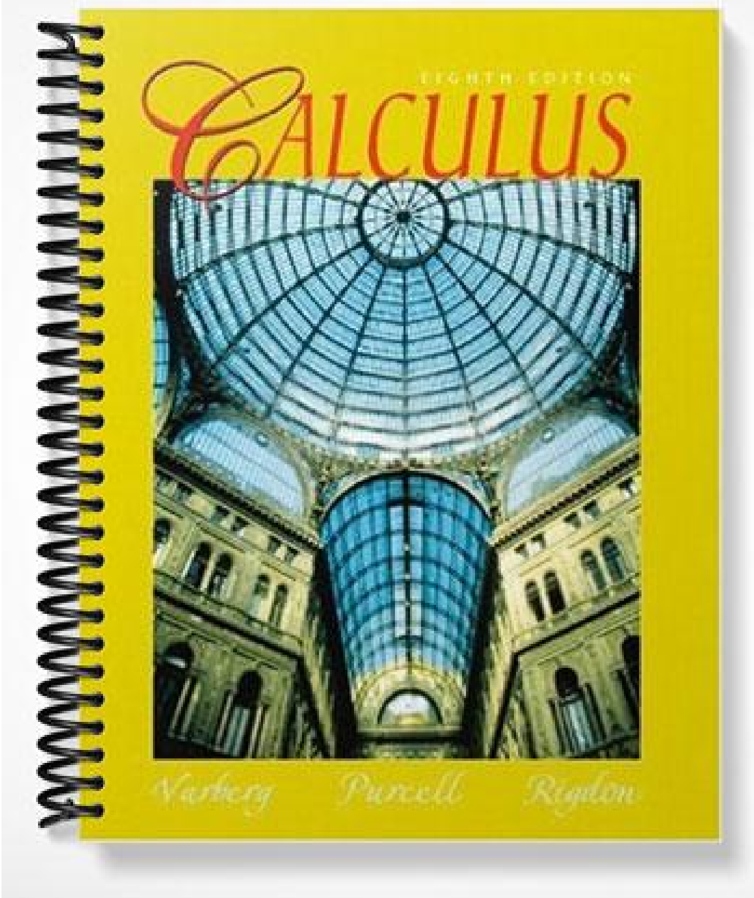
SOLUTIONS MANUAL



CHAPTER

2 Functions and Limits

2.1 Concepts Review

1. domain; range

2. $f(2u)=3(2u)^2=12u^2; f(x+h)=3(x+h)^2$

3. asymptote

4. even; odd; y-axis; origin

Problem Set 2.1

1. a. $f(1)=1-1^2=0$

 b. $f(-2)=1-(-2)^2=-3$

 c. $f(0)=1-0^2=1$

 d. $f(k)=1-k^2$

 e. $f(-5)=1-(-5)^2=-24$

 f. $f\left(\frac{1}{4}\right)=1-\left(\frac{1}{4}\right)^2=1-\frac{1}{16}=\frac{15}{16}$

 g. $f(3t)=1-(3t)^2=1-9t^2$

 h. $f(2x)=1-(2x)^2=1-4x^2$

 i. $f\left(\frac{1}{t}\right)=1-\left(\frac{1}{t}\right)^2=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}$

2. a. $F(1)=1^3+3\cdot 1=4$

 b. $F(\sqrt{2})=(\sqrt{2})^3+3(\sqrt{2})=2\sqrt{2}+3\sqrt{2}$
 $=5\sqrt{2}$

 c. $F\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^3+3\left(\frac{1}{4}\right)=\frac{1}{64}+\frac{3}{4}=\frac{49}{64}$

 d. $F(\pi)=\pi^3+3\pi$

 e. $F\left(\frac{1}{t}\right)=\left(\frac{1}{t}\right)^3+3\left(\frac{1}{t}\right)=\frac{1}{t^3}+\frac{3}{t}=\frac{1+3t^2}{t^3}$

 f. $F(2.718)=(2.718)^3+3(2.718)\approx 28.233$

3. a. $G(0)=\frac{1}{0-1}=-1$

 b. $G(0.999)=\frac{1}{0.999-1}=-1000$

 c. $G(1.01)=\frac{1}{1.01-1}=100$

 d. $G(y^2)=\frac{1}{y^2-1}$

 e. $G(-x)=\frac{1}{-x-1}=-\frac{1}{x+1}$

 f. $G\left(\frac{1}{x^2}\right)=\frac{1}{\frac{1}{x^2}-1}=\frac{x^2}{1-x^2}$

4. a. $\Phi(1)=\frac{1+1^2}{\sqrt{1}}=2$

 b. $\Phi(-t)=\frac{-t+(-t)^2}{\sqrt{-t}}=\frac{t^2-t}{\sqrt{-t}}$

 c. $\Phi\left(\frac{1}{2}\right)=\frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{2}}}=\frac{\frac{3}{4}}{\sqrt{\frac{1}{2}}}\approx 1.06$

 d. $\Phi(u+1)=\frac{(u+1)+(u+1)^2}{\sqrt{u+1}}=\frac{u^2+3u+2}{\sqrt{u+1}}$

 e. $\Phi(x^2)=\frac{(x^2)+(x^2)^2}{\sqrt{x^2}}=\frac{x^2+x^4}{|x|}$

f. $\Phi(x^2+x)=\dfrac{(x^2+x)+(x^2+x)^2}{\sqrt{x^2+x}}$

$=\dfrac{x^4+2x^3+2x^2+x}{\sqrt{x^2+x}}$

5. a. $f(0.25)=\dfrac{1}{\sqrt{0.25-3}}=\dfrac{1}{\sqrt{-2.75}}$ is not defined

b. $f(x)=\dfrac{1}{\sqrt{\pi-3}}\approx 2.658$

c. $f(3+\sqrt{2})=\dfrac{1}{\sqrt{3+\sqrt{2}-3}}=\dfrac{1}{\sqrt{\sqrt{2}}}$

$=2^{-0.25}\approx 0.841$

6. a. $f(0.79)=\dfrac{\sqrt{(0.79)^2+9}}{0.79-\sqrt{3}}\approx -3.293$

b. $f(12.26)=\dfrac{\sqrt{(12.26)^2+9}}{12.26-\sqrt{3}}\approx 1.199$

c. $f(\sqrt{3})=\dfrac{\sqrt{(\sqrt{3})^2+9}}{\sqrt{3}-\sqrt{3}}$; undefined

7. a. $x^2+y^2=1$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$; not a function

b. $xy+y+x=1$

$y(x+1)=1-x$

$y=\dfrac{1-x}{x+1}$; $f(x)=\dfrac{1-x}{x+1}$

c. $x=\sqrt{2y+1}$

$x^2=2y+1$

$y=\dfrac{x^2-1}{2}$; $f(x)=\dfrac{x^2-1}{2}$

d. $x=\dfrac{y}{y+1}$

$xy+x=y$

$x=y-xy$

$x=y(1-x)$

$y=\dfrac{x}{1-x}$; $f(x)=\dfrac{x}{1-x}$

8. The graphs on the left are not graphs of functions, the graphs on the right are graphs of functions.

9. $\dfrac{f(a+h)-f(a)}{h}=\dfrac{[2(a+h)^2-1]-(2a^2-1)}{h}$

$=\dfrac{4ah+2h^2}{h}=4a+2h$

10. $\dfrac{F(a+h)-F(a)}{h}=\dfrac{4(a+h)^3-4a^3}{h}$

$=\dfrac{4a^3+12a^2h+12ah^2+4h^3-4a^3}{h}$

$=\dfrac{12a^2h+12ah^2+4h^3}{h}$

$=12a^2+12ah+4h^2$

11. $\dfrac{g(x+h)-g(x)}{h}=\dfrac{\frac{3}{x+h-2}-\frac{3}{x-2}}{h}$

$=\dfrac{\frac{3x-6-3x-3h+6}{x^2-4x+hx-2h+4}}{h}=\dfrac{-3h}{h(x^2-4x+hx-2h+4)}$

$=-\dfrac{3}{x^2-4x+hx-2h+4}$

12. $\dfrac{G(a+h)-G(a)}{h}=\dfrac{\frac{a+h}{a+h+4}-\frac{a}{a+4}}{h}$

$=\dfrac{\frac{a^2+4a+ah+4h-a^2-ah-4a}{a^2+8a+ah+4h+16}}{h}=\dfrac{4h}{h(a^2+8a+ah+4h+16)}$

$=\dfrac{4}{a^2+8a+ah+4h+16}$

13. a. $F(z)=\sqrt{2z+3}$

$2z+3\ge 0$; $z\ge -\dfrac{3}{2}$

Domain: $\left\{z\in\mathbb{R}: z\ge -\dfrac{3}{2}\right\}$

b. $g(v)=\dfrac{1}{4v-1}$

$4v-1=0$; $v=\dfrac{1}{4}$

Domain: $\left\{v\in\mathbb{R}: v\ne \dfrac{1}{4}\right\}$

c. $\psi(x) = \sqrt{x^2 - 9}$

$x^2 - 9 \geq 0;\ x^2 \geq 9;\ |x| \geq 3$

Domain: $\{x \in \mathbb{R} : |x| \geq 3\}$

d. $H(y) = -\sqrt{625 - y^4}$

$625 - y^4 \geq 0; 625 \geq y^4; |y| \leq 5$

Domain: $\{y \in \mathbb{R} : |y| \leq 5\}$

14. a. $f(x) = \dfrac{4 - x^2}{x^2 - x - 6} = \dfrac{4 - x^2}{(x-3)(x+2)}$

Domain: $\{x \in \mathbb{R} : x \neq -2, 3\}$

b. $G(y) = \sqrt{(y+1)^{-1}}$

$\dfrac{1}{y+1} \geq 0;\ y > -1$

Domain: $\{y \in \mathbb{R} : y > -1\}$

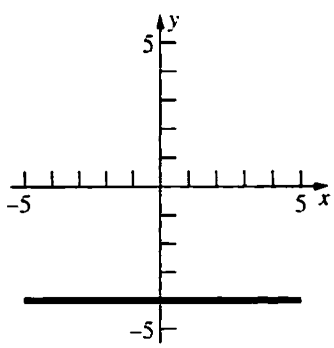
c. $\phi(u) = |2u + 3|$

Domain: $\mathbb{R}$

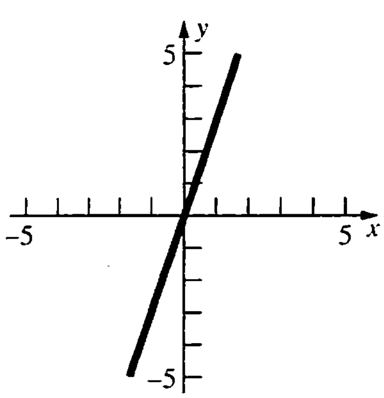
d. $F(t) = t^{2/3} - 4$

Domain: $\mathbb{R}$

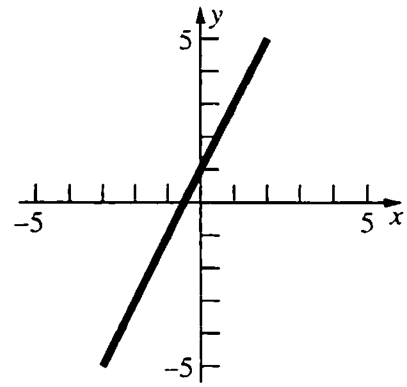
15. $f(x) = -4; f(-x) = -4$; even function



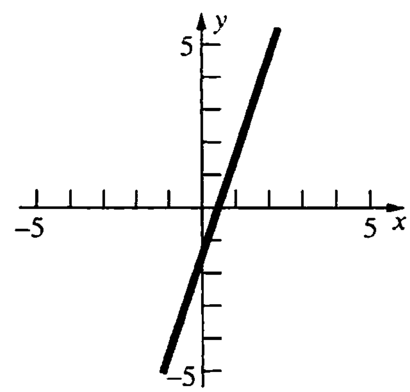
16. $f(x) = 3x; f(-x) = -3x$; odd function



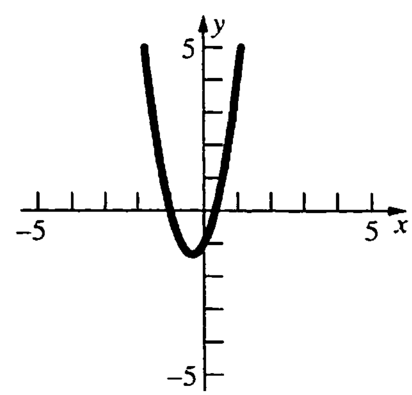
17. $F(x) = 2x + 1; F(-x) = -2x + 1$; neither



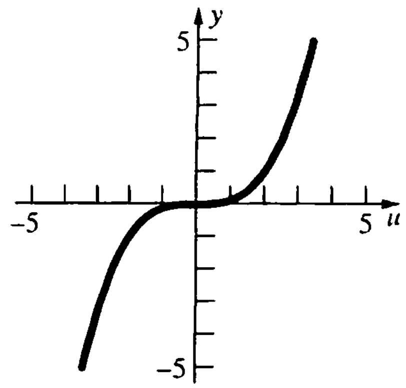
18. $F(x) = 3x - \sqrt{2}; F(-x) = -3x - \sqrt{2}$; neither



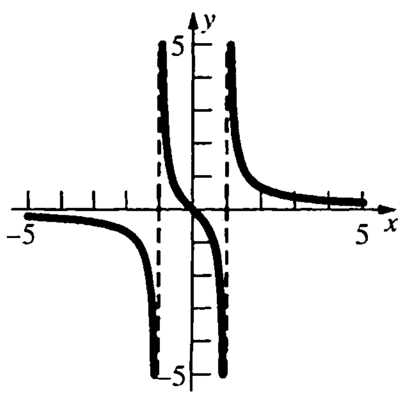
19. $g(x) = 3x^2 + 2x - 1; g(-x) = 3x^2 - 2x - 1$; neither



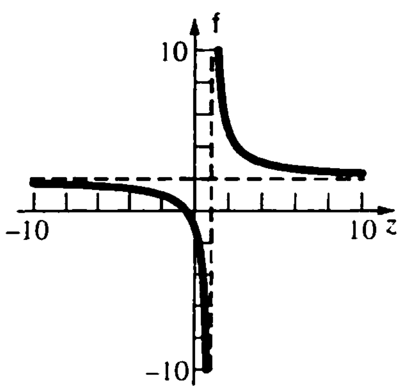
20. $g(u)=\frac{u^3}{8}; g(-u)=-\frac{u^3}{8}$; odd function



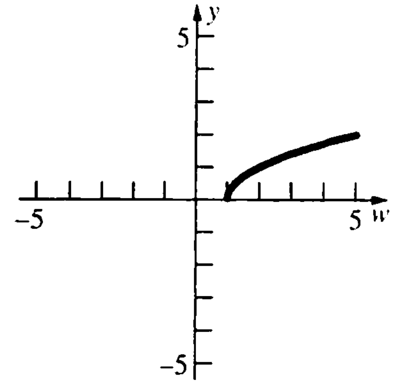
21. $g(x)=\frac{x}{x^2-1}; g(-x)=\frac{-x}{x^2-1}$; odd



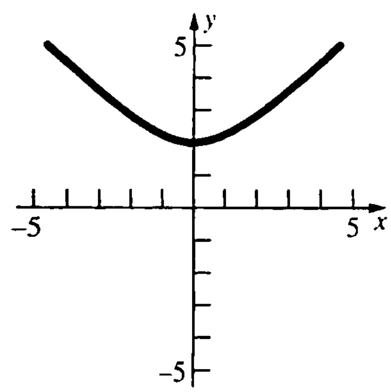
22. $\phi(z)=\frac{2z+1}{z-1}; \phi(-z)=\frac{-2z+1}{-z-1}$; neither



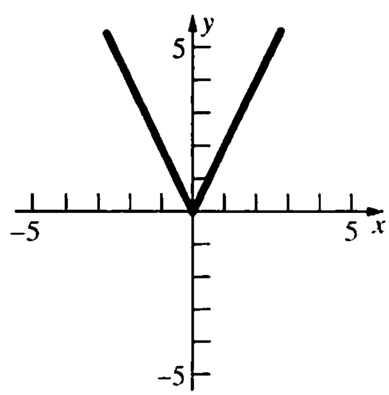
23. $f(w)=\sqrt{w-1}; f(-w)=\sqrt{-w-1}$; neither



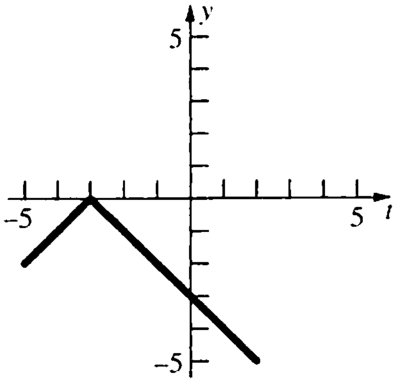
24. $h(x)=\sqrt{x^2+4}; h(-x)=\sqrt{x^2+4}$; even function



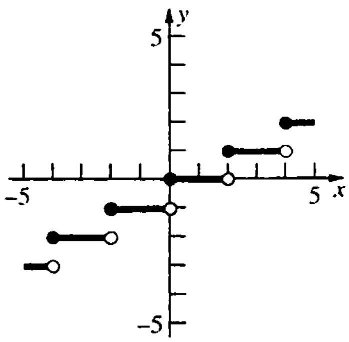
25. $f(x)=|2x|; f(-x)=|-2x|=|2x|$; even function



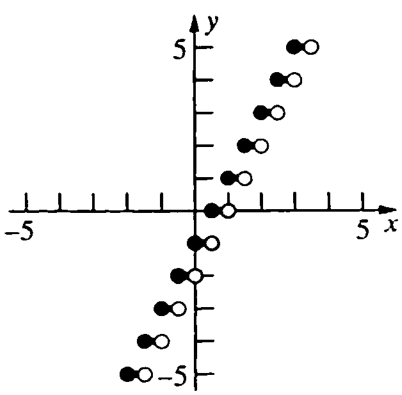
26. $F(t) = -|t+3|;\ F(-t) = -|-t+3|$; neither



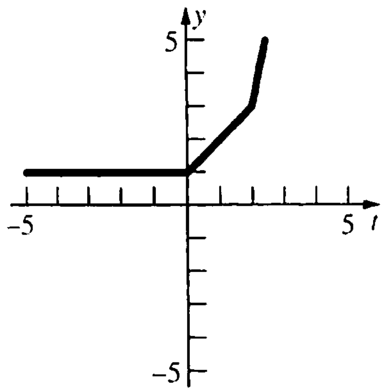
27. $g(x) = \left[\!\left[\frac{x}{2}\right]\!\right];\ g(-x) = \left[\!\left[-\frac{x}{2}\right]\!\right]$; neither



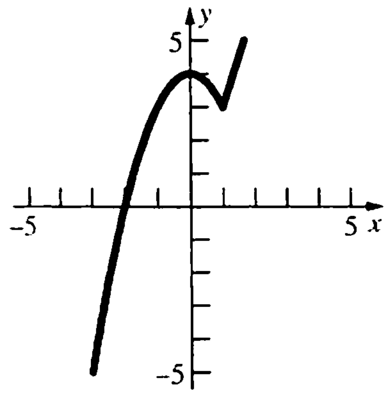
28. $G(x) = [\![2x-1]\!];\ G(-x) = [\![-2x+1]\!]$; neither



29. $g(t) = \begin{cases} 1 & \text{if } t \le 0 \\ t+1 & \text{if } 0 < t < 2 \\ t^2 - 1 & \text{if } t \ge 2 \end{cases}$ neither



30. $h(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$ neither



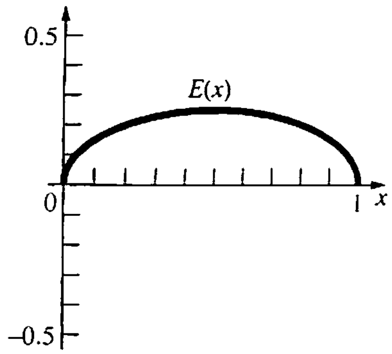
31. $T(x) = 5000 + 805x$
Domain: $\{x \in \text{integers}: 0 \le x \le 100\}$
$u(x) = \frac{T(x)}{x} = \frac{5000}{x} + 805$
Domain: $\{x \in \text{integers}: 0 < x \le 100\}$

32. a. $P(x) = 6x - (400 + 5\sqrt{x(x-4)})$
$= 6x - 400 - 5\sqrt{x(x-4)}$

b. $P(200) \approx -190$; $P(1000) = 610$

c. ABC breaks even when $P(x) = 0$;
$6x - 400 - 5\sqrt{x(x-4)} = 0;\ x \approx 390$

33. $E(x) = x - x^2$



$\frac{1}{2}$ exceeds its square by the maximum amount.

34. Each side has length $\frac{p}{3}$. The height of the triangle is $\frac{\sqrt{3}p}{6}$.

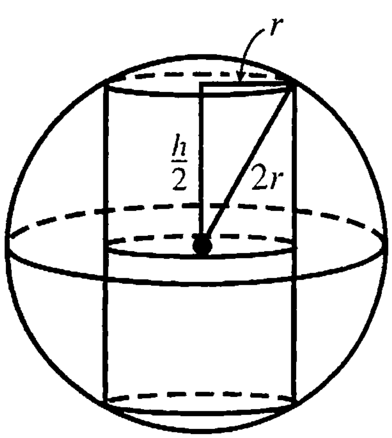
$$A(p) = \frac{1}{2}\left(\frac{p}{3}\right)\left(\frac{\sqrt{3}p}{6}\right) = \frac{\sqrt{3}p^2}{36}$$

35. a. $E(x) = 24 + 0.40x$

b. $120 = 24 + 0.40x$
$0.40x = 96;\ x = 240$ mi

36. The volume of the cylinder is $\pi r^2 h$, where h is the height of the cylinder. From the figure,

$$r^2 + \left(\frac{h}{2}\right)^2 = (2r)^2;\ \frac{h^2}{4} = 3r^2;$$
$$h = \sqrt{12r^2} = 2r\sqrt{3}.$$
$$V(r) = \pi r^2(2r\sqrt{3}) = 2\pi r^3\sqrt{3}$$



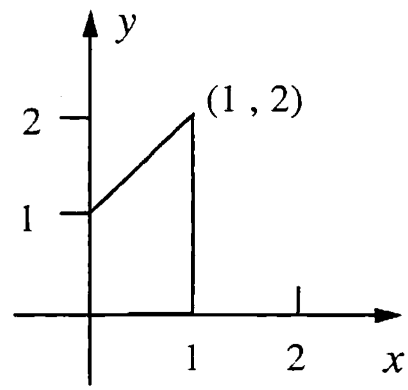
37. The area of the two semicircular ends is $\frac{\pi d^2}{4}$.

The length of each parallel side is $\frac{1-\pi d}{2}$.

$$A(d) = \frac{\pi d^2}{4} + d\left(\frac{1-\pi d}{2}\right) = \frac{\pi d^2}{4} + \frac{d - \pi d^2}{2}$$
$$= \frac{2d - \pi d^2}{4}$$

Since the track is one mile long, $\pi d < 1$, so $d < \frac{1}{\pi}$. Domain: $\left\{d \in \mathbb{R} : 0 < d < \frac{1}{\pi}\right\}$

38. a. $A(1) = 1(1) + \frac{1}{2}(1)(2-1) = \frac{3}{2}$

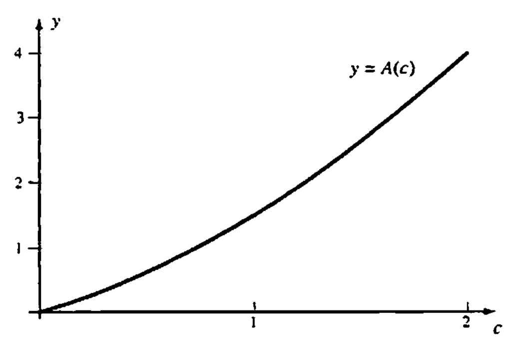


b. $A(2) = 2(1) + \frac{1}{2}(2)(3-1) = 4$

c. $A(0) = 0$

d. $A(c) = c(1) + \frac{1}{2}(c)(c+1-1) = \frac{1}{2}c^2 + c$

e.

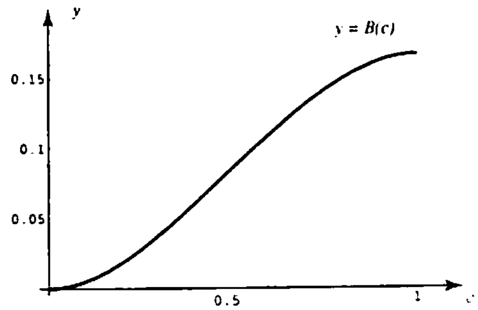


f. Domain: $\{c \in \mathbb{R} : c \geq 0\}$
Range: $\{y \in \mathbb{R} : y \geq 0\}$

39. a. $B(0) = 0$

b. $B\left(\frac{1}{2}\right)=\frac{1}{2}B(1)=\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}$

c.



