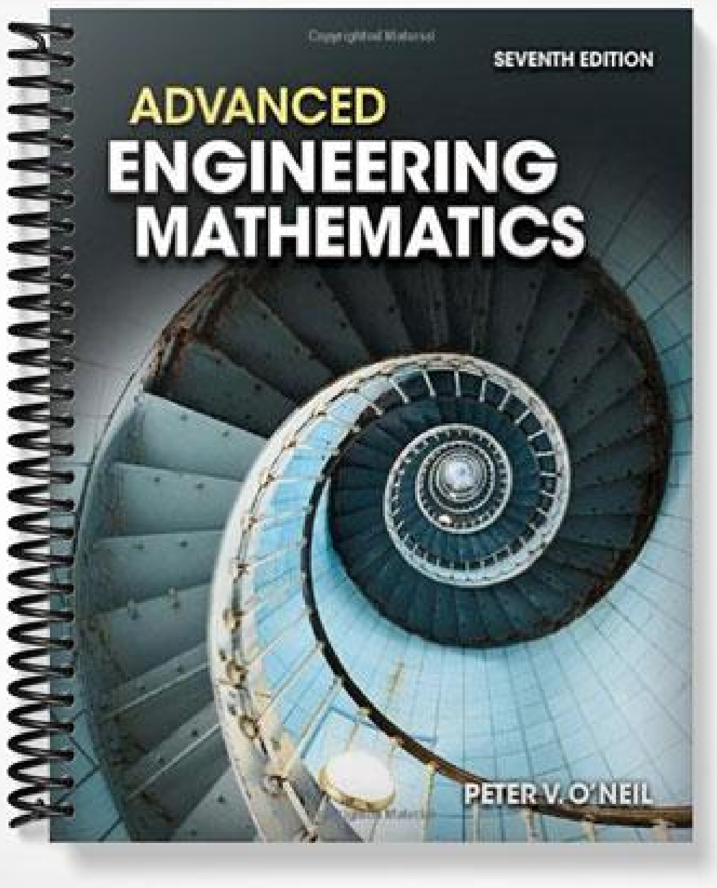
SOLUTIONS MANUAL



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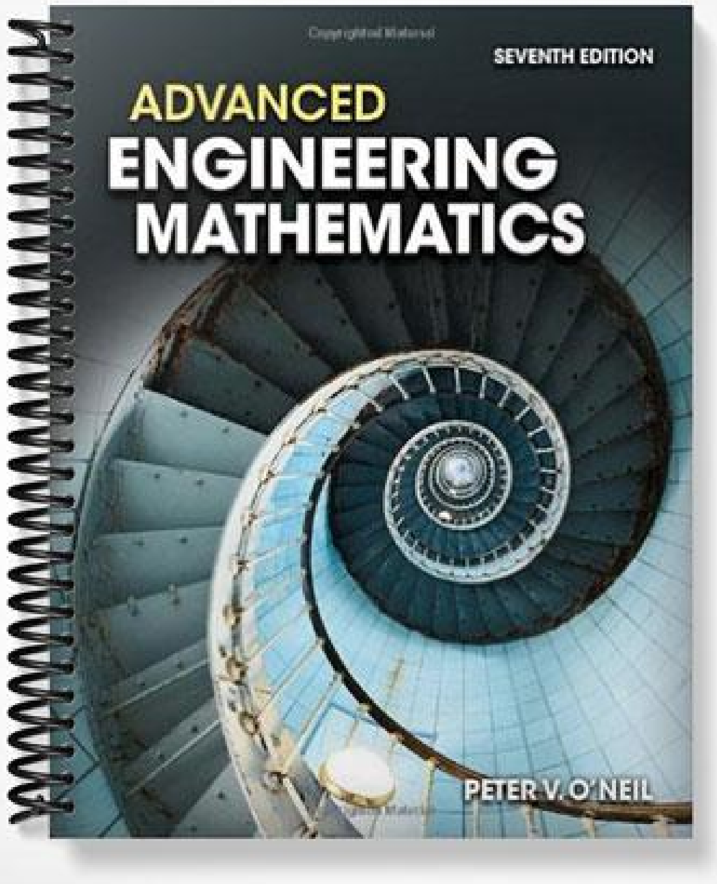

Chapter 2

Linear Second-Order Equations

2.1 Theory of the Linear Second-Order Equation

In Problems 1 - 5, verification that the given functions are solutions of the differential equation is a straightforward differentiation, which we omit.

