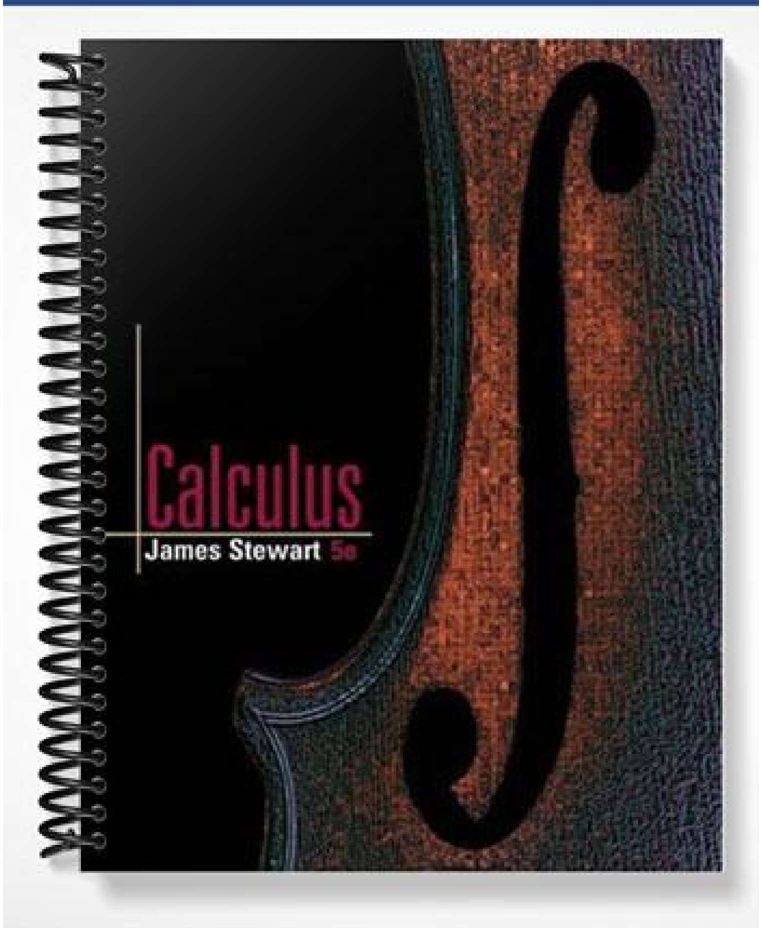
SOLUTIONS MANUAL



Chapter 2

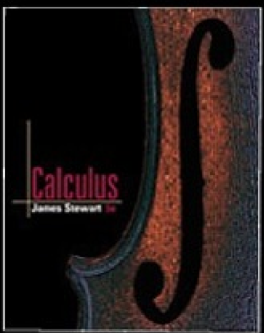
Adapted from the

Complete Solutions Manual

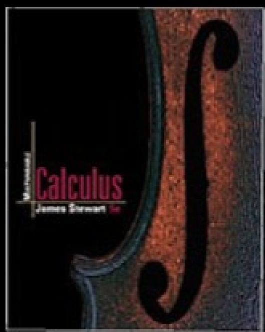
for

James Stewart's

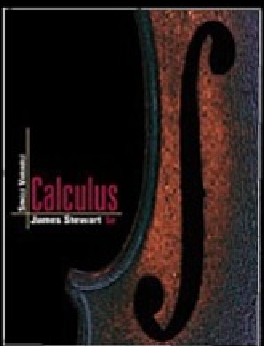
Calculus – 5th Edition



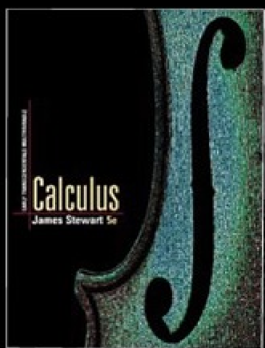
Calculus 5e
James Stewart
ISBN 0-534-39339-X



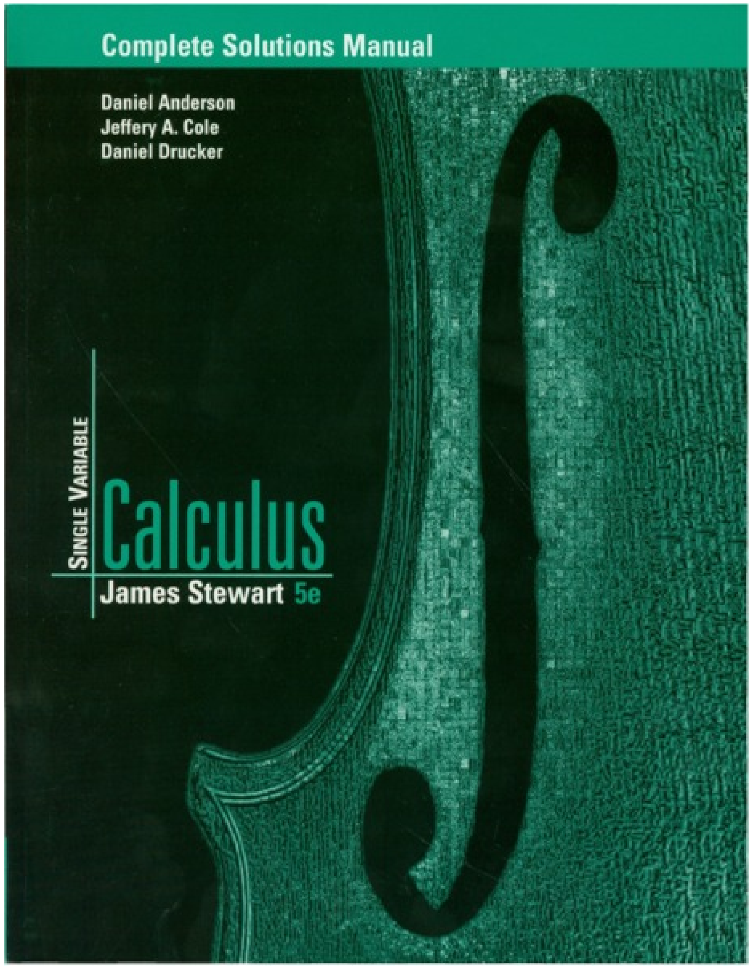
Calculus 5e - Multivariable
James Stewart
ISBN 0-534-39357-8



Calculus 5e - Single Variable
James Stewart
ISBN 0-534-39366-7



Calculus 5e - Multivariable
Early Transcendentals
James Stewart
ISBN 0-534-41778-7



2 ☐ LIMITS AND RATES OF CHANGE

2.1 The Tangent and Velocity Problems

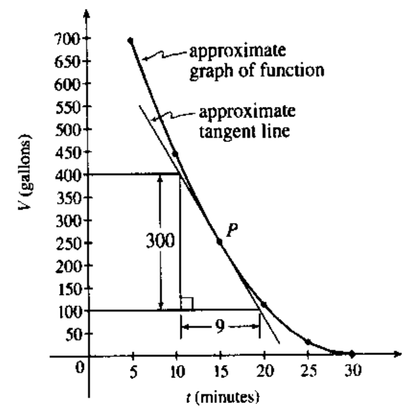
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$ (b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$ (d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. For the curve $y = x/(1 + x)$ and the point $P(1, \frac{1}{2})$:

(a)

	x	Q	m_{PQ}
(i)	0.5	$(0.5, 0.333333)$	0.333333
(ii)	0.9	$(0.9, 0.473684)$	0.263158
(iii)	0.99	$(0.99, 0.497487)$	0.251256
(iv)	0.999	$(0.999, 0.499750)$	0.250125
(v)	1.5	$(1.5, 0.6)$	0.2
(vi)	1.1	$(1.1, 0.523810)$	0.238095
(vii)	1.01	$(1.01, 0.502488)$	0.248756
(viii)	1.001	$(1.001, 0.500250)$	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ or $y = \frac{1}{4}x + \frac{1}{4}$.

4. For the curve $y = \sqrt{x+4}$ and the point $P(5,3)$:

(a)

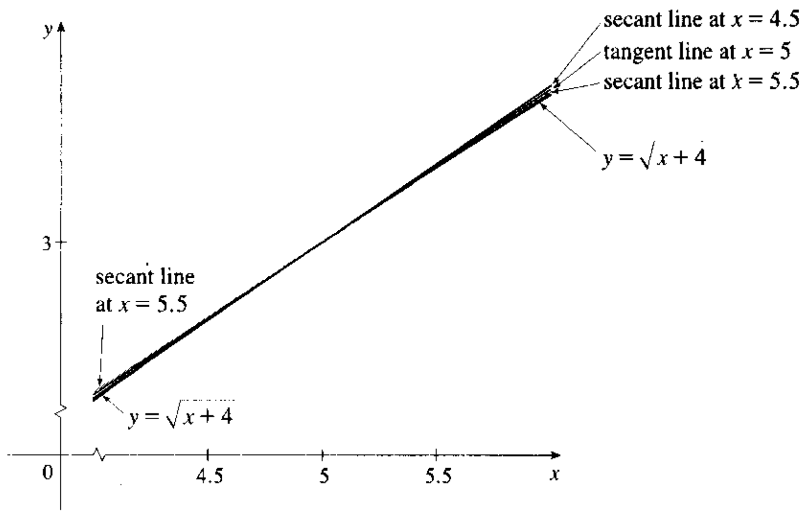
	x	Q	m_{PQ}
(i)	4.5	(4.5, 2.915476)	0.169048
(ii)	4.9	(4.9, 2.983287)	0.167132
(iii)	4.99	(4.99, 2.998333)	0.166713
(iv)	4.999	(4.999, 2.999833)	0.166671
(v)	5.5	(5.5, 3.082207)	0.164414
(vi)	5.1	(5.1, 3.016621)	0.166206
(vii)	5.01	(5.01, 3.001666)	0.166620
(viii)	5.001	(5.001, 3.000167)	0.166662

(b) The slope appears to be $\frac{1}{6}$.

(c) $y - 3 = \frac{1}{6}(x-5)$ or

$y = \frac{1}{6}x + \frac{13}{6}$

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2+h$ is $v_{\text{ave}} = \dfrac{y(2+h) - y(2)}{(2+h) - 2} = \dfrac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \dfrac{-24h - 16h^2}{h} = -24 - 16h$, if $h \neq 0$.

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s (ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s (iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. The average velocity between t and $t+h$ seconds is

$$\frac{58(t+h) - 0.83(t+h)^2 - \left(58t - 0.83t^2\right)}{h} = \frac{58h - 1.66th - 0.83h^2}{h} = 58 - 1.66t - 0.83h \text{ if } h \neq 0.$$

(a) Here $t = 1$, so the average velocity is $58 - 1.66 - 0.83h = 56.34 - 0.83h$.

(i) $[1, 2]$: $h = 1$, 55.51 m/s (ii) $[1, 1.5]$: $h = 0.5$, 55.925 m/s

(iii) $[1, 1.1]$: $h = 0.1$, 56.257 m/s (iv) $[1, 1.01]$: $h = 0.01$, 56.3317 m/s

(v) $[1, 1.001]$: $h = 0.001$, 56.33917 m/s

(b) The instantaneous velocity after 1 second is 56.34 m/s.

7. $s = s(t) = t^3/6$. Average velocity between times 1 and $1 + h$ is

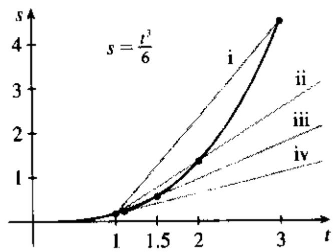
$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{(1+h)^3/6 - 1/6}{h} = \frac{h^3 + 3h^2 + 3h}{6h} = \frac{h^2 + 3h + 3}{6} \text{ if } h \neq 0.$$

(a) (i) $[1,3]$: $h = 2$, $v_{\text{ave}} = \frac{13}{6}$ ft/s (ii) $[1,2]$: $h = 1$, $v_{\text{ave}} = \frac{7}{6}$ ft/s

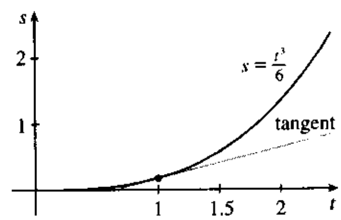
(iii) $[1,1.5]$: $h = 0.5$, $v_{\text{ave}} = \frac{19}{24}$ ft/s (iv) $[1,1.1]$: $h = 0.1$, $v_{\text{ave}} = \frac{331}{600}$ ft/s

(b) As h approaches 0, the velocity approaches $\frac{3}{6} = \frac{1}{2}$ ft/s.

(c)



(d)



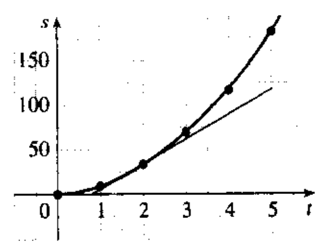
8. Average velocity between times $t = 2$ and $t = 2 + h$ is given by $\dfrac{s(2+h) - s(2)}{h}$.

(a) (i) $h = 3 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(5) - s(2)}{5 - 2} = \dfrac{178 - 32}{3} = \dfrac{146}{3} \approx 48.7$ ft/s

(ii) $h = 2 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{119 - 32}{2} = \dfrac{87}{2} = 43.5$ ft/s

(iii) $h = 1 \;\Rightarrow\; v_{\text{av}} = \dfrac{s(3) - s(2)}{3 - 2} = \dfrac{70 - 32}{1} = 38$ ft/s

(b) Using the points $(0.8, 0)$ and $(5, 118)$ from the approximate tangent line, the instantaneous velocity at $t = 2$ is about $\frac{118 - 0}{5 - 0.8} \approx 28$ ft/s.



9. For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

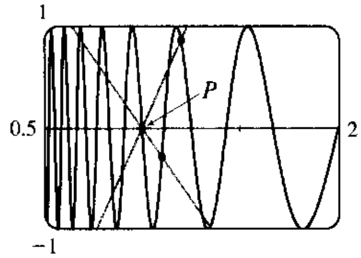
(a)

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	−1.0847
1.3	$(1.3, -0.8230)$	−2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	−2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	−2.1651
0.7	$(0.7, 0.7818)$	−2.6061
0.8	$(0.8, 1)$	−5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) $\lim\limits_{x \to 0} f(x) = 3$ (b) $\lim\limits_{x \to 3^-} f(x) = 4$ (c) $\lim\limits_{x \to 3^+} f(x) = 2$

(d) $\lim\limits_{x \to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.

(e) $f(3) = 3$

5. (a) $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim\limits_{x \to 1^-} f(x) = 2$.

(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim\limits_{x \to 1^+} f(x) = 3$.

(c) $\lim\limits_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim\limits_{x \to 5} f(x) = 4$.

(e) $f(5)$ is not defined, so it doesn't exist.

6. (a) $\lim\limits_{x \to -2^-} g(x) = -1$ (b) $\lim\limits_{x \to -2^+} g(x) = 1$ (c) $\lim\limits_{x \to -2} g(x)$ doesn't exist

(d) $g(-2) = 1$ (e) $\lim\limits_{x \to 2^-} g(x) = 1$ (f) $\lim\limits_{x \to 2^+} g(x) = 2$

(g) $\lim\limits_{x \to 2} g(x)$ doesn't exist (h) $g(2) = 2$ (i) $\lim\limits_{x \to 4^+} g(x)$ doesn't exist

(j) $\lim\limits_{x \to 4^-} g(x) = 2$ (k) $g(0)$ doesn't exist (l) $\lim\limits_{x \to 0} g(x) = 0$

7. (a) $\lim_{t\to 0^-} g(t) = -1$ (b) $\lim_{t\to 0^+} g(t) = -2$

(c) $\lim_{t\to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t\to 2^-} g(t) = 2$ (e) $\lim_{t\to 2^+} g(t) = 0$

(f) $\lim_{t\to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$ (h) $\lim_{t\to 4} g(t) = 3$

8. (a) $\lim_{x\to 2} R(x) = -\infty$ (b) $\lim_{x\to 5} R(x) = \infty$ (c) $\lim_{x\to -3^-} R(x) = -\infty$

(d) $\lim_{x\to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

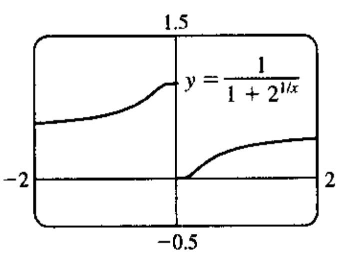
9. (a) $\lim_{x\to -7} f(x) = -\infty$ (b) $\lim_{x\to -3} f(x) = \infty$ (c) $\lim_{x\to 0} f(x) = \infty$

(d) $\lim_{x\to 6^-} f(x) = -\infty$ (e) $\lim_{x\to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11.

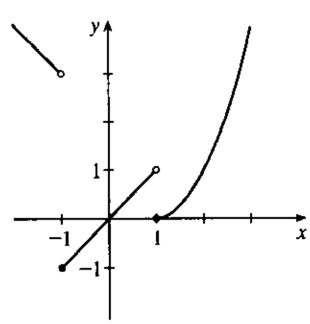


(a) $\lim_{x\to 0^-} f(x) = 1$

(b) $\lim_{x\to 0^+} f(x) = 0$

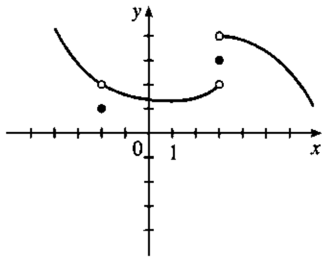
(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

12. $\lim_{x\to a} f(x)$ exists for all a except $a = \pm 1$.