40. a. $f(x+y)=2(x+y)=2x+2y=f(x)+f(y)$

b. $f(x+y)=(x+y)^2=x^2+2xy+y^2$
$\neq f(x)+f(y)$

c. $f(x+y)=2(x+y)+1=2x+2y+1$
$\neq f(x)+f(y)$

d. $f(x+y)=-3(x+y)=-3x-3y=f(x)+f(y)$

41. For any x, $x+0=x$, so
$f(x)=f(x+0)=f(x)+f(0)$, hence $f(0)=0$.
Let m be the value of $f(1)$. For p in N,
$p=p\cdot 1=1+1+\ldots+1$, so
$f(p)=f(1+1+\ldots+1)=f(1)+f(1)+\ldots+f(1)$
$=pf(1)=pm$.
$1=p\left(\frac{1}{p}\right)=\frac{1}{p}+\frac{1}{p}+\ldots+\frac{1}{p}$, so
$m=f(1)=f\left(\frac{1}{p}+\frac{1}{p}+\ldots+\frac{1}{p}\right)$
$=f\left(\frac{1}{p}\right)+f\left(\frac{1}{p}\right)+\ldots+f\left(\frac{1}{p}\right)=pf\left(\frac{1}{p}\right)$, hence
$f\left(\frac{1}{p}\right)=\frac{m}{p}$. Any rational number can be written as $\frac{p}{q}$ with p, q in N.

$\frac{p}{q}=p\left(\frac{1}{q}\right)=\frac{1}{q}+\frac{1}{q}+\ldots+\frac{1}{q}$,
so $f\left(\frac{p}{q}\right)=f\left(\frac{1}{q}+\frac{1}{q}+\ldots+\frac{1}{q}\right)$
$=f\left(\frac{1}{q}\right)+f\left(\frac{1}{q}\right)+\ldots+f\left(\frac{1}{q}\right)$
$=pf\left(\frac{1}{q}\right)=p\left(\frac{m}{q}\right)=m\left(\frac{p}{q}\right)$

42. The player has run $10t$ feet after t seconds. He reaches first base when $t=9$, second base when $t=18$, third base when $t=27$, and home plate when $t=36$. The player is $10t-90$ feet from first base when $9\le t\le 18$, hence $\sqrt{90^2+(10t-90)^2}$ feet from home plate. The player is $10t-180$ feet from second base when $18\le t\le 27$, thus he is $90-(10t-180)=270-10t$ feet from third base and $\sqrt{90^2+(270-10t)^2}$ feet from home plate. The player is $10t-270$ feet from third base when $27\le t\le 36$, thus he is $90-(10t-270)=360-10t$ feet from home plate.

a. $$s=\begin{cases}10t & \text{if } 0\le t\le 9\\ \sqrt{90^2+(10t-90)^2} & \text{if } 9<t\le 18\\ \sqrt{90^2+(270-10t)^2} & \text{if } 18<t\le 27\\ 360-10t & \text{if } 27<t\le 36\end{cases}$$

b. $$s=\begin{cases}180-|180-10t| & \text{if } 0\le t\le 9 \text{ or } 27<t\le 36\\ \sqrt{90^2+(10t-90)^2} & \text{if } 9<t\le 18\\ \sqrt{90^2+(270-10t)^2} & \text{if } 18<t\le 27\end{cases}$$

43. a. $f(1.38)\approx 0.2994$
$f(4.12)\approx 3.6852$

b.

x	$f(x)$
–4	–4.05
–3	–3.1538
–2	–2.375
–1	–1.8
0	–1.25
1	–0.2
2	1.125
3	2.3846
4	3.55

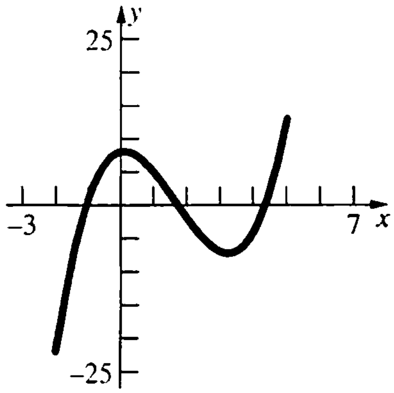
44. a. $f(1.38)\approx -76.8204$
$f(4.12)\approx 6.7508$

b.

x	$f(x)$
–4	–6.1902
–3	0.4118
–2	13.7651
–1	9.9579

0	0
1	-7.3369
2	-17.7388
3	-0.4521
4	4.4378

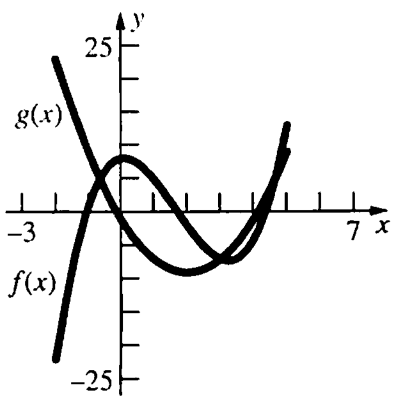
45.



a. Range: $\{y \in R: -22 \le y \le 13\}$

b. $f(x) = 0$ when $x \approx -1.1, 1.7, 4.3$
$f(x) \ge 0$ on $[-1.1, 1.7] \cup [4.3, 5]$

46.



a. $f(x) = g(x)$ at $x \approx -0.6, 3.0, 4.6$

b. $f(x) \ge g(x)$ on $[-0.6, 3.0] \cup [4.6, 5]$

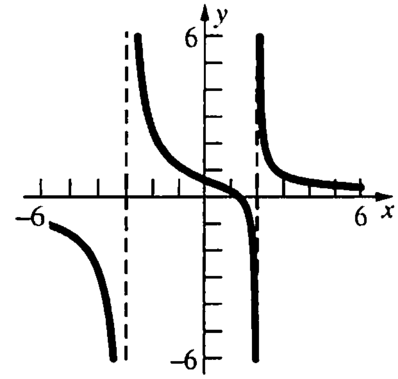
c. $|f(x)-g(x)|$

$= |x^3 - 5x^2 + x + 8 - 2x^2 + 8x + 1|$

$= |x^3 - 7x^2 + 9x + 9|$

Largest value $|f(-2) - g(-2)| = 45$

47.



a. x-intercept: $3x - 4 = 0$; $x = \frac{4}{3}$

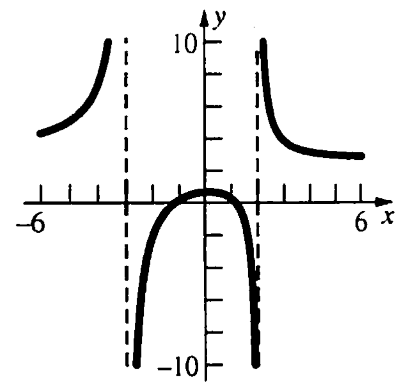
y-intercept: $\frac{3 \cdot 0 - 4}{0^2 + 0 - 6} = \frac{2}{3}$

b. $\mathbb{R}$

c. $x^2 + x - 6 = 0$; $(x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3$, $x = 2$

d. Horizontal asymptote at $y = 0$

48.



a. x-intercepts: $3x^2 - 4 = 0; x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2\sqrt{3}}{3}$

y-intercept: $\frac{2}{3}$

b. $\mathbb{R}$

c. $x^2 + x - 6 = 0$; $(x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3$, $x = 2$

d. Horizontal asymptote at $y = 3$

2.2 Concepts Review

1. $(x^2+1)^3$

2. $f(g(x))$

3. 2; left

4. a quotient of two polynomial functions

Problem Set 2.2

1. a. $(f+g)(2)=(2+3)+2^2=9$

 b. $(f\cdot g)(0)=(0+3)(0^2)=0$

 c. $(g/f)(3)=\dfrac{3^2}{3+3}=\dfrac{9}{6}=\dfrac{3}{2}$

 d. $(f\circ g)(1)=f(1^2)=1+3=4$

 e. $(g\circ f)(1)=g(1+3)=4^2=16$

 f. $(g\circ f)(-8)=g(-8+3)=(-5)^2=25$

2. a. $(f-g)(2)=(2^2+2)-\dfrac{2}{2+3}=6-\dfrac{2}{5}=\dfrac{28}{5}$

 b. $(f/g)(1)=\dfrac{1^2+1}{\frac{2}{1+3}}=\dfrac{2}{\frac{2}{4}}=4$

 c. $g^2(3)=\left[\dfrac{2}{3+3}\right]^2=\left(\dfrac{1}{3}\right)^2=\dfrac{1}{9}$

 d. $(f\circ g)(1)=f\left(\dfrac{2}{1+3}\right)=\left(\dfrac{1}{2}\right)^2+\dfrac{1}{2}=\dfrac{3}{4}$

 e. $(g\circ f)(1)=g(1^2+1)=\dfrac{2}{2+3}=\dfrac{2}{5}$

 f. $(g\circ g)(3)=g\left(\dfrac{2}{3+3}\right)=\dfrac{2}{\frac{1}{3}+3}=\dfrac{2}{\frac{10}{3}}=\dfrac{3}{5}$

3. a. $(\Phi+\Psi)(t)=t^3+1+\dfrac{1}{t}$

 b. $(\Phi\circ\Psi)(r)=\Phi\left(\dfrac{1}{r}\right)=\left(\dfrac{1}{r}\right)^3+1=\dfrac{1}{r^3}+1$

 c. $(\Psi\circ\Phi)(r)=\Psi(r^3+1)=\dfrac{1}{r^3+1}$

 d. $\Phi^3(z)=(z^3+1)^3$

 e. $(\Phi-\Psi)(5t)=[(5t)^3+1]-\dfrac{1}{5t}$
 $=125t^3+1-\dfrac{1}{5t}$

 f. $((\Phi-\Psi)\circ\Psi)(t)=(\Phi-\Psi)\left(\dfrac{1}{t}\right)$
 $=\left(\dfrac{1}{t}\right)^3+1-\dfrac{1}{\frac{1}{t}}=\dfrac{1}{t^3}+1-t$

4. a. $(f\cdot g)(x)=\dfrac{2\sqrt{x^2-1}}{x}$
 Domain: $(-\infty,\ -1]\cup[1,\ \infty)$

 b. $f^4(x)+g^4(x)=\left(\sqrt{x^2-1}\right)^4+\left(\dfrac{2}{x}\right)^4$
 $=(x^2-1)^2+\dfrac{16}{x^4}$
 Domain: $(-\infty,\ 0)\cup(0,\ \infty)$

 c. $(f\circ g)(x)=f\left(\dfrac{2}{x}\right)=\sqrt{\left(\dfrac{2}{x}\right)^2-1}=\sqrt{\dfrac{4}{x^2}-1}$
 Domain: $[-2,0)\cup(0,2]$

 d. $(g\circ f)(x)=g\left(\sqrt{x^2-1}\right)=\dfrac{2}{\sqrt{x^2-1}}$
 Domain: $(-\infty,-1)\cup(1,\infty)$

5. $(f\circ g)(x)=f(|1+x|)=\sqrt{|1+x|^2-4}$
 $=\sqrt{x^2+2x-3}$
 $(g\circ f)(x)=g\left(\sqrt{x^2-4}\right)=\left|1+\sqrt{x^2-4}\right|$
 $=1+\sqrt{x^2-4}$

6. $g^3(x)=(x^2+1)^3=(x^4+2x^2+1)(x^2+1)$
 $=x^6+3x^4+3x^2+1$
 $(g\circ g\circ g)(x)=(g\circ g)(x^2+1)$
 $=g[(x^2+1)^2+1]=g(x^4+2x^2+2)$
 $=(x^4+2x^2+2)^2+1$
 $=x^8+4x^6+8x^4+8x^2+5$

7. $g(3.141)\approx 1.188$

8. $g(2.03)\approx 0.000205$

9. $\left[g^2(\pi)-g(\pi)\right]^{1/3}=\left[(11-7\pi)^2-|11-7\pi|\right]^{1/3}$
 ≈ 4.789

10. $[g^3(\pi) - g(\pi)]^{1/3} = [(6\pi - 11)^3 - (6\pi - 11)]^{1/3}$
≈ 7.807

11. a. $g(x) = \sqrt{x}, f(x) = x + 7$

b. $g(x) = x^{15}, \ f(x) = x^2 + x$

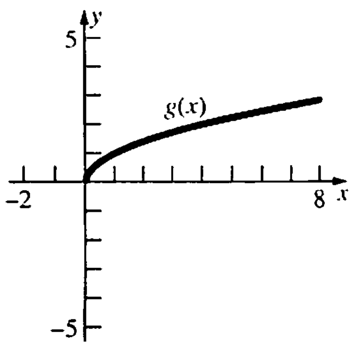
12. a. $f(x) = \dfrac{2}{x^3}, \ g(x) = x^2 + x + 1$

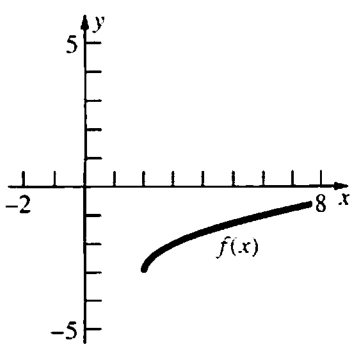
b. $f(x) = \log x, \ g(x) = x^3 + 3x$

13. $p = f \circ g \circ h$ if $f(x) = \log x, \ g(x) = \sqrt{x}$,
$h(x) = x^2 + 1$
$p = f \circ g \circ h$ if $f(x) = \log \sqrt{x}, \ g(x) = x + 1$,
$h(x) = x^2$

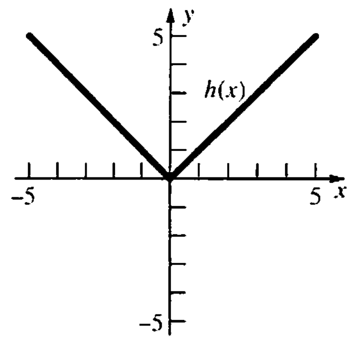
14. $p = f \circ g \circ h \circ l$ if $f(x) = \log x, \ g(x) = \sqrt{x}$,
$h(x) = x + 1, \ l(x) = x^2$

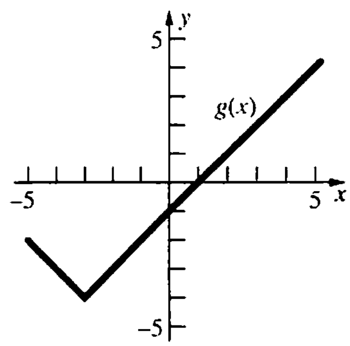
15. Translate the graph of $g(x) = \sqrt{x}$ to the right 2 units and down 3 units.



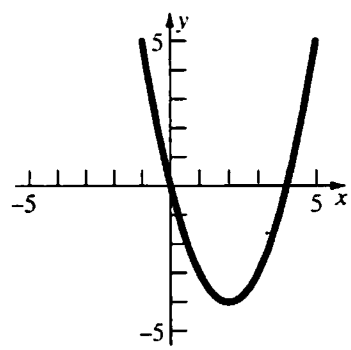


16. Translate the graph of $h(x) = |x|$ to the left 3 units and down 4 units.

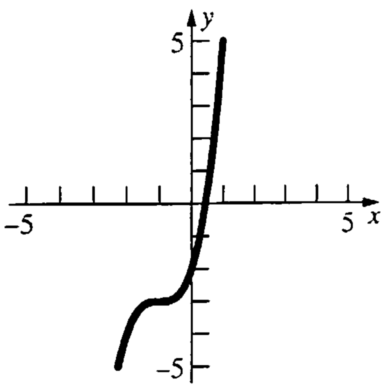




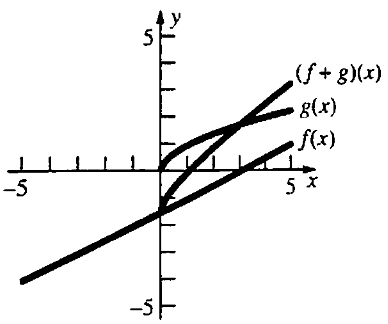
17. Translate the graph of $y = x^2$ to the right 2 units and down 4 units.



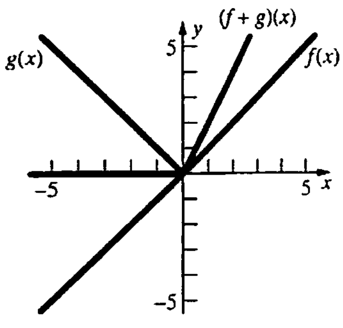
18. Translate the graph of $y = x^3$ to the left 1 unit and down 3 units.



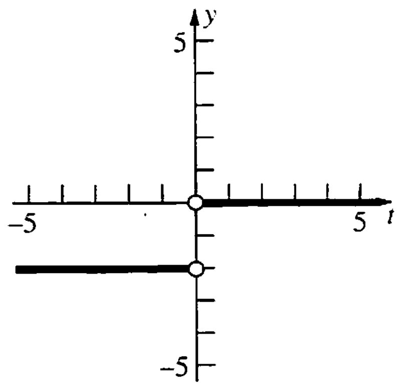
19. $(f+g)(x) = \frac{x-3}{2} + \sqrt{x}$



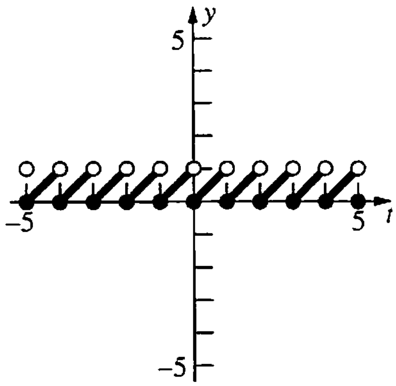
20. $(f+g)(x) = x + |x|$



21. $F(t) = \frac{|t| - t}{t}$



22. $G(t) = t - [\![t]\!]$



23. a. Even; $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x)$ $= (f+g)(x)$ if f and g are both even functions.

b. Odd; $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x)$ $= -(f+g)(x)$ if f and g are both odd functions.

c. Even; $(f \cdot g)(-x) = [f(-x)][g(-x)] = [f(x)][g(x)]$ $= (f \cdot g)(x)$ if f and g are both even functions.

d. Even; $(f \cdot g)(-x) = [f(-x)][g(-x)]$ $= [-f(x)][-g(x)] = [f(x)][g(x)] = (f \cdot g)(x)$

if f and g are both odd functions.

e. Odd; $(f \cdot g)(-x) = [f(-x)][g(-x)]$ $= [f(x)][-g(x)] = -[f(x)][g(x)] = -(f \cdot g)(x)$ if f is an even function and g is an odd function.

24. a. $F(x) - F(-x)$ is odd because
$F(-x) - F(x) = -[F(x) - F(-x)]$

b. $F(x) + F(-x)$ is even because
$F(-x) + F(-(-x)) = F(-x) + F(x)$
$= F(x) + F(-x)$

c. $\frac{F(x)-F(-x)}{2}$ is odd and $\frac{F(x)+F(-x)}{2}$ is even.
$$\frac{F(x)-F(-x)}{2}+\frac{F(x)+F(-x)}{2}=\frac{2F(x)}{2}=F(x)$$

25. Not every polynomial of even degree is an even function. For example $f(x)=x^2+x$ is neither even nor odd. Not every polynomial of odd degree is an odd function. For example $g(x)=x^3+x^2$ is neither even nor odd.

26. a. Neither

b. PF

c. RF

d. PF

e. RF

f. Neither

27. a. $P=\sqrt{29-3(2+\sqrt{t})+(2+\sqrt{t})^2}$
$=\sqrt{t+\sqrt{t}+27}$

b. When $t = 15$, $P=\sqrt{15+\sqrt{15}+27}\approx 7$

28. $R(t)=(120+2t+3t^2)(6000+700t)$
$=2100t^3+19,400t^2+96,000t+720,000$

29. $$D(t)=\begin{cases}400t & \text{if } 0\le t\le 1\\ \sqrt{(400t)^2+[300(t-1)]^2} & \text{if } t>1\end{cases}$$

$$D(t)=\begin{cases}400t & \text{if } 0\le t\le 1\\ \sqrt{250,000t^2-180,000t+90,000} & \text{if } t>1\end{cases}$$

30. $D(2.5)\approx 1097$ mi

31. $$f(f(x))=f\left(\frac{ax+b}{cx-a}\right)=\frac{a\left(\frac{ax+b}{cx-a}\right)+b}{c\left(\frac{ax+b}{cx-a}\right)-a}$$
$$=\frac{a^2x+ab+bcx-ab}{acx+bc-acx+a^2}=\frac{x(a^2+bc)}{a^2+bc}=x$$
If $a^2+bc=0$, $f(f(x))$ is undefined, while if $x=\frac{a}{c}$, $f(x)$ is undefined.

32. $$f(f(f(x)))=f\left(f\left(\frac{x-3}{x+1}\right)\right)=f\left(\frac{\frac{x-3}{x+1}-3}{\frac{x-3}{x+1}+1}\right)$$
$$=f\left(\frac{x-3-3x-3}{x-3+x+1}\right)=f\left(\frac{-2x-6}{2x-2}\right)=f\left(\frac{-x-3}{x-1}\right)$$
$$=\frac{\frac{-x-3}{x-1}-3}{\frac{-x-3}{x-1}+1}=\frac{-x-3-3x+3}{-x-3+x-1}=\frac{-4x}{-4}=x$$

If $x=-1$, $f(x)$ is undefined, while if $x=1$, $f(f(x))$ is undefined.

33. a. $$f\left(\frac{1}{x}\right)=\frac{\frac{1}{x}}{\frac{1}{x}-1}=\frac{1}{1-x}$$

b. $$f(f(x))=f\left(\frac{x}{x-1}\right)=\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}$$
$$=\frac{x}{x-x+1}=x$$

c. $$f\left(\frac{1}{f(x)}\right)=f\left(\frac{x-1}{x}\right)=\frac{\frac{x-1}{x}}{\frac{x-1}{x}-1}=\frac{x-1}{x-1-x}$$
$=1-x$

34. a. $$f(1/x)=\frac{1/x}{\sqrt{1/x}-1}=\frac{1}{\sqrt{x}-x}$$

b. $$f(f(x))=f(x/(\sqrt{x}-1))=\frac{x/(\sqrt{x}-1)}{\sqrt{\frac{x}{\sqrt{x}-1}}-1}$$
$$=\frac{x}{\sqrt{x(\sqrt{x}-1)}+1-\sqrt{x}}$$

35. $f_1(f_1(x))=x$;
$f_1(f_2(x))=\frac{1}{x}$;
$f_1(f_3(x))=1-x$;
$f_1(f_4(x))=\frac{1}{1-x}$;

$f_1(f_5(x)) = \frac{x-1}{x};$

$f_1(f_6(x)) = \frac{x}{x-1};$

$f_2(f_1(x)) = \frac{1}{x};$

$f_2(f_2(x)) = \frac{1}{\frac{1}{x}} = x;$

$f_2(f_3(x)) = \frac{1}{1-x};$

$f_2(f_4(x)) = \frac{1}{\frac{1}{1-x}} = 1-x;$

$f_2(f_5(x)) = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1};$

$f_2(f_6(x)) = \frac{1}{\frac{x}{x-1}} = \frac{x-1}{x};$

$f_3(f_1(x)) = 1-x;$

$f_3(f_2(x)) = 1-\frac{1}{x} = \frac{x-1}{x};$

$f_3(f_3(x)) = 1-(1-x) = x;$

$f_3(f_4(x)) = 1-\frac{1}{1-x} = \frac{x}{x-1};$

$f_3(f_5(x)) = 1-\frac{x-1}{x} = \frac{1}{x};$

$f_3(f_6(x)) = 1-\frac{x}{x-1} = \frac{1}{1-x};$

$f_4(f_1(x)) = \frac{1}{1-x};$

$f_4(f_2(x)) = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1};$

$f_4(f_3(x)) = \frac{1}{1-(1-x)} = \frac{1}{x};$

$f_4(f_4(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x};$

$f_4(f_5(x)) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{x-(x-1)} = x;$

$f_4(f_6(x)) = \frac{1}{1-\frac{x}{x-1}} = \frac{x-1}{x-1-x} = 1-x;$

$f_5(f_1(x)) = \frac{x-1}{x};$

$f_5(f_2(x)) = \frac{\frac{1}{x}-1}{\frac{1}{x}} = 1-x;$

$f_5(f_3(x)) = \frac{1-x-1}{1-x} = \frac{x}{x-1};$

$f_5(f_4(x)) = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-(1-x)}{1} = x;$

$f_5(f_5(x)) = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = \frac{x-1-x}{x-1} = \frac{1}{1-x};$

$f_5(f_6(x)) = \frac{\frac{x}{x-1}-1}{\frac{x}{x-1}} = \frac{x-(x-1)}{x} = \frac{1}{x};$

$f_6(f_1(x)) = \frac{x}{x-1};$

$f_6(f_2(x)) = \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{1}{1-x};$

$f_6(f_3(x)) = \frac{1-x}{1-x-1} = \frac{x-1}{x};$

$f_6(f_4(x)) = \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{1}{1-(1-x)} = \frac{1}{x};$

$f_6(f_5(x)) = \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x} = 1-x;$

$f_6(f_6(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = x$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

a. $f_3 \circ f_3 \circ f_3 \circ f_3 \circ f_3$
$= ((((f_3 \circ f_3) \circ f_3) \circ f_3) \circ f_3)$
$= (((f_1 \circ f_3) \circ f_3) \circ f_3)$
$= ((f_3 \circ f_3) \circ f_3)$
$= f_1 \circ f_3 = f_3$

b. $f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6$
$= (((((f_1 \circ f_2) \circ f_3) \circ f_4) \circ f_5) \circ f_6)$
$= ((((f_2 \circ f_3) \circ f_4) \circ f_5) \circ f_6)$
$= (f_4 \circ f_4) \circ (f_5 \circ f_6)$
$= f_5 \circ f_2 = f_3$

c. If $F \circ f_6 = f_1$, then $F = f_6$.

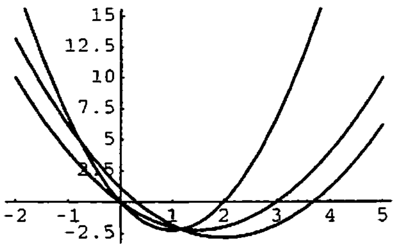
d. If $G \circ f_3 \circ f_6 = f_1$, then $G \circ f_4 = f_1$ so $G = f_5$.

e. If $f_2 \circ f_5 \circ H = f_5$, then $f_6 \circ H = f_5$ so $H = f_3$.