1. The general solution is $y(x) = c_1 \sin(6x) + c_2 \cos(6x)$. For the initial conditions, we need $y(0) = c_2 = -5$ and $y'(0) = 6c_1 = 2$. Then $c_1 = 1/3$ and the solution of the initial value problem is

$$y(x) = \frac{1}{3}\sin(6x) - 5\cos(6x).$$

2. The general solution is $y(x) = c_1 e^{4x} + c_2 e^{-4x}$. For the initial conditions, compute

$$y(0) = c_1 + c_2 = 12 \text{ and } y'(0) = 4c_1 - 4c_2 = 3.$$

   Solve these algebraic equations to obtain $c_1 = 51/8$ and $c_2 = 45/8$. The solution of the initial value problem is

$$y(x) = \frac{51}{8}e^{4x} + \frac{45}{8}e^{-4x}.$$

3. The general solution is $y(x) = c_1 e^{-2x} + c_2 e^{-x}$. For the initial conditions, we have

$$y(0) = c_1 + c_2 = -3 \text{ and } y'(0) = -2c_1 - c_2 = -1.$$

   Solve these to obtain $c_1 = 4$, $c_2 = -7$. The solution of the initial value problem is

$$y(x) = 4e^{-2x} - 7e^{-x}.$$

4. The general solution is $y(x) = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x)$. Compute

$$\begin{aligned} y'(x) &= 3c_1e^{3x}\cos(2x) - 2c_1e^{3x}\sin(2x) \\ &\quad + 3c_2e^{3x}\sin(2x) + 2c_2e^{3x}\cos(2x). \end{aligned}$$

From the initial conditions,

$$y(0) = c_1 = -1 \text{ and } y'(0) = 3c_1 + 2c_2 = 1.$$

Then $c_2 = 2$ and the solution of the initial value problem is

$$y(x) = -e^{3x}\cos(2x) + 2e^{3x}\sin(2x).$$

5. The general solution is $y(x) = c_1e^x\cos(x) + c_2e^x\sin(x)$. Then $y(0) = c_1 = 6$. We find that $y'(0) = c_1 + c_2 = 1$, so $c_2 = -5$. The initial value problem has solution

$$y(x) = 6e^x\cos(x) - 5e^x\sin(x).$$

6. The general solution is

$$y(x) = c_1\sin(6x) + c_2\cos(6x) + \frac{1}{36}(x-1).$$

7. The general solution is

$$y(x) = c_1e^{4x} + c_2e^{-4x} - \frac{1}{4}x^2 + \frac{1}{2}.$$

8. The general solution is

$$y(x) = c_1e^{-2x} + c_2e^{-x} + \frac{15}{2}.$$

9. The general solution is

$$y(x) = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x) - 8e^x.$$

10. The general solution is

$$y(x) = c_1e^x\cos(x) + c_2e^x\sin(x) - \frac{5}{2}x^2 - 5x - 4.$$

11. For conclusion (1), begin with the hint to the problem to write

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0, \\ y_2'' + py_2' + qy_2 &= 0. \end{aligned}$$

Multiply the first equation by y_2 and the second by $-y_1$ and add the resulting equations to obtain

$$y_1''y_2 - y_2''y_1 + p(y_1'y_2 - y_2'y_1) = 0.$$

Since $W = y_1 y_2 - y_2 y_1$, then

$$W' = y_1 y_2'' - y_1'' y_2,$$

so

$$W' + pW = y_1 y_2'' - y_1'' y_2 + p(y_1 y_2' - y_1' y_2) = 0.$$

Therefore the Wronskian satisfies the linear differential equation $W' + pW = 0$. This has integrating factor $e^{\int p(x)\,dx}$ and can be written

$$\left(We^{\int p(x)\,dx}\right)' = 0.$$

Upon integrating we obtain the general solution

$$W = ce^{-\int p(x)\,dx}.$$

If $c = 0$, then this Wronskian is zero for all x in I. If $c \neq 0$, then $W \neq 0$ for x in I because the exponential function does not vanish for any x.

Now turn to conclusion (2). Suppose first that $y_2(x) \neq 0$ on I. By the quotient rule for differentiation it is routine to verify that

$$y_2^2 \frac{d}{dx}\left(\frac{y_1}{y_2}\right) = -W(x).$$

If $W(x)$ vanishes, then the derivative of y_1/y_2 is identically zero on I, so y_1/y_2 is constant, hence y_1 is a constant multiple of y_1, making the two functions linearly dependent. Conversely, if the two functions are linearly independent, then one is a constant multiple of the other, say $y_1 = cy_2$, and then $W(x) = 0$.

If there are points in I at which $y_2(x) = 0$, then we have to use this argument on the open intervals between these points and then make use of the continuity of y_2 on the entire interval. This is a technical argument we will not pursue here.