13. $\lim_{x\to 3^+} f(x) = 4$, $\lim_{x\to 3^-} f(x) = 2$,

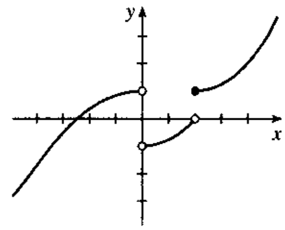
$\lim_{x\to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



14. $\lim_{x\to 0^-} f(x) = 1$, $\lim_{x\to 0^+} f(x) = -1$,

$\lim_{x\to 2^-} f(x) = 0$, $\lim_{x\to 2^+} f(x) = 1$, $f(2) = 1$,

$f(0)$ is undefined



15. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

16. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
0	0
−0.5	−1
−0.9	−9
−0.95	−19
−0.99	−99
−0.999	−999

x	$f(x)$
−2	2
−1.5	3
−1.1	11
−1.01	101
−1.001	1001

It appears that $\lim_{x\to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$ *does not exist* since $f(x) \to -\infty$ as $x \to -1^-$ and $f(x) \to \infty$ as $x \to -1^+$.

17. For $f(x) = \dfrac{\sin x}{x + \tan x}$:

x	$f(x)$
±1	0.329033
±0.5	0.458209
±0.2	0.493331
±0.1	0.498333
±0.05	0.499583
±0.01	0.499983

It appears that $\lim_{x\to 0} \dfrac{\sin x}{x + \tan x} = 0.5 = \dfrac{1}{2}$.

18. For $f(x) = \dfrac{\sqrt{x} - 4}{x - 16}$:

x	$f(x)$
17	0.123106
16.5	0.124038
16.1	0.124805
16.05	0.124902
16.01	0.124980

x	$f(x)$
15	0.127017
15.5	0.125992
15.9	0.125196
15.95	0.125098
15.99	0.125020

It appears that $\lim_{x\to 16} \dfrac{\sqrt{x} - 4}{x - 16} = 0.125 = \dfrac{1}{8}$.

19. For $f(x) = \dfrac{\sqrt{x+4}-2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x} = 0.25 = \frac{1}{4}$.

20. For $f(x) = \dfrac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\lim\limits_{x\to 0} \dfrac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

21. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim\limits_{x\to 1} \dfrac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

22. For $f(x) = \dfrac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
-0.5	0.227761
-0.1	0.485984
-0.05	0.534447
-0.01	0.576706
-0.001	0.586669

It appears that $\lim\limits_{x\to 0} \dfrac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

23. $\lim\limits_{x\to 5^+} \dfrac{6}{x-5} = \infty$ since $(x-5) \to 0$ as $x \to 5^+$ and $\dfrac{6}{x-5} > 0$ for $x > 5$.

24. $\lim\limits_{x\to 5^-} \dfrac{6}{x-5} = -\infty$ since $(x-5) \to 0$ as $x \to 5^-$ and $\dfrac{6}{x-5} < 0$ for $x < 5$.

25. $\lim\limits_{x\to 1} \dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

26. $\lim\limits_{x\to 0} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $0 < x < 1$ and for $-2 < x < 0$.

27. $\lim\limits_{x\to -2^+} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \to 0$ as $x \to -2^+$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $-2 < x < 0$.

28. $\lim\limits_{x\to\pi^-} \csc x = \lim\limits_{x\to\pi^-} (1/\sin x) = \infty$ since $\sin x \to 0$ as $x \to \pi^-$ and $\sin x > 0$ for $0 < x < \pi$.

29. $\lim\limits_{x\to(-\pi/2)^-} \sec x = \lim\limits_{x\to(-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \to 0$ as $x \to (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

30. $\lim\limits_{x\to1^+} \dfrac{x+1}{x\sin\pi x} = -\infty$ since $\dfrac{x+1}{x} \to 2$ as $x \to 1^+$ and $\sin\pi x \to 0$ through negative values as $x \to 1^+$.

31. (a) $f(x) = 1/(x^3 - 1)$

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	$-33{,}333.7$

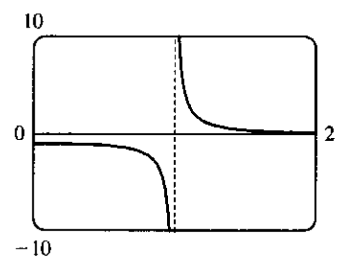
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

From these calculations, it seems that $\lim\limits_{x\to1^-} f(x) = -\infty$ and $\lim\limits_{x\to1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to1^-} f(x) = -\infty$.

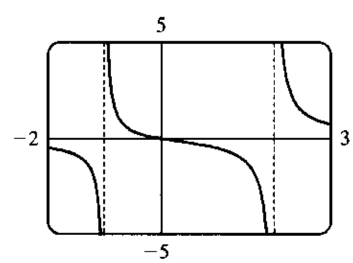
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim\limits_{x\to1^-} f(x) = -\infty$ and $\lim\limits_{x\to1^+} f(x) = \infty$.



32. (a) $y = \dfrac{x}{x^2 - x - 2} = \dfrac{x}{(x-2)(x+1)}$. Therefore, as $x \to -1^+$ or $x \to 2^+$, the denominator approaches 0, and $y > 0$ for $x < -1$ and for $x > 2$, so $\lim\limits_{x\to-1^+} y = \lim\limits_{x\to2^+} y = \infty$. Also, as $x \to -1^-$ or $x \to 2^-$, the denominator approaches 0 and $y < 0$ for $-1 < x < 2$, so $\lim\limits_{x\to-1^-} y = \lim\limits_{x\to2^-} y = -\infty$.

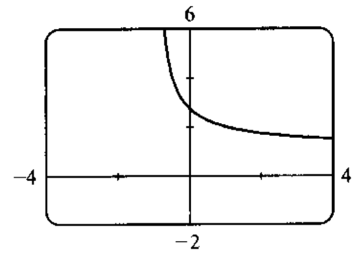
(b)



33. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

(b)



It appears that $\lim_{x\to 0}(1+x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 7.4 we will see that the value of the limit is exactly e.

34. For the curve $y = 2^x$ and the points $P(0,1)$ and $Q(x, 2^x)$:

x	Q	m_{PQ}
0.1	$(0.1, 1.0717735)$	0.71773
0.01	$(0.01, 1.0069556)$	0.69556
0.001	$(0.001, 1.0006934)$	0.69339
0.0001	$(0.0001, 1.0000693)$	0.69317

The slope appears to be about 0.693.

35. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x\to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	-0.000614
0.01	-0.000907
0.005	-0.000978
0.003	-0.000993
0.001	-0.001000

It appears that $\lim_{x\to 0} f(x) = -0.001$.

36. $h(x) = \dfrac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

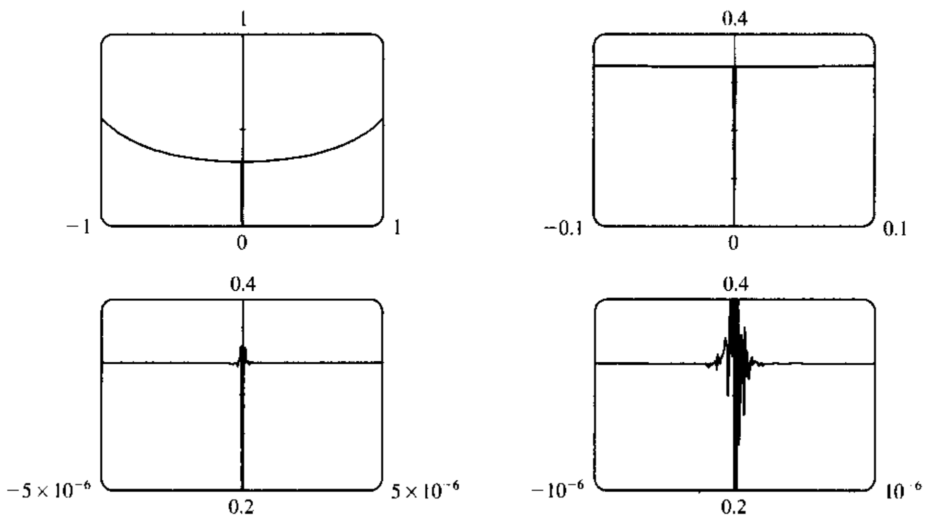
(b) It seems that $\lim\limits_{x \to 0} h(x) = \frac{1}{3}$.

(c)

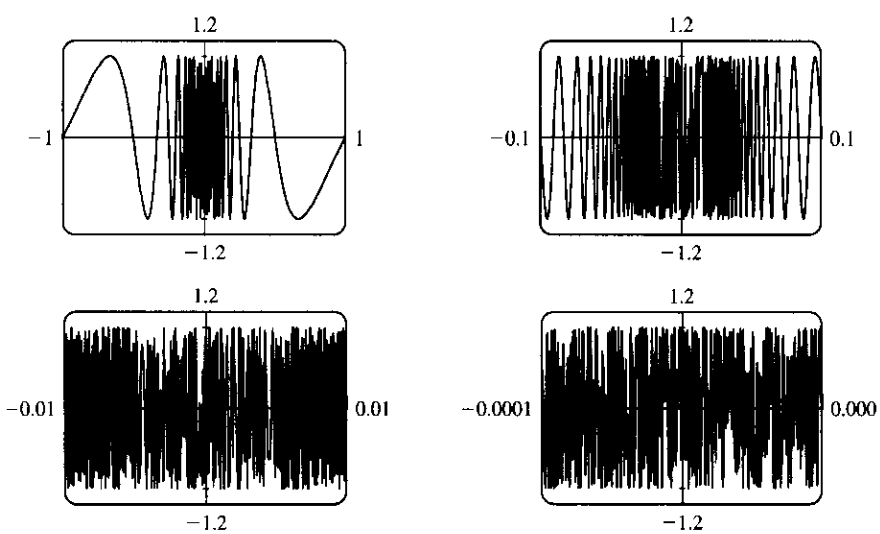
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

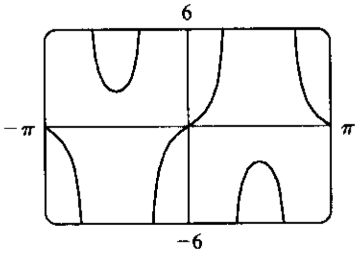


37. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.



38. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1-v^2/c^2}}$. As $v\to c^-$, $\sqrt{1-v^2/c^2}\to 0^+$, and $m\to\infty$.

39.

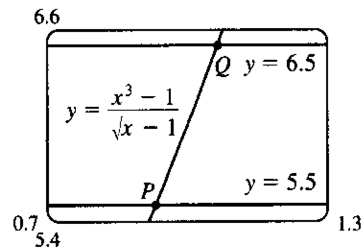


There appear to be vertical asymptotes of the curve $y=\tan(2\sin x)$ at $x\approx\pm0.90$ and $x\approx\pm2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x=\frac{\pi}{2}+\pi n$. Thus, we must have $2\sin x=\frac{\pi}{2}+\pi n$, or equivalently, $\sin x=\frac{\pi}{4}+\frac{\pi}{2}n$. Since $-1\le\sin x\le 1$, we must have $\sin x=\pm\frac{\pi}{4}$ and so $x=\pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x\approx\pm0.90$).

Just as $150°$ is the reference angle for $30°$, $\pi-\sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x=\pm\left(\pi-\sin^{-1}\frac{\pi}{4}\right)$ are also equations of the vertical asymptotes (corresponding to $x\approx\pm2.24$).

40. (a) Let $y=\left(x^3-1\right)/\left(\sqrt{x}-1\right)$.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

(b) We need to have $5.5<\dfrac{x^3-1}{\sqrt{x}-1}<6.5$. From the graph we obtain the approximate points of intersection $P(0.9313853, 5.5)$ and $Q(1.0649004, 6.5)$. Now $1-0.9313853\approx0.0686$ and $1.0649004-1\approx0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to a}[f(x)+h(x)]=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}h(x)$
$=-3+8=5$

(b) $\lim\limits_{x\to a}[f(x)]^2=\left[\lim\limits_{x\to a}f(x)\right]^2=(-3)^2=9$

(c) $\lim\limits_{x\to a}\sqrt[3]{h(x)}=\sqrt[3]{\lim\limits_{x\to a}h(x)}=\sqrt[3]{8}=2$

(d) $\lim\limits_{x\to a}\dfrac{1}{f(x)}=\dfrac{1}{\lim\limits_{x\to a}f(x)}=\dfrac{1}{-3}=-\dfrac{1}{3}$

(e) $\lim\limits_{x\to a}\dfrac{f(x)}{h(x)}=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}h(x)}=\dfrac{-3}{8}=-\dfrac{3}{8}$

(f) $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)}=\dfrac{\lim\limits_{x\to a}g(x)}{\lim\limits_{x\to a}f(x)}=\dfrac{0}{-3}=0$

(g) The limit does not exist, since $\lim\limits_{x\to a}g(x)=0$ but $\lim\limits_{x\to a}f(x)\neq0$.

(h) $\lim\limits_{x\to a}\dfrac{2f(x)}{h(x)-f(x)}=\dfrac{2\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}h(x)-\lim\limits_{x\to a}f(x)}=\dfrac{2(-3)}{8-(-3)}=-\dfrac{6}{11}$

2. (a) $\lim\limits_{x\to 2}[f(x)+g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 2+0 = 2$

(b) $\lim\limits_{x\to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x\to 0}[f(x)g(x)] = \lim\limits_{x\to 0} f(x)\cdot \lim\limits_{x\to 0} g(x) = 0\cdot 1.3 = 0$

(d) Since $\lim\limits_{x\to -1} g(x) = 0$ and g is in the denominator, but $\lim\limits_{x\to -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim\limits_{x\to 2} x^3 f(x) = \left[\lim\limits_{x\to 2} x^3\right]\left[\lim\limits_{x\to 2} f(x)\right] = 2^3\cdot 2 = 16$

(f) $\lim\limits_{x\to 1}\sqrt{3+f(x)} = \sqrt{3+\lim\limits_{x\to 1} f(x)} = \sqrt{3+1} = 2$

3.
$$\begin{aligned}
\lim_{x\to -2}(3x^4+2x^2-x+1) &= \lim_{x\to -2} 3x^4 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && \text{[Limit Laws 1 and 2]}\\
&= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && [3]\\
&= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && \text{[9, 8, and 7]}\\
&= 48+8+2+1 = 59
\end{aligned}$$

4.
$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4} &= \frac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(x^2+6x-4)} && \text{[Limit Law 5]}\\
&= \frac{2\lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2 + 6\lim\limits_{x\to 2} x - \lim\limits_{x\to 2} 4} && \text{[2, 1, and 3]}\\
&= \frac{2(2)^2+1}{(2)^2+6(2)-4} = \frac{9}{12} = \frac{3}{4} && \text{[9, 7, and 8]}
\end{aligned}$$

5.
$$\begin{aligned}
\lim_{x\to 3}(x^2-4)(x^3+5x-1) &= \lim_{x\to 3}(x^2-4)\cdot\lim_{x\to 3}(x^3+5x-1) && \text{[Limit Law 4]}\\
&= \left(\lim_{x\to 3} x^2 - \lim_{x\to 3} 4\right)\cdot\left(\lim_{x\to 3} x^3 + 5\lim_{x\to 3} x - \lim_{x\to 3} 1\right) && \text{[2, 1, and 3]}\\
&= (3^2-4)\cdot(3^3+5\cdot 3-1) && \text{[7, 8, and 9]}\\
&= 5\cdot 41 = 205
\end{aligned}$$

6.
$$\begin{aligned}
\lim_{t\to -1}(t^2+1)^3(t+3)^5 &= \lim_{t\to -1}(t^2+1)^3\cdot\lim_{t\to -1}(t+3)^5 && \text{[Limit Law 4]}\\
&= \left[\lim_{t\to -1}(t^2+1)\right]^3\cdot\left[\lim_{t\to -1}(t+3)\right]^5 && [6]\\
&= \left[\lim_{t\to -1} t^2 + \lim_{t\to -1} 1\right]^3\cdot\left[\lim_{t\to -1} t + \lim_{t\to -1} 3\right]^5 && [1]\\
&= \left[(-1)^2+1\right]^3\cdot[-1+3]^5 = 8\cdot 32 = 256 && \text{[9, 7, and 8]}
\end{aligned}$$

7. $\lim\limits_{x\to 1}\left(\dfrac{1+3x}{1+4x^2+3x^4}\right)^3 = \left(\lim\limits_{x\to 1}\dfrac{1+3x}{1+4x^2+3x^4}\right)^3$ [6]

$= \left[\dfrac{\lim\limits_{x\to 1}(1+3x)}{\lim\limits_{x\to 1}(1+4x^2+3x^4)}\right]^3$ [5]

$= \left[\dfrac{\lim\limits_{x\to 1}1+3\lim\limits_{x\to 1}x}{\lim\limits_{x\to 1}1+4\lim\limits_{x\to 1}x^2+3\lim\limits_{x\to 1}x^4}\right]^3$ [2, 1, and 3]

$= \left[\dfrac{1+3(1)}{1+4(1)^2+3(1)^4}\right]^3 = \left[\dfrac{4}{8}\right]^3 = \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$ [7, 8, and 9]

8. $\lim\limits_{u\to -2}\sqrt{u^4+3u+6} = \sqrt{\lim\limits_{u\to -2}(u^4+3u+6)}$ [11]

$= \sqrt{\lim\limits_{u\to -2}u^4+3\lim\limits_{u\to -2}u+\lim\limits_{u\to -2}6}$ [1, 2, and 3]

$= \sqrt{(-2)^4+3(-2)+6}$ [9, 8, and 7]

$= \sqrt{16-6+6} = \sqrt{16} = 4$

9. $\lim\limits_{x\to 4^-}\sqrt{16-x^2} = \sqrt{\lim\limits_{x\to 4^-}(16-x^2)}$ [11]

$= \sqrt{\lim\limits_{x\to 4^-}16-\lim\limits_{x\to 4^-}x^2}$ [2]

$= \sqrt{16-(4)^2} = 0$ [7 and 9]

10. (a) The left-hand side of the equation is not defined for $x=2$, but the right-hand side is.