36. $(f_1 \circ (f_2 \circ f_3))(x) = f_1((f_2 \circ f_3)(x))$
$= f_1(f_2(f_3(x)))$

$((f_1 \circ f_2) \circ f_3)(x) = (f_1 \circ f_2)(f_3(x))$
$= f_1(f_2(f_3(x)))$
$= (f_1 \circ (f_2 \circ f_3))(x)$

37.



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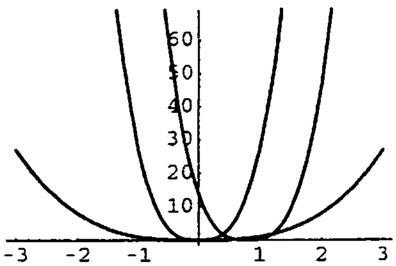

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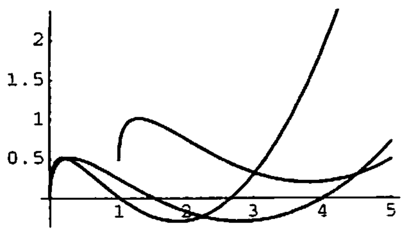

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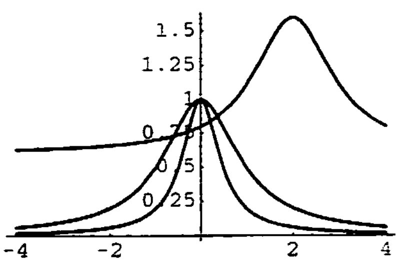
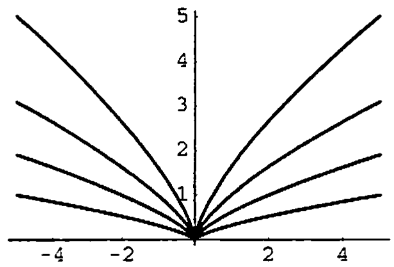
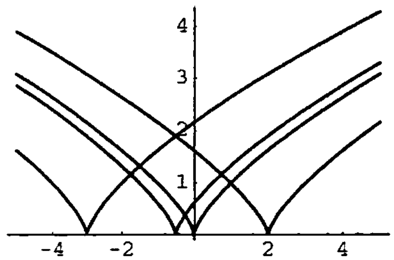


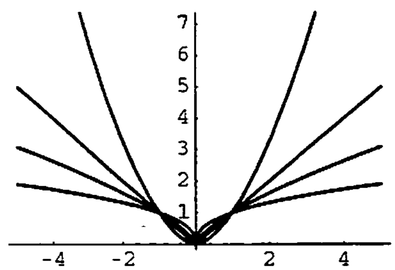
41. a.



b.



c.



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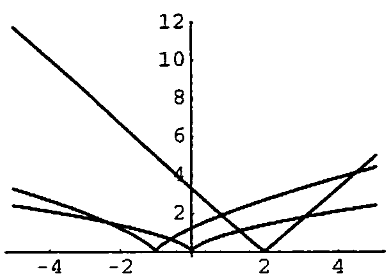


2.3 Concepts Review

1. $(-\infty, \infty)$; $[-1, 1]$

2. 2π ; 2π ; π

3. odd; even

4. $r = \sqrt{(-4)^2 + 3^2} = 5; \cos\theta = \frac{x}{r} = -\frac{4}{5}$

Problem Set 2.3

1. a. $30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$

   b. $45\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$

   c. $-60\left(\frac{\pi}{180}\right) = -\frac{\pi}{3}$

   d. $240\left(\frac{\pi}{180}\right) = \frac{4\pi}{3}$

   e. $-370\left(\frac{\pi}{180}\right) = -\frac{37\pi}{18}$

   f. $10\left(\frac{\pi}{180}\right) = \frac{\pi}{18}$

   g. $22\frac{1}{2}\left(\frac{\pi}{180}\right) = \frac{\pi}{8}$

   h. $600\left(\frac{\pi}{180}\right) = \frac{10\pi}{3}$

   i. $-120\left(\frac{\pi}{180}\right) = -\frac{2\pi}{3}$

2. a. $\frac{7}{6}\pi\left(\frac{180}{\pi}\right) = 210°$

   b. $\frac{3}{4}\pi\left(\frac{180}{\pi}\right) = 135°$

   c. $-\frac{1}{3}\pi\left(\frac{180}{\pi}\right) = -60°$

   d. $\frac{4}{3}\pi\left(\frac{180}{\pi}\right) = 240°$

   e. $-\frac{35}{18}\pi\left(\frac{180}{\pi}\right) = -350°$

   f. $\frac{3}{18}\pi\left(\frac{180}{\pi}\right) = 30°$

   g. $\frac{9}{8}\pi\left(\frac{180}{\pi}\right) = 202.5°$

   h. $\frac{10}{3}\pi\left(\frac{180}{\pi}\right) = 600°$

   i. $-\frac{4}{3}\pi\left(\frac{180}{\pi}\right) = -240°$

3. a. $33.3\left(\frac{\pi}{180}\right) \approx 0.5812$

   b. $46\left(\frac{\pi}{180}\right) \approx 0.8029$

   c. $-66.6\left(\frac{\pi}{180}\right) \approx -1.1624$

   d. $240.11\left(\frac{\pi}{180}\right) \approx 4.1907$

   e. $-369\left(\frac{\pi}{180}\right) \approx -6.4403$

   f. $11\left(\frac{\pi}{180}\right) \approx 0.1920$

   g. $22.5\left(\frac{\pi}{180}\right) \approx 0.3927$

   h. $359\left(\frac{\pi}{180}\right) \approx 6.2657$

   i. $-121.35\left(\frac{\pi}{180}\right) \approx -2.1180$

4. a. $3.141\left(\frac{180}{\pi}\right) \approx 180°$

   b. $6.28\left(\frac{180}{\pi}\right) \approx 359.8°$

   c. $5.00\left(\frac{180}{\pi}\right) \approx 286.5°$

   d. $0.001\left(\frac{180}{\pi}\right) \approx 0.057°$

   e. $-0.1\left(\frac{180}{\pi}\right) \approx -5.73°$

f. $36.0\left(\frac{180}{\pi}\right) \approx 2062.6°$

g. $-2.00\left(\frac{180}{\pi}\right) \approx -114.6°$

h. $1.234\left(\frac{180}{\pi}\right) \approx 70.70°$

i. $-10.0\left(\frac{180}{\pi}\right) \approx -573°$

5. a. $\dfrac{56.4 \tan 34.2°}{\sin 34.1°} \approx 68.37$

b. $\dfrac{5.34 \tan 21.3°}{\sin 3.1° + \cot 23.5°} \approx 0.8845$

c. $\tan(0.452) \approx 0.4855$

d. $\sin(-0.361) \approx -0.3532$

e. $\cos(-0.361) \approx 0.9355$

f. $\tan(-0.361) \approx -0.3775$

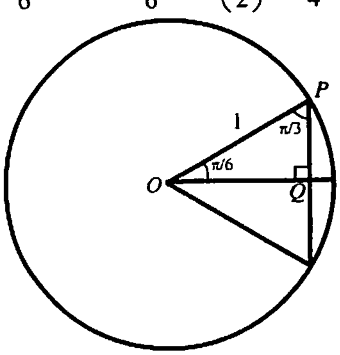
6. a. $\dfrac{234.1 \sin(1.56)}{\cos(0.34)} \approx 248.3$

b. $\sin^2(2.51) + \sqrt{\cos(0.51)} \approx 1.2828$

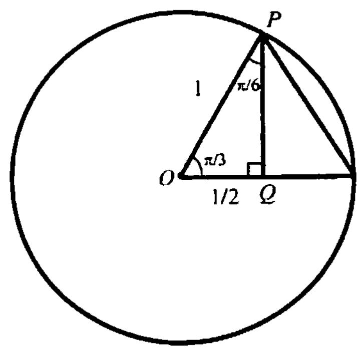
7. a. $\dfrac{56.3 \tan 34.2°}{\sin 56.1°} \approx 46.097$

b. $\left(\dfrac{\sin 35°}{\sin 26° + \cos 26°}\right)^3 \approx 0.0789$

8. Referring to Figure 2, it is clear that $\sin 0 = 0$ and $\cos 0 = 1$. If the angle is $\pi / 6$, then the triangle in the figure below is equilateral. Thus, $|PQ| = \frac{1}{2}|OP| = \frac{1}{2}$. This implies that $\sin\frac{\pi}{6} = \frac{1}{2}$. By the Pythagorean Identity, $\cos^2\frac{\pi}{6} = 1 - \sin^2\frac{\pi}{6} = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. Thus



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. The results $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ were derived in the text. If the angle is $\pi / 6$ then the triangle in the figure below is equilateral. Thus $\cos\frac{\pi}{6} = \frac{1}{2}$ and by the Pythagorean Identity, $\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.



Referring to Figure 2, it is clear that $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$. The rest of the values are obtained using the same kind of reasoning in the second quadrant.

9. a. $\tan\left(\frac{\pi}{6}\right) = \dfrac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \dfrac{\sqrt{3}}{3}$

b. $\sec(\pi) = \dfrac{1}{\cos(\pi)} = -1$

c. $\sec\left(\frac{3\pi}{4}\right) = \dfrac{1}{\cos\left(\frac{3\pi}{4}\right)} = -\sqrt{2}$

d. $\csc\left(\frac{\pi}{2}\right) = \dfrac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

e. $\cot\left(\frac{\pi}{4}\right) = \dfrac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = 1$

f. $\tan\left(-\frac{\pi}{4}\right) = \dfrac{\sin\left(-\frac{\pi}{4}\right)}{\cos\left(-\frac{\pi}{4}\right)} = -1$

10. a. $\tan\left(\frac{\pi}{3}\right)=\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}=\sqrt{3}$

b. $\sec\left(\frac{\pi}{3}\right)=\frac{1}{\cos\left(\frac{\pi}{3}\right)}=2$

c. $\cot\left(\frac{\pi}{3}\right)=\frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}=\frac{\sqrt{3}}{3}$

d. $\csc\left(\frac{\pi}{4}\right)=\frac{1}{\sin\left(\frac{\pi}{4}\right)}=\sqrt{2}$

e. $\tan\left(-\frac{\pi}{6}\right)=\frac{\sin\left(-\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{6}\right)}=-\frac{\sqrt{3}}{3}$

f. $\cos\left(-\frac{\pi}{3}\right)=\frac{1}{2}$

11. a. $(1+\sin z)(1-\sin z)=1-\sin^2 z$

$=\cos^2 z=\frac{1}{\sec^2 z}$

b. $(\sec t-1)(\sec t+1)=\sec^2 t-1=\tan^2 t$

c. $\sec t-\sin t\tan t=\frac{1}{\cos t}-\frac{\sin^2 t}{\cos t}$

$=\frac{1-\sin^2 t}{\cos t}=\frac{\cos^2 t}{\cos t}=\cos t$

d. $\frac{\sec^2 t-1}{\sec^2 t}=\frac{\tan^2 t}{\sec^2 t}=\frac{\frac{\sin^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}}=\sin^2 t$

12. a. $\sin^2 v+\frac{1}{\sec^2 v}=\sin^2 v+\cos^2 v=1$

b. $\cos 3t=\cos(2t+t)=\cos 2t\cos t-\sin 2t\sin t$

$=(2\cos^2 t-1)\cos t-2\sin^2 t\cos t$

$=2\cos^3 t-\cos t-2(1-\cos^2 t)\cos t$

$=2\cos^3 t-\cos t-2\cos t+2\cos^3 t$

$=4\cos^3 t-3\cos t$

c. $\sin 4x=\sin[2(2x)]=2\sin 2x\cos 2x$

$=2(2\sin x\cos x)(2\cos^2 x-1)$

$=2(4\sin x\cos^3 x-2\sin x\cos x)$

$=8\sin x\cos^3 x-4\sin x\cos x$

d. $(1+\cos\theta)(1-\cos\theta)=1-\cos^2\theta=\sin^2\theta$

13. a. $\frac{\sin u}{\csc u}+\frac{\cos u}{\sec u}=\sin^2 u+\cos^2 u=1$

b. $(1-\cos^2 x)(1+\cot^2 x)=(\sin^2 x)(\csc^2 x)$

$=\sin^2 x\left(\frac{1}{\sin^2 x}\right)=1$

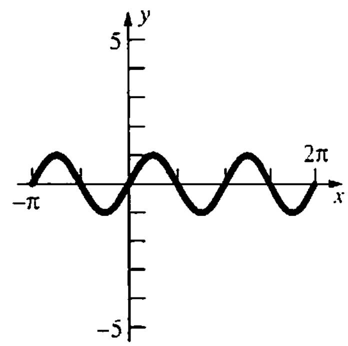
c. $\sin t(\csc t-\sin t)=\sin t\left(\frac{1}{\sin t}-\sin t\right)$

$=1-\sin^2 t=\cos^2 t$

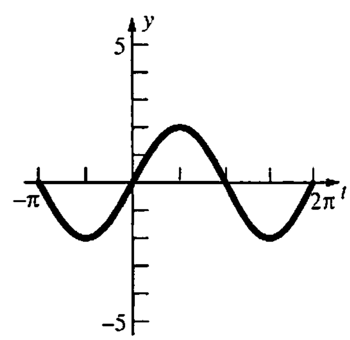
d. $\frac{1-\csc^2 t}{\csc^2 t}=-\frac{\cot^2 t}{\csc^2 t}=-\frac{\frac{\cos^2 t}{\sin^2 t}}{\frac{1}{\sin^2 t}}$

$=-\cos^2 t=-\frac{1}{\sec^2 t}$

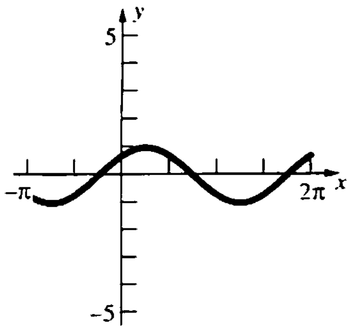
14. a. $y=\sin 2x$



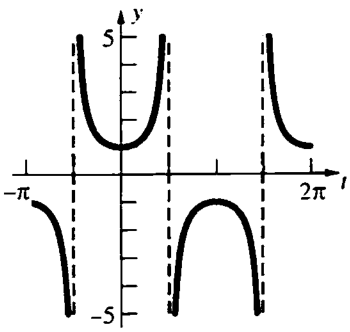
b. $y=2\sin t$



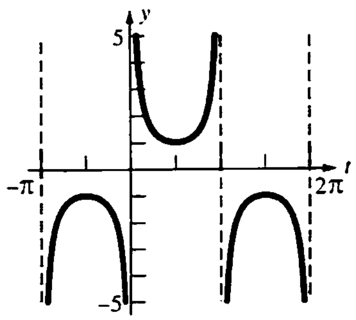
c. $y = \cos\left(x - \frac{\pi}{4}\right)$



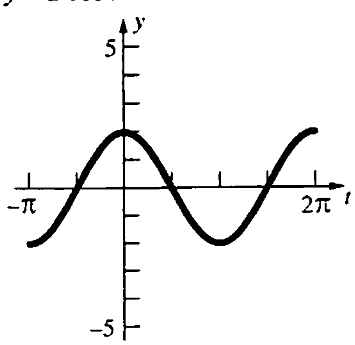
d. $y = \sec t$



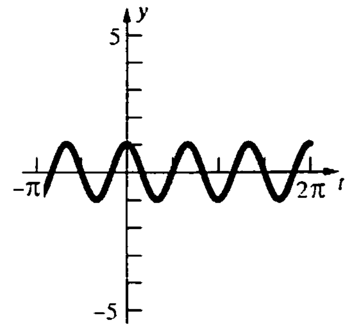
15. a. $y = \csc t$



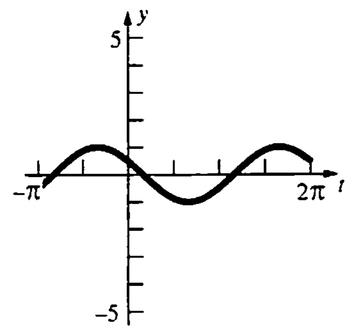
b. $y = 2 \cos t$



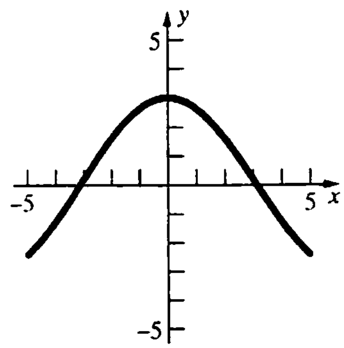
c. $y = \cos 3t$



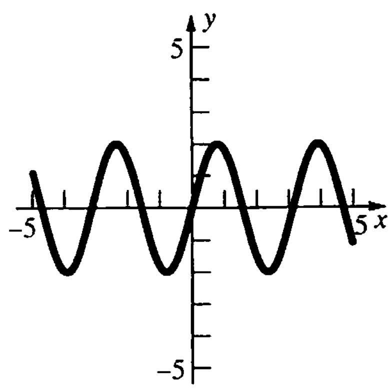
d. $y = \cos\left(t + \frac{\pi}{3}\right)$



16. $y = 3\cos\frac{x}{2}$
Period = 4π, amplitude = 3

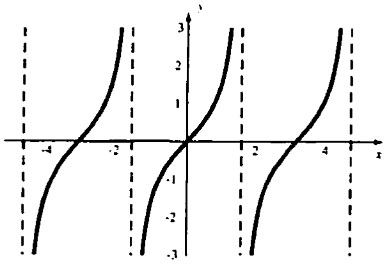


17. $y = 2 \sin 2x$
Period = π, amplitude = 2



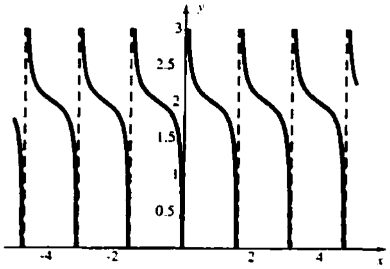
18. $y = \tan x$

Period = π



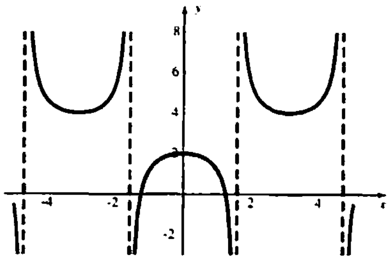
19. $y = 2 + \frac{1}{6}\cot(2x)$

Period = $\frac{\pi}{2}$, shift: 2 units up



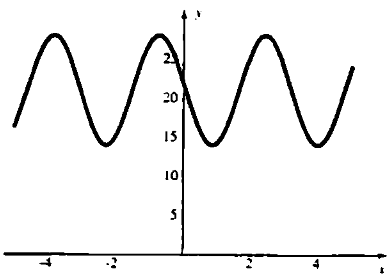
20. $y = 3 + \sec(x - \pi)$

Period = 2π, shift: 3 units up, π units right



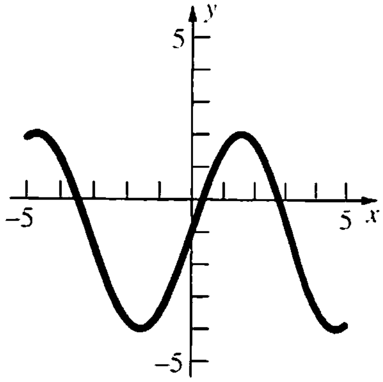
21. $y = 21 + 7\sin(2x + 3)$

Period = π, amplitude = 7, shift: 21 units up, $\frac{3}{2}$ units left



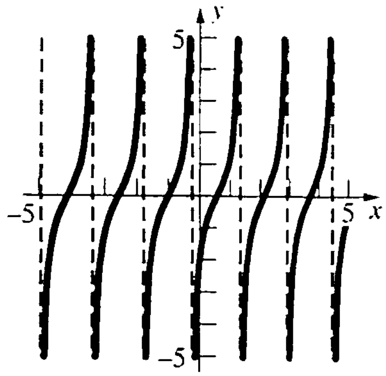
22. $y = 3\cos\left(x - \frac{\pi}{2}\right) - 1$

Period = 2π, amplitude = 3, shifts: $\frac{\pi}{2}$ units right and 1 unit down.



23. $y = \tan\left(2x - \frac{\pi}{3}\right)$

Period = $\frac{\pi}{2}$, shift: $\frac{\pi}{6}$ units right



24. **a.** and **g.**: $y = \sin\left(x + \frac{\pi}{2}\right) = \cos x = -\cos(\pi - x)$

b. and **e.**: $y = \cos\left(x + \frac{\pi}{2}\right) = \sin(x + \pi)$

$= -\sin(\pi - x)$

c. and **f.**: $y = \cos\left(x - \frac{\pi}{2}\right) = \sin x$

$= -\sin(x + \pi)$

d. and **h.**:

$y = \sin\left(x - \frac{\pi}{2}\right) = \cos(x + \pi) = \cos(x - \pi)$

25. **a.** $-t\sin(-t) = t\sin t$; even

b. $\sin^2(-t) = \sin^2 t$; even

c. $\csc(-t) = \frac{1}{\sin(-t)} = -\csc t$; odd

d. $|\sin(-t)| = |-\sin t| = |\sin t|$; even

e. $\sin(\cos(-t)) = \sin(\cos t)$; even

f. $-x + \sin(-x) = -x - \sin x = -(x + \sin x)$; odd

26. a. $\cot(-t) + \sin(-t) = -\cot t - \sin t$
$= -(\cot t + \sin t)$; odd

b. $\sin^3(-t) = -\sin^3 t$; odd

c. $\sec(-t) = \dfrac{1}{\cos(-t)} = \sec t$; even

d. $\sqrt{\sin^4(-t)} = \sqrt{\sin^4 t}$; even

e. $\cos(\sin(-t)) = \cos(-\sin t) = \cos(\sin t)$; even

f. $(-x)^2 + \sin(-x) = x^2 - \sin x$; neither

27. $\cos^2\dfrac{\pi}{3} = \dfrac{1+\cos 2\left(\frac{\pi}{3}\right)}{2} = \dfrac{1+\cos\frac{2\pi}{3}}{2} = \dfrac{1-\frac{1}{2}}{2} = \dfrac{1}{4}$

28. $\sin^2\dfrac{\pi}{6} = \dfrac{1-\cos 2\left(\frac{\pi}{6}\right)}{2} = \dfrac{1-\cos\frac{\pi}{3}}{2} = \dfrac{1-\frac{1}{2}}{2} = \dfrac{1}{4}$

29. $\sin^3\dfrac{\pi}{6} = \left(\sin^2\dfrac{\pi}{6}\right)\left(\sin\dfrac{\pi}{6}\right) = \dfrac{1}{4}\cdot\dfrac{1}{2} = \dfrac{1}{8}$

30. $\cos^2\dfrac{\pi}{12} = \dfrac{1+\cos 2\left(\frac{\pi}{12}\right)}{2} = \dfrac{1+\cos\frac{\pi}{6}}{2} = \dfrac{1+\frac{\sqrt{3}}{2}}{2}$
$= \dfrac{2+\sqrt{3}}{4}$

31. $\sin^2\dfrac{\pi}{8} = \dfrac{1-\cos 2\left(\frac{\pi}{8}\right)}{2} = \dfrac{1-\cos\frac{\pi}{4}}{2} = \dfrac{1-\frac{\sqrt{2}}{2}}{2}$
$= \dfrac{2-\sqrt{2}}{4}$

32. a. $\sin(x-y) = \sin x\cos(-y) + \cos x\sin(-y)$
$= \sin x\cos y - \cos x\sin y$

b. $\cos(x-y) = \cos x\cos(-y) - \sin x\sin(-y)$
$= \cos x\cos y + \sin x\sin y$

c. $\tan(x-y) = \dfrac{\tan x + \tan(-y)}{1-\tan x\tan(-y)}$
$= \dfrac{\tan x - \tan y}{1+\tan x\tan y}$

33. $\tan(t+\pi) = \dfrac{\tan t + \tan\pi}{1-\tan t\tan\pi} = \dfrac{\tan t + 0}{1-(\tan t)(0)}$
$= \tan t$

34. $\cos(x-\pi) = \cos x\cos(-\pi) - \sin x\sin(-\pi)$
$= -\cos x - 0\cdot\sin x = -\cos x$

35. $s = rt = (2.5\text{ ft})(2\pi\text{ rad}) = 5\pi$ ft, so the tire goes 5π feet per revolution, or $\frac{1}{5\pi}$ revolutions per foot.
$$\left(\frac{1}{5\pi}\frac{\text{rev}}{\text{ft}}\right)\left(60\frac{\text{mi}}{\text{hr}}\right)\left(\frac{1}{60}\frac{\text{hr}}{\text{min}}\right)\left(5280\frac{\text{ft}}{\text{mi}}\right)$$
≈ 336 rev/min

36. $s = rt = (2\text{ ft})(150\text{ rev})(2\pi\text{ rad/rev}) \approx 1885$ ft

37. $r_1t_1 = r_2t_2; 6(2\pi)t_1 = 8(2\pi)(21)$
$t_1 = 28$ rev/sec

38. $\Delta y = \sin\alpha$ and $\Delta x = \cos\alpha$
$m = \dfrac{\Delta y}{\Delta x} = \dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$

39. a. $\tan\alpha = \sqrt{3}$
$\alpha = \dfrac{\pi}{3}$

b. $\sqrt{3}x + 3y = 6$
$3y = -\sqrt{3}x + 6$
$y = -\dfrac{\sqrt{3}}{3}x + 2; m = -\dfrac{\sqrt{3}}{3}$
$\tan\alpha = -\dfrac{\sqrt{3}}{3}$
$\alpha = -\dfrac{\pi}{6} = \dfrac{5\pi}{6}$

40. $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$
$\tan\theta = \tan(\theta_2 - \theta_1) = \dfrac{\tan\theta_2 + \tan(-\theta_1)}{1-\tan\theta_2\tan(-\theta_1)}$
$= \dfrac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2\tan\theta_1} = \dfrac{m_2 - m_1}{1+m_1m_2}$

41. a. $\tan\theta = \dfrac{3-2}{1+3(2)} = \dfrac{1}{7}$
$\theta \approx 0.1419$

b. $\tan\theta = \dfrac{-1-\frac{1}{2}}{1+\left(\frac{1}{2}\right)(-1)} = -3$
$\theta \approx 1.8925$

c. $2x - 6y = 12$ $\quad 2x + y = 0$

$-6y = -2x + 12 \quad y = -2x$

$y = \frac{1}{3}x - 2$

$m_1 = \frac{1}{3},\ m_2 = -2$

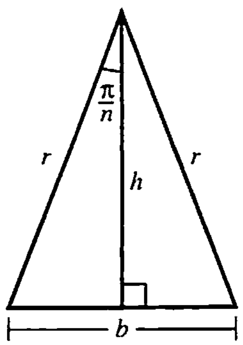
$$\tan\theta = \frac{-2-\frac{1}{3}}{1+\left(\frac{1}{3}\right)(-2)} = -7; \theta \approx 1.7127$$

42. Recall that the area of the circle is πr^2. The measure of the vertex angle of the circle is 2π. Observe that the ratios of the vertex angles must equal the ratios of the areas. Thus, $\frac{t}{2\pi} = \frac{A}{\pi r^2}$, so

$A = \frac{1}{2}r^2 t$.