12.
$$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.$$

Then $W(0) = 0$, while $W(x) \neq 0$ if $x \neq 0$. However, the theorem only applies to solutions of a linear second-order differential equation on an interval containing the point at which the Wronskian is evaluated. x^2 and x^3 are not solutions of such a second-order linear equation on an open interval containing 0.

13. It is routine to verify by substitution that x and x^2 are solutions of the given differential equation. The Wronskian is

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2,$$

which vanishes at $x = 0$, but at no other points. However, the theorem only applies to solutions of linear second order differential equations. To write the given differential equation in standard linear form, we must write

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0,$$

which is not defined at $x = 0$. Thus the theorem does not apply.

14. If y_1 and y_2 have relative extrema at some point x_0 within the interval,

$$y_1'(x_0) = y_2'(x_0) = 0.$$

Then

$$W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix} = 0.$$

Therefore y_1 and y_2 are linearly dependent.

15. Suppose $\varphi'(x_0) = 0$. Then φ is the unique solution of the initial value problem

$$y'' + py' + qy = 0; y(x_0) = y'(x_0) = 0$$

on I. But the functions that is identically zero on I is also a solution of this problem. Therefore $\varphi(x) = 0$ for all x in I.

2.2 The Constant Coefficient Case

1. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$, with roots $-2, 3$. The general solution is

$$y = c_1e^{-2x} + c_2e^{3x}.$$

2. The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$, with roots $1 \pm 3i$. The general solution is

$$y = c_1e^x \cos(3x) + c_2e^x \sin(3x).$$

3. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, with repeated root -3. The general solution is

$$y = c_1e^{-3x} + c_2xe^{-3x}.$$

4. The characteristic equation is $\lambda^2 - 3\lambda = 0$, with roots $0, 3$. The general solution is

$$y = c_1 + c_2e^{3x}.$$

5. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$. The general solution is

$$y = c_1e^{-5x} \cos(x) + c_2e^{-5x} \sin(x).$$

6. The characteristic equation is $\lambda^2 + 6\lambda - 40 = 0$, with roots $-10, 4$. The general solution is
$$y = c_1 e^{-10x} + c_2 e^{4x}.$$

7. The characteristic equation is $\lambda^2 + 3\lambda + 18 = 0$, with roots $-3/2 \pm 3\sqrt{7}i/2$. The general solution is
$$y = e^{-3x/2}\left[c_1 \cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2 \sin\left(\frac{3\sqrt{7}x}{2}\right)\right].$$

8. The characteristic equation is $\lambda^2 + 16\lambda + 64 = 0$, with repeated root -8. The general solution is
$$y = e^{-8x}(c_1 + c_2 x).$$

9. The characteristic equation is $\lambda^2 - 14\lambda + 49 = 0$, with repeated root 7. The general solution is
$$y = e^{7x}(c_1 + c_2 x).$$

10. The characteristic equation is $\lambda^2 - 6\lambda + 7 = 0$, with roots $3 \pm \sqrt{2}i$. The general solution is
$$y = e^{3x}[c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)].$$

In each of Problems 11 through 20, the solution is obtained by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions. We provide the details only for Problems 11 and 12, the other problems proceeding similarly.

11. The characteristic equation is $\lambda^2 + 3\lambda = 0$, with roots $0, -3$. The general solution of the differential equation is $y = c_1 + c_2 e^{-3x}$. To find a solution satisfying the initial conditions, we need
$$y(0) = c_1 + c_2 = 3 \text{ and } y'(0) = -3c_2 = 6.$$
Then $c_1 = 5$ and $c_2 = -2$, so the solution of the initial value problem is $y = 5 - 2e^{-3x}$.

12. The characteristic equation is $\lambda^2 + 2\lambda - 3 = 0$, with roots $1, -3$. The general solution of the differential equation is
$$y(x) = c_1 e^{x} + c_2 e^{-3x}.$$
Now we need
$$y(0) = c_1 + c_2 = 6 \text{ and } y'(0) = c_1 - 3c_2 = -2.$$
Then $c_1 = 4$ and $c_2 = 2$, so the solution of the initial value problem is
$$y(x) = 4e^{x} + 2e^{-3x}.$$

13. $y = 0$ for all x

14. $y = e^{2x}(3 - x)$

15.
$$y = \frac{1}{7}[9e^{3(x-2)} + 5e^{-4(x-2)}]$$

16.
$$y = \frac{\sqrt{6}}{4}e^{x}\left[e^{\sqrt{6}x} - e^{-\sqrt{6}x}\right]$$

17. $y = e^{x-1}(29 - 17x)$

18.
$$y = -4(5 - \sqrt{23})e^{5(x-2)/7}\sin\left(\frac{\sqrt{23}}{2}(x-2)\right)$$