(b) Since the equation holds for all $x\neq 2$, it follows that both sides of the equation approach the same limit as $x\to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a}f(x)$, we never consider $x=a$.

11. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2} = \lim\limits_{x\to 2}\dfrac{(x+3)(x-2)}{x-2} = \lim\limits_{x\to 2}(x+3) = 2+3 = 5$

12. $\lim\limits_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4} = \lim\limits_{x\to -4}\dfrac{(x+4)(x+1)}{(x+4)(x-1)} = \lim\limits_{x\to -4}\dfrac{x+1}{x-1} = \dfrac{-4+1}{-4-1} = \dfrac{-3}{-5} = \dfrac{3}{5}$

13. $\lim\limits_{x\to 2}\dfrac{x^2-x+6}{x-2}$ does not exist since $x-2\to 0$ but $x^2-x+6\to 8$ as $x\to 2$.

14. $\lim\limits_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4} = \lim\limits_{x\to 4}\dfrac{x(x-4)}{(x-4)(x+1)} = \lim\limits_{x\to 4}\dfrac{x}{x+1} = \dfrac{4}{4+1} = \dfrac{4}{5}$

15. $\lim\limits_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3} = \lim\limits_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim\limits_{t\to -3}\dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

16. $\lim\limits_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to 0$ but $x^2-4x\to 5$ as $x\to -1$.

17. $\lim\limits_{h\to 0}\dfrac{(4+h)^2-16}{h} = \lim\limits_{h\to 0}\dfrac{(16+8h+h^2)-16}{h} = \lim\limits_{h\to 0}\dfrac{8h+h^2}{h} = \lim\limits_{h\to 0}\dfrac{h(8+h)}{h} = \lim\limits_{h\to 0}(8+h) = 8+0 = 8$

18. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1} = \lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim\limits_{x\to 1}\dfrac{x^2+x+1}{x+1} = \dfrac{1^2+1+1}{1+1} = \dfrac{3}{2}$

19. $\lim\limits_{h\to 0}\dfrac{(1+h)^4-1}{h}=\lim\limits_{h\to 0}\dfrac{(1+4h+6h^2+4h^3+h^4)-1}{h}=\lim\limits_{h\to 0}\dfrac{4h+6h^2+4h^3+h^4}{h}$

$=\lim\limits_{h\to 0}\dfrac{h(4+6h+4h^2+h^3)}{h}=\lim\limits_{h\to 0}(4+6h+4h^2+h^3)=4+0+0+0=4$

20. $\lim\limits_{h\to 0}\dfrac{(2+h)^3-8}{h}=\lim\limits_{h\to 0}\dfrac{(8+12h+6h^2+h^3)-8}{h}=\lim\limits_{h\to 0}\dfrac{12h+6h^2+h^3}{h}$

$=\lim\limits_{h\to 0}(12+6h+h^2)=12+0+0=12$

21. $\lim\limits_{t\to 9}\dfrac{9-t}{3-\sqrt{t}}=\lim\limits_{t\to 9}\dfrac{(3+\sqrt{t})(3-\sqrt{t})}{3-\sqrt{t}}=\lim\limits_{t\to 9}(3+\sqrt{t})=3+\sqrt{9}=6$

22. $\lim\limits_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}\cdot\dfrac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim\limits_{h\to 0}\dfrac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}=\lim\limits_{h\to 0}\dfrac{h}{h\left(\sqrt{1+h}+1\right)}$

$=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{1+h}+1}=\dfrac{1}{\sqrt{1}+1}=\dfrac{1}{2}$

23. $\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}=\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}\cdot\dfrac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim\limits_{x\to 7}\dfrac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)}$

$=\lim\limits_{x\to 7}\dfrac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}=\lim\limits_{x\to 7}\dfrac{1}{\sqrt{x+2}+3}=\dfrac{1}{\sqrt{9}+3}=\dfrac{1}{6}$

24. $\lim\limits_{x\to 2}\dfrac{x^4-16}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+2)(x-2)(x^2+4)}{x-2}=\lim\limits_{x\to 2}(x+2)(x^2+4)=\lim\limits_{x\to 2}(x+2)\lim\limits_{x\to 2}(x^2+4)$

$=(2+2)(2^2+4)=32$

25. $\lim\limits_{x\to -4}\dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim\limits_{x\to -4}\dfrac{\frac{x+4}{4x}}{4+x}=\lim\limits_{x\to -4}\dfrac{x+4}{4x(4+x)}=\lim\limits_{x\to -4}\dfrac{1}{4x}=\dfrac{1}{4(-4)}=-\dfrac{1}{16}$

26. $\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)=\lim\limits_{t\to 0}\dfrac{(t^2+t)-t}{t(t^2+t)}=\lim\limits_{t\to 0}\dfrac{t^2}{t\cdot t(t+1)}=\lim\limits_{t\to 0}\dfrac{1}{t+1}=\dfrac{1}{0+1}=1$

27. $\lim\limits_{x\to 9}\dfrac{x^2-81}{\sqrt{x}-3}=\lim\limits_{x\to 9}\dfrac{(x-9)(x+9)}{\sqrt{x}-3}=\lim\limits_{x\to 9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x}-3}$ [factor $x-9$ as a difference of squares]

$=\lim\limits_{x\to 9}[(\sqrt{x}+3)(x+9)]=(\sqrt{9}+3)(9+9)=6\cdot 18=108$

28. $\lim\limits_{h\to 0}\dfrac{(3+h)^{-1}-3^{-1}}{h}=\lim\limits_{h\to 0}\dfrac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim\limits_{h\to 0}\dfrac{3-(3+h)}{h(3+h)3}=\lim\limits_{h\to 0}\dfrac{-h}{h(3+h)3}$

$=\lim\limits_{h\to 0}\left[-\dfrac{1}{3(3+h)}\right]=-\dfrac{1}{\lim\limits_{h\to 0}[3(3+h)]}=-\dfrac{1}{3(3+0)}=-\dfrac{1}{9}$

29. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

30. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\left(1-x^{3/2}\right)}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\,(1-\sqrt{x})(1+\sqrt{x}+x)}{1-\sqrt{x}}$ [difference of cubes]

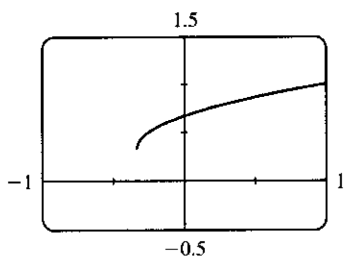
$$=\lim_{x\to 1}\left[\sqrt{x}\,(1+\sqrt{x}+x)\right]=\lim_{x\to 1}\left[1(1+1+1)\right]=3$$

Another method: We "add and subtract" 1 in the numerator, and then split up the fraction:

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{(\sqrt{x}-1)+\left(1-x^2\right)}{1-\sqrt{x}}=\lim_{x\to 1}\left[-1+\frac{(1-x)(1+x)}{1-\sqrt{x}}\right]$$

$$=\lim_{x\to 1}\left[-1+\frac{(1-\sqrt{x})(1+\sqrt{x})(1+x)}{1-\sqrt{x}}\right]=-1+\left(1+\sqrt{1}\right)(1+1)=3$$

31. (a)



$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)\quad\text{[Limit Law 3]}$$

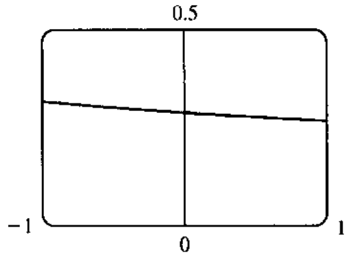
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]\quad\text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)\quad\text{[1, 3, and 7]}$$

$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)\quad\text{[7 and 8]}$$

$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

32. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

(b)

x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

(c) $$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

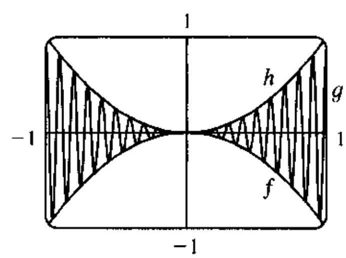
$$=\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\sqrt{3+x}+\lim\limits_{x\to 0}\sqrt{3}}\qquad\text{[Limit Laws 5 and 1]}$$

$$=\frac{1}{\sqrt{\lim\limits_{x\to 0}(3+x)}+\sqrt{3}}\qquad\text{[7 and 11]}$$

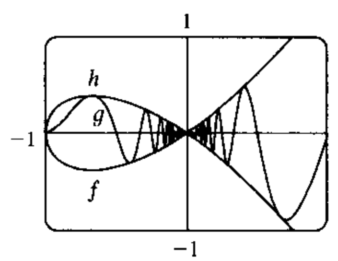
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}\qquad\text{[1, 7, and 8]}$$

$$=\frac{1}{2\sqrt{3}}$$

33. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le\cos 20\pi x\le 1\;\Rightarrow\;-x^2\le x^2\cos 20\pi x\le x^2\;\Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



34. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le\sin(\pi/x)\le 1\;\Rightarrow$ $-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\;\Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



35. $1\le f(x)\le x^2+2x+2$ for all x. Now $\lim\limits_{x\to -1}1=1$ and $\lim\limits_{x\to -1}(x^2+2x+2)=\lim\limits_{x\to -1}x^2+2\lim\limits_{x\to -1}x+\lim\limits_{x\to -1}2=(-1)^2+2(-1)+2=1$. Therefore, by the Squeeze Theorem, $\lim\limits_{x\to -1}f(x)=1$.

36. $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$. Now $\lim\limits_{x \to 1} 3x = 3$ and $\lim\limits_{x \to 1} (x^3 + 2) = \lim\limits_{x \to 1} x^3 + \lim\limits_{x \to 1} 2 = 1^3 + 2 = 3$.
Therefore, by the Squeeze Theorem, $\lim\limits_{x \to 1} f(x) = 3$.

37. $-1 \le \cos(2/x) \le 1 \quad \Rightarrow \quad -x^4 \le x^4 \cos(2/x) \le x^4$. Since $\lim\limits_{x \to 0} (-x^4) = 0$ and $\lim\limits_{x \to 0} x^4 = 0$, we have $\lim\limits_{x \to 0} \left[x^4 \cos(2/x)\right] = 0$ by the Squeeze Theorem.

38. $-1 \le \sin(2\pi/x) \le 1 \quad \Rightarrow \quad 0 \le \sin^2(2\pi/x) \le 1 \quad \Rightarrow \quad 1 \le 1 + \sin^2(2\pi/x) \le 2 \quad \Rightarrow$
$\sqrt{x} \le \sqrt{x}\left[1 + \sin^2(2\pi/x)\right] \le 2\sqrt{x}$. Since $\lim\limits_{x \to 0^+} \sqrt{x} = 0$ and $\lim\limits_{x \to 0^+} 2\sqrt{x} = 0$, we have
$\lim\limits_{x \to 0^+} \left[\sqrt{x}\left(1 + \sin^2(2\pi/x)\right)\right] = 0$ by the Squeeze Theorem.

39. If $x > -4$, then $|x + 4| = x + 4$, so $\lim\limits_{x \to -4^+} |x + 4| = \lim\limits_{x \to -4^+} (x + 4) = -4 + 4 = 0$.
If $x < -4$, then $|x + 4| = -(x + 4)$, so $\lim\limits_{x \to -4^-} |x + 4| = \lim\limits_{x \to -4^-} -(x + 4) = -(-4 + 4) = 0$.
Since the right and left limits are equal, $\lim\limits_{x \to -4} |x + 4| = 0$.

40. If $x < -4$, then $|x + 4| = -(x + 4)$, so $\lim\limits_{x \to -4^-} \dfrac{|x + 4|}{x + 4} = \lim\limits_{x \to -4^-} \dfrac{-(x + 4)}{x + 4} = \lim\limits_{x \to -4^-} (-1) = -1$.

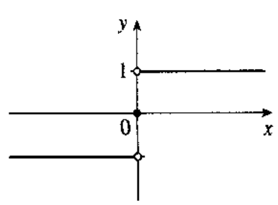
41. If $x > 2$, then $|x - 2| = x - 2$, so $\lim\limits_{x \to 2^+} \dfrac{|x - 2|}{x - 2} = \lim\limits_{x \to 2^+} \dfrac{x - 2}{x - 2} = \lim\limits_{x \to 2^+} 1 = 1$. If $x < 2$, then
$|x - 2| = -(x - 2)$, so $\lim\limits_{x \to 2^-} \dfrac{|x - 2|}{x - 2} = \lim\limits_{x \to 2^-} \dfrac{-(x - 2)}{x - 2} = \lim\limits_{x \to 2^-} -1 = -1$. The right and left limits are different, so $\lim\limits_{x \to 2} \dfrac{|x - 2|}{x - 2}$ does not exist.

42. If $x > \dfrac{3}{2}$, then $|2x - 3| = 2x - 3$, so
$\lim\limits_{x \to 1.5^+} \dfrac{2x^2 - 3x}{|2x - 3|} = \lim\limits_{x \to 1.5^+} \dfrac{2x^2 - 3x}{2x - 3} = \lim\limits_{x \to 1.5^+} \dfrac{x(2x - 3)}{2x - 3} = \lim\limits_{x \to 1.5^+} x = 1.5$. If $x < \dfrac{3}{2}$, then
$|2x - 3| = 3 - 2x$, so $\lim\limits_{x \to 1.5^-} \dfrac{2x^2 - 3x}{|2x - 3|} = \lim\limits_{x \to 1.5^-} \dfrac{2x^2 - 3x}{-(2x - 3)} = \lim\limits_{x \to 1.5^-} \dfrac{x(2x - 3)}{-(2x - 3)} = \lim\limits_{x \to 1.5^-} -x = -1.5$.
The right and left limits are different, so $\lim\limits_{x \to 1.5} \dfrac{2x^2 - 3x}{|2x - 3|}$ does not exist.

43. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x \to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x \to 0^+} 0 = 0$.

45. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x \to 0^+} \operatorname{sgn} x = \lim\limits_{x \to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x \to 0^-} \operatorname{sgn} x = \lim\limits_{x \to 0^-} -1 = -1$.

(iii) Since $\lim\limits_{x \to 0^-} \operatorname{sgn} x \ne \lim\limits_{x \to 0^+} \operatorname{sgn} x$, $\lim\limits_{x \to 0} \operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim\limits_{x \to 0} |\operatorname{sgn} x| = \lim\limits_{x \to 0} 1 = 1$.

46. (a) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(4 - x^2\right) = \lim\limits_{x\to 2^-} 4 - \lim\limits_{x\to 2^-} x^2$

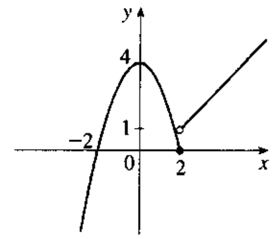
$= 4 - 4 = 0$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x-1) = \lim\limits_{x\to 2^+} x - \lim\limits_{x\to 2^+} 1$

$= 2 - 1 = 1$

(b) No, $\lim\limits_{x\to 2} f(x)$ does not exist since $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$.

(c)

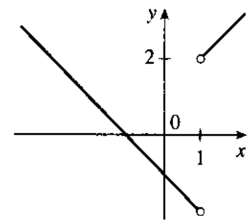


47. (a) (i) $\lim\limits_{x\to 1^+} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to 1^+} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1^+} (x+1) = 2$

(ii) $\lim\limits_{x\to 1^-} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to 1^-} \dfrac{x^2-1}{-(x-1)} = \lim\limits_{x\to 1^-} -(x+1) = -2$

(b) No, $\lim\limits_{x\to 1} F(x)$ does not exist since $\lim\limits_{x\to 1^+} F(x) \neq \lim\limits_{x\to 1^-} F(x)$.

(c)



48. (a) (i) $\lim\limits_{x\to 0^+} h(x) = \lim\limits_{x\to 0^+} x^2 = 0^2 = 0$

(ii) $\lim\limits_{x\to 0^-} h(x) = \lim\limits_{x\to 0^-} x = 0$, so $\lim\limits_{x\to 0} h(x) = 0$.

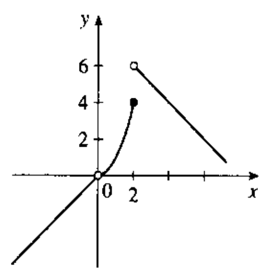
(iii) $\lim\limits_{x\to 1} h(x) = \lim\limits_{x\to 1} x^2 = 1^2 = 1$

(iv) $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^-} x^2 = 2^2 = 4$

(v) $\lim\limits_{x\to 2^+} h(x) = \lim\limits_{x\to 2^+} (8-x) = 8 - 2 = 6$

(vi) Since $\lim\limits_{x\to 2^-} h(x) \neq \lim\limits_{x\to 2^+} h(x)$, $\lim\limits_{x\to 2} h(x)$ does not exist.

(b)



49. (a) (i) $[\![x]\!] = -2$ for $-2 \leq x < -1$, so $\lim\limits_{x\to -2^+} [\![x]\!] = \lim\limits_{x\to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \leq x < -2$, so $\lim\limits_{x\to -2^-} [\![x]\!] = \lim\limits_{x\to -2^-} (-3) = -3$. The right and left limits are different, so $\lim\limits_{x\to -2} [\![x]\!]$ does not exist.