43. $A = \frac{1}{2}(2)(5)^2 = 25\text{cm}^2$

44. Divide the polygon into n isosceles triangles by drawing lines from the center of the circle to the corners of the polygon. If the base of each triangle is on the perimeter of the polygon, then the angle opposite each base has measure $\frac{2\pi}{n}$. Bisect this angle to divide the triangle into two right triangles (See figure).



$\sin\frac{\pi}{n} = \frac{b}{2r}$ so $b = 2r\sin\frac{\pi}{n}$ and $\cos\frac{\pi}{n} = \frac{h}{r}$ so

$h = r\cos\frac{\pi}{n}$.

$P = nb = 2rn\sin\frac{\pi}{n}$

$A = n\left(\frac{1}{2}bh\right) = nr^2\cos\frac{\pi}{n}\sin\frac{\pi}{n}$

45. The base of the triangle is the side opposite the angle t. Then the base has length $2r\sin\frac{t}{2}$ (similar to Problem 45). The radius of the semicircle is $r\sin\frac{t}{2}$ and the height of the triangle is $r\cos\frac{t}{2}$.

$$A = \frac{1}{2}\left(2r\sin\frac{t}{2}\right)\left(r\cos\frac{t}{2}\right) + \frac{\pi}{2}\left(r\sin\frac{t}{2}\right)^2$$
$$= r^2\sin\frac{t}{2}\cos\frac{t}{2} + \frac{\pi r^2}{2}\sin^2\frac{t}{2}$$

46.
$$\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}$$
$$= \frac{1}{2}\left[\cos\frac{3}{4}x + \cos\frac{1}{4}x\right]\frac{1}{2}\left[\cos\frac{3}{16}x + \cos\frac{1}{16}x\right]$$
$$= \frac{1}{4}\left[\cos\frac{3}{4}x + \cos\frac{1}{4}x\right]\left[\cos\frac{3}{16}x + \cos\frac{1}{16}x\right]$$
$$= \frac{1}{4}\left[\cos\frac{3}{4}x\cos\frac{3}{16}x + \cos\frac{3}{4}x\cos\frac{1}{16}x\right.$$
$$\left. + \cos\frac{1}{4}x\cos\frac{3}{16}x + \cos\frac{1}{4}x\cos\frac{1}{16}x\right]$$
$$= \frac{1}{4}\left[\frac{1}{2}\left(\cos\frac{15}{16} + \cos\frac{9}{16}x\right) + \frac{1}{2}\left(\cos\frac{13}{16}x + \cos\frac{11}{16}x\right)\right.$$
$$\left. + \frac{1}{2}\left(\cos\frac{7}{16}x + \cos\frac{1}{16}x\right) + \frac{1}{2}\left(\cos\frac{5}{16}x + \cos\frac{3}{16}x\right)\right]$$
$$= \frac{1}{8}\left[\cos\frac{15}{16}x + \cos\frac{13}{16}x + \cos\frac{11}{16}x + \cos\frac{9}{16}x\right.$$
$$\left. + \cos\frac{7}{16}x + \cos\frac{5}{16}x + \cos\frac{3}{16}x + \cos\frac{1}{16}x\right]$$

47.
$$S_1(n) = 1 + 2 + 3 + \ldots + n$$
$$+ \quad S_1(n) = n + (n-1) + (n-2) + \ldots + 1$$
$$2 \cdot S_1(n) = (n+1) + (n+1) + (n+1) + \ldots + (n+1)$$

Thus, $2 \cdot S_1(n) = n(n+1)$, so $S_1(n) = \frac{n(n+1)}{2}$

48. a. Add up $(x+1)^3 - x^3$ for $x = 0, 1, 2, \ldots, n$ to obtain

$1^3 + 0^3 + 2^3 - 1^3 + 3^2 - 2^3 + \ldots + (n+1)^3 - n^3$
$= (n+1)^3$

Now add up $3x^2 + 3x + 1$ for $x = 0, 1, 2, \ldots, n$ to obtain

$3 \cdot 0^2 + 3 \cdot 0 + 1 + 3 \cdot 1^2 + 3 \cdot 1 + 1 + 3 \cdot 2^2 + 3 \cdot 2$
$+ 1 + \ \ldots\ + 3 \cdot n^2 + 3 \cdot n + 1$

$= 3(0^2 + 1^2 + 2^2 + ... + n^2)$
$+3(0 + 1 + 2 + ... + n) + 1 + 1 + 1 + ... + 1$
$= 3S_2(n) + 3S_1(n) + n + 1$

Thus, $(n+1)^3 = 3S_2(n) + 3S_1(n) + n + 1$

b. Solve the equation in part a. for $S_2(n)$, and use the formula for $S_1(n)$, to get:

$$S_2(n) = \frac{1}{3}\left[(n+1)^3 - 3S_1(n) - (n+1)\right]$$
$$= \frac{1}{3}\left[(n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)\right]$$
$$= \frac{1}{3}(n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right]$$
$$= \frac{1}{6}(n+1)[2(n+1)^2 - 3n - 2]$$
$$= \frac{1}{6}(n+1)[2n^2 + 4n + 2 - 3n - 2]$$
$$= \frac{1}{6}(n+1)(2n^2 + n) = \frac{1}{6}n(n+1)(2n+1)$$

49. $(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Add up $(x+1)^4 - x^4$ for $x = 0, 1, 2, ..., n$ to obtain

$1^4 - 0^4 + 2^4 - 1^4 + 3^4 - 2^4 + ...$
$+(n+1)^4 - n^4 = (n+1)^4$

Now add up $4x^3 + 6x^2 + 4x + 1$ for $x = 0, 1, 2, ..., n$ to obtain

$4(0^3 + 1^3 + 2^3 + ... + n^3)$
$+6(0^2 + 1^2 + 2^2 + ... + n^2)$
$+4(0 + 1 + 2 + ... + n) + 1 + 1 + 1 + ... + 1$
$= 4S_3(n) + 6S_2(n) + 4S_1(n) + n + 1$

Thus, since $(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$,
$(n+1)^4 = 4S_3(n) + 6S_2(n) + 4S_1(n) + n + 1$

Solve this equation for $S_3(n)$:

$$S_3(n) = \frac{1}{4}[(n+1)^4 - 6S_2(n) - 4S_1(n) - (n+1)]$$
$$= \frac{1}{4}[(n+1)^4 - 6\cdot\frac{1}{6}n(n+1)(2n+1)$$
$$-4\cdot\frac{1}{2}n(n+1) - (n+1)]$$
$$= \frac{1}{4}(n+1)[(n+1)^3 - n(2n+1) - 2n - 1]$$
$$= \frac{1}{4}(n+1)(n^3 + 3n^2 + 3n + 1 - 2n^2 - n - 2n - 1)$$
$$= \frac{1}{4}(n+1)(n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$$

50. $(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Add up $(x+1)^5 - x^5$ for $x = 0, 1, 2, ..., n$ to obtain

$1^5 - 0^5 + 2^5 - 1^5 + 3^5 - 2^5 + ... + (n+1)^5 - n^5$
$= (n+1)^5$

Now add up $5x^4 + 10x^3 + 10x^2 + 5x + 1$ for $x = 0, 1, 2, ..., n$ to obtain

$5(0^4 + 1^4 + 2^4 + ... + n^4)$
$+10(0^3 + 1^3 + 2^3 + ... + n^3)$
$+10(0^2 + 1^2 + 2^2 + ... + n^2)$
$+5(0 + 1 + 2 + ... + n) + 1 + 1 + 1 + ... + 1$
$= 5S_4(n) + 10S_3(n) + 10S_2(n) + 5S_1(n) + n + 1$

Thus, since
$(x+1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1$,
$(n+1)^5 = 5S_4(n) + 10S_3(n) + 10S_2(n) + 5S_1(n) + n + 1$

Solve this equation for $S_4(n)$:

$$S_4(n) = \frac{1}{5}[(n+1)^5 - 10S_3(n) - 10S_2(n) - 5S_1(n) - (n+1)]$$
$$= \frac{1}{5}\left[(n+1)^5 - 10\cdot\frac{1}{4}n^2(n+1)^2 - 10\cdot\frac{1}{6}n(n+1)(2n+1)\right.$$
$$\left.-5\cdot\frac{1}{2}n(n+1) - (n+1)\right]$$
$$= \frac{1}{5}(n+1)\left[(n+1)^4 - \frac{5}{2}n^2(n+1) - \frac{5}{3}n(2n+1) - \frac{5}{2}n - 1\right]$$
$$= \frac{1}{30}(n+1)[6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6]$$
$$= \frac{1}{30}(n+1)(6n^4 + 24n^3 + 36n^2 + 24n + 6 - 15n^3$$
$$-15n^2 - 20n^2 - 10n - 15n - 6)$$
$$= \frac{1}{30}(n+1)(6n^4 + 9n^3 + n^2 - n)$$
$$= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1)$$

51. The temperature function is

$$T(t) = 80 + 25\sin\left(\frac{2\pi}{12}\left(t - \frac{7}{2}\right)\right).$$

The normal high temperature for November 15[th] is then $T(10.5) = 67.5$ °F.

52. The water level function is

$$F(t) = 8.5 + 3.5\sin\left(\frac{2\pi}{12}(t-9)\right).$$

The water level at 5:30 P.M. is then $F(17.5) \approx 5.12$ ft .

53. As t increases, the point on the rim of the wheel will move around the circle of radius 2.

a. $x(2) \approx 1.902$
$y(2) \approx 0.618$

$x(6) \approx -1.176$
$y(6) \approx -1.618$
$x(10) = 0$
$y(10) = 2$
$x(0) = 0$
$y(0) = 2$

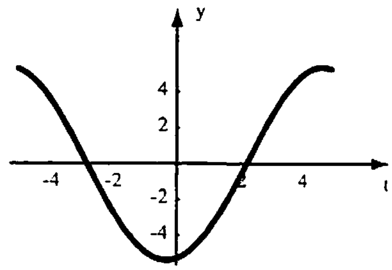
b. $x(t) = -2\sin\left(\frac{\pi}{5}t\right), y(t) = 2\cos\left(\frac{\pi}{5}t\right)$

c. The point is at (2, 0) when $\frac{\pi}{5}t = \frac{\pi}{2}$; that is ,

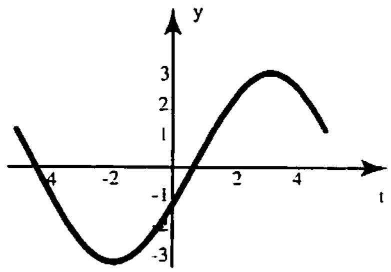
when $t = \frac{5}{2}$

54. Both functions have frequency $\frac{2\pi}{10}$. When you add functions that have the same frequency, the sum has the same frequency.

a. $y(t) = 3\sin(\pi t/5) - 5\cos(\pi t/5) + 2\sin((\pi t/5) - 3)$



b. $y(t) = 3\cos(\pi t/5 - 2) + \cos(\pi t/5) + \cos((\pi t/5) - 3)$



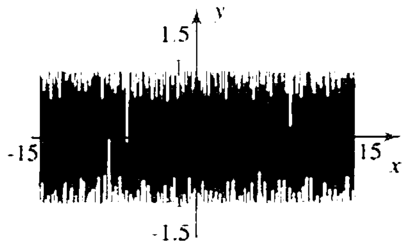
55. a. $C\sin(\omega t + \phi) = (C\sin\omega t)\cos\phi + (C\cos\omega t)\sin\phi$. Thus $A = C\cos\omega t$ and $B = C\sin\omega t$.

b. $A^2 + B^2 = (C\cos\omega t)^2 + (C\sin\omega t)^2$
$= C^2(\cos^2\omega t) + C^2(\sin^2\omega t) = C^2$

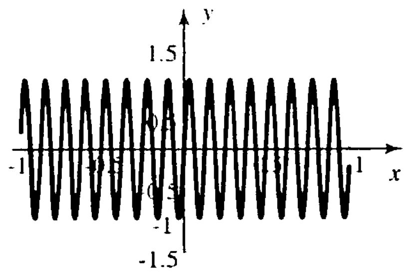
c. $A_1\sin(\omega t + \phi_1) + A_2\sin(\omega t + \phi_2) + A_3(\sin\omega t + \phi_3)$
$= A_1(\sin\omega t\cos\phi_1 + \cos\omega t\sin\phi_1)$
$+ A_2(\sin\omega t\cos\phi_2 + \cos\omega t\sin\phi_2)$
$+ A_3(\sin\omega t\cos\phi_3 + \cos\omega t\sin\phi_3)$
$= (A_1\cos\phi_1 + A_2\cos\phi_2 + A_3\cos\phi_3)\sin\omega t$
$+ (A_1\sin\phi_1 + A_2\sin\phi_2 + A_3\sin\phi_3)\cos\omega t$

d. Written response. Answers will vary.

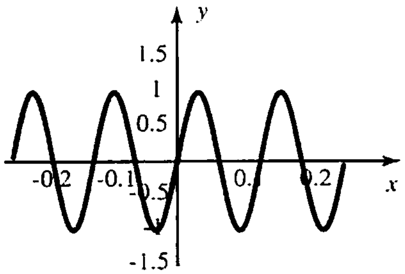
56. (**a.**), (**b.**), and (**c.**) all look similar to this:



d.

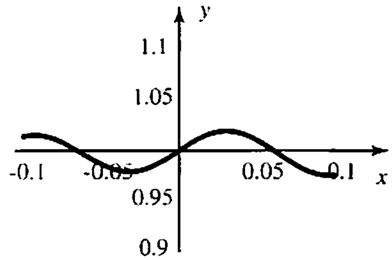


e.



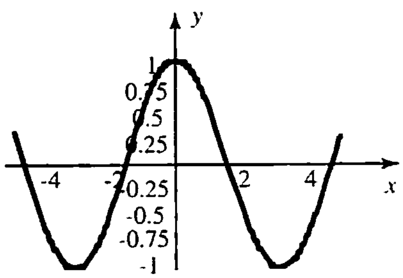
The windows in (**a**)-(**c**) are not helpful because the function oscillates too much over the domain plotted. Plots in (**d**) or (**e**) show the behavior of the function.

c.

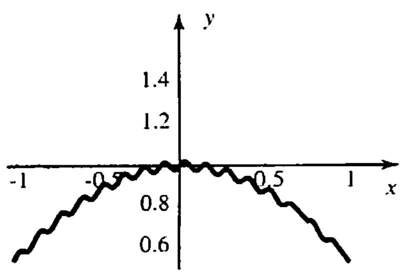


The plot in (**a**) shows the long term behavior of the function, but not the short term behavior, whereas the plot in (**c**) shows the short term behavior, but not the long term behavior. The plot in (**b**) shows a little of each.

57. a.



b.

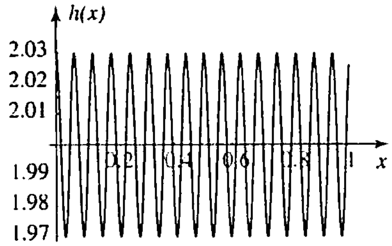


58. a.

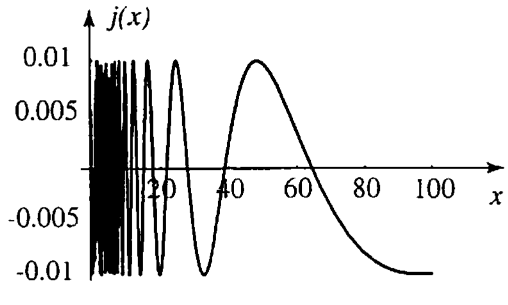
$$h(x) = (f \circ g)(x) = \frac{\frac{3}{100}\cos(100x) + 2}{\left(\frac{1}{100}\right)^2 \cos^2(100x) + 1}$$

$$j(x) = (g \circ f)(x) = \frac{1}{100}\cos\left(100\frac{3x+2}{x^2+1}\right)$$

b.



c.



2.4 Concepts Review

1. L; c

2. 6

3. L; right

4. $\lim_{x \to c} f(x) = M$

Problem Set 2.4

1. $\lim_{x \to 3}(x - 5) = -2$

2. $\lim_{t \to -1}(1 - 2t) = 3$

3. $\lim_{x \to -2}(x^2 + 2x - 1) = (-2)^2 + 2(-2) - 1 = -1$

4. $\lim_{x \to -2}(x^2 + 2t - 1) = (2)^2 + 2t - 1 = 3 + 2t$

5. $\lim_{t \to -1} \frac{1-2t}{\sqrt{3t+21}} = \frac{1-2(-1)}{\sqrt{3(-1)+21}} = \frac{3}{\sqrt{18}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

6. $\lim_{t \to -1} \frac{\sqrt{1-2t}}{(3t+2)^3} = \frac{\sqrt{1-2(-1)}}{[3(-1)+2]^3} = \frac{\sqrt{3}}{(-1)^3} = -\sqrt{3}$

7. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x+2)$
$= 2 + 2 = 4$

8. $\lim_{t \to -7} \frac{t^2+4t-21}{t+7} = \lim_{t \to -7} \frac{(t+7)(t-3)}{t+7}$
$= \lim_{t \to -7}(t-3) = -7 - 3 = -10$

9. $\lim_{x \to -1} \frac{x^3 - 4x^2 + x + 6}{x+1}$
$= \lim_{x \to -1} \frac{(x+1)(x^2-5x+6)}{x+1}$
$= \lim_{x \to -1}(x^2 - 5x + 6) = (-1)^2 - 5(-1) + 6 = 12$

10. $\lim_{x \to 0} \frac{x^4 + 2x^3 - x^2}{x^2} = \lim_{x \to 0}(x^2 + 2x - 1) = -1$

11. $\lim_{x \to -t} \frac{x^2 - t^2}{x+t} = \lim_{x \to -t} \frac{(x+t)(x-t)}{x+t} = \lim_{x \to -t}(x-t)$
$= -t - t = -2t$

12. $\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3)$
$= 3 + 3 = 6$

13. $\lim_{t \to 2} \frac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2} = \lim_{t \to 2} \frac{(t-2)^2\sqrt{t+4}}{9(t-2)^2}$
$= \lim_{t \to 2} \frac{\sqrt{t+4}}{9} = \frac{\sqrt{2+4}}{9} = \frac{\sqrt{6}}{9}$

14. $\lim_{t \to 7} \frac{\sqrt{(t-7)^3}}{t-7} = \lim_{t \to 7} \frac{(t-7)\sqrt{t-7}}{t-7} = \lim_{t \to 7}\sqrt{t-7}$
$= \sqrt{7-7} = 0$

15. $\lim_{x \to 3} \frac{x^4 - 18x^2 + 81}{(x-3)^2} = \lim_{x \to 3} \frac{(x^2-9)^2}{(x-3)^2}$
$= \lim_{x \to 3} \frac{(x-3)^2(x+3)^2}{(x-3)^2} = \lim_{x \to 3}(x+3)^2 = (3+3)^2$
$= 36$

16. $\lim_{u \to 1} \frac{(3u+4)(2u-2)^3}{(u-1)^2} = \lim_{u \to 1} \frac{8(3u+4)(u-1)^3}{(u-1)^2}$
$= \lim_{u \to 1} 8(3u+4)(u-1) = 8[3(1)+4](1-1) = 0$

17. $\lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$
$= \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0}(h+4) = 4$

18. $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$
$= \lim_{h \to 0} \frac{h^2 + 2xh}{h} = \lim_{h \to 0}(h + 2x) = 2x$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.5
–1.	0.420735
–0.1	0.499167
–0.01	0.499992
–0.001	0.5

$\lim_{x \to 0} \frac{\sin x}{2x} = 0.5$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.00025
–1.	–0.229849
–0.1	–0.0249792
–0.01	–0.00249998
–0.001	–0.00025

$$\lim_{t\to 0}\frac{1-\cos t}{2t}=0$$

21.

x	$(x-\sin x)^2/x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.77775×10^{-10}
0.001	2.77778×10^{-14}
–1.	0.0251314
–0.1	2.775×10^{-6}
–0.01	2.77775×10^{-10}
–0.001	2.77778×10^{-14}

$$\lim_{x\to 0}\frac{(x-\sin x)^2}{x^2}=0$$

22.

x	$(1-\cos x)^2/x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
–1.	0.211322
–0.1	0.00249584
–0.01	0.0000249996
–0.001	2.5×10^{-7}

$$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2}=0$$

23.

t	$(t^2-1)/(\sin(t-1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$$\lim_{t\to 1}\frac{t^2-1}{\sin(t-1)}=2$$

24.

x	$\frac{x-\sin(x-3)-3}{x-3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x\to 3}\frac{x-\sin(x-3)-3}{x-3}=0$$

25.

x	$(1+\sin(x-3\pi/2))/(x-\pi)$
$1.+\pi$	0.4597
$0.1+\pi$	0.0500
$0.01+\pi$	0.0050
$0.001+\pi$	0.0005
$-1.+\pi$	–0.4597
$-0.1+\pi$	–0.0500
$-0.01+\pi$	–0.0050
$-0.001+\pi$	–0.0005

$$\lim_{x\to\pi}\frac{1+\sin\left(x-\frac{3\pi}{2}\right)}{x-\pi}=0$$

26.

t	$(1-\cot t)/(1/t)$
1.	0.357907
0.1	–0.896664
0.01	–0.989967
0.001	–0.999
–1.	–1.64209
–0.1	–1.09666
–0.01	–1.00997
–0.001	–1.001

$$\lim_{t\to 0}\frac{1-\cot t}{\frac{1}{t}} = -1$$

27.

x	$(x-\pi/4)^2/(\tan x-1)^2$
$1.+\frac{\pi}{4}$	0.0320244
$0.1+\frac{\pi}{4}$	0.201002
$0.01+\frac{\pi}{4}$	0.245009
$0.001+\frac{\pi}{4}$	0.2495
$-1.+\frac{\pi}{4}$	0.674117
$-0.1+\frac{\pi}{4}$	0.300668
$-0.01+\frac{\pi}{4}$	0.255008
$-0.001+\frac{\pi}{4}$	0.2505

$$\lim_{x\to\frac{\pi}{4}}\frac{\left(x-\frac{\pi}{4}\right)^2}{(\tan x-1)^2} = 0.25$$

28.

u	$(2-2\sin u)/3u$
$1.+\frac{\pi}{2}$	0.11921
$0.1+\frac{\pi}{2}$	0.00199339
$0.01+\frac{\pi}{2}$	0.0000210862
$0.001+\frac{\pi}{2}$	2.12072×10^{-7}
$-1.+\frac{\pi}{2}$	0.536908
$-0.1+\frac{\pi}{2}$	0.00226446
$-0.01+\frac{\pi}{2}$	0.0000213564
$-0.001+\frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u\to\frac{\pi}{2}}\frac{2-2\sin u}{3u} = 0$$

29. a. $\lim_{x\to -3} f(x) = 2$

b. $f(-3) = 1$

c. $f(-1)$ does not exist.

d. $\lim_{x\to -1} f(x) = \frac{5}{2}$

e. $f(1) = 2$

f. $\lim_{x\to 1} f(x)$ does not exist.

g. $\lim_{x\to 1^-} f(x) = 2$

h. $\lim_{x\to 1^+} f(x) = 1$

30. a. $\lim_{x\to -3} f(x)$ does not exist.

b. $f(-3) = 1$

c. $f(-1) = 1$

d. $\lim_{x\to -1} f(x) = 2$

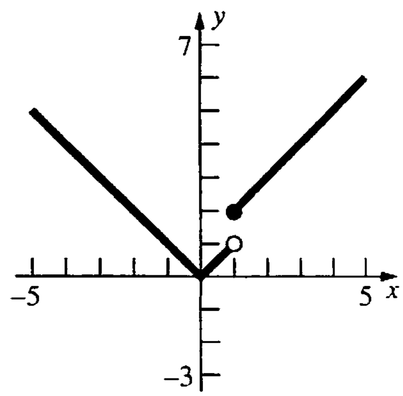
e. $f(1) = 1$

f. $\lim_{x\to 1} f(x)$ does not exist.

g. $\lim_{x\to 1^-} f(x) = 1$

h. $\lim_{x\to 1^+} f(x)$ does not exist.