19.
$$y = e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right) + \frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$$

20.
$$y = ae^{(-1+\sqrt{5})x/2} + be^{(-1-\sqrt{5})x/2},$$

where
$$a = \frac{(9 + 7\sqrt{5})}{2\sqrt{5}}e^{-2+\sqrt{5}}$$

and
$$b = \frac{(7\sqrt{5} - 9)}{2\sqrt{5}}e^{-2-\sqrt{5}}$$

21. (a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$, with repeated roots $\lambda = \alpha$. The general solution is
$$y(x) = \varphi(x) = (c_1 + c_2 x)e^{\alpha x}.$$

(b) The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \epsilon^2) = 0$, with roots $\alpha \pm \epsilon$. The general solution is
$$y_\epsilon(x) = \varphi_\epsilon(x) = e^{\alpha x}(c_1 e^{\epsilon x} + c_2 e^{-\epsilon x}).$$

(c) In general,
$$\lim_{\epsilon \to 0} y_\epsilon(x) = e^{\alpha x}(c_1 + c_2) \neq y(x).$$

22. (a) We find

$$y = \psi(x) = e^{\alpha x}(c + (d - ac)x).$$

(b) We obtain

$$y_\epsilon = \psi_\epsilon(x) = \frac{1}{2\epsilon} e^{\alpha x}\left[(d - ac + \epsilon c)e^{\epsilon x} + (ac - d + \epsilon c)e^{-\epsilon x}\right].$$

(c) Using l'Hospital's rule, take the limit

$$\lim_{\epsilon \to 0} \psi_\epsilon(x) =$$
$$\frac{1}{2} e^{\alpha x} \lim_{\epsilon \to 0}\left[(d - ac + \epsilon c)xe^{\epsilon x} - (ac - d + \epsilon c)xe^{-\epsilon x} + ce^(\epsilon x + ce^{-\epsilon x})\right]$$
$$= e^{\alpha x}(c + (d - ac)x) = \psi(x).$$

23. The characteristic equation has roots

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

As we have seen, there are three cases.

If $a^2 = 4b$, then

$$y = e^{-ax/2}(c_1 + c_2 x) \to 0 \text{ as } x \to \infty,$$

because $a > 0$.

If $a^2 > 4b$, then $a^2 - 4b < a^2$ and λ_1 and λ_2 are both negative, so

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \to 0 \text{ as } x \to \infty.$$

Finally, if $a^2 < 4b$, then the general solution has the form

$$y(x) = e^{-ax/2}(c_1 \cos(\beta x) + c_2 \sin(\beta x)),$$

where $\beta = \sqrt{4b - a^2}/2$. Because $a > 0$, this solution also has limit zero as $x \to \infty$.

24. We will use the fact that, for any positive integer n,

$$i^{2n} = (i^2)^n = (-1)^n \text{ and } i^{2n+1} = i^{2n}i = (-1)^n i.$$

Now suppose a is real and split the exponential series into two series, one for even values of the summation index, and the other for odd values:

$$\begin{aligned}
e^{ia} &= \sum_{n=0}^{\infty} \frac{1}{n!} i^n a^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} i^{2n} a^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} i^{2n+1} a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} a^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} i a^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a^{2n+1} \\
&= \cos(a) + i \sin(a).
\end{aligned}$$

2.3 The Nonhomogeneous Equation

1. Two independent solutions of $y'' + y = 0$ are $y_1 = \cos(x)$ and $y_2 = \sin(x)$. The Wronskian is

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

To use variation of parameters, seek a particular solution of the differential equation of the form

$$y = u_1 y_1 + u_2 y_2.$$

Let $f(x) = \tan(x)$. We found that we can choose

$$\begin{aligned}
u_1(x) &= -\int \frac{y_2(x) f(x)}{W(x)}\, dx = -\int \tan(x)\sin(x)\, dx \\
&= -\int \frac{\sin^2(x)}{\cos(x)}\, dx \\
&= -\int \frac{1 - \cos^2(x)}{\cos(x)}\, dx \\
&= \int \cos(x)\, dx - \int \sec(x)\, dx \\
&= \sin(x) - \ln|\sec(x) + \tan(x)|
\end{aligned}$$

and

$$\begin{aligned}
u_2(x) &= \int \frac{y_1(x) f(x)}{W(x)}\, dx = \int \cos(x)\tan(x)\, dx \\
&= \int \sin(x)\, dx = -\cos(x).
\end{aligned}$$