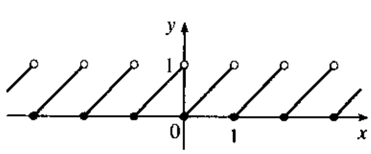
(iii) $[\![x]\!] = -3$ for $-3 \leq x < -2$, so $\lim\limits_{x\to -2.4} [\![x]\!] = \lim\limits_{x\to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \leq x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n-1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \leq x < n + 1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

50. (a)



(b) (i) $\lim\limits_{x\to n^-} f(x) = \lim\limits_{x\to n^-} (x - [\![x]\!]) = \lim\limits_{x\to n^-} [x - (n-1)] = n - (n-1) = 1$

(ii) $\lim\limits_{x\to n^+} f(x) = \lim\limits_{x\to n^+} (x - [\![x]\!]) = \lim\limits_{x\to n^+} (x - n) = n - n = 0$

(c) $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow$ a is not an integer.

51. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim\limits_{x\to 2^-} f(x) = -1$ and $\lim\limits_{x\to 2^+} f(x) = -1$, so $\lim\limits_{x\to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim\limits_{x\to 2} f(x) \neq f(2)$.

52. $\lim\limits_{v\to c^-}\left(L_0\sqrt{1-\dfrac{v^2}{c^2}}\right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0. A left-hand limit is necessary since L is not defined for $v > c$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\\ &= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n\\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim\limits_{x\to a} p(x) = p(a)$.

54. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 53]} \quad = r(a).$$

55. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim\limits_{x\to 0} 0 = 0 = \lim\limits_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim\limits_{x\to 0} f(x) = 0$.

56. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ do not exist (Example 10) but $\lim\limits_{x\to 3}\,[f(x) + g(x)] = \lim\limits_{x\to 3}\left([\![x]\!] - [\![x]\!]\right) = \lim\limits_{x\to 3} 0 = 0$.

57. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59. Thus, either f or g is 0 for any value of x. Then $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0}\,[f(x)g(x)] = \lim\limits_{x\to 0} 0 = 0$.

58.
$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)\\ &= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)\\ &= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}\end{aligned}$$

59. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x\to -2}\left(3x^2 + ax + a + 3\right) = 0 \quad\Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad\Leftrightarrow\quad 12 - 2a + a + 3 = 0 \quad\Leftrightarrow\quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

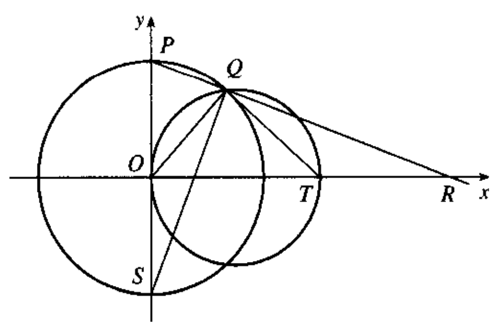
60. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$.
The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad \Leftrightarrow \quad r^2 = 1 + 2x - 1 \quad \Leftrightarrow \quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad \Leftrightarrow \quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad \Leftrightarrow \quad y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right).$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r \to 0^+} x = \lim\limits_{r \to 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim\limits_{r \to 0^+} 2\left(\sqrt{1} + 1\right) = 4$.
So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

1. (a) To have $5x + 3$ within a distance of 0.1 of 13, we must have $12.9 \le 5x + 3 \le 13.1 \quad \Rightarrow \quad 9.9 \le 5x \le 10.1 \quad \Rightarrow \quad 1.98 \le x \le 2.02$. Thus, x must be within 0.02 units of 2 so that $5x + 3$ is within 0.1 of 13.

(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. (a) To have $6x - 1$ within a distance of 0.01 of 29, we must have $28.99 \le 6x - 1 \le 29.01 \quad \Rightarrow \quad 29.99 \le 6x \le 30.01 \quad \Rightarrow \quad 4.998\overline{3} \le x \le 5.001\overline{6}$. Thus, x must be within $0.001\overline{6}$ units of 5 so that $6x - 1$ is within 0.01 of 29.

(b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\overline{6}$.

(c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\overline{6}$.

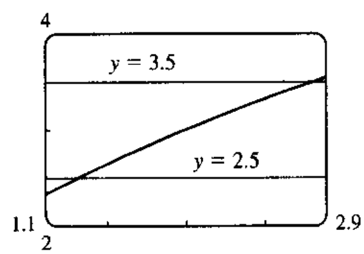
3. On the left side of $x = 2$, we need $|x - 2| < \left|\frac{10}{7} - 2\right| = \frac{4}{7}$. On the right side, we need $|x - 2| < \left|\frac{10}{3} - 2\right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x - 2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

4. On the left side, we need $|x - 5| < |4 - 5| = 1$. On the right side, we need $|x - 5| < |5.7 - 5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x - 5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.

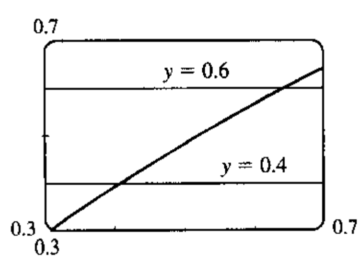
5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

6. The left-hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right-hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

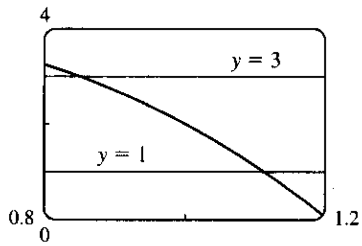
7. $\left|\sqrt{4x+1} - 3\right| < 0.5 \quad \Leftrightarrow \quad 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen and identify the x-coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125 \le x \le 2.8125$. Since $|2 - 1.3125| = 0.6875$ and $|2 - 2.8125| = 0.8125$, we choose $0 < \delta < \min\{0.6875, 0.8125\} = 0.6875$.

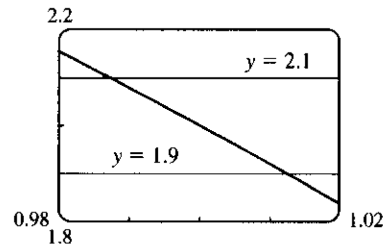


8. $\left|\sin x - \frac{1}{2}\right| < 0.1 \quad \Leftrightarrow \quad 0.4 < \sin x < 0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \le x \le 0.64$. So since $|0.5 - 0.42| = 0.08$ and $|0.5 - 0.64| = 0.14$, we choose $0 < \delta \le \min\{0.08, 0.14\} = 0.08$.

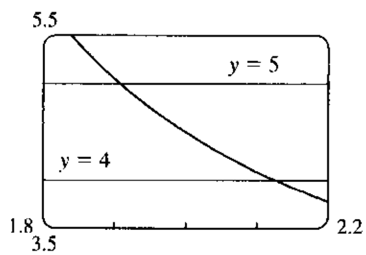


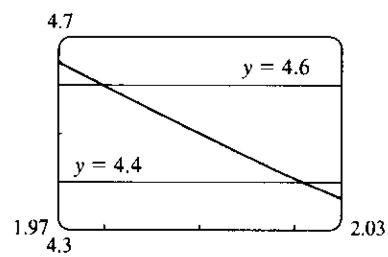
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $\left|\left(4 + x - 3x^3\right) - 2\right| < 1 \quad \Leftrightarrow$ $1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left|\left(4 + x - 3x^3\right) - 2\right| < 0.1 \quad \Leftrightarrow \quad 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



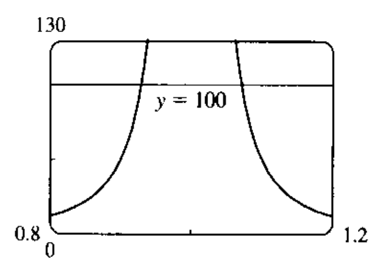


10. For $\varepsilon = 0.5$, we need $1.91 \leq x \leq 2.125$. So since $|2 - 1.91| = 0.09$ and $|2 - 2.125| = 0.125$, we can take $0 < \delta \leq 0.09$. For $\varepsilon = 0.1$, we need $1.980 \leq 2.021$. So since $|2 - 1.980| = 0.02$ and $|2 - 2.021| = 0.021$, we can take $\delta = 0.02$ (or any smaller positive number).

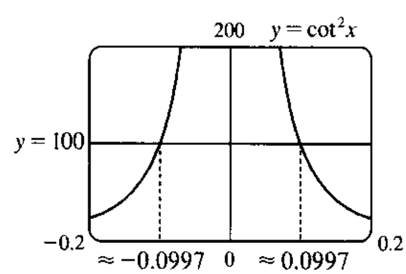




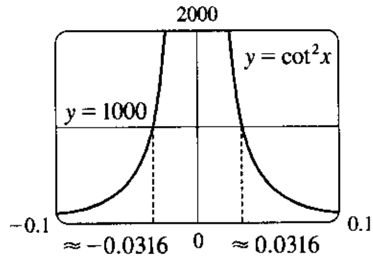
11. From the graph, we see that $\dfrac{x}{(x^2+1)(x-1)^2} > 100$ whenever $0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M = 100$, we need $-0.0997 < x < 0$ or $0 < x < 0.0997$. Thus, we choose $\delta = 0.0997$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 100$.



For $M = 1000$, we need $-0.0316 < x < 0$ or $0 < x < 0.0316$. Thus, we choose $\delta = 0.0316$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 1000$.

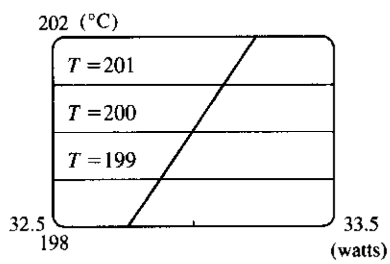


13. (a) $A = \pi r^2$ and $A = 1000 \text{ cm}^2 \;\Rightarrow\; \pi r^2 = 1000 \;\Rightarrow\; r^2 = \frac{1000}{\pi} \;\Rightarrow$
$r = \sqrt{\frac{1000}{\pi}} \quad [r > 0] \quad \approx 17.8412$ cm.

(b) $|A - 1000| \le 5 \;\Rightarrow\; -5 \le \pi r^2 - 1000 \le 5 \;\Rightarrow\; 1000 - 5 \le \pi r^2 \le 1000 + 5 \;\Rightarrow$
$\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \;\Rightarrow\; 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and
$\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

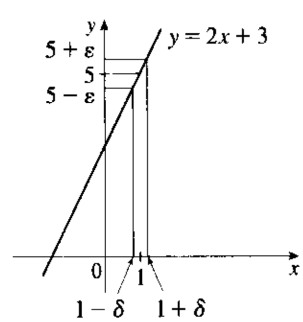
14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \;\Rightarrow$
$0.1w^2 + 2.155w + 20 = 200 \;\Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts ($w > 0$)



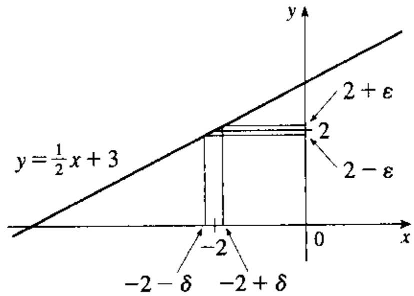
(b) From the graph, $199 \le T \le 201 \;\Rightarrow\; 32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

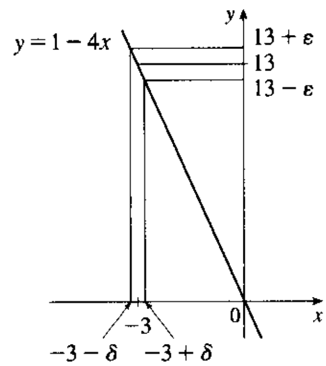
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x + 3) - 5| < \varepsilon$. But $|(2x + 3) - 5| < \varepsilon \;\Leftrightarrow\; |2x - 2| < \varepsilon \;\Leftrightarrow\; 2|x - 1| < \varepsilon \;\Leftrightarrow\; |x - 1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 1| < \delta \;\Rightarrow\; |(2x + 3) - 5| < \varepsilon$. Thus, $\lim_{x \to 1} (2x + 3) = 5$ by the definition of a limit.



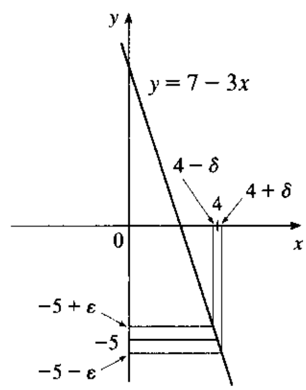
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon \;\Leftrightarrow\; \left|\frac{1}{2}x + 1\right| < \varepsilon \;\Leftrightarrow\; \frac{1}{2}|x + 2| < \varepsilon \;\Leftrightarrow\; |x - (-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \;\Rightarrow\; \left|\left(\frac{1}{2}x + 3\right) - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to -2}\left(\frac{1}{2}x + 3\right) = 2$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \;\Leftrightarrow\; |-4x - 12| < \varepsilon \;\Leftrightarrow\; |-4|\,|x + 3| < \varepsilon \;\Leftrightarrow\; |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \;\Rightarrow\; |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim\limits_{x \to -3}(1 - 4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(7 - 3x) - (-5)| < \varepsilon$. But $|(7 - 3x) - (-5)| < \varepsilon \;\Leftrightarrow\; |-3x + 12| < \varepsilon \;\Leftrightarrow\; |-3|\,|x - 4| < \varepsilon \;\Leftrightarrow\; |x - 4| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x - 4| < \delta \;\Rightarrow\; |(7 - 3x) - (-5)| < \varepsilon$. Thus, $\lim\limits_{x \to 4}(7 - 3x) = -5$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon \;\Leftrightarrow\; \frac{1}{5}|x - 3| < \varepsilon \;\Leftrightarrow\; |x - 3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x - 3| < \delta \;\Rightarrow\; |x - 3| < 5\varepsilon \;\Rightarrow\; \dfrac{|x - 3|}{5} < \varepsilon \;\Rightarrow\; \left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 3}\dfrac{x}{5} = \dfrac{3}{5}$.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 6| < \delta$, then $\left|\left(\frac{x}{4} + 3\right) - \frac{9}{2}\right| < \varepsilon \;\Leftrightarrow\; \left|\frac{x}{4} - \frac{3}{2}\right| < \varepsilon \;\Leftrightarrow\; \frac{1}{4}|x - 6| < \varepsilon \;\Leftrightarrow\; |x - 6| < 4\varepsilon$. So choose $\delta = 4\varepsilon$. Then $0 < |x - 6| < \delta \;\Rightarrow\; |x - 6| < 4\varepsilon \;\Rightarrow\; \dfrac{|x - 6|}{4} < \varepsilon \;\Rightarrow\; \left|\frac{x}{4} - \frac{6}{4}\right| < \varepsilon \;\Rightarrow\; \left|\left(\frac{x}{4} + 3\right) - \frac{9}{2}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 6}\left(\frac{x}{4} + 3\right) = \frac{9}{2}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-5)| < \delta$, then $\left|\left(4 - \frac{3}{5}x\right) - 7\right| < \varepsilon \;\Leftrightarrow\; \left|-\frac{3}{5}x - 3\right| < \varepsilon \;\Leftrightarrow\; \frac{3}{5}|x + 5| < \varepsilon \;\Leftrightarrow\; |x - (-5)| < \frac{5}{3}\varepsilon$. So choose $\delta = \frac{5}{3}\varepsilon$. Then $|x - (-5)| < \delta \;\Rightarrow\; \left|\left(4 - \frac{3}{5}x\right) - 7\right| < \varepsilon$. Thus, $\lim\limits_{x \to -5}\left(4 - \frac{3}{5}x\right) = 7$ by the definition of a limit.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon$. Notice that if $0 < |x-3|$, then $x \neq 3$, so $\dfrac{x^2+x-12}{x-3} = \dfrac{(x+4)(x-3)}{x-3} = x+4$. Thus, when $0 < |x-3|$, we have $\left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon \quad\Leftrightarrow\quad |(x+4)-7| < \varepsilon \quad\Leftrightarrow\quad |x-3| < \varepsilon$. We take $\delta = \varepsilon$ and see that $0 < |x-3| < \delta$ $\Rightarrow \quad \left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon$. By the definition of a limit, $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3} = 7$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x-a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|c-c| < \varepsilon$. But $|c-c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^2-0\right| < \varepsilon \quad\Leftrightarrow\quad x^2 < \varepsilon \quad\Leftrightarrow\quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-0| < \delta \quad\Rightarrow\quad \left|x^2-0\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^3-0\right| < \varepsilon \quad\Leftrightarrow\quad |x|^3 < \varepsilon \quad\Leftrightarrow\quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x-0| < \delta \quad\Rightarrow\quad \left|x^3-0\right| < \delta^3 = \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\big||x|-0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9 - \delta < x < 9$, then $\left|\sqrt[4]{9-x} - 0\right| < \varepsilon \quad\Leftrightarrow\quad \sqrt[4]{9-x} < \varepsilon \quad\Leftrightarrow\quad 9 - x < \varepsilon^4 \quad\Leftrightarrow\quad 9 - \varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9 - \delta < x < 9 \quad\Rightarrow\quad \left|\sqrt[4]{9-x} - 0\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 9^-} \sqrt[4]{9-x} = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|(x^2-4x+5) - 1\right| < \varepsilon \quad\Leftrightarrow\quad \left|x^2-4x+4\right| < \varepsilon \quad\Leftrightarrow\quad \left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \quad\Leftrightarrow\quad |x-2| < \sqrt{\varepsilon} \quad\Leftrightarrow\quad \left|(x-2)^2\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 2}(x^2-4x+5) = 1$ by the definition of a limit.