31.



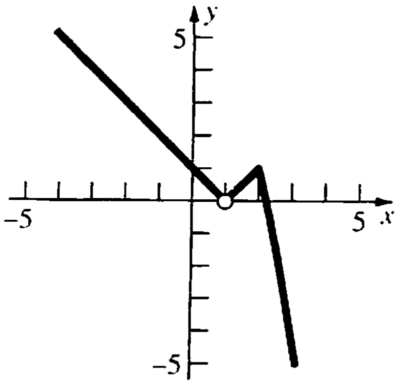
a. $\lim_{x\to 0} f(x) = 0$

b. $\lim_{x\to 1} f(x)$ does not exist.

c. $f(1) = 2$

d. $\lim_{x\to 1^+} f(x) = 2$

32.



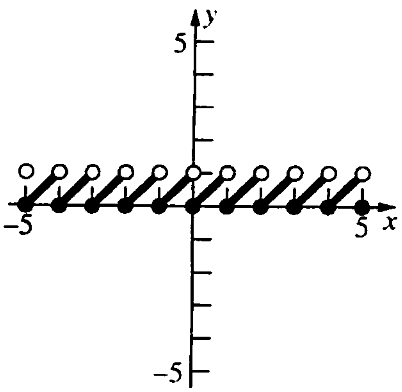
a. $\lim_{x\to 1} g(x) = 0$

b. $g(1)$ does not exist.

c. $\lim_{x\to 2} g(x) = 1$

d. $\lim_{x\to 2^+} g(x) = 1$

33. $f(x) = x - [\![x]\!]$



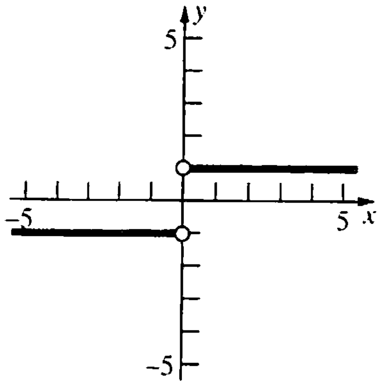
a. $f(0) = 0$

b. $\lim_{x\to 0} f(x)$ does not exist.

c. $\lim_{x\to 0^-} f(x) = 1$

d. $\lim_{x\to \frac{1}{2}} f(x) = \frac{1}{2}$

34. $f(x) = \frac{x}{|x|}$



a. $f(0)$ does not exist.

b. $\lim_{x\to 0} f(x)$ does not exist.

c. $\lim_{x\to 0^-} f(x) = -1$

d. $\lim_{x\to \frac{1}{2}} f(x) = 1$

35. $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ does not exist.

$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = -2$ and $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = 2$

36. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})}$

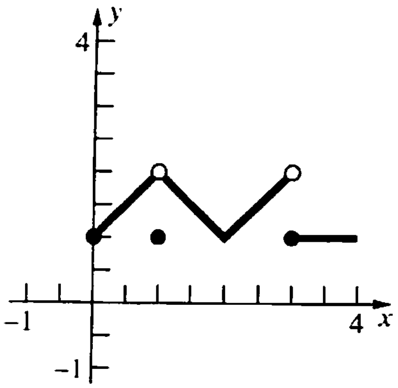
$= \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{0+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

37. a. $\lim_{x\to 1} f(x)$ does not exist.

b. $\lim_{x\to 0} f(x) = 0$

38.



39. $\lim_{x\to a} f(x)$ exists for $a = -1, 0, 1$.

40. The changed values should not change $\lim_{x\to a} f(x)$ at any a as long as the changed points are not all together. As x approaches a, the limit should still be a^2.

41. a. $\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1$ and $\lim_{x\to 1^+} \frac{|x-1|}{x-1} = 1$

b. $\lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1$

c. $\lim_{x\to 1^-} \frac{x^2 - |x-1| - 1}{|x-1|} = -3$

d. $\lim_{x\to 1^-} \left[\frac{1}{x-1} - \frac{1}{|x-1|}\right]$ does not exist.

42. a. $\lim_{x\to 1^+} \sqrt{x - [\![x]\!]} = 0$

b. $\lim_{x\to 0^+} \frac{1}{x}$ does not exist.

c. $\lim_{x\to 0^+} x(-1)^{[\![1/x]\!]}$ does not exist.

d. $\lim_{x\to 0^+} [\![x]\!](-1)^{[\![1/x]\!]} = 0$

e. $\lim_{x\to 0^+} x\left(\frac{1}{x}\right) = 1$

f. $\lim_{x\to 0^+} x^2\left(\frac{1}{x}\right) = 0$

43. $\lim_{x\to 0} \sqrt{x}$ does not exist since $\sqrt{x}$ is not defined for $x < 0$.

44. $\lim_{x\to 0^+} x^x = 1$

45. $\lim_{x\to 0} \sqrt{|x|} = 0$

46. $\lim_{x\to 0} |x|^x = 1$

47. $\lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2}$

48. $\lim_{x\to 0} \frac{\sin 5x}{3x} = \frac{5}{3}$

49. $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist.

50. $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

51. $\lim_{x\to 1} \frac{x^3 - 1}{\sqrt{2x+2} - 2} = 6$

52. $\lim_{x\to 0} \frac{x \sin 2x}{\sin(x^2)} = 2$

53. $\lim_{x\to 2^-} \frac{x^2 - x - 2}{|x-2|} = -3$

54. $\lim_{x\to 1^+} \frac{2}{1 + 2^{1/(x-1)}} = 0$

55. $\lim_{x\to 0} \sqrt{x}$; The computer gives a value of 0, but $\lim_{x\to 0^-} \sqrt{x}$ does not exist.

2.5 Concepts Review

1. $L-\varepsilon;\ L+\varepsilon$

2. $0<|x-a|<\delta;\ |f(x)-L|<\varepsilon$

3. $\dfrac{\varepsilon}{3}$

4. $ma+b$

Problem Set 2.5

1. $0<|t-a|<\delta \Rightarrow |f(t)-M|<\varepsilon$

2. $0<|u-b|<\delta \Rightarrow |g(u)-L|<\varepsilon$

3. $0<|z-d|<\delta \Rightarrow |h(z)-P|<\varepsilon$

4. $0<|y-e|<\delta \Rightarrow |\phi(y)-B|<\varepsilon$

5. $0<c-x<\delta \Rightarrow |f(x)-L|<\varepsilon$

6. $0<t-a<\delta \Rightarrow |g(t)-D|<\varepsilon$

7. $0<|x-0|<\delta \Rightarrow |(2x-1)-(-1)|<\varepsilon$
$|2x-1+1|<\varepsilon \Leftrightarrow |2x|<\varepsilon$
$\Leftrightarrow 2|x|<\varepsilon$
$\Leftrightarrow |x|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-0|<\delta$
$|(2x-1)-(-1)|=|2x|=2|x|<2\delta=\varepsilon$

8. $0<|x+21|<\delta \Rightarrow |(3x-1)-(-64)|<\varepsilon$
$|3x-1+64|<\varepsilon \Leftrightarrow |3x+63|<\varepsilon$
$\Leftrightarrow |3(x+21)|<\varepsilon$
$\Leftrightarrow 3|x+21|<\varepsilon$
$\Leftrightarrow |x+21|<\dfrac{\varepsilon}{3}$

$\delta=\dfrac{\varepsilon}{3};\ 0<|x+21|<\delta$
$|(3x-1)-(-64)|=|3x+63|=3|x+21|<3\delta=\varepsilon$

9. $0<|x-5|<\delta \Rightarrow \left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon$
$\left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon \Leftrightarrow \left|\dfrac{(x-5)(x+5)}{x-5}-10\right|<\varepsilon$
$\Leftrightarrow |x+5-10|<\varepsilon$
$\Leftrightarrow |x-5|<\varepsilon$

$\delta=\varepsilon;\ 0<|x-5|<\delta$
$\left|\dfrac{x^2-25}{x-5}-10\right|=\left|\dfrac{(x-5)(x+5)}{x-5}-10\right|=|x+5-10|$
$=|x-5|<\delta=\varepsilon$

10. $0<|x-0|<\delta \Rightarrow \left|\dfrac{2x^2-x}{x}-(-1)\right|<\varepsilon$
$\left|\dfrac{2x^2-x}{x}+1\right|<\varepsilon \Leftrightarrow \left|\dfrac{x(2x-1)}{x}+1\right|<\varepsilon$
$\Leftrightarrow |2x-1+1|<\varepsilon$
$\Leftrightarrow |2x|<\varepsilon$
$\Leftrightarrow 2|x|<\varepsilon$
$\Leftrightarrow |x|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-0|<\delta$
$\left|\dfrac{2x^2-x}{x}-(-1)\right|=\left|\dfrac{x(2x-1)}{x}+1\right|=|2x-1+1|$
$=|2x|=2|x|<2\delta=\varepsilon$

11. $0<|x-5|<\delta \Rightarrow \left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon$
$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon \Leftrightarrow \left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|<\varepsilon$
$\Leftrightarrow |2x-1-9|<\varepsilon$
$\Leftrightarrow |2(x-5)|<\varepsilon$
$\Leftrightarrow |x-5|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-5|<\delta$
$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|=\left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|$
$=|2x-1-9|=|2(x-5)|=2|x-5|<2\delta=\varepsilon$

12. $0<|x-1|<\delta \Rightarrow \left|\sqrt{2x}-\sqrt{2}\right|<\varepsilon$

$\left|\sqrt{2x}-\sqrt{2}\right|<\varepsilon$

$\Leftrightarrow \left|\frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow \left|\frac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow 2\left|\frac{x-1}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\delta=\frac{\sqrt{2}\varepsilon}{2}; 0<|x-1|<\delta$

$\left|\sqrt{2x}-\sqrt{2}\right|=\left|\frac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|$

$=\left|\frac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|$

$\frac{2|x-1|}{\sqrt{2x}+\sqrt{2}}\le\frac{2|x-1|}{\sqrt{2}}<\frac{2\delta}{\sqrt{2}}=\varepsilon$

13. $0<|x-4|<\delta \Rightarrow \left|\frac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon$

$\left|\frac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon \Leftrightarrow \left|\frac{\sqrt{2x-1}-\sqrt{7(x-3)}}{\sqrt{x-3}}\right|<\varepsilon$

$\Leftrightarrow \left|\frac{(\sqrt{2x-1}-\sqrt{7(x-3)})(\sqrt{2x-1}+\sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\frac{2x-1-(7x-21)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\frac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow |x-4|\left|\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

To bound $\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree that $\delta\le\frac{1}{2}$. If $\delta\le\frac{1}{2}$, then $\frac{7}{2}<x<\frac{9}{2}$, so $0.65<\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}<1.65$ and

hence $|x-4|\left|\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow |x-4|<\frac{\varepsilon}{1.65}$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$\delta=\min\left\{\frac{1}{2},\ \frac{\varepsilon}{1.65}\right\},\ 0<|x-4|<\delta$

$\left|\frac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|=|x-4|\left|\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|$

$<|x-4|(1.65)<1.65\delta\le\varepsilon$

since $\delta=\frac{1}{2}$ only when $\frac{1}{2}\le\frac{\varepsilon}{1.65}$ so $1.65\delta\le\varepsilon$.

14. $0<|x-1|<\delta \Rightarrow \left|\frac{14x^2-20x+6}{x-1}-8\right|<\varepsilon$

$\left|\frac{14x^2-20x+6}{x-1}-8\right|<\varepsilon \Leftrightarrow \left|\frac{2(7x-3)(x-1)}{x-1}-8\right|<\varepsilon$

$\Leftrightarrow |2(7x-3)-8|<\varepsilon$

$\Leftrightarrow |14(x-1)|<\varepsilon$

$\Leftrightarrow 14|x-1|<\varepsilon$

$\Leftrightarrow |x-1|<\frac{\varepsilon}{14}$

$\delta=\frac{\varepsilon}{14}; 0<|x-1|<\delta$

$\left|\frac{14x^2-20x+6}{x-1}-8\right|=\left|\frac{2(7x-3)(x-1)}{x-1}-8\right|$

$=|2(7x-3)-8|$

$=|14(x-1)|=14|x-1|<14\delta=\varepsilon$

15. $0<|x-1|<\delta \Rightarrow \left|\frac{10x^3-26x^2+22x-6}{(x-1)^2}-4\right|<\varepsilon$

$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2}-4\right|<\varepsilon$

$\Leftrightarrow \left|\frac{(10x-6)(x-1)^2}{(x-1)^2}-4\right|<\varepsilon$

$\Leftrightarrow |10x-6-4|<\varepsilon$

$\Leftrightarrow |10(x-1)|<\varepsilon$

$\Leftrightarrow 10|x-1|<\varepsilon$

$\Leftrightarrow |x-1|<\frac{\varepsilon}{10}$

$\delta=\frac{\varepsilon}{10}; 0<|x-1|<\delta$

$$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2}-4\right|$$
$$=\left|\frac{(10x-6)(x-1)^2}{(x-1)^2}-4\right|$$
$$=|10x-6-4|=|10(x-1)|$$
$$=10|x-1|<10\delta=\varepsilon$$

16. $0<|x-1|<\delta \Rightarrow \left|(2x^2+1)-3\right|<\varepsilon$

$\left|2x^2+1-3\right|=\left|2x^2-2\right|=2|x+1||x-1|$

To bound $|2x+2|$, agree that $\delta \le 1$.

$|x-1|<\delta$ implies

$|2x+2|=|2x-2+4|$

$\le|2x-2|+|4|$

$<2+4=6$

$\delta \le \frac{\varepsilon}{6}; \delta=\min\left\{1,\frac{\varepsilon}{6}\right\}; 0<|x-1|<\delta$

$\left|(2x^2+1)-3\right|=\left|2x^2-2\right|$

$=|2x+2||x-1|<6\cdot\left(\frac{\varepsilon}{6}\right)=\varepsilon$

17. $0<|x+1|<\delta \Rightarrow \left|(x^2-2x-1)-2\right|<\varepsilon$

$\left|x^2-2x-1-2\right|=\left|x^2-2x-3\right|=|x+1||x-3|$

To bound $|x-3|$, agree that $\delta \le 1$.

$|x+1|<\delta$ implies

$|x-3|=|x+1-4|$

$\le|x+1|+|-4|$

$<1+4=5$

$\delta \le \frac{\varepsilon}{5}; \delta=\min\left\{1,\frac{\varepsilon}{5}\right\}; 0<|x+1|<\delta$

$\left|(x^2-2x-1)-2\right|=\left|x^2-2x-3\right|$

$=|x+1||x-3|<5\cdot\frac{\varepsilon}{5}=\varepsilon$

18. $0<|x|<\delta \Rightarrow \left|x^4-0\right|=\left|x^4\right|<\varepsilon$

$\left|x^4\right|=|x|\left|x^3\right|$. To bound $\left|x^3\right|$, agree that $\delta \le 1$. $|x|<\delta\le 1$ implies $\left|x^3\right|=|x|^3\le 1$ so $\delta \le \varepsilon$.

$\delta=\min\{1,\varepsilon\}: 0<|x|<\delta \Rightarrow \left|x^4\right|=|x|\left|x^3\right|<\varepsilon\cdot 1$

$=\varepsilon$

19. Choose $\varepsilon>0$. Then since $\lim_{x\to c} f(x)=L$, there is some $\delta_1>0$ such that

$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$.

Since $\lim_{x\to c} f(x)=M$, there is some $\delta_2>0$ such that $0<|x-c|<\delta_2 \Rightarrow |f(x)-L|<\varepsilon$.

Let $\delta=\min\{\delta_1,\delta_2\}$ and choose x_0 such that $0<|x_0-c|<\delta$.

Thus, $|f(x_0)-L|<\varepsilon \Rightarrow -\varepsilon<f(x_0)-L<\varepsilon$

$\Rightarrow -f(x_0)-\varepsilon<-L<-f(x_0)+\varepsilon$

$\Rightarrow f(x_0)-\varepsilon<L<f(x_0)+\varepsilon$.

Similarly,

$f(x_0)-\varepsilon<M<f(x_0)+\varepsilon$.

Thus,

$-2\varepsilon<L-M<2\varepsilon$. As $\varepsilon \Rightarrow 0$, $L-M\to 0$, so $L=M$.

20. Since $\lim_{x\to c} G(x)=0$, then given any $\varepsilon>0$, we can find $\delta>0$ such that whenever $|x-c|<\delta, |G(x)|<\varepsilon$.

Take any $\varepsilon>0$ and the corresponding δ that works for $G(x)$, then $|x-c|<\delta$ implies

$|F(x)-0|=|F(x)|\le|G(x)|<\varepsilon$ since $\lim_{x\to c} G(x)=0$.

Thus, $\lim_{x\to c} F(x)=0$.

21. For all $x\ne 0$, $0\le\sin^2\left(\frac{1}{x}\right)\le 1$ so

$x^4\sin^2\left(\frac{1}{x}\right)\le x^4$ for all $x\ne 0$. By Problem 18,

$\lim_{x\to 0} x^4=0$, so, by Problem 20,

$\lim_{x\to 0} x^4\sin^2\left(\frac{1}{x}\right)=0$.

22. $0 < x < \delta \Rightarrow \left|\sqrt{x} - 0\right| = \left|\sqrt{x}\right| = \sqrt{x} < \varepsilon$

For $x > 0$, $(\sqrt{x})^2 = x$.

$\sqrt{x} < \varepsilon \Leftrightarrow (\sqrt{x})^2 = x < \varepsilon^2$

$\delta = \varepsilon^2; 0 < x < \delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

23. $\lim_{x \to 0^+} |x| : 0 < x < \delta \Rightarrow \left||x| - 0\right| < \varepsilon$

For $x \geq 0$, $|x| = x$.

$\delta = \varepsilon: 0 < x < \delta \Rightarrow \left||x| - 0\right| = |x| = x < \delta = \varepsilon$

Thus, $\lim_{x \to 0^+} |x| = 0$.

$\lim_{x \to 0^-} |x| : 0 < 0 - x < \delta \Rightarrow \left||x| - 0\right| < \varepsilon$

For $x < 0$, $|x| = -x$; note also that $\left||x|\right| = |x|$ since $|x| \geq 0$.

$\delta = \varepsilon: 0 < -x < \delta \Rightarrow \left||x|\right| = |x| = -x < \delta = \varepsilon$

Thus, $\lim_{x \to 0^-} |x| = 0$,

since $\lim_{x \to 0^+} |x| = \lim_{x \to 0^-} |x| = 0$, $\lim_{x \to 0} |x| = 0$.

24. Choose $\varepsilon > 0$. Since $\lim_{x \to a} g(x) = 0$ there is some $\delta_1 > 0$ such that

$0 < |x - a| < \delta_1 \Rightarrow |g(x) - 0| < \frac{\varepsilon}{B}$.

Let $\delta = \min\{1, \delta_1\}$, then $|f(x)| < B$ for $|x - a| < \delta$ or $|x - a| < \delta \Rightarrow |f(x)| < B$. Thus,

$|x - a| < \delta \Rightarrow |f(x)g(x) - 0| = |f(x)g(x)|$

$= |f(x)||g(x)| < B \cdot \frac{\varepsilon}{B} = \varepsilon$ so $\lim_{x \to a} f(x)g(x) = 0$.

25. Choose $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.

That is, for

$a - \delta < x < a$ or $a < x < a + \delta$,

$L - \varepsilon < f(x) < L + \varepsilon$.

Let $f(a) = A$,

$M = \max\{|L - \varepsilon|, |L + \varepsilon|, |A|\}$, $c = a - \delta$,

$d = a + \delta$. Then for x in (c, d), $|f(x)| \leq M$, since either $x = a$, in which case

$|f(x)| = |f(a)| = |A| \leq M$ or $0 < |x - a| < \delta$ so

$L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

26. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where

$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and

$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.

Thus, for $0 < |x - a| < \delta$,

$L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.

Combine the inequalities and use the fact that $f(x) \leq g(x)$ to get

$L - \varepsilon < f(x) \leq g(x) < M + \varepsilon$ which leads to

$L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.

However,

$L - M = \alpha > 2\varepsilon$

which is a contradiction.

Thus $L \leq M$.

27. (b) and (c) are equivalent to the definition of limit.

28. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with $0 < |x - c| < \delta$ such that $|f(x) - L| > \varepsilon$.

29. a. $g(x) = \dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\dfrac{x + 6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has an asymptote at $x \approx 3.49$.

c. If $\delta \leq \frac{1}{4}$, then $2.75 < x < 3$ or $3 < x < 3.25$ and by graphing

$y = |g(x)| = \left|\dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right|$

on the interval $[2.75, 3.25]$, we see that

$0 < \left|\dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right| < 3$

so m must be at least three.

2.6 Concepts Review

1. 48

2. 4

3. $-8; -4+5c$

4. $0; L; 2L$

Problem Set 2.6

1. $\lim_{x\to 1}(2x+1)$ 4

$= \lim_{x\to 1} 2x + \lim_{x\to 1} 1$ 3

$= 2\lim_{x\to 1} x + \lim_{x\to 1} 1$ 2,1

$= 2(1) + 1 = 3$

2. $\lim_{x\to -1}(3x^2-1)$ 5

$= \lim_{x\to -1} 3x^2 - \lim_{x\to -1} 1$ 3

$= 3\lim_{x\to -1} x^2 - \lim_{x\to -1} 1$ 8

$= 3\left(\lim_{x\to -1} x\right)^2 - \lim_{x\to -1} 1$ 2, 1

$= 3(-1)^2 - 1 = 2$

3. $\lim_{x\to 0}[(2x+1)(x-3)]$ 6

$= \lim_{x\to 0}(2x+1)\cdot\lim_{x\to 0}(x-3)$ 4, 5

$= \left(\lim_{x\to 0} 2x + \lim_{x\to 0} 1\right)\cdot\left(\lim_{x\to 0} x - \lim_{x\to 0} 3\right)$ 3

$= \left(2\lim_{x\to 0} x + \lim_{x\to 0} 1\right)\cdot\left(\lim_{x\to 0} x - \lim_{x\to 0} 3\right)$ 2, 1

$= [2(0)+1](0-3) = -3$

4. $\lim_{x\to\sqrt{2}}[(2x^2+1)(7x^2+13)]$ 6

$= \lim_{x\to\sqrt{2}}(2x^2+1)\cdot\lim_{x\to\sqrt{2}}(7x^2+13)$ 4, 3

$= \left(2\lim_{x\to\sqrt{2}} x^2 + \lim_{x\to\sqrt{2}} 1\right)\cdot\left(7\lim_{x\to\sqrt{2}} x^2 + \lim_{x\to\sqrt{2}} 13\right)$ 8,1

$= \left[2\left(\lim_{x\to\sqrt{2}} x\right)^2 + 1\right]\left[7\left(\lim_{x\to\sqrt{2}} x\right)^2 + 13\right]$ 2

$= [2(\sqrt{2})^2+1][7(\sqrt{2})^2+13] = 135$

5. $\lim_{x\to 2}\frac{2x+1}{5-3x}$ 7

$= \frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(5-3x)}$ 4, 5

$= \frac{\lim_{x\to 2} 2x + \lim_{x\to 2} 1}{\lim_{x\to 2} 5 - \lim_{x\to 2} 3x}$ 3, 1

$= \frac{2\lim_{x\to 2} x + 1}{5 - 3\lim_{x\to 2} x}$ 2

$= \frac{2(2)+1}{5-3(2)} = -5$

6. $\lim_{x\to -3}\frac{4x^3+1}{7-2x^2}$ 7

$= \frac{\lim_{x\to -3}(4x^3+1)}{\lim_{x\to -3}(7-2x^2)}$ 4, 5

$= \frac{\lim_{x\to -3} 4x^3 + \lim_{x\to -3} 1}{\lim_{x\to -3} 7 - \lim_{x\to -3} 2x^2}$ 3, 1

$= \frac{4\lim_{x\to -3} x^3 + 1}{7 - 2\lim_{x\to -3} x^2}$ 8

$= \frac{4\left(\lim_{x\to -3} x\right)^3 + 1}{7 - 2\left(\lim_{x\to -3} x\right)^2}$ 2

$= \frac{4(-3)^3+1}{7-2(-3)^2} = \frac{107}{11}$

7. $\lim_{x\to 3}\sqrt{3x-5}$ 9

$= \sqrt{\lim_{x\to 3}(3x-5)}$ 5, 3

$= \sqrt{3\lim_{x\to 3} x - \lim_{x\to 3} 5}$ 2, 1

$= \sqrt{3(3)-5} = 2$

8. $\lim_{x\to -3}\sqrt{5x^2+2x}$ 9

$= \sqrt{\lim_{x\to -3}(5x^2+2x)}$ 4, 3

$= \sqrt{5\lim_{x\to -3} x^2 + 2\lim_{x\to -3} x}$ 8

$= \sqrt{5\left(\lim_{x\to -3} x\right)^2 + 2\lim_{x\to -3} x}$ 2

$= \sqrt{5(-3)^2+2(-3)} = \sqrt{39}$

9. $\lim_{t\to -2}(2t^3+15)^{13}$ 8

$= \left[\lim_{t\to -2}(2t^3+15)\right]^{13}$ 4, 3

$= \left[2\lim_{t\to -2} t^3 + \lim_{t\to -2} 15\right]^{13}$ 8

$= \left[2\left(\lim_{t\to -2} t\right)^3 + \lim_{t\to -2} 15\right]^{13}$ 2, 1

$= [2(-2)^3+15]^{13} = -1$

10. $\lim_{w\to -2}\sqrt{-3w^3+7w^2}$ 9

$= \sqrt{\lim_{w\to -2}(-3w^3+7w^2)}$ 4, 3

$= \sqrt{-3\lim_{w\to -2} w^3 + 7\lim_{w\to -2} w^2}$ 8

$= \sqrt{-3\left(\lim_{w\to -2} w\right)^3 + 7\left(\lim_{w\to -2} w\right)^2}$ 2