The general solution can be written

$$\begin{aligned} y &= c_1\cos(x) + c_2\sin(x) + \sin(x)\cos(x) \\ &\quad - \cos(x)\ln|\sec(x)+\tan(x)| - \sin(x)\cos(x) \\ &= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x)+\tan(x)| \end{aligned}$$

2. Two independent solutions of the associated homogeneous equation are $y_1(x) = e^{3x}$ and $y_2(x) = e^x$. These have Wronskian $W(x) = -2e^{4x}$. Then

$$\begin{aligned} u_1(x) &= -\int \frac{2e^x\cos(x+3)}{-2e^{4x}}\,dx \\ &= \int e^{-3x}\cos(x+3)\,dx \\ &= -\frac{3}{10}e^{-3x}\cos(x+3) + \frac{1}{10}e^{-3x}\sin(x+3) \end{aligned}$$

and

$$\begin{aligned} v(x) &= \int \frac{2e^{3x}\cos(x+3)}{-2e^{4x}}\,dx \\ &= \int e^{-x}\cos(x+3)\,dx \\ &= \frac{1}{2}e^{-x}\cos(x+3) - \frac{1}{2}e^{-x}\sin(x+3). \end{aligned}$$

The general solution is

$$\begin{aligned} y(x) &= c_1e^{3x} + c_2e^x \\ &\quad - \frac{3}{10}\cos(x+3) + \frac{1}{10}\sin(x+3) \\ &\quad + \frac{1}{2}\cos(x+3) - \frac{1}{2}\sin(x+3). \end{aligned}$$

This can be written

$$\begin{aligned} y(x) &= c_1e^{3x} + c_2e^x \\ &\quad + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3). \end{aligned}$$

For Problems 3 through 6 we will omit some of the details and give an outline of the solution.

3. $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$ are linearly independent solutions of the associated homogeneous equation. Their Wronskian is $W = 3$. With $f(x) = 12\sec(3x)$, carry out the integrations in the equations for u_1 and u_2 to obtain the general solution

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\cos(3x)\ln|\cos(3x)|.$$

4. $y_1 = e^{3x}$ and $y_2 = e^{-x}$, with Wronskian $-4e^{-2x}$. With $f(x) = 2\sin^2(x) = 1 - \cos(2x)$, obtain u_1 and u_2 to write the general solution
$$y(x) = c_1e^{3x} + c_2e^{-x} - \frac{1}{3} + \frac{7}{65}\cos(2x) + \frac{4}{65}\sin(2x).$$

5. $y_1 = e^x$ and $y_2 = e^{2x}$, with Wronskian $W = e^{3x}$. With $f(x) = \cos(e^{-x})$, carry out the integrations to obtain u_1 and u_2 to write the general solution
$$y(x) = c_1e^x + c_2e^{2x} - e^{2x}\cos(e^{-x})$$

6. $y_1 = e^{3x}$ and $y_2 = e^{2x}$, with Wronskian $W = -e^{5x}$. Use the identity $8\sin^2(4x) = 4\cos(8x) - 4$ to help find u_1 and u_2 and write the general solution
$$y = c_1e^{3x} + c_2e^{2x} + \frac{2}{3} + \frac{58}{1241}\cos(8x) + \frac{40}{1241}\sin(8x).$$

In Problems 7 - 16 we use the method of undetermined coefficients in writing the general solution. For Problems 7 and 8 all the details are included, while for Problems 9 through 16 the important details of the solution are outlined.

7. Two independent solutions of the associated homogeneous equation are $y_1 = e^{2x}$ and $y_2 = e^{-x}$. Since $2x^2 + 5$ is a second degree polynomial, we attempt such a polynomial as a particular solution:
$$y_p(x) = Ax^2 + Bx + C.$$
Substitute this into the (nonhomogeneous) differential equation to obtain
$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 + 5.$$
Then
$$\begin{aligned} 2A - B - 2C &= 5, \\ -2A - 2B &= 0, \\ -2A &= 2. \end{aligned}$$
Then $A = -1, B = 1$ and $C = -4$. The general solution is
$$y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4.$$

8. We find $y_1 = e^{3x}$ and $y_2 = e^{-2x}$. Since $f(x) = 8e^{2x}$, which is not a constant multiple of y_1 or y_2, try $y_p(x) = Ae^{2x}$ to obtain
$$y = c_1e^{3x} + c_2e^{-2x} - 2e^{2x}.$$