30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|(x^2+x-4) - 8\right| < \varepsilon \quad\Leftrightarrow\quad \left|x^2+x-12\right| < \varepsilon \quad\Leftrightarrow\quad |(x-3)(x+4)| < \varepsilon$. Notice that if $|x-3| < 1$, then $-1 < x-3 < 1 \quad\Rightarrow\quad 6 < x+4 < 8 \quad\Rightarrow\quad |x+4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $0 < |x-3| < \delta \quad\Leftrightarrow\quad |(x-3)(x+4)| \le |8(x-3)| = 8 \cdot |x-3| < 8\delta \le \varepsilon$. Thus, $\displaystyle\lim_{x\to 3}(x^2+x-4) = 8$ by the definition of a limit.

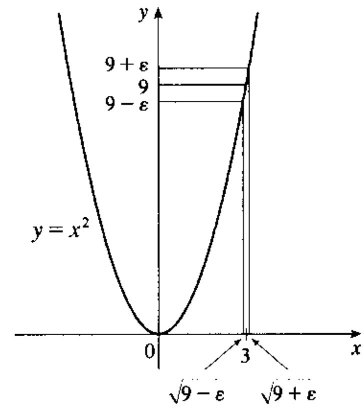
31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|(x^2-1) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2-4\right| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x+2 < 1 \quad\Rightarrow\quad -5 < x-2 < -3 \quad\Rightarrow\quad |x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \quad\Rightarrow\quad |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|(x^2-1) - 3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\displaystyle\lim_{x\to -2}(x^2-1) = 3$.

32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|x^3-8\right| < \varepsilon$. Now $\left|x^3-8\right| = \left|(x-2)(x^2+2x+4)\right|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2+2x+4 < 3^2+2(3)+4 = 19$ and so $\left|x^3-8\right| = |x-2|\,(x^2+2x+4) < 19\,|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then

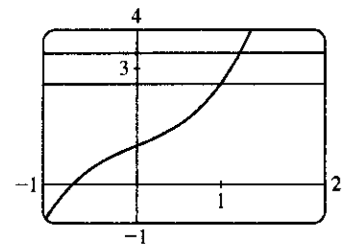
$0 < |x-2| < \delta \quad \Rightarrow \quad |x^3 - 8| = |x-2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \quad \Rightarrow \quad -2 < x - 3 < 2 \quad \Rightarrow$ $4 < x + 3 < 8 \quad \Rightarrow \quad |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2 - 9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x \to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9 - \varepsilon}$ and $\delta_2 = \sqrt{9 + \varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9 + \varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \quad \Rightarrow$ $\frac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \quad \Rightarrow \quad 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \quad \Rightarrow \quad \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \quad \Rightarrow \quad \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \quad \Rightarrow$ $1 < x < 3 \quad \Rightarrow \quad \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x \to 2} (1/x) = \frac{1}{2}$.

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \;\Rightarrow\; \frac{1}{2}a < x < \frac{3}{2}a$ $\Rightarrow\; \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then $|x - a| < \frac{1}{2}a \;\Rightarrow\; \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon.$$ Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim\limits_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \;\Rightarrow\; |H(t) - L| < \frac{1}{2} \;\Leftrightarrow\; L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \;\Rightarrow\; L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \;\Rightarrow\; L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim\limits_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \;\Rightarrow\; |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \;\Rightarrow\; 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a + \delta \;\Rightarrow\; 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \;\Rightarrow\; |f(x) - L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2$ $\Rightarrow\; |f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \;\Rightarrow\; a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \;\Leftrightarrow\; \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \;\Leftrightarrow\; (x+3)^4 < \dfrac{1}{10{,}000} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \;\Leftrightarrow\; |x - (-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \Leftrightarrow$ $(x+3)^4 < \dfrac{1}{M} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \Rightarrow$ $\dfrac{1}{(x+3)^4} > M$, so $\displaystyle\lim_{x\to -3} \frac{1}{(x+3)^4} = \infty$.

43. Let $N < 0$ be given. Then, for $x < -1$, we have $\dfrac{5}{(x+1)^3} < N \Leftrightarrow \dfrac{5}{N} < (x+1)^3 \Leftrightarrow \sqrt[3]{\dfrac{5}{N}} < x+1$. Let $\delta = -\sqrt[3]{\dfrac{5}{N}}$. Then $-1-\delta < x < -1 \Rightarrow \sqrt[3]{\dfrac{5}{N}} < x+1 < 0 \Rightarrow \dfrac{5}{(x+1)^3} < N$, so $\displaystyle\lim_{x\to -1^-} \frac{5}{(x+1)^3} = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow$ $f(x) > M + 1 - c$. Since $\lim\limits_{x\to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow |g(x) - c| < 1$ $\Rightarrow g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow$ $f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim\limits_{x\to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x\to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow$ $|g(x) - c| < c/2 \Rightarrow g(x) > c/2$. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ $\Rightarrow f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x\to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x\to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow$ $|g(x) - c| < -c/2 \Rightarrow g(x) < c/2$. Since $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ $\Rightarrow f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)\,g(x) < \dfrac{2N}{c}\cdot\dfrac{c}{2} = N$, so $\lim\limits_{x\to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim\limits_{x\to 4} f(x) = f(4)$.

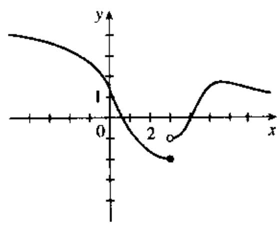
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).

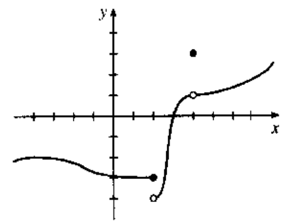
(b) f is continuous from the left at -2 since $\lim\limits_{x\to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x\to 2^+} f(x) = f(2)$ and $\lim\limits_{x\to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

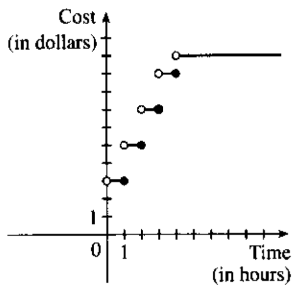
5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim\limits_{x \to 3^-} f(x) = f(3)$.



6.



7. (a)



(b) There are discontinuities at times $t = 1, 2, 3$, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values—at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned} \lim_{x \to 3} [2f(x) - g(x)] &= 2 \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x) && \text{[by Limit Laws 2 and 3]} \\ &= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3] \\ &= 2 \cdot 5 - g(3) = 10 - g(3) \end{aligned}$$

Since it is given that $\lim\limits_{x \to 3} [2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

10. $\lim\limits_{x \to 4} f(x) = \lim\limits_{x \to 4} \left(x^2 + \sqrt{7 - x} \right) = \lim\limits_{x \to 4} x^2 + \sqrt{\lim\limits_{x \to 4} 7 - \lim\limits_{x \to 4} x} = 4^2 + \sqrt{7 - 4} = 16 + \sqrt{3} = f(4)$.

By the definition of continuity, f is continuous at $a = 4$.

11. $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \left(x + 2x^3 \right)^4 = \left(\lim\limits_{x \to -1} x + 2 \lim\limits_{x \to -1} x^3 \right)^4 = \left[-1 + 2(-1)^3 \right]^4 = (-3)^4 = 81 = f(-1)$.

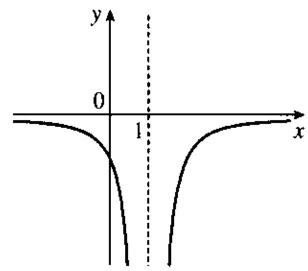
By the definition of continuity, f is continuous at $a = -1$.

12. $\lim\limits_{x \to 4} g(x) = \lim\limits_{x \to 4} \dfrac{x + 1}{2x^2 - 1} = \dfrac{\lim\limits_{x \to 4} x + \lim\limits_{x \to 4} 1}{2 \lim\limits_{x \to 4} x^2 - \lim\limits_{x \to 4} 1} = \dfrac{4 + 1}{2(4)^2 - 1} = \dfrac{5}{31} = g(4)$. So g is continuous at 4.

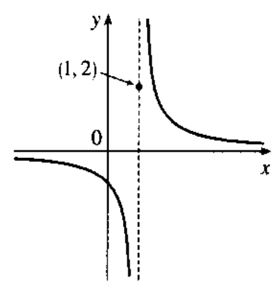
13. For $a > 2$, we have $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \dfrac{2x+3}{x-2} = \dfrac{\lim\limits_{x\to a}(2x+3)}{\lim\limits_{x\to a}(x-2)}$ [Limit Law 5] $= \dfrac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2}$ [1, 2, and 3] $= \dfrac{2a+3}{a-2}$ [7 and 8] $= f(a)$. Thus, f is continuous at $x = a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

14. For $a < 3$, we have $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} 2\sqrt{3-x} = 2\lim\limits_{x\to a}\sqrt{3-x}$ [Limit Law 3] $= 2\sqrt{\lim\limits_{x\to a}(3-x)}$ [11] $= 2\sqrt{\lim\limits_{x\to a} 3 - \lim\limits_{x\to a} x}$ [2] $= 2\sqrt{3-a}$ [7 and 8] $= g(a)$, so g is continuous at $x = a$ for every a in $(-\infty, 3)$. Also, $\lim\limits_{x\to 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3. Thus, g is continuous on $(-\infty, 3]$.

15. $f(x) = -\dfrac{1}{(x-1)^2}$ is discontinuous at 1 since $f(1)$ is not defined.

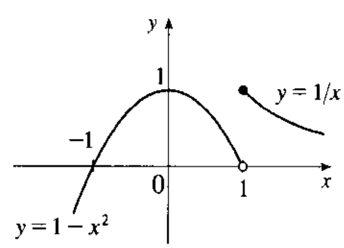


16. $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ is discontinuous at 1 because $\lim\limits_{x\to 1} f(x)$ does not exist.



17. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$

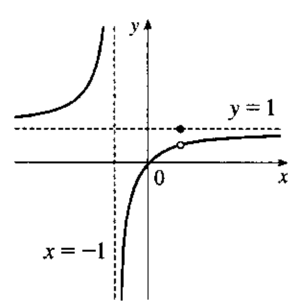
The left-hand limit of f at $a = 1$ is $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(1-x^2) = 0$. The right-hand limit of f at $a = 1$ is $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(1/x) = 1$. Since these limits are not equal, $\lim\limits_{x\to 1} f(x)$ does not exist and f is discontinuous at 1.



18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-x}{x^2-1} = \lim_{x\to 1}\frac{x(x-1)}{(x+1)(x-1)}$$
$$= \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2},$$

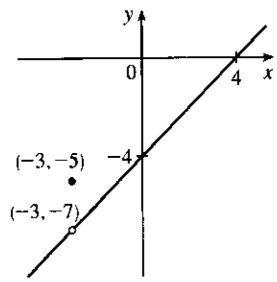
but $f(1) = 1$, so f is discontinous at 1.



19. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x+3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} = \begin{cases} x - 4 & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$

So $\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} (x - 4) = -7$ and $f(-3) = -5$.

Since $\lim\limits_{x \to -3} f(x) \neq f(-3)$, f is discontinuous at -3.

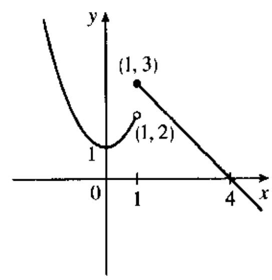


20. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (1 + x^2) = 1 + 1^2 = 2$ and

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (4 - x) = 4 - 1 = 3$.

Thus, f is discontinuous at 1 because $\lim\limits_{x \to 1} f(x)$ does not exist.



21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x+3)(x+2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}\,(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x-1}$ is continuous on $[\frac{1}{2}, \infty)$.

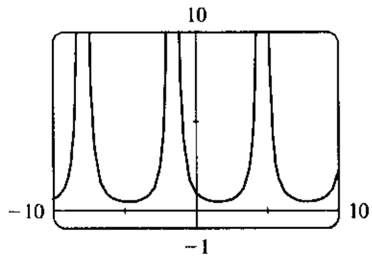
24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x+1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

25. By Theorem 5, the polynomial $1 - x^2$ is continuous on $(-\infty, \infty)$. By Theorem 7, cos is continuous on its domain, $\mathbb{R}$. By Theorem 9, $\cos(1 - x^2)$ is continuous on its domain, which is $\mathbb{R}$.

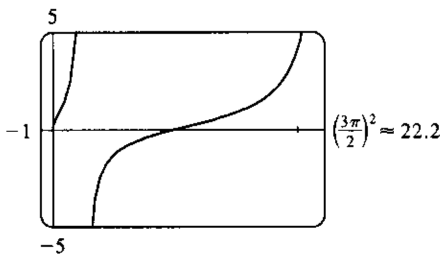
26. By Theorem 5, the polynomial $2x$ is continuous on $(-\infty, \infty)$. By Theorem 7, tan is continuous at every number in its domain; that is, $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. By Theorem 9, $\tan 2x$ is continuous on its domain, which is $\{x \mid 2x \neq \frac{\pi}{2} + \pi n\} = \{x \mid x \neq \frac{\pi}{4} + \frac{\pi}{2} n\}$ (the odd multiples of $\frac{\pi}{4}$).

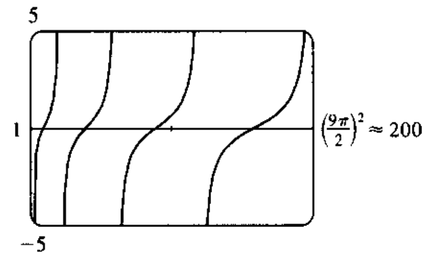
27. By Theorem 7, the root function $\sqrt{x}$ and the trigonometric function $\sin x$ are continuous on their domains, $[0, \infty)$ and $(-\infty, \infty)$, respectively. Thus, the product $F(x) = \sqrt{x} \sin x$ is continuous on the intersection of those domains, $[0, \infty)$, by part 4 of Theorem 4.

28. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

29. 

$y = \dfrac{1}{1+\sin x}$ is undefined and hence discontinuous when $1+\sin x = 0 \;\Leftrightarrow\; \sin x = -1 \;\Leftrightarrow\; x = -\frac{\pi}{2} + 2\pi n$, n an integer. The figure shows discontinuities for $n = -1$, 0, and 1; that is, $-\dfrac{5\pi}{2} \approx -7.85$, $-\dfrac{\pi}{2} \approx -1.57$, and $\dfrac{3\pi}{2} \approx 4.71$.

30. 



The function $y = f(x) = \tan\sqrt{x}$ is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain $[0,\infty)$ and the tangent function has domain $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. So f is discontinuous when $x < 0$ and when $\sqrt{x} = \frac{\pi}{2} + \pi n \;\Rightarrow\; x = \left(\frac{\pi}{2} + \pi n\right)^2$, where n is a nonnegative integer. Note that as x increases, the distance between discontinuities increases.

31. Because we are dealing with root functions, $5+\sqrt{x}$ is continuous on $[0,\infty)$, $\sqrt{x+5}$ is continuous on $[-5,\infty)$, so the quotient $f(x) = \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0,\infty)$. Since f is continuous at $x=4$, $\lim\limits_{x\to 4} f(x) = f(4) = \frac{7}{3}$.

32. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x+\sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

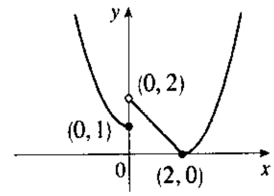
33. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty,1)$, f is continuous on $(-\infty,1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1,\infty)$, f is continuous on $(1,\infty)$. At $x=1$, $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} x^2 = 1$ and $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x\to1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x=1$. We conclude that f is continuous on $(-\infty,\infty)$.

34. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

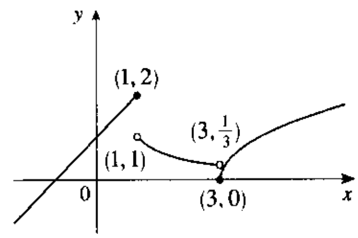
By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty,\pi/4)$ and $f(x) = \cos x$ on $(\pi/4,\infty)$, f is continuous on $(-\infty,\pi/4)\cup(\pi/4,\infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty,\infty)$.

35. $f(x) = \begin{cases} 1+x^2 & \text{if } x \le 0 \\ 2-x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$



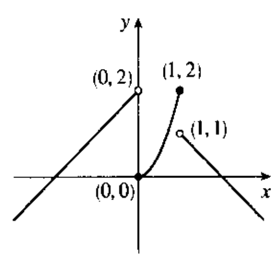
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(1+x^2) = 1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(2-x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(2-x) = 0$, $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

36. $f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \ge 3 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x+1) = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(1/x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1. Also, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(1/x) = 1/3$, and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}\sqrt{x-3} = 0 = f(3)$, so f is discontinuous at 3, but it is continuous from the right at 3.

37. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ since on each of these intervals it is a polynomial. Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(x+2) = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x^2 = 0$, so f is discontinuous at 0. Since $f(0) = 0$, f is continuous from the right at 0. Also $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x^2 = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2-x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1.

38. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$. $\lim\limits_{r\to R^-} F(r) = \lim\limits_{r\to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r\to R^+} F(r) = \lim\limits_{r\to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r\to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

39. f is continuous on $(-\infty, 3)$ and $(3, \infty)$. Now $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(cx+1) = 3c+1$ and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(cx^2-1) = 9c-1$. So f is continuous $\Leftrightarrow$ $3c+1 = 9c-1$ $\Leftrightarrow$ $6c = 2$ $\Leftrightarrow$ $c = \frac{1}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{1}{3}$.