$= \sqrt{-3(-2)^3+7(-2)^2} = 2\sqrt{13}$

11. $\lim_{y\to 2}\left(\frac{4y^3+8y}{y+4}\right)^{1/3}$ 9

$= \left(\lim_{y\to 2}\frac{4y^3+8y}{y+4}\right)^{1/3}$ 7

$= \left[\frac{\lim_{y\to 2}(4y^3+8y)}{\lim_{y\to 2}(y+4)}\right]^{1/3}$ 4, 3

$= \left(\frac{4\lim_{y\to 2} y^3 + 8\lim_{y\to 2} y}{\lim_{y\to 2} y + \lim_{y\to 2} 4}\right)^{1/3}$ 8, 1

$= \left[\frac{4\left(\lim_{y\to 2} y\right)^3 + 8\lim_{y\to 2} y}{\lim_{y\to 2} y+4}\right]^{1/3}$ 2

$= \left[\frac{4(2)^3+8(2)}{2+4}\right]^{1/3} = 2$

12. $\lim_{w\to 5}(2w^4-9w^3+19)^{-1/2}$

$= \lim_{w\to 5}\frac{1}{\sqrt{2w^4-9w^3+19}}$ 7

$= \frac{\lim_{w\to 5} 1}{\lim_{w\to 5}\sqrt{2w^4-9w^3+19}}$ 1

$= \frac{1}{\sqrt{\lim_{w\to 5}(2w^4-9w^3+19)}}$ 4.5

$= \frac{1}{\sqrt{\lim_{w\to 5} 2w^4 - \lim_{w\to 5} 9w^3 + \lim_{w\to 5} 19}}$ 1,3

$= \frac{1}{\sqrt{2\lim_{w\to 5} w^4 - 9\lim_{w\to 5} w^3 + 19}}$ 8

$= \frac{1}{\sqrt{2\left(\lim_{w\to 5} w\right)^4 - 9\left(\lim_{w\to 5} w\right)^3 + 19}}$ 2

$= \frac{1}{\sqrt{2(5)^4-9(5)^3+19}}$

$= \frac{1}{\sqrt{144}} = \frac{1}{12}$

13. $\lim_{x\to -1}\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x\to -1}\frac{x-3}{x+7}$

$= \frac{-1-3}{-1+7} = -\frac{2}{3}$

14. $\lim_{x\to 2}\frac{x^2+7x+10}{x+2} = \lim_{x\to 2}\frac{(x+2)(x+5)}{x+2}$

$= \lim_{x\to 2}(x+5) = 7$

15. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(x+1)(x-1)}$

$= \lim_{x\to 1}\frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$

16. $\lim_{x\to -3}\frac{x^2-14x-51}{x^2-4x-21}=\lim_{x\to -3}\frac{(x+3)(x-17)}{(x+3)(x-7)}$
$=\lim_{x\to -3}\frac{x-17}{x-7}=\frac{-3-17}{-3-7}=2$

17. $\lim_{u\to -2}\frac{u^2-ux+2u-2x}{u^2-u-6}=\lim_{u\to -2}\frac{(u+2)(u-x)}{(u+2)(u-3)}$
$=\lim_{u\to -2}\frac{u-x}{u-3}=\frac{x+2}{5}$

18. $\lim_{x\to 1}\frac{x^2+ux-x-u}{x^2+2x-3}=\lim_{x\to 1}\frac{(x-1)(x+u)}{(x-1)(x+3)}$
$=\lim_{x\to 1}\frac{x+u}{x+3}=\frac{1+u}{1+3}=\frac{u+1}{4}$

19. $\lim_{x\to \pi}\frac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2}=\lim_{x\to \pi}\frac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)}$
$=\lim_{x\to \pi}\frac{2(x-2\pi)}{x+\pi}=\frac{2(\pi-2\pi)}{\pi+\pi}=-1$

20. $\lim_{w\to -2}\frac{(w+2)(w^2-w-6)}{w^2+4w+4}$
$=\lim_{w\to -2}\frac{(w+2)^2(w-3)}{(w+2)^2}=\lim_{w\to -2}(w-3)$
$=-2-3=-5$

21. $\lim_{x\to a}\sqrt{f^2(x)+g^2(x)}$
$=\sqrt{\lim_{x\to a}f^2(x)+\lim_{x\to a}g^2(x)}$
$=\sqrt{\left(\lim_{x\to a}f(x)\right)^2+\left(\lim_{x\to a}g(x)\right)^2}$
$=\sqrt{(3)^2+(-1)^2}=\sqrt{10}$

22. $\lim_{x\to a}\frac{2f(x)-3g(x)}{f(x)+g(x)}=\frac{\lim_{x\to a}[2f(x)-3g(x)]}{\lim_{x\to a}[f(x)+g(x)]}$
$=\frac{2\lim_{x\to a}f(x)-3\lim_{x\to a}g(x)}{\lim_{x\to a}f(x)+\lim_{x\to a}g(x)}=\frac{2(3)-3(-1)}{3+(-1)}=\frac{9}{2}$

23. $\lim_{x\to a}\sqrt[3]{g(x)}[f(x)+3]=\lim_{x\to a}\sqrt[3]{g(x)}\cdot\lim_{x\to a}[f(x)+3]$
$=\sqrt[3]{\lim_{x\to a}g(x)}\cdot\left[\lim_{x\to a}f(x)+\lim_{x\to a}3\right]=\sqrt[3]{-1}\cdot(3+3)$
$=-6$

24. $\lim_{x\to a}[f(x)-3]^4=\left[\lim_{x\to a}(f(x)-3)\right]^4$
$=\left[\lim_{x\to a}f(x)-\lim_{x\to a}3\right]^4=(3-3)^4=0$

25. $\lim_{t\to a}\left[|f(t)|+|3g(t)|\right]=\lim_{t\to a}|f(t)|+3\lim_{t\to a}|g(t)|$
$=\left|\lim_{t\to a}f(t)\right|+3\left|\lim_{t\to a}g(t)\right|$
$=|3|+3|-1|=6$

26. $\lim_{u\to a}[f(u)+3g(u)]^3=\left(\lim_{u\to a}[f(u)+3g(u)]\right)^3$
$=\left[\lim_{u\to a}f(u)+3\lim_{u\to a}g(u)\right]^3=[3+3(-1)]^3=0$

27. $\lim_{x\to 2}\frac{3x^2-12}{x-2}=\lim_{x\to 2}\frac{3(x-2)(x+2)}{x-2}$
$=3\lim_{x\to 2}(x+2)=3(2+2)=12$

28. $\lim_{x\to 2}\frac{(3x^2+2x+1)-17}{x-2}=\lim_{x\to 2}\frac{3x^2+2x-16}{x-2}$
$=\lim_{x\to 2}\frac{(3x+8)(x-2)}{x-2}=\lim_{x\to 2}(3x+8)$
$=3\lim_{x\to 2}x+8=3(2)+8=14$

29. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2}=\lim_{x\to 2}\frac{-\frac{x-2}{2x}}{x-2}$
$=\lim_{x\to 2}-\frac{1}{2x}=\frac{-1}{2\lim_{x\to 2}x}=\frac{-1}{2(2)}=-\frac{1}{4}$

30. $\lim_{x\to 2}\frac{\frac{3}{x^2}-\frac{3}{4}}{x-2}=\lim_{x\to 2}\frac{\frac{3(4-x^2)}{4x^2}}{x-2}=\lim_{x\to 2}\frac{\frac{-3(x+2)(x-2)}{4x^2}}{x-2}$

$$=\lim_{x\to 2}\frac{-3(x+2)}{4x^2}=\frac{-3\left(\lim_{x\to 2}x+2\right)}{4\left(\lim_{x\to 2}x\right)^2}=\frac{-3(2+2)}{4(2)^2}$$

$$=-\frac{3}{4}$$

31. Suppose $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}g(x)=M$.
$|f(x)g(x)-LM|\le|g(x)||f(x)-L|+|L||g(x)-M|$ as shown in the text. Choose $\varepsilon_1=1$. Since $\lim_{x\to c}g(x)=M$, there is some $\delta_1>0$ such that if $0<|x-c|<\delta_1$, $|g(x)-M|<\varepsilon_1=1$ or $M-1<g(x)<M+1$, $|M-1|\le|M|+1$ and $|M+1|\le|M|+1$ so for $0<|x-c|<\delta_1, |g(x)|<|M|+1$. Choose $\varepsilon>0$. Since $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}g(x)=M$, there exist δ_2 and δ_3 such that $0<|x-c|<\delta_2\Rightarrow$

$|f(x)-L|<\frac{\varepsilon}{|L|+|M|+1}$ and $0<|x-c|<\delta_3\Rightarrow$

$|g(x)-M|<\frac{\varepsilon}{|L|+|M|+1}$. Let

$\delta=\min\{\delta_1,\delta_2,\delta_3\}$, then $0<|x-c|<\delta\Rightarrow$

$|f(x)g(x)-LM|\le|g(x)||f(x)-L|+|L||g(x)-M|$

$$<(|M|+1)\frac{\varepsilon}{|L|+|M|+1}+|L|\frac{\varepsilon}{|L|+|M|+1}=\varepsilon$$

Hence,

$$\lim_{x\to c}f(x)g(x)=LM=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}g(x)\right)$$

32. Say $\lim_{x\to c}g(x)=M$ and choose $\varepsilon_1=1$. There is some $\delta_1>0$ such that

$0<|x-c|<\delta_1\Rightarrow|g(x)-M|<\varepsilon_1=1$ or

$M-1<g(x)<M+1$

$|M-1|\ge|M|-1$ and $|M+1|\ge|M|-1$

so $|g(x)|>|M|-1$ and $\frac{1}{|g(x)|}<\frac{1}{|M|-1}$

Choose $\varepsilon>0$.

Since $\lim_{x\to c}g(x)=M$ there is $\delta_2>0$ such that

$0<|x-c|<\delta_2\Rightarrow|g(x)-M|<\varepsilon|M|(|M|-1)$

Let $\delta=\min\{\delta_1,\ \delta_2\}$, then

$$0<|x-c|<\delta\Rightarrow\left|\frac{1}{g(x)}-\frac{1}{M}\right|=\left|\frac{M-g(x)}{g(x)M}\right|$$

$$=\frac{1}{|M||g(x)|}|g(x)-M|<\frac{1}{|M|(|M|-1)}\varepsilon|M|(|M|-1)$$

$$=\varepsilon$$

Thus, $\lim_{x\to c}\frac{1}{g(x)}=\frac{1}{M}=\frac{1}{\lim_{x\to c}g(x)}$.

Using statement 6 and the above result,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}f(x)\cdot\lim_{x\to c}\frac{1}{g(x)}$$

$$=\lim_{x\to c}f(x)\cdot\frac{1}{\lim_{x\to c}g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}.$$

33. $\lim_{x\to c}f(x)=L\Leftrightarrow\lim_{x\to c}f(x)=\lim_{x\to c}L$

$\Leftrightarrow\lim_{x\to c}f(x)-\lim_{x\to c}L=0$

$\Leftrightarrow\lim_{x\to c}[f(x)-L]=0$

34. $\lim_{x\to c}f(x)=0\Leftrightarrow\left[\lim_{x\to c}f(x)\right]^2=0$

$\Leftrightarrow\lim_{x\to c}f^2(x)=0$

$\Leftrightarrow\sqrt{\lim_{x\to c}f^2(x)}=0$

$\Leftrightarrow\lim_{x\to c}\sqrt{f^2(x)}=0$

$\Leftrightarrow\lim_{x\to c}|f(x)|=0$

35. $\lim_{x\to c}|x|=\sqrt{\left(\lim_{x\to c}|x|\right)^2}=\sqrt{\lim_{x\to c}|x|^2}=\sqrt{\lim_{x\to c}x^2}$

$$=\sqrt{\left(\lim_{x\to c}x\right)^2}=\sqrt{c^2}=|c|$$

36. a. If $f(x)=\frac{x+1}{x-2}, g(x)=\frac{x-5}{x-2}$ and $c=2$, then $\lim_{x\to c}[f(x)+g(x)]$ exists, but neither $\lim_{x\to c}f(x)$ nor $\lim_{x\to c}g(x)$ exists.

b. If $f(x)=\frac{2}{x}, g(x)=x$, and $c=0$, then $\lim_{x\to c}[f(x)\cdot g(x)]$ exists, but $\lim_{x\to c}f(x)$ does not exist.

37. $\lim_{x\to -3^+} \frac{\sqrt{3+x}}{x} = \frac{\sqrt{3-3}}{-3} = 0$

38. $\lim_{x\to -\pi^+} \frac{\sqrt{\pi^3+x^3}}{x} = \frac{\sqrt{\pi^3+(-\pi)^3}}{-\pi} = 0$

39. $\lim_{x\to 3^+} \frac{x-3}{\sqrt{x^2-9}} = \lim_{x\to 3^+} \frac{(x-3)\sqrt{x^2-9}}{x^2-9}$

$= \lim_{x\to 3^+} \frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)} = \lim_{x\to 3^+} \frac{\sqrt{x^2-9}}{x+3}$

$= \frac{\sqrt{3^2-9}}{3+3} = 0$

40. $\lim_{x\to 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{1+1}}{4+4(1)} = \frac{\sqrt{2}}{8}$

41. $\lim_{x\to 2^+} \frac{(x^2+1)[\![x]\!]}{(3x-1)^2} = \frac{(2^2+1)[\![2]\!]}{(3\cdot 2-1)^2} = \frac{5\cdot 2}{5^2} = \frac{2}{5}$

42. $\lim_{x\to 3^-} (x-[\![x]\!]) = \lim_{x\to 3^-} x - \lim_{x\to 3^-} [\![x]\!] = 3-2 = 1$

43. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$

44. $\lim_{x\to 3^+} [\![x^2+2x]\!] = [\![3^2+2\cdot 3]\!] = 15$

45. $f(x)g(x) = 1; g(x) = \frac{1}{f(x)}$

$\lim_{x\to a} g(x) = 0 \Leftrightarrow \lim_{x\to a} \frac{1}{f(x)} = 0$

$\Leftrightarrow \frac{1}{\lim_{x\to a} f(x)} = 0$

No value satisfies this equation, so $\lim_{x\to a} f(x)$ must not exist.

46. R has the vertices $\left(\pm\frac{x}{2}, \pm\frac{1}{2}\right)$

Each side of Q has length $\sqrt{x^2+1}$ so the perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of length 1 and two sides of length $\sqrt{x^2}$ so the perimeter of R is $2+2\sqrt{x^2}$.

$\lim_{x\to 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q} = \lim_{x\to 0^+} \frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}}$

$= \frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}} = \frac{2}{4} = \frac{1}{2}$

47. a. $NO = \sqrt{(0-0)^2+(1-0)^2} = 1$

$OP = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$

$= \sqrt{x^2+x}$

$NP = \sqrt{(x-0)^2+(y-1)^2} = \sqrt{x^2+y^2-2y+1}$

$= \sqrt{x^2+x-2\sqrt{x}+1}$

$MO = \sqrt{(1-0)^2+(0-0)^2} = 1$

$MP = \sqrt{(x-1)^2+(y-0)^2} = \sqrt{y^2+x^2-2x+1}$

$= \sqrt{x^2-x+1}$

$\lim_{x\to 0^+} \frac{\text{perimeter of } \Delta NOP}{\text{perimeter of } \Delta MOP}$

$= \lim_{x\to 0^+} \frac{1+\sqrt{x^2+x}+\sqrt{x^2+x-2\sqrt{x}+1}}{1+\sqrt{x^2+x}+\sqrt{x^2-x+1}}$

$= \frac{1+\sqrt{1}}{1+\sqrt{1}} = 1$

b. Area of $\Delta NOP = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area of $\Delta MOP = \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2}$

$\lim_{x\to 0^+} \frac{\text{area of } \Delta NOP}{\text{area of } \Delta MOP} = \lim_{x\to 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}}$

$= \lim_{x\to 0^+} \sqrt{x} = 0$

2.7 Concepts Review

1. 0

2. 1

3. the denominator is 0 when $t = 0$.

4. 1

Problem Set 2.7

1. $\lim_{x\to 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$

2. $\lim_{\theta\to\pi/2} \theta\cos\theta = \frac{\pi}{2}\cdot 0 = 0$

3. $\lim_{t\to 0} \frac{\cos^2 t}{1+\sin t} = \lim_{t\to 0} \frac{1-\sin^2 t}{1+\sin t}$
$= \lim_{t\to 0} \frac{(1-\sin t)(1+\sin t)}{1+\sin t}$
$= \lim_{t\to 0}(1-\sin t) = 1$

4. $\lim_{x\to 0} \frac{3x\tan x}{\sin x} = \lim_{x\to 0} \frac{3x(\sin x/\cos x)}{\sin x} = \lim_{x\to 0} \frac{3x}{\cos x}$
$= \frac{0}{1} = 0$

5. $\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$

6. $\lim_{\theta\to 0} \frac{\sin 3\theta}{2\theta} = \lim_{\theta\to 0} \frac{3}{2}\cdot\frac{\sin 3\theta}{3\theta} = \frac{3}{2}\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}$
$= \frac{3}{2}\cdot 1 = \frac{3}{2}$

7. $\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan\theta} = \lim_{\theta\to 0} \frac{\sin 3\theta}{\frac{\sin\theta}{\cos\theta}} = \lim_{\theta\to 0} \frac{\cos\theta\sin 3\theta}{\sin\theta}$
$= \lim_{\theta\to 0}\left[\cos\theta\cdot 3\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right]$
$= 3\lim_{\theta\to 0}\left[\cos\theta\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right]$
$= 3\cdot 1\cdot 1\cdot 1 = 3$

8. $\lim_{\theta\to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\sin 5\theta}{\cos 5\theta\sin 2\theta}$
$= \lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot 5\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{1}{2}\cdot\frac{2\theta}{\sin 2\theta}\right]$
$= \frac{5}{2}\lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{2\theta}{\sin 2\theta}\right]$
$= \frac{5}{2}\cdot 1\cdot 1\cdot 1 = \frac{5}{2}$

9. $\lim_{\theta\to 0} \frac{\cot\pi\theta\sin\theta}{2\sec\theta} = \lim_{\theta\to 0} \frac{\frac{\cos\pi\theta}{\sin\pi\theta}\sin\theta}{\frac{2}{\cos\theta}}$
$= \lim_{\theta\to 0} \frac{\cos\pi\theta\sin\theta\cos\theta}{2\sin\pi\theta}$
$= \lim_{\theta\to 0}\left[\frac{\cos\pi\theta\cos\theta}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\pi}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$
$= \frac{1}{2\pi}\lim_{\theta\to 0}\left[\cos\pi\theta\cos\theta\cdot\frac{\sin\theta}{\theta}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$
$= \frac{1}{2\pi}\cdot 1\cdot 1\cdot 1\cdot 1 = \frac{1}{2\pi}$

10. $\lim_{t\to 0} \frac{\sin^2 3t}{2t} = \lim_{t\to 0} \frac{9t}{2}\cdot\frac{\sin 3t}{3t}\cdot\frac{\sin 3t}{3t}$
$= 0\cdot 1\cdot 1 = 0$

11. $\lim_{t\to 0} \frac{\tan^2 3t}{2t} = \lim_{t\to 0} \frac{\sin^2 3t}{(2t)(\cos^2 3t)}$
$= \lim_{t\to 0} \frac{3(\sin 3t)}{2\cos^2 3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1 = 0$

12. $\lim_{t\to 0} \frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$

13. $\lim_{t\to 0} \frac{\sin(3t)+4t}{t\sec t} = \lim_{t\to 0}\left(\frac{\sin 3t}{t\sec t} + \frac{4t}{t\sec t}\right)$
$= \lim_{t\to 0}\frac{\sin 3t}{t\sec t} + \lim_{t\to 0}\frac{4t}{t\sec t}$
$= \lim_{t\to 0} 3\cos t\cdot\frac{\sin 3t}{3t} + \lim_{t\to 0} 4\cos t$
$= 3\cdot 1 + 4 = 7$

14. The result that $\lim_{t\to 0}\cos t = 1$ was established in the proof of the theorem. Then

$$\lim_{t\to c}\cos t = \lim_{h\to 0}\cos(c+h)$$
$$= \lim_{h\to 0}(\cos c\cosh - \sin c\sinh)$$
$$= \lim_{h\to 0}\cos c \lim_{h\to 0}\cos h - \sin c \lim_{h\to 0}\sin h$$
$$= \cos c$$

15. $$\lim_{t\to c}\tan t = \lim_{t\to c}\frac{\sin t}{\cos t} = \frac{\lim_{t\to c}\sin t}{\lim_{t\to c}\cos t} = \frac{\sin c}{\cos c} = \tan c$$
$$\lim_{t\to c}\cot t = \lim_{t\to c}\frac{\cos t}{\sin t} = \frac{\lim_{t\to c}\cos t}{\lim_{t\to c}\sin t} = \frac{\cos c}{\sin c} = \cot c$$

16. $$\lim_{t\to c}\sec t = \lim_{t\to c}\frac{1}{\cos t} = \frac{1}{\cos c} = \sec c$$
$$\lim_{t\to c}\csc t = \lim_{t\to c}\frac{1}{\sin t} = \frac{1}{\sin c} = \csc c$$

17. $\overline{BP} = \sin t, \overline{OB} = \cos t$

area$(\Delta OBP) \le$ area (sector OAP)

$\le$ area (ΔOBP) + area$(ABPQ)$

$$\frac{1}{2}\overline{OB}\cdot\overline{BP} \le \frac{1}{2}t(1)^2 \le \frac{1}{2}\overline{OB}\cdot\overline{BP} + (1-\overline{OB})\overline{BP}$$
$$\frac{1}{2}\sin t\cos t \le \frac{1}{2}t \le \frac{1}{2}\sin t\cos t + (1-\cos t)\sin t$$
$$\cos t \le \frac{t}{\sin t} \le 2 - \cos t$$
$$\frac{1}{2-\cos t} \le \frac{\sin t}{t} \le \frac{1}{\cos t}$$
$$\lim_{t\to 0}\frac{1}{2-\cos t} \le \lim_{t\to 0}\frac{\sin t}{t} \le \lim_{t\to 0}\frac{1}{\cos t}$$
$$1 \le \lim_{t\to 0}\frac{\sin t}{t} \le 1$$

Thus, $\lim_{t\to 0}\frac{\sin t}{t} = 1.$

18. a. Written response

b. $$D = \frac{1}{2}\overline{AB}\cdot\overline{BP} = \frac{1}{2}(1-\cos t)\sin t$$
$$= \frac{\sin t(1-\cos t)}{2}$$
$$E = \frac{1}{2}t(1)^2 - \frac{1}{2}\overline{OB}\cdot\overline{BP} = \frac{t}{2} - \frac{\sin t\cos t}{2}$$
$$\frac{D}{E} = \frac{\sin t(1-\cos t)}{t - \sin t\cos t}$$

c. $$\lim_{t\to 0^+}\left(\frac{D}{E}\right) \approx 0.75$$

2.8 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound

2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left

3. $y = 6$; horizontal

4. $x = 6$; vertical

Problem Set 2.8

1. $$\lim_{x\to\infty}\frac{x}{x-5} = \lim_{x\to\infty}\frac{1}{1-\frac{5}{x}} = 1$$

2. $$\lim_{x\to\infty}\frac{x^2}{5-x^3} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{5}{x^3}-1} = 0$$

3. $$\lim_{t\to-\infty}\frac{t^2}{7-t^2} = \lim_{t\to-\infty}\frac{1}{\frac{7}{t^2}-1} = -1$$

4. $$\lim_{t\to-\infty}\frac{t}{t-5} = \lim_{t\to-\infty}\frac{1}{1-\frac{5}{t}} = 1$$

5. $$\lim_{x\to\infty}\frac{x^2}{(x-5)(3-x)} = \lim_{x\to\infty}\frac{x^2}{-x^2+8x-15}$$
$$= \lim_{x\to\infty}\frac{1}{-1+\frac{8}{x}-\frac{15}{x^2}} = -1$$

6. $$\lim_{x\to\infty}\frac{x^2}{x^2-8x+15} = \lim_{x\to\infty}\frac{1}{1-\frac{8}{x}+\frac{15}{x^2}} = 1$$

7. $$\lim_{x\to\infty}\frac{x^3}{2x^3-100x^2} = \lim_{x\to\infty}\frac{1}{2-\frac{100}{x}} = \frac{1}{2}$$

8. $\lim_{\theta\to-\infty}\frac{\pi\theta^5}{\theta^5-5\theta^4}=\lim_{\theta\to-\infty}\frac{\pi}{1-\frac{5}{\theta}}=\pi$

9. $\lim_{x\to\infty}\frac{3x^3-x^2}{\pi x^3-5x^2}=\lim_{x\to\infty}\frac{3-\frac{1}{x}}{\pi-\frac{5}{x}}=\frac{3}{\pi}$

10. $\lim_{\theta\to\infty}\frac{\sin^2\theta}{\theta^2-5}$; $0\le\sin^2\theta\le 1$ for all θ and

$\lim_{\theta\to\infty}\frac{1}{\theta^2-5}=\lim_{\theta\to\infty}\frac{\frac{1}{\theta^2}}{1-\frac{5}{\theta^2}}=0$ so $\lim_{\theta\to\infty}\frac{\sin^2\theta}{\theta^2-5}=0$

11. $\lim_{x\to\infty}\frac{3\sqrt{x^3}+3x}{\sqrt{2x^3}}=\lim_{x\to\infty}\frac{3x^{3/2}+3x}{\sqrt{2}x^{3/2}}$

$=\lim_{x\to\infty}\frac{3+\frac{3}{\sqrt{x}}}{\sqrt{2}}=\frac{3}{\sqrt{2}}$

12. $\lim_{x\to\infty}\sqrt[3]{\frac{\pi x^3+3x}{\sqrt{2}x^3+7x}}=\sqrt[3]{\lim_{x\to\infty}\frac{\pi x^3+3x}{\sqrt{2}x^3+7x}}$

$=\sqrt[3]{\lim_{x\to\infty}\frac{\pi+\frac{3}{x^2}}{\sqrt{2}+\frac{7}{x^2}}}=\sqrt[3]{\frac{\pi}{\sqrt{2}}}$

13. $\lim_{x\to\infty}\sqrt[3]{\frac{1+8x^2}{x^2+4}}=\sqrt[3]{\lim_{x\to\infty}\frac{1+8x^2}{x^2+4}}$

$=\sqrt[3]{\lim_{x\to\infty}\frac{\frac{1}{x^2}+8}{1+\frac{4}{x^2}}}=\sqrt[3]{8}=2$

14. $\lim_{x\to\infty}\sqrt{\frac{x^2+x+3}{(x-1)(x+1)}}=\sqrt{\lim_{x\to\infty}\frac{x^2+x+3}{x^2-1}}$

$=\sqrt{\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{3}{x^2}}{1-\frac{1}{x^2}}}=\sqrt{1}=1$

15. For $x>0$, $x=\sqrt{x^2}$.

$\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\frac{\sqrt{x^2+3}}{\sqrt{x^2}}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{3}{x^2}}}$

$=\frac{2}{\sqrt{1}}=2$

16. $\lim_{x\to\infty}\frac{\sqrt{2x+1}}{x+4}=\lim_{x\to\infty}\frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1+\frac{4}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{2}{x}+\frac{1}{x^2}}}{1+\frac{4}{x}}=0$

17. $\lim_{x\to\infty}\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)$

$=\lim_{x\to\infty}\frac{\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)\left(\sqrt{2x^2+3}+\sqrt{2x^2-5}\right)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$=\lim_{x\to\infty}\frac{2x^2+3-(2x^2-5)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$=\lim_{x\to\infty}\frac{8}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$=\lim_{x\to\infty}\frac{\frac{8}{x}}{\frac{\sqrt{2x^2+3}+\sqrt{2x^2-5}}{\sqrt{x^2}}}=\lim_{x\to\infty}\frac{\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}+\sqrt{2-\frac{5}{x^2}}}=0$

18.