9. $y_1 = e^x \cos(3x)$ and $y_2 = e^x \sin(3x)$. With $f(x)$ a second degree polynomial, try $y_p(x) = Ax^2 + Bx + C$ to obtain

$$y = e^x[c_1 \cos(3x) + c_2 \sin(3x)] + 2x^2 + x - 1.$$

10. $y_1 = e^{2x} \cos(x)$ and $y_2 = e^{2x} \sin(x)$. With $f(x) = 21e^{2x}$, try $y_p(x) = Ae^{2x}$ to obtain

$$y = e^{2x}[c_1 \cos(x) + c_2 \sin(x)] + 21e^{2x}.$$

11. $y_1 = e^{2x}$ and $y_2 = e^{4x}$. With $f(x) = 3e^x$, try $y_p(x) = Ae^x$, noting that e^x is not a solution of the associated homogeneous equation. Obtain the general solution

$$y = c_1e^{2x} + c_2e^{4x} + e^x.$$

12. $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$. Because $f(x) = 9\cos(3x)$, try $y_p(x) = A\cos(x)+B\sin(x)$, obtaining both a $\cos(3x)$ and a $\sin(3x)$ term, to obtain

$$y = e^{-3x}[c_1 + c_2x] + \frac{1}{2}\sin(3x).$$

Although the general solution does not contain a $\cos(3x)$ term, this does not automatically follow and in general both the sine and cosine term must be included in our attempt at $y_p(x)$.

13. $y_1 = e^x$ and $y_2 = e^{2x}$. With $f(x) = 10\sin(x)$, try $y_p(x) = A\cos(x) + B\sin(x)$ to obtain

$$y = c_1e^x + c_2e^{2x} + 3\cos(x) + \sin(x).$$

14. $y_1 = 1$ and $y_2 = e^{-4x}$. With $f(x) = 8x^2 + 2e^{3x}$, try $y_p(x) = Ax^2 + Bx + C + De^{3x}$, since e^{3x} is not a solution of the homogeneous equation. This gives us the general solution

$$y = c_1 + c_2e^{-4x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{2}{3}e^{3x}.$$

15. $y_1 = e^{2x}\cos(3x)$ and $y_2 = e^{2x}\sin(3x)$. Since neither e^{2x} nor e^{3x} is a solution of the homogeneous equation, try $y_p(x) = Ae^{2x} + Be^{3x}$ to obtain the general solution

$$y = e^{2x}[c_1 \cos(3x) + c_2 \sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}.$$

16. $y_1 = e^x$ and $y_2 = xe^x$. Because $f(x)$ is a first degree polynomial plus a $\sin(3x)$ term, try

$$y_p(x) = Ax + B + C\sin(3x) + D\cos(3x)$$

to obtain the general solution

$$y = e^x[c_1 + c_2x] + 3x + 6 + \frac{3}{2}\cos(3x) - 2\sin(3x).$$

Notice that the solution contains both a $\sin(3x)$ term and a $\cos(3x)$ term, even though $f(x)$ has just a $\sin(3x)$ term.

In Problems 17 through 24, we first find the general solution of the differential equation, then solve for the constants to satisfy the initial conditions. Problems 17 through 22 are well suited to the method of undetermined coefficients, while Problems 23 and 24 can be solved fairly directly by variation of parameters.

17. $y_1 = e^{2x}$ and $y_2 = e^{-2x}$. Since e^{2x} is a solution of the homogeneous equation, try $y_p(x) = Axe^{2x} + Bx + C$ to obtain the general solution

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

Now

$$y(0) = c_1 + c_2 = 1 \text{ and } y'(0) = 2c_1 - 2c_2 - \frac{7}{4} = 3.$$

Then $c_1 = 7/4$ and $c_2 = -3/4$. The solution of the initial value problem is

$$y = -\frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

18. Two independent solutions of the homogeneous equation are $y_1 = 1$ and $y_2 = e^{-4x}$. For a particular solution we might try $A + B\cos(x) + C\sin(x)$, but A is a solution of the homogeneous equation, so try $y_p(x) = Ax + B\cos(x) + C\sin(x)$. The general solution is

$$y(x) = c_1 + c_2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x.$$

Now

$$y(0) = c_1 + c_2 - 2 = 3 \text{ and } y'(0) = -4c_2 + 8 + 2 = 2.$$

These lead to the solution of the initial value problem:

$$y = 3 + 2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x.$$

19. We find the general solution

$$y(x) = c_1e^{-2x} + c_2e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

Solve for the constants to obtain the solution

$$y(x) = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$$

20. The general solution is

$$y(x) = c_1 + c_2 e^{3x} - \frac{1}{5}e^{2x}(\cos(x) + 3\sin(x)).$$

The solution of the initial value problem is

$$y = \frac{1}{5} + e^{3x} - \frac{1}{5}e^{2x}[\cos(x) + 3\sin(x)].$$

21. The general solution is

$$y(x) = c_1 e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

The initial value problem has the solution

$$y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

22. The general solution is

$$y = e^{x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1$$

To make it easier to fit the initial conditions specified at $x = 1$, we can also write this general solution as

$$y = e^{x/2}\left[d_1 \cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2 \sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