40. The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c. So the only possible discontinuity is at $x = 4$. For the function to be continuous at $x = 4$, the left-hand and right-hand limits must be the same. Now $\lim\limits_{x\to 4^-} g(x) = \lim\limits_{x\to 4^-} (x^2 - c^2) = 16 - c^2$ and $\lim\limits_{x\to 4^+} g(x) = \lim\limits_{x\to 4^+} (cx + 20) = 4c + 20 = g(4)$. Thus, $16 - c^2 = 4c + 20 \quad\Leftrightarrow\quad c^2 + 4c + 4 = 0 \quad\Leftrightarrow$ $c = -2$.

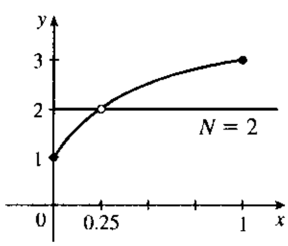
41. (a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2} = \dfrac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x) = x - 4$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -2$. [The discontinuity is removed by defining $f(-2) = -6$.]

(b) $f(x) = \dfrac{x - 7}{|x - 7|} \quad\Rightarrow\quad \lim\limits_{x\to 7^-} f(x) = -1$ and $\lim\limits_{x\to 7^+} f(x) = 1$. Thus, $\lim\limits_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

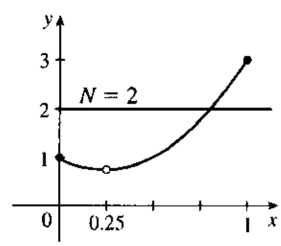
(c) $f(x) = \dfrac{x^3 + 64}{x + 4} = \dfrac{(x+4)(x^2 - 4x + 16)}{x + 4}$ has a removable discontinuity at -4 because $g(x) = x^2 - 4x + 16$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x} = \dfrac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = \dfrac{1}{3 + \sqrt{x}}$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

42.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

43. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.

44. $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$, and $f(2) = 4$. Since $1 < 2 < 4$, there is a number c in $(1, 2)$ such that $f(c) = c^2 = 2$ by the Intermediate Value Theorem.

45. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

46. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

47. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

48. $f(x) = \tan x - 2x$ is continuous on the interval $[0, 1.4]$, $f(1) = \tan 1 - 2 \approx -0.44$, and $f(1.4) = \tan 1.4 - 2.8 \approx 3.00$. Since $-0.44 < 0 < 3.00$, there is a number c in $(0, 1.4)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\tan x - 2x = 0$, or $\tan x = 2x$, in the interval $(0, 1.4)$.

49. (a) $f(x) = \sin x - 2 + x$ is continuous on $[0, 2]$, $f(0) = -2$, and $f(2) = \sin 2 \approx 0.91$. Since $-2 < 0 < 0.91$, there is a number c in $(0, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - 2 + x = 0$, or $\sin x = 2 - x$, in the interval $(0, 2)$.

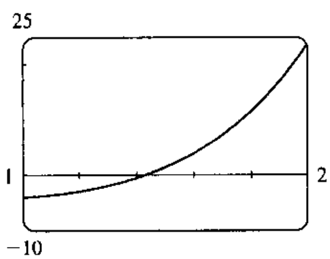
(b) $f(1.10) \approx -0.009 < 0$ and $f(1.11) \approx 0.006 > 0$, so there is a root between 1.10 and 1.11.

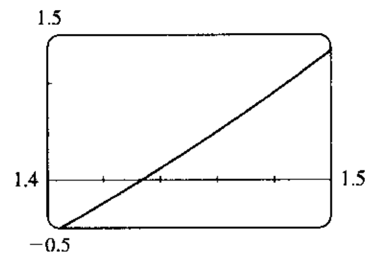
50. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between -0.88 and -0.87.

51. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.

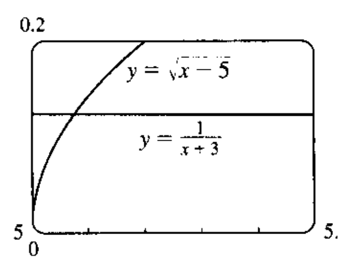
(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.





52. (a) Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c+3} = \sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



53. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \to 0} f(a+h) = f\left(\lim_{h \to 0}(a+h)\right) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim\limits_{h \to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \quad \Rightarrow$

$|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x - a| < \delta$, then $|f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon$.

Thus, $\lim\limits_{x \to a} f(x) = f(a)$ and so f is continuous at a.

54. $\lim\limits_{h\to 0}\sin(a+h) = \lim\limits_{h\to 0}(\sin a\cos h + \cos a\sin h) = \lim\limits_{h\to 0}(\sin a\cos h) + \lim\limits_{h\to 0}(\cos a\sin h)$

$$= \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right) + \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right)$$

$$= (\sin a)(1) + (\cos a)(0) = \sin a$$

55. As in the previous exercise, we must show that $\lim\limits_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h - \sin a\sin h)\\ &= \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)\\ &= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)\\ &= (\cos a)(1) - (\sin a)(0) = \cos a\end{aligned}$$

56. (a) Since f is continuous at a, $\lim\limits_{x\to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have $\lim\limits_{x\to a}(cf)(x) = \lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim\limits_{x\to a} f(x) = f(a)$ and $\lim\limits_{x\to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law of Limits: $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x) = \lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

57. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim\limits_{x\to a} f(x) \neq f(a)$. [In fact, $\lim\limits_{x\to a} f(x)$ does not even exist.]

58. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0. To see why, note that $-|x| \le g(x) \le |x|$, so by the Squeeze Theorem $\lim\limits_{x\to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim\limits_{x\to a} g(x) \neq g(a)$.

59. If there is such a number, it satisfies the equation $x^3 + 1 = x \;\Leftrightarrow\; x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

60. (a) $\lim_{x \to 0^+} F(x) = 0$ and $\lim_{x \to 0^-} F(x) = 0$, so $\lim_{x \to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} x = a = F(a)$. For $a < 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} (-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

61. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow \quad u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

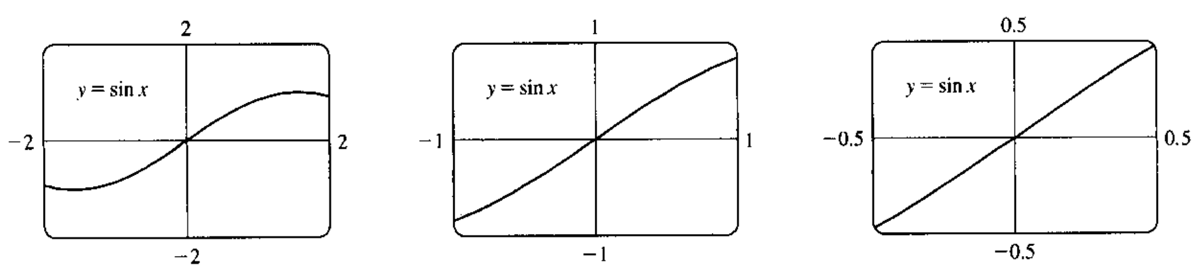
(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. (a) Average velocity $= \dfrac{\Delta s}{\Delta t} = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$

(b) Instantaneous velocity $= \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E. The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A, and B.

4. The curve looks more like a line as the viewing rectangle gets smaller.



5. (a) (i) Using Definition 1,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)} = \lim_{x \to -3} \frac{(x^2 + 2x) - (3)}{x - (-3)} = \lim_{x \to -3} \frac{(x+3)(x-1)}{x+3}$$

$$= \lim_{x \to -3} (x - 1) = -4$$

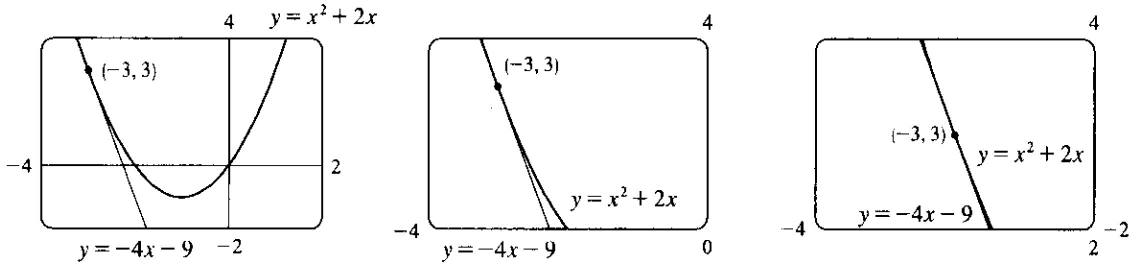
(ii) Using Equation 2,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \frac{\left[(-3+h)^2 + 2(-3+h)\right] - (3)}{h}$$

$$= \lim_{h \to 0} \frac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4$$

(b) Using the point-slope form of the equation of a line, an equation of the tangent line is $y - 3 = -4(x + 3)$. Solving for y gives us $y = -4x - 9$, which is the slope-intercept form of the equation of the tangent line.

(c)



6. (a) (i) $m = \lim_{x \to -1} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \dfrac{x^3 - (-1)}{x + 1} = \lim_{x \to -1} \dfrac{(x+1)(x^2 - x + 1)}{x + 1}$

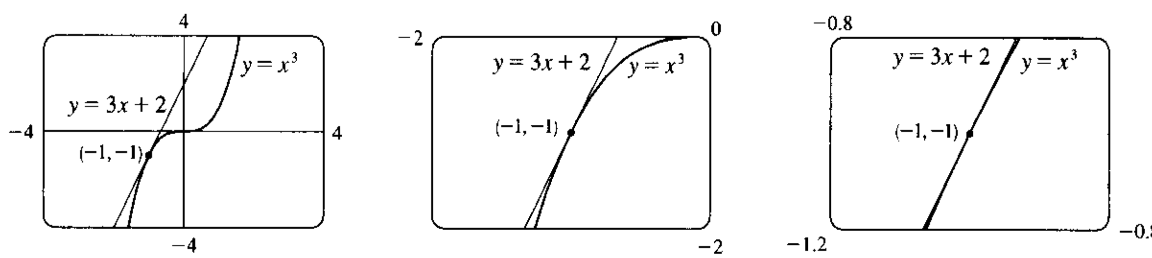
$$= \lim_{x \to -1} (x^2 - x + 1) = 3$$

(ii) $m = \lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \dfrac{(-1+h)^3 - (-1)}{h} = \lim_{h \to 0} \dfrac{h^3 - 3h^2 + 3h - 1 + 1}{h}$

$$= \lim_{h \to 0} (h^2 - 3h + 3) = 3$$

(b) $y - (-1) = 3\,[x - (-1)] \quad \Leftrightarrow \quad y + 1 = 3x + 3 \quad \Leftrightarrow \quad y = 3x + 2$

(c)



7. Using (2) with $f(x) = 1 + 2x - x^3$ and $P(1, 2)$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[1 + 2(1+h) - (1+h)^3\right] - 2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2 + 2h - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(-h^2 - 3h - 1)}{h} = \lim_{h \to 0} (-h^2 - 3h - 1) = -1$$

Tangent line: $y - 2 = -1(x - 1) \quad \Leftrightarrow \quad y - 2 = -x + 1 \quad \Leftrightarrow \quad y = -x + 3$

8. Using (1),

$$m = \lim_{x\to 4} \frac{\sqrt{2x+1}-\sqrt{2(4)+1}}{x-4} = \lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}$$

$$= \lim_{x\to 4} \frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)}$$

$$= \lim_{x\to 4} \frac{2}{(\sqrt{2x+1}+3)} = \frac{2}{3+3} = \frac{1}{3}.$$

Tangent line: $y-3=\frac{1}{3}(x-4) \quad \Leftrightarrow \quad y-3=\frac{1}{3}x-\frac{4}{3} \quad \Leftrightarrow \quad y=\frac{1}{3}x+\frac{5}{3}$

9. Using (1) with $f(x)=\dfrac{x-1}{x-2}$ and $P(3,2)$,

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 3} \frac{\frac{x-1}{x-2}-2}{x-3} = \lim_{x\to 3} \frac{\frac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x\to 3} \frac{3-x}{(x-2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-1}{x-2} = \frac{-1}{1} = -1.$$

Tangent line: $y-2=-1(x-3) \quad \Leftrightarrow \quad y-2=-x+3 \quad \Leftrightarrow \quad y=-x+5$

10. Using (1), $m = \lim\limits_{x\to 0} \dfrac{\frac{2x}{(x+1)^2}-0}{x-0} = \lim\limits_{x\to 0} \dfrac{2x}{x(x+1)^2} = \lim\limits_{x\to 0} \dfrac{2}{(x+1)^2} = \dfrac{2}{1^2} = 2.$

Tangent line: $y-0=2(x-0) \quad \Leftrightarrow \quad y=2x$

11. (a) $m = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{2/(x+3)-2/(a+3)}{x-a} = \lim\limits_{x\to a} \dfrac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$

$$= \lim_{x\to a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$$

(b) (i) $a=-1 \quad \Rightarrow \quad m=\dfrac{-2}{(-1+3)^2}=-\dfrac{1}{2}$ (ii) $a=0 \quad \Rightarrow \quad m=\dfrac{-2}{(0+3)^2}=-\dfrac{2}{9}$

(iii) $a=1 \quad \Rightarrow \quad m=\dfrac{-2}{(1+3)^2}=-\dfrac{1}{8}$

12. (a) Using (1),

$$m = \lim_{x\to a} \frac{(1+x+x^2)-(1+a+a^2)}{x-a} = \lim_{x\to a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x\to a} \frac{x-a+(x-a)(x+a)}{x-a}$$

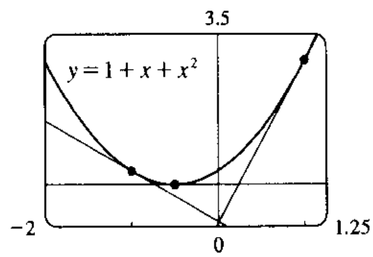
$$= \lim_{x\to a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x\to a} (1+x+a) = 1+2a$$

(b) (i) $x=-1 \quad \Rightarrow \quad m=1+2(-1)=-1$

(ii) $x=-\frac{1}{2} \quad \Rightarrow \quad m=1+2(-\frac{1}{2})=0$

(iii) $x=1 \quad \Rightarrow \quad m=1+2(1)=3$

(c)

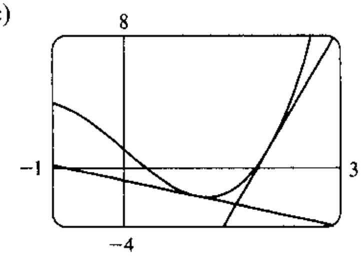


13. (a) Using (1),

$$m=\lim_{x\to a}\frac{(x^3-4x+1)-(a^3-4a+1)}{x-a}=\lim_{x\to a}\frac{(x^3-a^3)-4(x-a)}{x-a}$$
$$=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)-4(x-a)}{x-a}=\lim_{x\to a}(x^2+ax+a^2-4)=3a^2-4$$

(b) At $(1,-2)$: $m=3(1)^2-4=-1$, so an equation of the tangent line is $y-(-2)=-1(x-1)\quad\Leftrightarrow\quad y=-x-1$. At $(2,1)$: $m=3(2)^2-4=8$, so an equation of the tangent line is $y-1=8(x-2)\quad\Leftrightarrow\quad y=8x-15$.

(c)



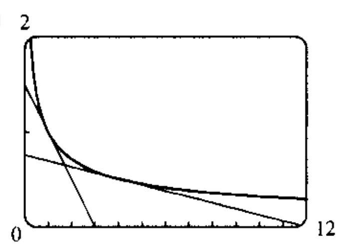
14. (a) Using (1),

$$m=\lim_{x\to a}\frac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{a}}}{x-a}=\lim_{x\to a}\frac{\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a}=\lim_{x\to a}\frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{\sqrt{ax}\,(x-a)\,(\sqrt{a}+\sqrt{x})}$$
$$=\lim_{x\to a}\frac{a-x}{\sqrt{ax}\,(x-a)\,(\sqrt{a}+\sqrt{x})}=\lim_{x\to a}\frac{-1}{\sqrt{ax}\,(\sqrt{a}+\sqrt{x})}=\frac{-1}{\sqrt{a^2}\,(2\sqrt{a})}=-\frac{1}{2a^{3/2}}\text{ or }-\frac{1}{2}a^{-3/2}$$

(b) At $(1,1)$: $m=-\frac{1}{2}$, so an equation of the tangent line is $y-1=-\frac{1}{2}(x-1)\quad\Leftrightarrow\quad y=-\frac{1}{2}x+\frac{3}{2}$.

At $\left(4,\frac{1}{2}\right)$: $m=-\frac{1}{16}$, so an equation of the tangent line is $y-\frac{1}{2}=-\frac{1}{16}(x-4)\quad\Leftrightarrow\quad y=-\frac{1}{16}x+\frac{3}{4}$.