$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}$

$=\lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$

$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\frac{2}{2}=1$

19. $\lim_{y\to-\infty}\frac{9y^3+1}{y^2-2y+2}=\lim_{y\to-\infty}\frac{9y+\frac{1}{y^2}}{1-\frac{2}{y}+\frac{2}{y^2}}=-\infty$

20. $\lim_{x\to\infty}\frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$

$=\lim_{x\to\infty}\frac{a_0+\frac{a_1}{x}+\ldots+\frac{a_{n-1}}{x^{n-1}}+\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\ldots+\frac{b_{n-1}}{x^{n-1}}+\frac{b_n}{x^n}}=\frac{a_0}{b_0}$

21. As $x\to 4^+, x\to 4$ while $x-4\to 0^+$.

$\lim_{x\to 4^+}\frac{x}{x-4}=\infty$

22. $\lim_{t\to-3^+}\frac{t^2-9}{t+3}=\lim_{t\to-3^+}\frac{(t+3)(t-3)}{t+3}$

$=\lim_{t\to-3^+}(t-3)=-6$

23. As $t \to 3^-, t^2 \to 9$ while $9 - t^2 \to 0^+$.

$$\lim_{t \to 3^-} \frac{t^2}{9 - t^2} = \infty$$

24. As $x \to \sqrt[3]{5}^+, x^2 \to 5^{2/3}$ while $5 - x^3 \to 0^-$.

$$\lim_{t \to \sqrt[3]{5}^+} \frac{x^2}{5 - x^3} = -\infty$$

25. As $x \to 5^-, x^2 \to 25, x - 5 \to 0^-$, and $3 - x \to -2$.

$$\lim_{x \to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty$$

26. As $\theta \to \pi^+, \theta^2 \to \pi^2$ while $\sin\theta \to 0^-$.

$$\lim_{\theta \to \pi^+} \frac{\theta^2}{\sin\theta} = -\infty$$

27. As $x \to 3^-, x^3 \to 27$, while $x - 3 \to 0^-$.

$$\lim_{x \to 3^-} \frac{x^3}{x-3} = -\infty$$

28. As $\theta \to \frac{\pi}{2}^+, \pi\theta \to \frac{\pi^2}{2}$ while $\cos\theta \to 0^-$.

$$\lim_{\theta \to \frac{\pi}{2}^+} \frac{\pi\theta}{\cos\theta} = -\infty$$

29. $$\lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3^-} \frac{(x+2)(x-3)}{x-3} = \lim_{x \to 3^-} (x+2) = 5$$

30. $$\lim_{x \to 2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to 2^+} \frac{(x+4)(x-2)}{(x+2)(x-2)} = \lim_{x \to 2^+} \frac{x+4}{x+2} = \frac{6}{4} = \frac{3}{2}$$

31. For $0 \le x < 1$, $[\![x]\!] = 0$, so for $0 < x < 1$, $\frac{[\![x]\!]}{x} = 0$ thus $\lim_{x \to 0^+} \frac{[\![x]\!]}{x} = 0$

32. For $-1 \le x < 0$, $[\![x]\!] = -1$, so for $-1 < x < 0$, $\frac{[\![x]\!]}{x} = -\frac{1}{x}$ thus $\lim_{x \to 0^-} \frac{[\![x]\!]}{x} = \infty$.
(Since $x < 0$, $-\frac{1}{x} > 0$.)

33. For $x < 0$, $|x| = -x$, thus

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

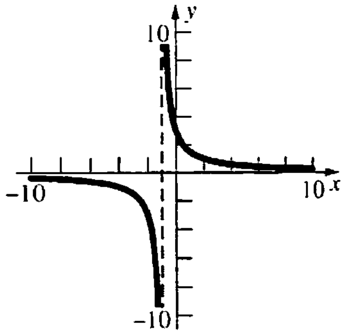
34. For $x > 0$, $|x| = x$, thus $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

35. As $x \to 0^-, 1 + \cos x \to 2$ while $\sin x \to 0^-$.

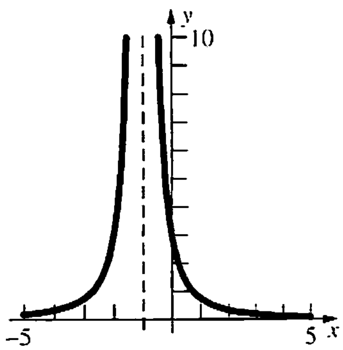
$$\lim_{x \to 0^-} \frac{1 + \cos x}{\sin x} = -\infty$$

36. $-1 \le \sin x \le 1$ for all x, and $\lim_{x \to \infty} \frac{1}{x} = 0$, so $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

37. $\lim_{x \to \infty} \frac{3}{x+1} = 0$, $\lim_{x \to -\infty} \frac{3}{x+1} = 0$;
Horizontal asymptote $y = 0$.
$\lim_{x \to -1^+} \frac{3}{x+1} = \infty$, $\lim_{x \to -1^-} \frac{3}{x+1} = -\infty$;
Vertical asymptote $x = -1$



38. $\lim_{x \to \infty} \frac{3}{(x+1)^2} = 0$, $\lim_{x \to -\infty} \frac{3}{(x+1)^2} = 0$;
Horizontal asymptote $y = 0$.
$\lim_{x \to -1^+} \frac{3}{(x+1)^2} = \infty$, $\lim_{x \to -1^-} \frac{3}{(x+1)^2} = \infty$;
Vertical asymptote $x = -1$



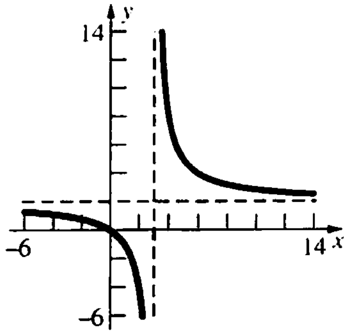
39. $\lim\limits_{x\to\infty} \frac{2x}{x-3} = \lim\limits_{x\to\infty} \frac{2}{1-\frac{3}{x}} = 2,$

$\lim\limits_{x\to-\infty} \frac{2x}{x-3} = \lim\limits_{x\to-\infty} \frac{2}{1-\frac{3}{x}} = 2.$

Horizontal asymptote $y = 2$

$\lim\limits_{x\to3^+} \frac{2x}{x-3} = \infty, \lim\limits_{x\to3^-} \frac{2x}{x-3} = -\infty;$

Vertical asymptote $x = 3$



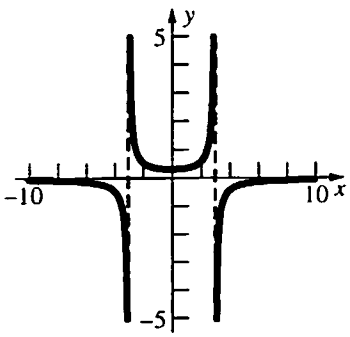
40. $\lim\limits_{x\to\infty} \frac{3}{9-x^2} = 0, \lim\limits_{x\to-\infty} \frac{3}{9-x^2} = 0;$

Horizontal asymptote $y = 0$

$\lim\limits_{x\to3^+} \frac{3}{9-x^2} = -\infty, \lim\limits_{x\to3^-} \frac{3}{9-x^2} = \infty,$

$\lim\limits_{x\to-3^+} \frac{3}{9-x^2} = \infty, \lim\limits_{x\to-3^-} \frac{3}{9-x^2} = -\infty;$

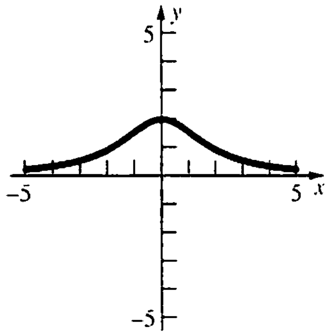
Vertical asymptotes $x = -3, x = 3$



41. $\lim\limits_{x\to\infty} \frac{14}{2x^2+7} = 0, \lim\limits_{x\to-\infty} \frac{14}{2x^2+7} = 0;$

Horizontal asymptote $y = 0$

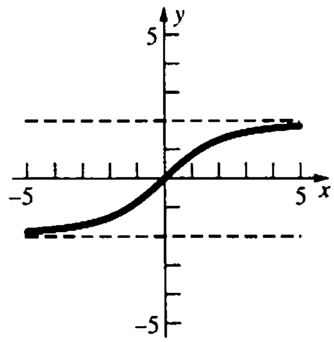
Since $2x^2 + 7 > 0$ for all x, $g(x)$ has no vertical asymptotes.



42. $\lim\limits_{x\to\infty} \frac{2x}{\sqrt{x^2+5}} = \lim\limits_{x\to\infty} \frac{2}{\sqrt{1+\frac{5}{x^2}}} = \frac{2}{\sqrt{1}} = 2,$

$\lim\limits_{x\to-\infty} \frac{2x}{\sqrt{x^2+5}} = \lim\limits_{x\to-\infty} \frac{2}{-\sqrt{1+\frac{5}{x^2}}} = \frac{2}{-\sqrt{1}} = -2$

Since $\sqrt{x^2+5} > 0$ for all x, $g(x)$ has no vertical asymptotes.



43. $f(x) = 2x + 3 - \frac{1}{x^3-1}$, thus

$\lim\limits_{x\to\infty} [f(x) - (2x+3)] = \lim\limits_{x\to\infty} \left[-\frac{1}{x^3-1}\right] = 0$

The oblique asymptote is $y = 2x + 3$.

44. $f(x) = 3x + 4 - \frac{4x+3}{x^2+1}$, thus

$\lim\limits_{x\to\infty} [f(x) - (3x+4)] = \lim\limits_{x\to\infty} \left[-\frac{4x+3}{x^2+1}\right]$

$= \lim\limits_{x\to\infty} \left[-\frac{\frac{4}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}}\right] = 0.$

The oblique asymptote is $y = 3x + 4$.

45. **a.** We say that $\lim_{x\to c^+} f(x) = -\infty$ if for each negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

b. We say that $\lim_{x\to c^-} f(x) = \infty$ if for each positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

46. **a.** We say that $\lim_{x\to\infty} f(x) = \infty$ if for each positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

b. We say that $\lim_{x\to-\infty} f(x) = \infty$ if for each positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

47. Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = A$, there is a corresponding number M_1 such that

$x > M_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$. Similarly, there is a number M_2 such that $x > M_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$.

Let $M = \max\{M_1, M_2\}$, then

$$x > M \Rightarrow |f(x) + g(x) - (A + B)|$$
$$= |f(x) - A + g(x) - B| \le |f(x) - A| + |g(x) - B|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\lim_{x\to\infty} [f(x) + g(x)] = A + B$

48. Written response

49. **a.** $\lim_{x\to\infty} \sin x$ does not exist as $\sin x$ oscillates between -1 and 1 as x increases.

b. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{u\to 0^+} \sin u = 0$$

c. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{u\to 0^+} \frac{1}{u}\sin u = \lim_{u\to 0^+} \frac{\sin u}{u} = 1$$

d. Let $u = \frac{1}{x}$, then

$$\lim_{x\to\infty} x^{3/2}\sin\frac{1}{x} = \lim_{u\to 0^+} \left(\frac{1}{u}\right)^{3/2} \sin u$$
$$= \lim_{u\to 0^+} \left[\left(\frac{1}{\sqrt{u}}\right)\left(\frac{\sin u}{u}\right)\right] = \infty$$

e. As $x \to \infty$, $\sin x$ oscillates between -1 and 1, while $x^{-1/2} = \frac{1}{\sqrt{x}} \to 0$.

$$\lim_{x\to\infty} x^{-1/2}\sin x = 0$$

f. Let $u = \frac{1}{x}$, then

$$\lim_{x\to\infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right) = \lim_{u\to 0^+} \sin\left(\frac{\pi}{6} + u\right)$$
$$= \sin\frac{\pi}{6} = \frac{1}{2}$$

g. As $x \to \infty, x + \frac{1}{x} \to \infty$, so $\lim_{x\to\infty} \sin\left(x + \frac{1}{x}\right)$ does not exist. (See part a.)

h. $\sin\left(x + \frac{1}{x}\right) = \sin x \cos\frac{1}{x} + \cos x \sin\frac{1}{x}$

$$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$$
$$= \lim_{x\to\infty} \left[\sin x\left(\cos\frac{1}{x} - 1\right) + \cos x \sin\frac{1}{x}\right]$$

As $x \to \infty, \cos\frac{1}{x} \to 1$ so $\cos\frac{1}{x} - 1 \to 0$.

From part **b.**, $\lim_{x\to\infty} \sin\frac{1}{x} = 0$.

As $x \to \infty$ both $\sin x$ and $\cos x$ oscillate between -1 and 1.

$$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right] = 0.$$

50. $\lim_{v\to c^-} m(v) = \lim_{v\to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$

51. $\lim_{x\to\infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$

52. $\lim_{x\to-\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$

53. $\lim_{x\to-\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5}\right) = -\frac{3}{2\sqrt{2}}$

54. $\lim_{x\to\infty} \frac{2x+1}{\sqrt{3x^2+1}} = \frac{2}{\sqrt{3}}$

55. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{10} = 1$

56. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e \approx 2.718$

57. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x^2} = \infty$

58. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{\sin x} = 1$

59. $\lim_{x\to 3^-} \frac{\sin|x-3|}{x-3} = -1$

60. $\lim_{x\to 3^-} \frac{\sin|x-3|}{\tan(x-3)} = -1$

61. $\lim_{x\to 3^-} \frac{\cos(x-3)}{x-3} = -\infty$

62. $\lim_{x\to\frac{\pi}{2}^+} \frac{\cos x}{x-\frac{\pi}{2}} = -1$

63. $\lim_{x\to 0^+} (1+\sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$

64. $\lim_{x\to 0^+} (1+\sqrt{x})^{1/x} = \infty$

65. $\lim_{x\to 0^+} (1+\sqrt{x})^{x} = 1$

2.9 Concepts Review

1. $\lim_{x\to c} f(x)$

2. every integer

3. $\lim_{x\to a^+} f(x) = f(a)$; $\lim_{x\to b^-} f(x) = f(b)$

4. a; b; $f(c) = W$

Problem Set 2.9

1. $\lim_{x\to 3}[(x-3)(x-4)] = 0 = f(3)$; continuous

2. $\lim_{x\to 3}(x^2-9) = 0 = g(3)$; continuous

3. $\lim_{x\to 3} \frac{3}{x-3}$ and $h(3)$ do not exist, so $h(x)$ is not continuous at 3.

4. $\lim_{t\to 3} \sqrt{t-4}$ and $g(3)$ do not exist, so $g(t)$ is not continuous at 3.

5. $\lim_{t\to 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist, so $h(t)$ is not continuous at 3.

6. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

7. $\lim_{t\to 3}|t| = 3 = f(3)$; continuous

8. $\lim_{t\to 3}|t-2| = 1 = g(3)$; continuous

9. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

10. $f(3)$ does not exist, so $f(x)$ is not continuous at 3.

11. $\lim_{t\to 3} \frac{t^3-27}{t-3} = \lim_{t\to 3} \frac{(t-3)(t^2+3t+9)}{t-3}$
$= \lim_{t\to 3}(t^2+3t+9) = (3)^2+3(3)+9 = 27 = r(3)$
continuous

12. From Problem 11, $\lim_{t\to 3} r(t) = 27$, so $r(t)$ is not continuous at 3 because $\lim_{t\to 3} r(t) \neq r(3)$.

13. $\lim_{t\to 3^+} f(t) = \lim_{t\to 3^+} (3-t) = 0$
$\lim_{t\to 3^-} f(t) = \lim_{t\to 3^-} (t-3) = 0$
$\lim_{t\to 3} f(t) = f(3)$; continuous

14. $\lim_{t\to 3^+} f(t) = \lim_{t\to 3^+} (3-t)^2 = 0$

$\lim_{t\to 3^-} f(t) = \lim_{t\to 3^-} (t^2-9) = 0$

$\lim_{t\to 3} f(t) = f(3)$; continuous

15. $\lim_{t\to 3} f(x) = -2 = f(3)$; continuous

16. g is discontinuous at $x = -3, 4, 6, 8$; g is left continuous at $x = 4, 8$; g is right continuous at $x = -3, 6$

17. h is continuous on the intervals

$(-\infty,-5)\cup\left[\frac{-5}{2},4\right]\cup(4,6)\cup[6,8]\cup(8,\infty)$

18. $\lim_{x\to 7} \frac{x^2-49}{x-7} = \lim_{x\to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x\to 7}(x+7)$
$= 7+7 = 14$
Define $f(7) = 14$.

19. $\lim_{x\to 3} \frac{2x^2-18}{3-x} = \lim_{x\to 3} \frac{2(x+3)(x-3)}{3-x}$
$= \lim_{x\to 3}[-2(x+3)] = -2(3+3) = -12$
Define $f(3) = -12$.

20. $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$
Define $g(0) = 1$

21. $\lim_{t\to 1} \frac{\sqrt{t}-1}{t-1} = \lim_{t\to 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}$

$= \lim_{t\to 1} \frac{t-1}{(t-1)(\sqrt{t}+1)} = \lim_{t\to 1} \frac{1}{\sqrt{t}+1} = \frac{1}{2}$

Define $H(1) = \frac{1}{2}$.

22. $\lim_{x\to -1} \frac{x^4+2x^2-3}{x+1} = \lim_{x\to -1} \frac{(x^2-1)(x^2+3)}{x+1}$

$= \lim_{x\to -1} \frac{(x+1)(x-1)(x^2+3)}{x+1}$

$= \lim_{x\to -1} [(x-1)(x^2+3)]$

$= (-1-1)[(-1)^2+3] = -8$

Define $\phi(-1) = -8$.

23. $\lim_{x\to -1} \sin\left(\frac{x^2-1}{x+1}\right) = \lim_{x\to -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$

$= \lim_{x\to -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$

Define $F(-1) = -\sin 2$.

24. Discontinuous at $x = \pi, 30$

25. $f(x) = \frac{33-x^2}{(\pi-x)(x-3)}$

Discontinuous at $x = 3, \pi$

26. Continuous at all points

27. Discontinuous at all $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

28. Discontinuous at all $u \le -5$

29. Discontinuous at $u = -1$

30. Continuous at all points

31. $G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$

Discontinuous on $(-\infty,-2]\cup[2,\infty)$

32. Continuous at all points since $\lim_{x\to 0} f(x) = 0 = f(0)$ and $\lim_{x\to 1} f(x) = 1 = f(1)$.

33. $\lim_{x\to 0} g(x) = 0 = g(0)$

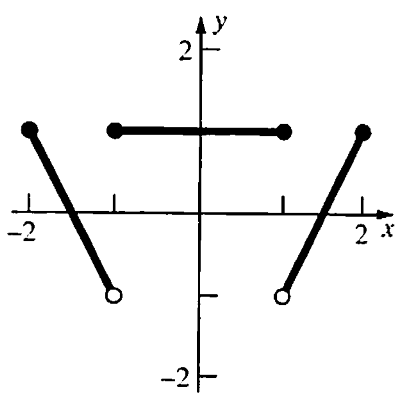
$\lim_{x\to 1^+} g(x) = 1$, $\lim_{x\to 1^-} g(x) = -1$

$\lim_{x\to 1} g(x)$ does not exist, so $g(x)$ is discontinuous at
$x = 1$.

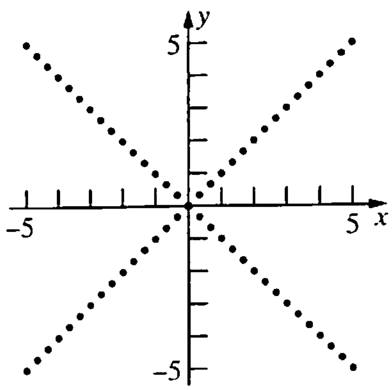
34. Discontinuous at every integer

35. Discontinuous at $t = n + \frac{1}{2}$ where n is a y integer

36.



37.



Discontinuous at all points except $x = 0$, because $\lim_{x \to c} f(x) \neq f(c)$ for $c \neq 0$. $\lim_{x \to c} f(x)$ exists only at $c = 0$ and $\lim_{x \to 0} f(x) = 0 = f(0)$.

38. Let $f(x) = x^3 + 3x - 2$. f is continuous on [0, 1]. $f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

39. Because the function is continuous on $[0, 2\pi]$ and $(\cos 0)0^3 + 6\sin^5 0 - 3 = -3 < 0$, $(\cos 2\pi)(2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0$, there is at least one number c between 0 and 2π such that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$.

40. Let $f(x) = x^5 + 4x^3 - 7x + 14$
$f(x)$ is continuous at all values of x.
$f(-2) = -36, f(0) = 14$
Because 0 is between –36 and 14, there is at least one number c between –2 and 0 such that
$f(x) = x^5 + 4x^3 - 7x + 14 = 0$.

41. Suppose that f is continuous at c, so $\lim_{x \to c} f(x) = f(c)$. Let $x = t + c$, so $t = x - c$, then as $x \to c$, $t \to 0$ and the statement $\lim_{x \to c} f(x) = f(c)$ becomes $\lim_{t \to 0} f(t + c) = f(c)$.
Suppose that $\lim_{t \to 0} f(t + c) = f(c)$ and let $x = t + c$, so $t = x - c$. Since c is fixed, $t \to 0$ means that $x \to c$ and the statement $\lim_{t \to 0} f(t + c) = f(c)$ becomes $\lim_{x \to c} f(x) = f(c)$, so f is continuous at c.

42. Since $f(x)$ is continuous at c, $\lim_{x \to c} f(x) = f(c) > 0$. Thus, $f(x) > 0$ for all x in some deleted interval $(c - \delta, c + \delta)$ about c. Since also $f(c) > 0$, $f(x) > 0$ for all x in $(c - \delta, c + \delta)$.

43. Let $g(x) = x - f(x)$. Then,
$g(0) = 0 - f(0) = -f(0) \le 0$ and $g(1) = 1 - f(1) \ge 0$ since $0 \le f(x) \le 1$ on [0, 1]. If $g(0) = 0$, then $f(0) = 0$ and $c = 0$ is a fixed point of f. If $g(1) = 0$, then $f(1) = 1$ and $c = 1$ is a fixed point of f. If neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and $g(1) > 0$ so there is some c in [0, 1] such that $g(c) = 0$. If $g(c) = 0$ then $c - f(c) = 0$ or $f(c) = c$ and c is a fixed point of f.

44. For $f(x)$ to be continuous everywhere,
$f(1) = a(1) + b = 2$ and $f(2) = 6 = a(2) + b$

$$\begin{aligned} a + b &= 2 \\ \underline{2a + b} &\underline{= 6} \\ -a &= -4 \\ a = 4,\ & b = -2 \end{aligned}$$

45. For x in [0, 1], let $f(x)$ indicate where the string originally at x ends up. Thus $f(0) = a$, $f(1) = b$. $f(x)$ is continuous since the string is unbroken. Since $0 \le a,\ b \le 1$, $f(x)$ satisfies the conditions of Problem 43, so there is some c in [0, 1] with $f(c) = c$, i.e., the point of string originally at c ends up at c.

