}$ is equivalent to $a(ax+b)+b = x$ for all $x \in \mathbb{R}$. Hence, $(a^2-1)x + ab + b = 0$ for all $x \in \mathbb{R}$. The latter holds if and only if $a^2 - 1 = 0$ and $ab + b = 0$. The solutions for this system in a and b are $a = 1$, $b = 0$ or $a = -1$, b arbitrary real number. Consequently, $f(x) = x$ or $f(x) = -x + b$.