(c)



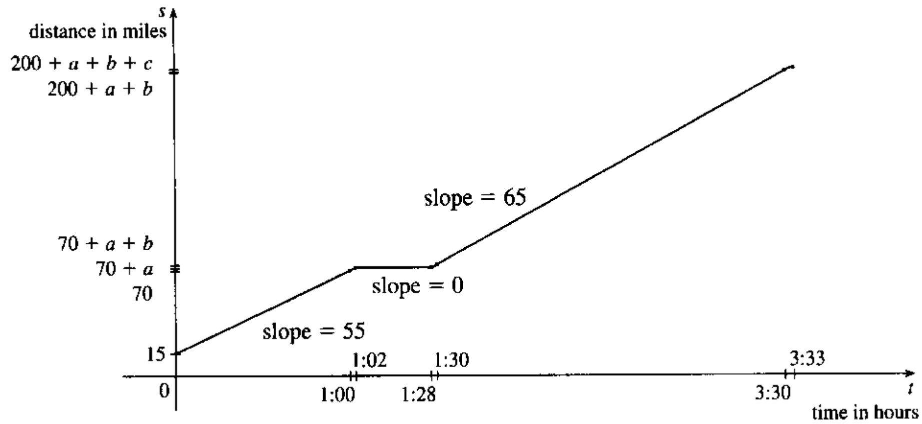
15. (a) Since the slope of the tangent at $t=0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B, so the car was going faster at C.

(c) Near A, the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B, the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C, so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E, the slope of the tangent is 0, so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t = 0$ at the beginning of the trip.



17. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t) - s(2)}{t-2} = \lim_{t\to 2} \frac{(40t - 16t^2) - 16}{t-2} = \lim_{t\to 2} \frac{-16t^2 + 40t - 16}{t-2} = \lim_{t\to 2} \frac{-8(2t^2 - 5t + 2)}{t-2}$$

$$= \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8 \lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

18. (a) $v(1) = \lim\limits_{h\to 0} \dfrac{H(1+h) - H(1)}{h}$

$$= \lim_{h\to 0} \frac{(58 + 58h - 0.83 - 1.66h - 0.83h^2) - 57.17}{h} = \lim_{h\to 0}(56.34 - 0.83h) = 56.34 \text{ m/s}$$

(b) $v(a) = \lim\limits_{h\to 0} \dfrac{H(a+h) - H(a)}{h}$

$$= \lim_{h\to 0} \frac{(58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2) - (58a - 0.83a^2)}{h}$$

$$= \lim_{h\to 0} (58 - 1.66a - 0.83h) = 58 - 1.66a \text{ m/s}$$

(c) The arrow strikes the moon when the height is 0, that is, $58t - 0.83t^2 = 0 \quad\Leftrightarrow\quad t(58 - 0.83t) = 0 \quad\Leftrightarrow$ $t = \dfrac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\dfrac{58}{0.83}\right) = 58 - 1.66\left(\dfrac{58}{0.83}\right) = -58$ m/s. Thus, the arrow will have a velocity of -58 m/s.

19. $v(a) = \lim\limits_{h\to 0} \dfrac{s(a+h) - s(a)}{h} = \lim\limits_{h\to 0} \dfrac{4(a+h)^3 + 6(a+h) + 2 - (4a^3 + 6a + 2)}{h}$

$$= \lim_{h\to 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h}$$

$$= \lim_{h\to 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h\to 0} (12a^2 + 12ah + 4h^2 + 6) = (12a^2 + 6) \text{ m/s}$$

So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

20. (a) The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t} = \frac{(t+h)^2 - 8(t+h) + 18 - (t^2 - 8t + 18)}{h}$$

$$= \frac{t^2 + 2th + h^2 - 8t - 8h + 18 - t^2 + 8t - 18}{h} = \frac{2th + h^2 - 8h}{h}$$

$$= (2t + h - 8) \ \ \text{m/s}$$

(i) $[3,4]$: $t=3$, $h=4-3=1$, so the average velocity is $2(3)+1-8=-1$ m/s.

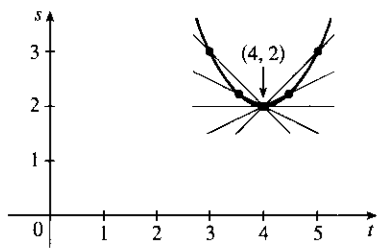
(ii) $[3.5,4]$: $t=3.5$, $h=0.5$, so the average velocity is $2(3.5)+0.5-8=-0.5$ m/s.

(iii) $[4,5]$: $t=4$, $h=1$, so the average velocity is $2(4)+1-8=1$ m/s.

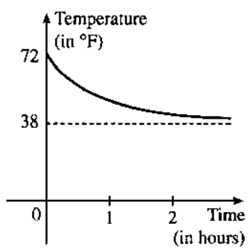
(iv) $[4,4.5]$: $t=4$, $h=0.5$, so the average velocity is $2(4)+0.5-8=0.5$ m/s.

(b) $v(t) = \lim\limits_{h\to 0} \dfrac{s(t+h)-s(t)}{h} = \lim\limits_{h\to 0}(2t+h-8) = 2t-8$, so $v(4)=0$.

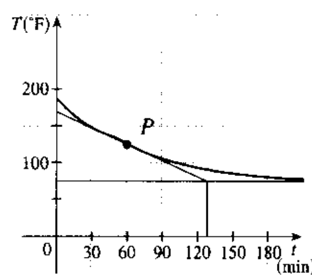
(c)



21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,°\text{F/min}$.

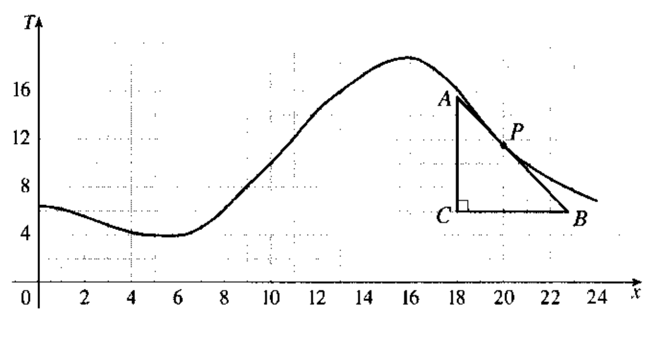


23. (a) (i) $[20,23]$: $\dfrac{7.9-11.5}{23-20} = -1.2\,°\text{C/h}$

(ii) $[20,22]$: $\dfrac{9.0-11.5}{22-20} = -1.25\,°\text{C/h}$

(iii) $[20,21]$: $\dfrac{10.2-11.5}{21-20} = -1.3\,°\text{C/h}$

(b) In the figure, we estimate A to be $(18, 15.5)$ and B as $(23, 6)$. So the slope is $\dfrac{6-15.5}{23-18} = -1.9\,°\text{C/h}$ at 8:00 P.M.



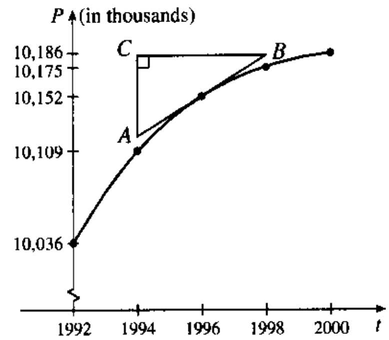
24. (a) (i) $[1992, 1996]$: $\dfrac{P(1996) - P(1992)}{1996 - 1992} = \dfrac{10{,}152 - 10{,}036}{4} = \dfrac{116}{4} = 29$ thousand/year

(ii) $[1994, 1996]$: $\dfrac{P(1996) - P(1994)}{1996 - 1994} = \dfrac{10{,}152 - 10{,}109}{2} = \dfrac{43}{2} = 21.5$ thousand/year

(iii) $[1996, 1998]$: $\dfrac{P(1998) - P(1996)}{1998 - 1996} = \dfrac{10{,}175 - 10{,}152}{2} = \dfrac{23}{2} = 11.5$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{21.5 + 11.5}{2} = 16.5$ thousand/year.

(c) Estimating A as $(1994, 10{,}125)$ and B as $(1998, 10{,}182)$, the slope at 1996 is $\dfrac{10{,}182 - 10{,}125}{1998 - 1994} = \dfrac{57}{4} = 14.25$ thousand/year.



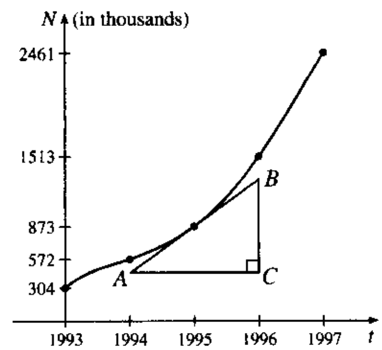
25. (a) (i) $[1995, 1997]$: $\dfrac{N(1997) - N(1995)}{1997 - 1995} = \dfrac{2461 - 873}{2} = \dfrac{1588}{2} = 794$ thousand/year

(ii) $[1995, 1996]$: $\dfrac{N(1996) - N(1995)}{1996 - 1995} = \dfrac{1513 - 873}{1} = 640$ thousand/year

(iii) $[1994, 1995]$: $\dfrac{N(1995) - N(1994)}{1995 - 1994} = \dfrac{873 - 572}{1} = 301$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{640 + 301}{2} = \dfrac{941}{2} = 470.5$ thousand/year.

(c) Estimating A as $(1994, 420)$ and B as $(1996, 1275)$, the slope at 1995 is $\dfrac{1275 - 420}{1996 - 1994} = \dfrac{855}{2} = 427.5$ thousand/year



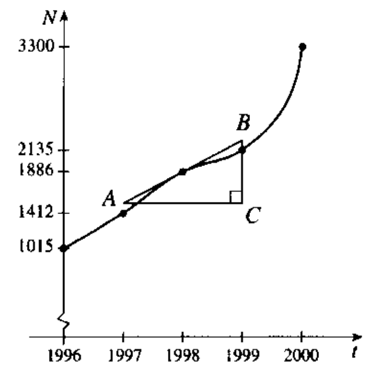
26. (a) (i) $[1996, 1998]$: $\dfrac{N(1998) - N(1996)}{1998 - 1996} = \dfrac{1886 - 1015}{2} = \dfrac{871}{2} = 435.5$ locations/year

(ii) $[1997, 1998]$: $\dfrac{N(1998) - N(1997)}{1998 - 1997} = \dfrac{1886 - 1412}{1} = 474$ locations/year

(iii) $[1998, 1999]$: $\dfrac{N(1999) - N(1998)}{1999 - 1998} = \dfrac{2135 - 1886}{1} = 249$ locations/year

(b) Using the values from (ii) and (iii), we have $\dfrac{474 + 249}{2} = \dfrac{723}{2} = 361.5 \approx 362$ locations/year.

(c) Estimating A as $(1997, 1525)$ and B as $(1999, 2250)$, the slope at 1998 is $\dfrac{2250-1525}{1999-1997}=\dfrac{725}{2}=362.5$ locations/year.



27. (a) (i) $\dfrac{\Delta C}{\Delta x}=\dfrac{C(105)-C(100)}{105-100}=\dfrac{6601.25-6500}{5}=\$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x}=\dfrac{C(101)-C(100)}{101-100}=\dfrac{6520.05-6500}{1}=\$20.05/\text{unit}.$

(b) $$\frac{C(100+h)-C(100)}{h}=\frac{\left[5000+10(100+h)+0.05(100+h)^2\right]-6500}{h}=\frac{20h+0.05h^2}{h}$$
$$=20+0.05h,\ h\neq 0$$

So the instantaneous rate of change is $\displaystyle\lim_{h\to 0}\frac{C(100+h)-C(100)}{h}=\lim_{h\to 0}(20+0.05h)=\$20/\text{unit}.$

28. $$\Delta V=V(t+h)-V(t)=100{,}000\left(1-\frac{t+h}{60}\right)^2-100{,}000\left(1-\frac{t}{60}\right)^2$$
$$=100{,}000\left[\left(1-\frac{t+h}{30}+\frac{(t+h)^2}{3600}\right)-\left(1-\frac{t}{30}+\frac{t^2}{3600}\right)\right]=100{,}000\left(-\frac{h}{30}+\frac{2th}{3600}+\frac{h^2}{3600}\right)$$
$$=\frac{100{,}000}{3600}h\,(-120+2t+h)=\frac{250}{9}h\,(-120+2t+h)$$

Dividing ΔV by h and then letting $h\to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t-60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	100,000
10	$-2777.\overline{7}$	$69{,}444.\overline{4}$
20	$-2222.\overline{2}$	$44{,}444.\overline{4}$
30	$-1666.\overline{6}$	25,000
40	$-1111.\overline{1}$	$11{,}111.\overline{1}$
50	$-555.\overline{5}$	$2{,}777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

2 Review

CONCEPT CHECK

1. (a) $\lim_{x \to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

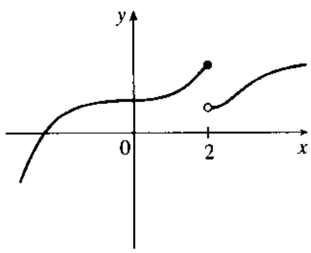
(b) $\lim_{x \to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim_{x \to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

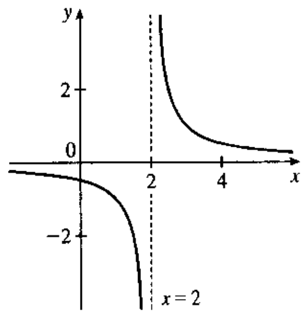
(d) $\lim_{x \to a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.

(e) $\lim_{x \to a} f(x) = -\infty$: See Definition 2.2.5 and Figure 13 in Section 2.2.

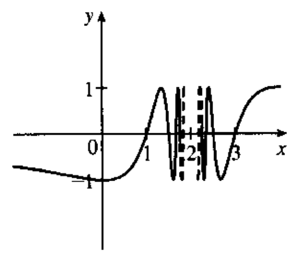
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

3. See Definition 2.2.6 and Figure 14 in Section 2.2.

4. (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

5. See Theorem 3 in Section 2.3.

6. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

7. See Theorem 2.5.10.

8. See Definition 2.6.1.

9. See the paragraph containing Formula 3 in Section 2.6.

10. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

5. False. Consider $\lim\limits_{x\to 5}\dfrac{x(x-5)}{x-5}$ or $\lim\limits_{x\to 5}\dfrac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim\limits_{x\to 6}[f(x)g(x)] = \lim\limits_{x\to 6}\left[(x-6)\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6)=0$ and $g(6)$ does not exist, so $f(6)g(6)\neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim\limits_{x\to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim\limits_{x\to 0}[f(x)-g(x)] = \lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. False. Consider $f(x)=\begin{cases} 1/(x-1) & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$

10. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x)=\begin{cases} 1 & \text{if } 0\le x<3 \\ -1 & \text{if } x=3 \end{cases}$ There is no number $c\in[0,3]$ with $f(c)=0$.

11. True. Use Theorem 2.5.8 with $a=2$, $b=5$, and $g(x)=4x^2-11$. Note that $f(4)=3$ is not needed.

12. True. Use the Intermediate Value Theorem with $a=-1$, $b=1$, and $N=\pi$, since $3<\pi<4$.

13. True, by the definition of a limit with $\varepsilon=1$.

14. False. For example, let $f(x)=\begin{cases} x^2+1 & \text{if } x\neq 0 \\ 2 & \text{if } x=0 \end{cases}$

Then $f(x)>1$ for all x, but $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(x^2+1) = 1$.

EXERCISES

1. (a) (i) $\lim\limits_{x\to 2^+} f(x)=3$

(ii) $\lim\limits_{x\to -3^+} f(x)=0$

(iii) $\lim\limits_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim\limits_{x\to 4} f(x)=2$

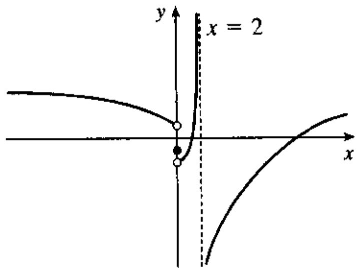
(v) $\lim\limits_{x\to 0} f(x)=\infty$

(vi) $\lim\limits_{x\to 2^-} f(x)=-\infty$

(b) The equations of the vertical asymptotes are $x=0$ and $x=2$.

(c) f is discontinuous at $x=-3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2.



3. $\lim_{x\to 0}\cos(x+\sin x) = \cos\left[\lim_{x\to 0}(x+\sin x)\right]$ [by Theorem 2.5.8] $=\cos 0 = 1$

4. Since rational functions are continuous, $\lim_{x\to 3}\dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0$.

5. $\lim_{x\to -3}\dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to -3}\dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3}\dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+}\dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3\to 0$ as $x\to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1<x<3$.

7. $\lim_{h\to 0}\dfrac{(h-1)^3+1}{h} = \lim_{h\to 0}\dfrac{(h^3-3h^2+3h-1)+1}{h} = \lim_{h\to 0}\dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0}\left(h^2-3h+3\right) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\lim_{h\to 0}\frac{(h-1)^3+1}{h} = \lim_{h\to 0}\frac{(h-1)^3+1^3}{h} = \lim_{h\to 0}\frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}$$
$$= \lim_{h\to 0}\left[(h-1)^2-h+2\right] = 1-0+2 = 3$$

8. $\lim_{t\to 2}\dfrac{t^2-4}{t^3-8} = \lim_{t\to 2}\dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2}\dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9}\dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4\to 0$ as $r\to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r\neq 9$.