Now

$$y(1) = e^{1/2}d_1 + 1 = 4 \text{ and } y'(1) = \frac{1}{2}e^{1/2}d_1 + \frac{\sqrt{3}}{2}e^{1/2}d_2 = -2.$$

Solve these to get $d_1 = 3e^{-1/2}$ and $d_2 = -7e^{-1/2}/\sqrt{3}$. The solution of the initial value problem is

$$y = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

23. We find the general solution

$$y(x) = c_1 e^{x} + c_2 e^{-x} - \sin^2(x) - 2.$$

The initial value problem has the solution

$$y = 4e^{-x} - \sin^2(x) - 2.$$

24. The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|.$$

The solution of the initial value problem is

$$y = 4\cos(x) + 4\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|.$$

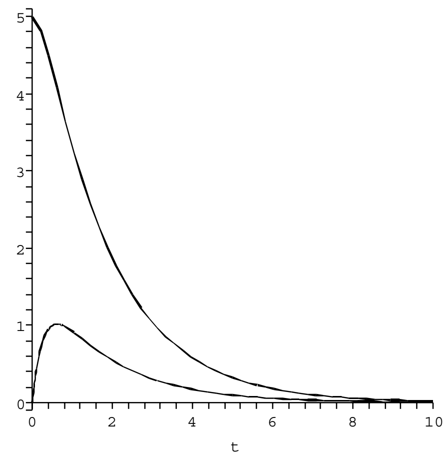

Figure 2.1: Solutions to Problem 1, Section 2.4.

2.4 Spring Motion

1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is

$$y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)].$$

With initial conditions $y(0) = 0, y'(0) = 5$, we obtain

$$y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t).$$

Graphs of these solutions are shown in Figure 2.1.

2. With $y(0) = 5$ and $y' = 0$, $y_1(t) = 5e^{-2t}(1 + 2t)$; with $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = 5te^{-2t}$. Graphs are given in Figure 2.2.

3. With $y(0) = 5$ and $y' = 0$,

$$y_1(t) = \frac{5}{2}e^{-t}[2\cos(2t) + \sin(2t)].$$

With $y(0) = 0$ and $y'(0) = 5$, $y_2(t) = \frac{5}{2}e^{-t}\sin(2t)$. Graphs are given in Figure 2.3.

4. The solution is

$$y(t) = Ae^{-t}[\cosh(\sqrt{2}t) + \sqrt{(2)}\sinh(\sqrt{2}t)].$$

Graphs for $A = 1, 3, 6, 10, -4$ and -7 are given in Figure 2.4.

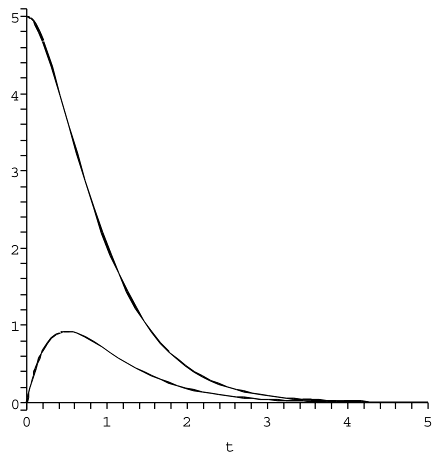

Figure 2.2: Solutions to Problem 2, Section 2.4.

5. The solution is
$$y(t) = \frac{A}{\sqrt{2}} e^{-2t} \sinh(\sqrt{(2)}t)$$
and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.5.

6. The solution is $y(t) = Ae^{-2t}(1 + 2t)$ and is graphed for $A = 1, 3, 6, 10, -4, -7$ in Figure 2.6.

7. The solution is $y(t) = Ate^{-2t}$, graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.7.

8. The solution is
$$y(t) = \frac{A}{2} e^{-t}[2\cos(2t) + \sin(2t)],$$
graphed in Figure 2.8 for $A = 1, 3, 6, 10, -4$ and -7.