46. The Intermediate Value Theorem does not imply the existence of a number c between –2 and 2 such that $f(c) = 0$. The reason is that the function $f(x)$ is not continuous on $[-2, 2]$.

47. Let $f(x)$ be the difference in times on the hiker's watch where x is a point on the path, and suppose $x = 0$ at the bottom and $x = 1$ at the top of the mountain.
So $f(x)$ = (time on watch on the way up) – (time on watch on the way down).
$f(0) = 4 - 11 = -7$, $f(1) = 12 - 5 = 7$. Since time is continuous, $f(x)$ is continuous, hence there is some c between 0 and 1 where $f(c) = 0$. This c is the point where the hiker's watch showed the same time on both days.

48. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is the length of the side of the rectangle which makes angle θ with the x-axis minus the length of the sides perpendicular to it. f is continuous on $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is circumscribed by a square. If $f(0) \neq 0$, then observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the Intermediate Value Theorem, there is an angle θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$. Hence, D can be circumscribed by a square.

49. a. $f(x) = f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. We want to prove that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, $\lim_{x\to c}[f(x) - f(c)] = 0$. But $f(x) - f(c) = f(x - c)$, so $\lim_{x\to c}[f(x) - f(c)] = \lim_{x\to c} f(x-c)$. Let $h = x - c$ then as $x \to c$, $h \to 0$ and $\lim_{x\to c} f(x-c) = \lim_{h\to 0} f(h) = f(0) = 0$. Hence $\lim_{x\to c} f(x) = f(c)$ and f is continuous at c. Thus, f is continuous everywhere, since c was arbitrary.

b. By Problem 41 of Section 2.1, $f(t) = mt$ for all t in Q. Since $g(t) = mt$ is a polynomial function, it is continuous for all real numbers. $f(t) = g(t)$ for all t in Q, thus $f(t) = g(t)$ for all t in R, i.e. $f(t) = mt$.

50. $y_1(x) = m_1x + b_1,\ y_2(x) = m_2x + b_2$
$y_1(x) + y_2(x) = m_1x + b_1 + m_2x + b_2$
$= (m_1 + m_2)x + (b_1 + b_2)$
This is a linear function.

51. $y_1(x) = m_1x + b_1,\ y_2(x) = m_2x + b_2$
$y_1(y_2(x)) = y_1(m_2x + b_2) = m_1(m_2x + b_2) + b_1$
$= m_1m_2x + m_1b_2 + b_1 = (m_1m_2)x + (m_1b_2 + b_1)$
This is a linear function.

52. $y_1(x) = m_1x + b_1,\ y_2(x) = m_2x + b_2$
$y_1(x)y_2(x) = (m_1x + b_1)(m_2x + b_2)$
$= m_1m_2x^2 + m_1b_2x + b_1m_2x + b_1b_2$
This is not a linear function unless $m_1m_2 = 0$.
Thus, the product of two linear functions is not linear unless at least one of the functions is a constant function.

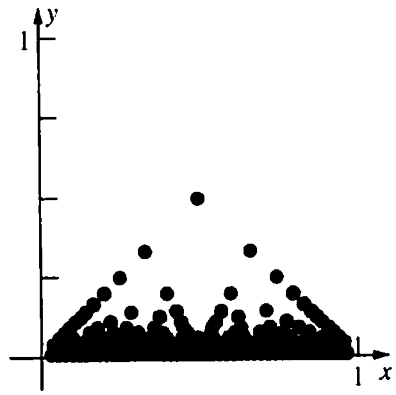
53. $y_1(x) = m_1x + b_1,\ y_2(x) = m_2x + b_2$
$\frac{y_1(x)}{y_2(x)} = \frac{m_1x + b_1}{m_2x + b_2}$
This is a linear function only when $m_2 = 0$ and $b_2 \neq 0$.

54. If $f(x)$ is continuous on an interval then $\lim_{x\to c} f(x) = f(c)$ for all points in the interval:
$\lim_{x\to c} f(x) = f(c) \Rightarrow \lim_{x\to c} |f(x)|$
$= \lim_{x\to c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x\to c} f(x)\right)^2}$
$= \sqrt{(f(c))^2} = |f(c)|$

55. Suppose $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

56. If f is continuous at c, $\lim_{x\to c} f(x) = f(c)$
$= f\left(\lim_{x\to c} x\right)$ since $\lim_{x\to c} x = c$.

57. a.



b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q}$. Hence, $\lim_{x\to r} f(x)$ does not exist.
If c is any irrational number in (0, 1), then as $x = \frac{p}{q} \to c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \to \infty$, so

$f(x) \to 0$ as $x \to c$. Thus, $\lim_{x \to c} f(x) = 0 = f(c)$ for any irrational number c.

58. a. Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

Domain: $\left[-\frac{3}{4}, \frac{3}{4}\right]$;

Range: $\left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$

b. At $x = 0$

c. If $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, \frac{3}{4}$ are fixed points of f.

2.10 Chapter Review

Concepts Test

1. True:
$$xy + x^2 = 3y$$
$$xy - 3y = -x^2$$
$$y = -\frac{x^2}{x-3}$$

2. False:
$$xy^2 + x^2 = 3x$$
$$xy^2 = 3x - x^2$$
$$y^2 = \frac{3x - x^2}{x}$$
$$y = \pm\sqrt{\frac{3x - x^2}{x}}$$

3. True:
$$\theta \sin\theta + t - \cos\theta = 0$$
$$t = \cos\theta - \theta\sin\theta$$

4. False: $\Phi + \Psi = |\Phi + \Psi|$ cannot be written in the form $\Phi = f(\Psi)$.

5. False: The equation determines T as a function of θ.

6. True: $f(x) = \sqrt{\frac{x}{4-x}}$; $\frac{x}{4-x}$ cannot be negative and $4 - x \neq 0$, so the domain is [0, 4).

7. True:
$$f(x) = \sqrt{-(x^2 + 4x + 3)}$$
$$= \sqrt{-(x+3)(x+1)}$$
$-(x^2 + 4x + 3) \geq 0$ on $-3 \leq x \leq -1$.

8. False: The domain does not include $n\pi + \frac{\pi}{2}$ where n is an integer.

9. True: The domain is $(-\infty, \infty)$ and the range is $[-6, \infty)$.

10. False: The range is $(-\infty, \infty)$.

11. False: The range $(-\infty, \infty)$.

12. True: If $f(x)$ and $g(x)$ are even functions, $f(x) + g(x)$ is even.
$f(-x) + g(-x) = f(x) + g(x)$

13. True: If $f(x)$ and $g(x)$ are odd functions,
$f(-x) + g(-x) = -f(x) - g(x)$
$= -[f(x) + g(x)]$, so $f(x) + g(x)$ is odd

14. False: If $f(x)$ and $g(x)$ are odd functions,
$f(-x)g(-x) = -f(x)[-g(x)] = f(x)g(x)$, so $f(x)g(x)$ is even.

15. True: If $f(x)$ is even and $g(x)$ is odd,
$f(-x)g(-x) = f(x)[-g(x)]$
$= -f(x)g(x)$, so $f(x)g(x)$ is odd.

16. False: If $f(x)$ is even and $g(x)$ is odd,
$f(g(-x)) = f(-g(x)) = f(g(x))$; while if $f(x)$ is odd and $g(x)$ is even,
$f(g(-x)) = f(g(x))$; so $f(g(x))$ is even.

17. False: If $f(x)$ and $g(x)$ are odd functions,
$f(g(-x)) = f(-g(x)) = -f(g(x))$, so $f(g(x))$ is odd.

18. True: $f(-x)=\frac{2(-x)^3+(-x)}{(-x)^2+1}=\frac{-2x^3-x}{x^2+1}$
$=-\frac{2x^3+x}{x^2+1}$

19. True: $f(-t)=\frac{(\sin(-t))^2+\cos(-t)}{\tan(-t)\csc(-t)}$
$=\frac{(-\sin t)^2+\cos t}{-\tan t(-\csc t)}=\frac{(\sin t)^2+\cos t}{\tan t\csc t}$

20. False: $f(x) = c$ has domain $(-\infty, \infty)$ and the only value of the range is c.

21. False: $f(x) = c$ has domain $(-\infty, \infty)$, yet the range has only one value, c.

22. True: $g(-1.8)=\left[\!\left[\frac{-1.8}{2}\right]\!\right]=[\![-0.9]\!]=-1$

23. True: $(f\circ g)(x)=(x^3)^2=x^6$
$(g\circ f)(x)=(x^2)^3=x^6$

24. False: $(f\circ g)(x)=(x^3)^2=x^6$
$f(x)\cdot g(x)=x^2x^3=x^5$

25. False: The domain of $\frac{f}{g}$ excludes any values where $g = 0$.

26. True: $f(a) = 0$
Let $F(x) = f(x + h)$, then
$F(a - h) = f(a - h + h) = f(a) = 0$

27. True: $\cot x=\frac{\cos x}{\sin x}$
$\cot(-x)=\frac{\cos(-x)}{\sin(-x)}=\frac{\cos x}{-\sin x}=-\cot x$

28. False: The domain of the tangent function excludes all $n\pi+\frac{\pi}{2}$ where n is an integer.

29. False: The cosine function is periodic, so $\cos s = \cos t$ does not necessarily imply $s = t$; e.g., $\cos 0 = \cos 2\pi = 1$, but $0 \neq 2\pi$.

30. False: c may not be in the domain of $f(x)$, or it may be defined separately.

31. False: If $f(c)$ is not defined, $\lim_{x\to c} f(x)$ might exist; e.g., $f(x)=\frac{x^2-4}{x+2}$.
$f(-2)$ does not exist,
but $\lim_{x\to -2}\frac{x^2-4}{x+2}=-4$.

32. True: $\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+5)}{x-5}$
$=\lim_{x\to 5}(x+5)=5+5=10$

33. True: Substitution Theorem

34. False: $\lim_{x\to 0}\frac{\sin x}{x}=1$

35. False: The tangent function is not defined for all values of c.

36. True: Since both $\sin x$ and $\cos x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x)=2\sin^2 x-\cos x$ is also continuous for all real numbers.

37. False: Since $-1\le \sin x\le 1$ for all x and $\lim_{x\to\infty}\frac{1}{x}=0$, we get $\lim_{x\to\infty}\frac{\sin x}{x}=0$.

38. False: Consider $f(x) = \sin x$.

39. True: As $x\to 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.

40. False: $\lim_{x\to c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.

41. True: $\lim_{x\to c} f(x)=f\left(\lim_{x\to c} x\right)=f(c)$, so f is continuous at $x = c$.

42. True: $\lim_{x\to 2.3}\left[\!\left[\frac{x}{2}\right]\!\right]=1=f(2.3)$

43. True: Choose $\varepsilon = 0.001f(2)$ then since $\lim_{x\to 2} f(x)=f(2)$, there is some δ such that $0<|x-2|<\delta\Rightarrow$
$|f(x)-f(2)|<0.001f(2)$, or

$-0.001f(2) < f(x) - f(2)$
$< 0.001f(2)$
Thus, $0.999f(2) < f(x) < 1.001f(2)$ and $f(x) < 1.001f(2)$ for $0 < |x-2| < \delta$. Since $f(2) < 1.001f(2)$, as $f(2) > 0$, $f(x) < 1.001f(2)$ on $(2-\delta,\ 2+\delta)$.

44. False: That $\lim_{x\to c}[f(x)+g(x)]$ exists does not imply that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and $g(x) = \frac{x+7}{x+2}$.

45. True: Squeeze Theorem

46. True: A function has only one limit at a point, so if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, $L = M$

47. False: That $f(x) \neq g(x)$ for all x does not imply that $\lim_{x\to c} f(x) \neq \lim_{x\to c} g(x)$. For example, if $f(x) = \frac{x^2+x-6}{x-2}$ and $g(x) = \frac{5}{2}x$, then $f(x) \neq g(x)$ for all x, but $\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 5$.

48. False: If $f(x) < 10$, $\lim_{x\to 2} f(x)$ could equal 10 if there is a discontinuity point (2, 10). For example,
$f(x) = \frac{-x^3+6x^2-2x-12}{x-2} < 10$ for all x, but $\lim_{x\to 2} f(x) = 10$.

49. True: $\lim_{x\to a} |f(x)| = \lim_{x\to a} \sqrt{f^2(x)}$
$= \sqrt{\left[\lim_{x\to a} f(x)\right]^2} = \sqrt{(b)^2} = |b|$

50. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

1. a. $f(1) = \frac{1}{1+1} - \frac{1}{1} = -\frac{1}{2}$

b. $f\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}+1} - \frac{1}{-\frac{1}{2}} = 4$

c. $f(-1)$ does not exist.

d. $f(t-1) = \frac{1}{t-1+1} - \frac{1}{t-1} = \frac{1}{t} - \frac{1}{t-1}$

e. $f\left(\frac{1}{t}\right) = \frac{1}{\frac{1}{t}+1} - \frac{1}{\frac{1}{t}} = \frac{t}{1+t} - t$

2. a. $g(2) = \frac{2+1}{2} = \frac{3}{2}$

b. $g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2}} = 3$

c. $g\left(\frac{1}{10}\right) = \frac{\frac{1}{10}+1}{\frac{1}{10}} = 11$

d. $\frac{g(2+h)-g(2)}{h} = \frac{\frac{2+h+1}{2+h} - \frac{2+1}{2}}{h}$
$= \frac{\frac{2h+6-3h-6}{2(h+2)}}{h} = \frac{-\frac{h}{2(h+2)}}{h} = \frac{-1}{2(h+2)}$

3. a. $\{x \in \mathbb{R} : x \neq -1, 1\}$

b. $\{x \in \mathbb{R} : |x| \le 2\}$

c. $\left\{x \in \mathbb{R} : x \neq -\frac{3}{2}\right\}$

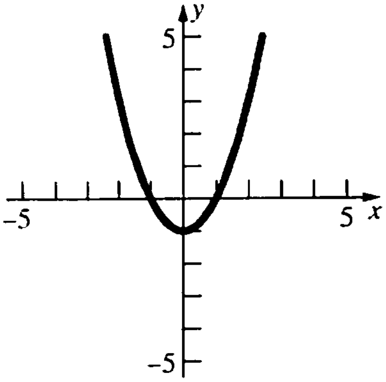
4. a. $f(-x) = \frac{3(-x)}{(-x)^2+1} = -\frac{3x}{x^2+1}$; odd

b. $g(-x) = |\sin(-x)| + \cos(-x)$
$= |-\sin x| + \cos x = |\sin x| + \cos x$; even

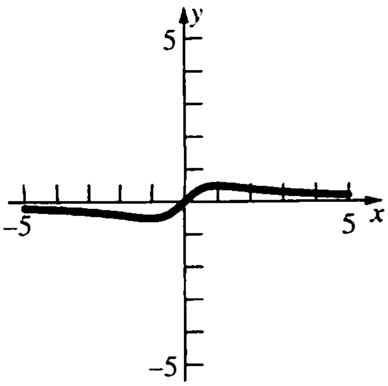
c. $h(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x$; odd

d. $k(-x) = \frac{(-x)^2+1}{|-x|+(-x)^4} = \frac{x^2+1}{|x|+x^4}$; even

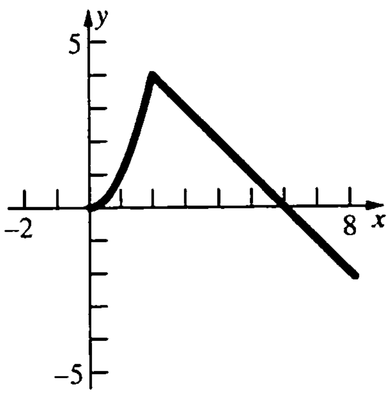
5. a. $f(x) = x^2 - 1$



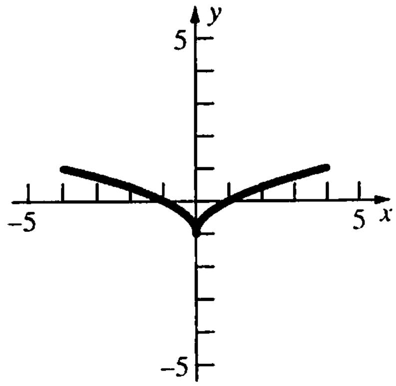
b. $g(x) = \dfrac{x}{x^2 + 1}$



c. $h(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 2 \\ 6 - x & \text{if } x > 2 \end{cases}$



6.



7. $V(x) = x(32 - 2x)(24 - 2x)$
Domain [0, 12]

8. a. $(f + g)(2) = \left(2 - \dfrac{1}{2}\right) + (2^2 + 1) = \dfrac{13}{2}$

b. $(f \cdot g)(2) = \left(\dfrac{3}{2}\right)(5) = \dfrac{15}{2}$

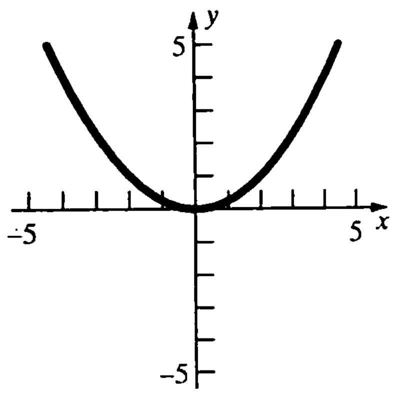
c. $(f \circ g)(2) = f(5) = 5 - \dfrac{1}{5} = \dfrac{24}{5}$

d. $(g \circ f)(2) = g\left(\dfrac{3}{2}\right) = \left(\dfrac{3}{2}\right)^2 + 1 = \dfrac{13}{4}$

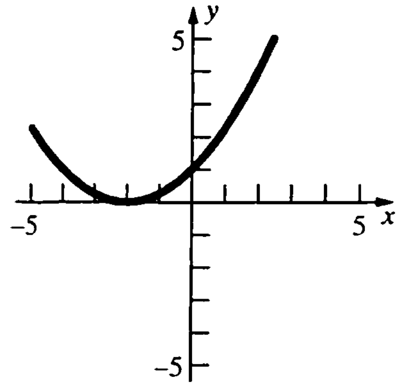
e. $f^3(-1) = \left(-1 + \dfrac{1}{1}\right)^3 = 0$

f. $f^2(2) + g^2(2) = \left(\dfrac{3}{2}\right)^2 + (5)^2 = \dfrac{9}{4} + 25 = \dfrac{109}{4}$

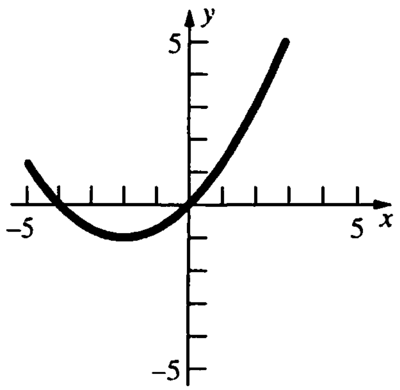
9. a. $y = \dfrac{1}{4}x^2$



b. $y=\frac{1}{4}(x+2)^2$



c. $y=-1+\frac{1}{4}(x+2)^2$



10. a. $(-\infty,16]$

b. $f\circ g=\sqrt{16-x^4}$; domain $[-2, 2]$

c. $g\circ f=(\sqrt{16-x})^4=(16-x)^2$; domain $(-\infty,16]$

11. $f(x)=\sqrt{x}, g(x)=1+x, h(x)=x^2,$
$k(x)=\sin x,\ F(x)=\sqrt{1+\sin^2 x}=f\circ g\circ h\circ k$

12. a. $\sin(570°)=\sin(210°)=-\frac{1}{2}$

b. $\cos\left(\frac{9\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$

c. $\sin^2(5)+\cos^2(5)=1$

d. $\cos\left(-\frac{13\pi}{6}\right)=\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

13. a. $\sin(-t)=-\sin t=-0.8$

b. $\sin^2 t+\cos^2 t=1$
$\cos^2 t=1-(0.8)^2=0.36$
$\cos t=-0.6$

c. $\sin 2t=2\sin t\cos t=2(0.8)(-0.6)=-0.96$

d. $\tan t=\frac{\sin t}{\cos t}=\frac{0.8}{-0.6}=-\frac{4}{3}\approx -1.333$

e. $\cos\left(\frac{\pi}{2}-t\right)=\sin t=0.8$

f. $\sin(\pi+t)=-\sin t=-0.8$

14. $\sin 3t=\sin(2t+t)=\sin 2t\cos t+\cos 2t\sin t$
$=2\sin t\cos^2 t+(1-2\sin^2 t)\sin t$
$=2\sin t(1-\sin^2 t)+\sin t-2\sin^3 t$
$=2\sin t-2\sin^3 t+\sin t-2\sin^3 t$
$=3\sin t-4\sin^3 t$

15. $s=rt$
$=9\left(20\frac{\text{rev}}{\text{min}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right)(1\text{ sec})=6\pi$
≈ 18.85 in.

16. $\lim_{u\to 1}\frac{u^2-1}{u+1}=\frac{1^2-1}{1+1}=0$

17. $\lim_{u\to 1}\frac{u^2-1}{u-1}=\lim_{u\to 1}\frac{(u-1)(u+1)}{u-1}=\lim_{u\to 1}(u+1)$
$=1+1=2$

18. $\lim_{u\to 1}\frac{u+1}{u^2-1}=\lim_{u\to 1}\frac{u+1}{(u+1)(u-1)}=\lim_{u\to 1}\frac{1}{u-1}$; does not exist

19. $\lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}=\lim_{x\to 2}\frac{\frac{x-2}{x}}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x(x+2)}$
$=\frac{1}{2(2+2)}=\frac{1}{8}$

20. $\lim_{z\to 2}\frac{z^2-4}{z^2+z-6}=\lim_{z\to 2}\frac{(z+2)(z-2)}{(z+3)(z-2)}$
$=\lim_{z\to 2}\frac{z+2}{z+3}=\frac{2+2}{2+3}=\frac{4}{5}$

21. $\lim_{x\to 0}\frac{\tan x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}=\lim_{x\to 0}\frac{1}{2\cos^2 x}$
$=\frac{1}{2\cos^2 0}=\frac{1}{2}$

22. $\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$
$=\lim_{y\to 1}\frac{y^2+y+1}{y+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$

23. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}$
$=\lim_{x\to 4}(\sqrt{x}+2)=\sqrt{4}+2=4$

24. $\lim_{x\to 0}\frac{\cos x}{x}$ does not exist.

25. $\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$

26. $\lim_{x\to (1/2)^+}[\![4x]\!]=2$

27. $\lim_{t\to 2^-}([\![t]\!]-t)=\lim_{t\to 2^-}[\![t]\!]-\lim_{t\to 2^-}t=1-2=-1$

28. a. $f(1)=0$

b. $\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(1-x)=0$

c. $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}x=1$

d. $\lim_{x\to -1}f(x)=-1$ because

$\lim_{x\to -1^-}f(x)=\lim_{x\to -1^-}x^3=-1$ and
$\lim_{x\to -1^+}f(x)=\lim_{x\to -1^+}x=-1$

29. a. f is discontinuous at $x=1$ because $f(1)=0$, but $\lim_{x\to 1}f(x)$ does not exist. f is discontinuous at $x=-1$ because $f(-1)$ does not exist.

b. Define $f(-1)=-1$

30. a. $0<|u-a|<\delta \Rightarrow |g(u)-M|<\varepsilon$

b. $0<a-x<\delta \Rightarrow |f(x)-L|<\varepsilon$

31. a. $\lim_{x\to 3}[2f(x)-4g(x)]$
$=2\lim_{x\to 3}f(x)-4\lim_{x\to 3}g(x)$
$=2(3)-4(-2)=14$

b. $\lim_{x\to 3}g(x)\frac{x^2-9}{x-3}=\lim_{x\to 3}g(x)(x+3)$
$=\lim_{x\to 3}g(x)\cdot\lim_{x\to 3}(x+3)=-2\cdot(3+3)=-12$

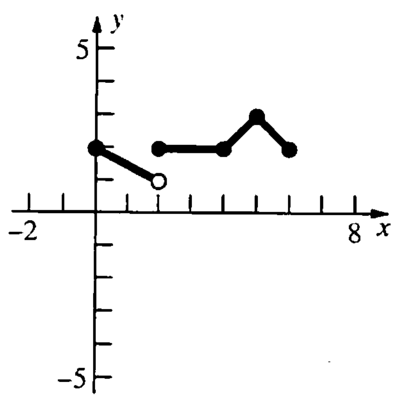
c. $g(3)=-2$

d. $\lim_{x\to 3}g(f(x))=g\left(\lim_{x\to 3}f(x)\right)=g(3)=-2$

e. $\lim_{x\to 3}\sqrt{f^2(x)-8g(x)}$
$=\sqrt{\left[\lim_{x\to 3}f(x)\right]^2-8\lim_{x\to 3}g(x)}$
$=\sqrt{(3)^2-8(-2)}=5$

f. $\lim_{x\to 3}\frac{|g(x)-g(3)|}{f(x)}=\frac{|-2-g(3)|}{3}=\frac{|-2-(-2)|}{3}$
$=0$

32.



33. $a(0)+b=-1$ and $a(1)+b=1$
$b=-1;\ \ a+b=1$
$a-1=1$
$a=2$

34. Let $f(x)=x^5-4x^3-3x+1$
$f(2)=-5, f(3)=127$
Because $f(x)$ is continuous on [2, 3] and $f(2)<0<f(3)$, there exists some number c between 2 and 3 such that $f(c)=0$.