10. $\lim_{v\to 4^+}\dfrac{4-v}{|4-v|} = \lim_{v\to 4^+}\dfrac{4-v}{-(4-v)} = \lim_{v\to 4^+}\dfrac{1}{-1} = -1$

11. $\lim_{s\to 16}\dfrac{4-\sqrt{s}}{s-16} = \lim_{s\to 16}\dfrac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)} = \lim_{s\to 16}\dfrac{-1}{\sqrt{s}+4} = \dfrac{-1}{\sqrt{16}+4} = -\dfrac{1}{8}$

12. $\lim_{v\to 2}\dfrac{v^2+2v-8}{v^4-16} = \lim_{v\to 2}\dfrac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)} = \lim_{v\to 2}\dfrac{v+4}{(v+2)(v^2+4)} = \dfrac{2+4}{(2+2)(2^2+4)} = \dfrac{3}{16}$

13. $\dfrac{|x-8|}{x-8} = \begin{cases}\dfrac{x-8}{x-8} & \text{if } x-8>0\\[2ex] \dfrac{-(x-8)}{x-8} & \text{if } x-8<0\end{cases} = \begin{cases}1 & \text{if } x>8\\ -1 & \text{if } x<8\end{cases}$

Thus, $\lim_{x\to 8^-}\dfrac{|x-8|}{x-8} = \lim_{x\to 8^-}(-1) = -1$.

14. $\lim_{x\to 9^+}\left(\sqrt{x-9}+[\![x+1]\!]\right) = \lim_{x\to 9^+}\sqrt{x-9}+\lim_{x\to 9^+}[\![x+1]\!] = \sqrt{9-9}+10 = 10$

15. $\lim_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{x}\cdot\dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x\to 0}\dfrac{1-\left(1-x^2\right)}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0}\dfrac{x^2}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0}\dfrac{x}{1+\sqrt{1-x^2}} = 0$

16. $\lim_{x\to 2}\dfrac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)}\cdot\dfrac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2}\dfrac{-(x-2)}{x(x-2)\left(\sqrt{x+2}+\sqrt{2x}\right)} = \lim_{x\to 2}\dfrac{-1}{x\left(\sqrt{x+2}+\sqrt{2x}\right)} = -\dfrac{1}{8}$

17. Since $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim\limits_{x\to 1}(2x-1) = 1 = \lim\limits_{x\to 1} x^2$, we have $\lim\limits_{x\to 1} f(x) = 1$ by the Squeeze Theorem.

18. Let $f(x) = -x^2$, $g(x) = x^2\cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have $f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0 \Rightarrow \lim\limits_{x\to 0} g(x) = 0$ by the Squeeze Theorem.

19. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-5| < \delta$, then $|(7x-27) - 8| < \varepsilon \;\Leftrightarrow\; |7x - 35| < \varepsilon \;\Leftrightarrow\; |x-5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0 < |x-5| < \delta \;\Rightarrow\; |(7x-27) - 8| < \varepsilon$. Thus, $\lim\limits_{x\to 5}(7x-27) = 8$ by the definition of a limit.

20. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \;\Rightarrow\; |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x - 0| = |x| < \varepsilon^3 \;\Rightarrow\; |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim\limits_{x\to 0}\sqrt[3]{x} = 0$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-2| < \delta$, then $\left|x^2 - 3x - (-2)\right| < \varepsilon$. First, note that if $|x-2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \;\Rightarrow\; |x-1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x-2| < \delta$ $\Rightarrow\; \left|x^2 - 3x - (-2)\right| = |(x-2)(x-1)| = |x-2|\,|x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim\limits_{x\to 2}\left(x^2 - 3x\right) = -2$ by the definition of a limit.

22. Given $M > 0$, we need $\delta > 0$ such that if $0 < x - 4 < \delta$, then $2/\sqrt{x-4} > M$. This is true $\Leftrightarrow$ $\sqrt{x-4} < 2/M \;\Leftrightarrow\; x - 4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x - 4 < \delta \;\Rightarrow\; 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim\limits_{x\to 4^+}\left(2/\sqrt{x-4}\right) = \infty$.

23. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \le x < 3$, $f(x) = (x-3)^2$ if $x > 3$.

(i) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(3 - x) = 3$

(ii) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim\limits_{x\to 0} f(x)$ does not exist.

(iv) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(3 - x) = 0$

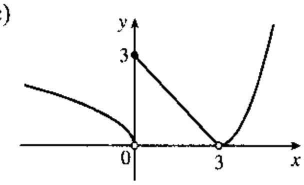
(v) $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim\limits_{x\to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim\limits_{x\to 0} f(x)$ does not exist.

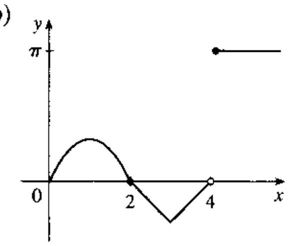
f is discontinuous at 3 since $f(3)$ does not exist.

(c)



24. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$. Therefore, $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-}\left(2x - x^2\right) = 0$ and $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+}(2 - x) = 0$. Thus, $\lim\limits_{x\to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim\limits_{x\to 3^-} g(x) = \lim\limits_{x\to 3^-}(2 - x) = -1$ and $\lim\limits_{x\to 3^+} g(x) = \lim\limits_{x\to 3^+}(x - 4) = -1$. Thus, $\lim\limits_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3. $\lim\limits_{x\to 4^-} g(x) = \lim\limits_{x\to 4^-}(x - 4) = 0$ and $\lim\limits_{x\to 4^+} g(x) = \lim\limits_{x\to 4^+}\pi = \pi$. Thus, $\lim\limits_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim\limits_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



25. x^3 is continuous on $\mathbb{R}$ since it is a polynomial and $\cos x$ is also continuous on $\mathbb{R}$, so the product $x^3 \cos x$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{x}$ is continuous on its domain, $[0, \infty)$, and so the sum, $h(x) = \sqrt[4]{x} + x^3 \cos x$, is continuous on its domain, $[0, \infty)$.

26. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.

27. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

28. Let $f(x) = 2\sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2\sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2\sin x = 3 - 2x$ has a root in $(0, 1)$.

29. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x\to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x\to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x\to 2} \frac{-2(x - 2)(x + 2)}{x - 2}$$
$$= \lim_{x\to 2} [-2(x + 2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

30. For a general point with x-coordinate a, we have

$$m = \lim_{x\to a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x\to a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)}$$
$$= \lim_{x\to a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x\to a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x - 0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x + 1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

31. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1 + h) - s(1)}{(1 + h) - 1} = \frac{1 + 2(1 + h) + (1 + h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}.$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\displaystyle\lim_{h\to 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h\to 0} \frac{10 + h}{4} = \frac{10}{4} = 2.5$ m/s.

32. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change is

$\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016 \, \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

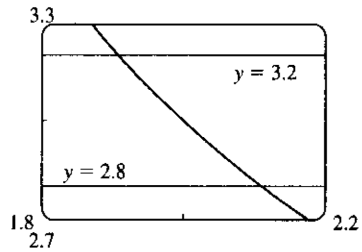
$$\lim_{h\to 0}\frac{\Delta V}{\Delta P} = \lim_{h\to 0}\frac{V(P+h)-V(P)}{h} = \lim_{h\to 0}\frac{800/(P+h)-800/P}{h}$$

$$= \lim_{h\to 0}\frac{800\,[P-(P+h)]}{h(P+h)P} = \lim_{h\to 0}\frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

which is inversely proportional to the square of P.

33. $\left|\dfrac{x+1}{x-1} - 3\right| < 0.2 \;\Rightarrow\; -0.2 < \dfrac{x+1}{x-1} - 3 < 0.2 \;\Rightarrow$

$2.8 < \dfrac{x+1}{x-1} < 3.2$. Graphing the functions $y = 2.8$, $y = (x+1)/(x-1)$, and $y = 3.2$ on the interval $[1.8, 2.2]$, we see that the inequality holds whenever $1.91 < x < 2.11$ (approximately). So since $|2 - 1.91| = 0.09$ and $|2 - 2.15| = 0.15$, any positive $\delta \le 0.09$ will do.



34. The slope of the tangent to $y = \dfrac{x+1}{x-1}$ is

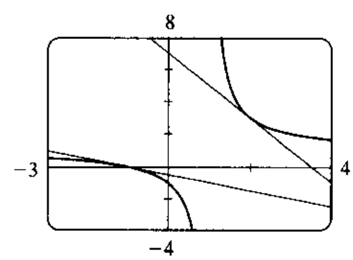
$$\lim_{h\to 0}\frac{\dfrac{(x+h)+1}{(x+h)-1} - \dfrac{x+1}{x-1}}{h} = \lim_{h\to 0}\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h\to 0}\frac{-2h}{h(x-1)(x+h-1)} = -\frac{2}{(x-1)^2}$$

So at $(2,3)$, $m = -\dfrac{2}{(2-1)^2} = -2 \;\Rightarrow\; y - 3 = -2(x-2) \;\Rightarrow$

$y = -2x + 7$. At $(-1, 0)$, $m = -\dfrac{2}{(-1-1)^2} = -\dfrac{1}{2} \;\Rightarrow$

$y = -\frac{1}{2}(x+1) \;\Rightarrow\; y = -\frac{1}{2}x - \frac{1}{2}$.

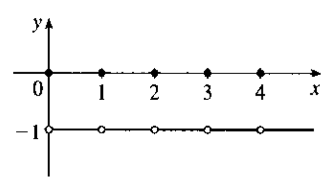


35. $|f(x)| \le g(x) \;\Leftrightarrow\; -g(x) \le f(x) \le g(x)$ and $\lim\limits_{x\to a} g(x) = 0 = \lim\limits_{x\to a} -g(x)$. Thus, by the Squeeze Theorem, $\lim\limits_{x\to a} f(x) = 0$.

36. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer,

$[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim\limits_{x\to a} f(x)$ exists

(and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.

PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\displaystyle\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}\cdot\frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to 0}\frac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes

$$\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} = 1 \quad\Rightarrow\quad \frac{a}{\sqrt{4}+2} = 1 \quad\Rightarrow\quad a=4. \text{ Therefore, } a=b=4.$$

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0}\frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0}\frac{-4x}{x} = \lim_{x\to 0}(-4) = -4.$

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q=(0,a)$. Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But

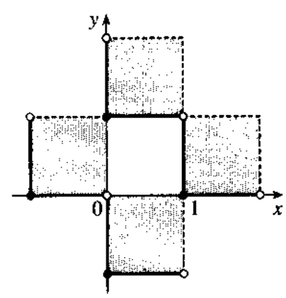
$$m_{QR} = \frac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \frac{x^2-2a}{x}, \text{ so we conclude that } -1 = x^2-2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \tfrac{1}{2}x^2+\tfrac{1}{2}.$$

As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. Since $[\![x]\!] \le x < [\![x]\!]+1$, we have $\dfrac{[\![x]\!]}{[\![x]\!]} \le \dfrac{x}{[\![x]\!]} < \dfrac{[\![x]\!]+1}{[\![x]\!]} \;\Rightarrow\; 1 \le \dfrac{x}{[\![x]\!]} < 1+\dfrac{1}{[\![x]\!]}$ for $x \ge 1$. As $x \to \infty$, $[\![x]\!] \to \infty$, so $\dfrac{1}{[\![x]\!]} \to 0$ and $1+\dfrac{1}{[\![x]\!]} \to 1$. Thus, $\displaystyle\lim_{x\to\infty}\frac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

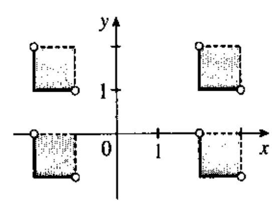
Case (i):	$[\![x]\!]=1,\ [\![y]\!]=0$	$\Rightarrow$	$1 \le x < 2$ and $0 \le y < 1$
Case (ii):	$[\![x]\!]=-1,\ [\![y]\!]=0$	$\Rightarrow$	$-1 \le x < 0$ and $0 \le y < 1$
Case (iii):	$[\![x]\!]=0,\ [\![y]\!]=1$	$\Rightarrow$	$0 \le x < 1$ and $1 \le y < 2$
Case (iv):	$[\![x]\!]=0,\ [\![y]\!]=-1$	$\Rightarrow$	$0 \le x < 1$ and $-1 \le y < 0$



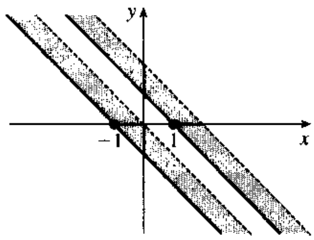
(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

$\{(x,y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} =$

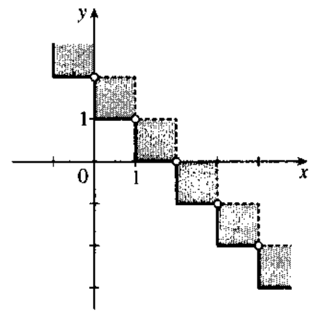
$$\left\{(x,y)\ \middle|\ \begin{matrix} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{matrix}\right\}.$$



(c) $[\![x+y]\!]^2 = 1 \quad\Rightarrow\quad [\![x+y]\!] = \pm 1$
$\Rightarrow\quad 1 \le x+y < 2$ or
$-1 \le x+y < 0$

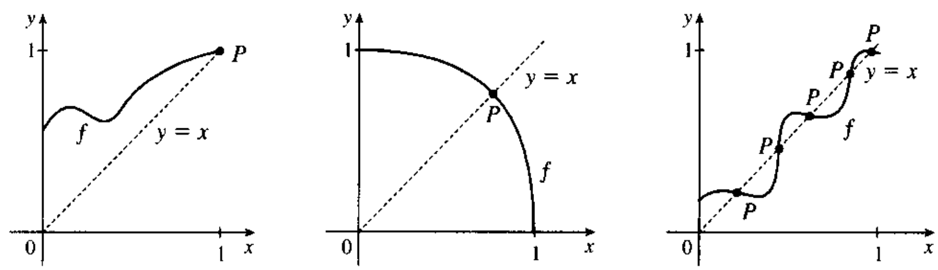


(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \quad\Rightarrow$
$[\![y]\!] = 1 - n \quad\Rightarrow\quad 1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.



7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,
$\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) \quad\Rightarrow\quad \lim\limits_{x\to a^+} x^2 = \lim\limits_{x\to a^-} (x+1) \quad\Rightarrow\quad a^2 = a+1 \quad\Rightarrow\quad a^2 - a - 1 = 0 \quad\Rightarrow$
[by the quadratic formula] $\quad a = \left(1 \pm \sqrt{5}\,\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point.
Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \ne 0$. For the same reason we can assume that $f(1) \ne 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \left(\frac{1}{2}[f(x)+g(x)] + \frac{1}{2}[f(x)-g(x)]\right)$

$$= \tfrac{1}{2} \lim_{x\to a} [f(x)+g(x)] + \tfrac{1}{2} \lim_{x\to a} [f(x)-g(x)]$$

$$= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 1 = \tfrac{3}{2}, \text{ and}$$

$\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} \Big([f(x)+g(x)] - f(x)\Big) = \lim\limits_{x\to a}[f(x)+g(x)] - \lim\limits_{x\to a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}$.

So $\lim\limits_{x\to a} [f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

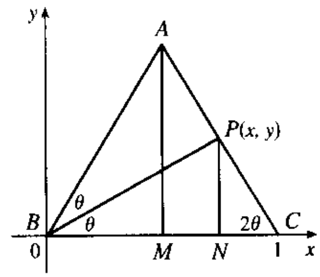
Another solution: Since $\lim\limits_{x\to a}[f(x)+g(x)]$ and $\lim\limits_{x\to a}[f(x)-g(x)]$ exist, we must have

$$\lim_{x\to a}[f(x)+g(x)]^2=\left(\lim_{x\to a}[f(x)+g(x)]\right)^2 \text{ and } \lim_{x\to a}[f(x)-g(x)]^2=\left(\lim_{x\to a}[f(x)-g(x)]\right)^2, \text{ so}$$

$$\lim_{x\to a}[f(x)\,g(x)]=\lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2-[f(x)-g(x)]^2\right) \quad [\text{because all of the } f^2 \text{ and } g^2 \text{ cancel}]$$

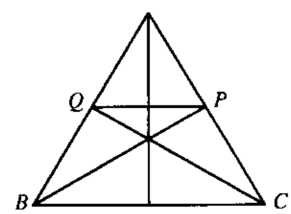
$$=\tfrac{1}{4}\left(\lim_{x\to a}[f(x)+g(x)]^2-\lim_{x\to a}[f(x)-g(x)]^2\right)=\tfrac{1}{4}(2^2-1^2)=\tfrac{3}{4}.$$

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta=\dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta=y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x}=\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2(y/x)}{1-(y/x)^2}$.

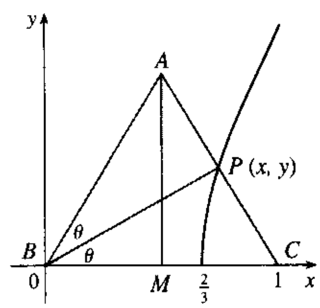


After a bit of simplification, this becomes $\dfrac{1}{1-x}=\dfrac{2x}{x^2-y^2}$ $\Leftrightarrow$ $y^2=x(3x-2)$. As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y\to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2)=0$. Obviously it is not $x=0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x-2=0$, that is, $x=\frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ=\angle PBC$ (alternate angles; $QP\parallel BC$ by symmetry) and similarly $\angle CQP=\angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM|\to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2=x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM|\to\infty$, $2\theta\to\frac{\pi}{2}$, $\theta\to\frac{\pi}{4}$, $x\to 1$, and since $\tan\theta=y/x$, $y\to 1$. Thus, P only traces out the part of the curve with $0\le y<1$.



11. (a) Consider $G(x)=T(x+180^\circ)-T(x)$. Fix any number a. If $G(a)=0$, we are done: Temperature at a = Temperature at $a+180^\circ$. If $G(a)>0$, then $G(a+180^\circ)=T(a+360^\circ)-T(a+180^\circ)$ $=T(a)-T(a+180^\circ)=-G(a)<0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a,a+180^\circ]$. If $G(a)<0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.