9. The solution is
$$y(t) = \frac{A}{2} e^{-t} \sin(2t)$$
and is graphed for $A = 1, 3, 6, 10, -4$ and -7 in Figure 2.9.

10. From Newton's second law of motion,
$$y'' = \text{ sum of the external forces } = -29y - 10y'$$
so the motion is described by the solution of
$$y'' + 10y' + 29y = 0; y(0) = 3, y'(0) = -1.$$

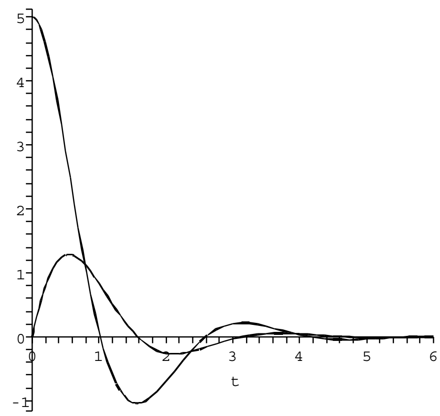

Figure 2.3: Solutions to Problem 3, Section 2.4.

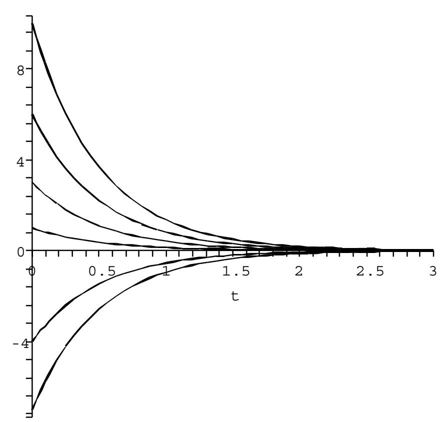

Figure 2.4: Solutions to Problem 4, Section 2.4.

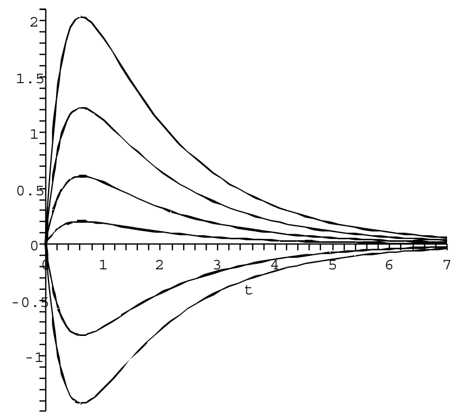

Figure 2.5: Solutions to Problem 5, Section 2.4.

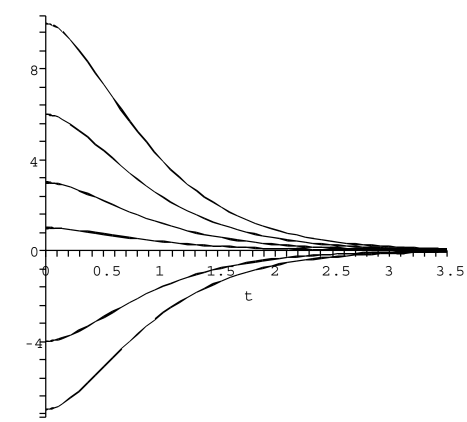

Figure 2.6: Solutions to Problem 6, Section 2.4.

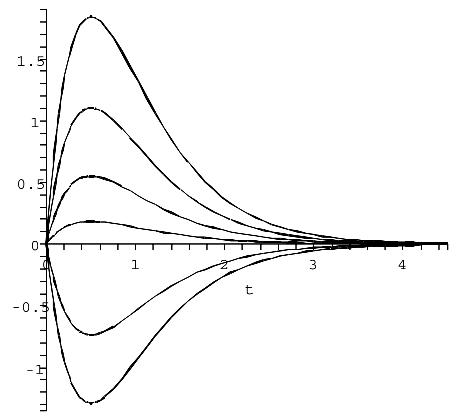

Figure 2.7: Solutions to Problem 7, Section 2.4.

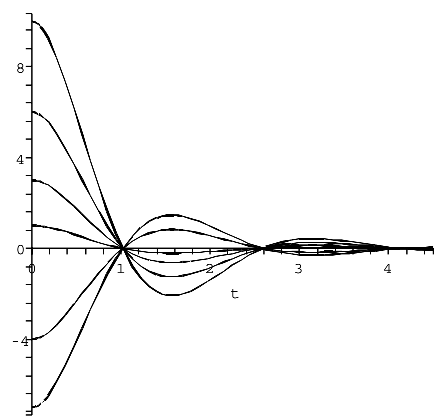

Figure 2.8: Solutions to Problem 8, Section 2.4.

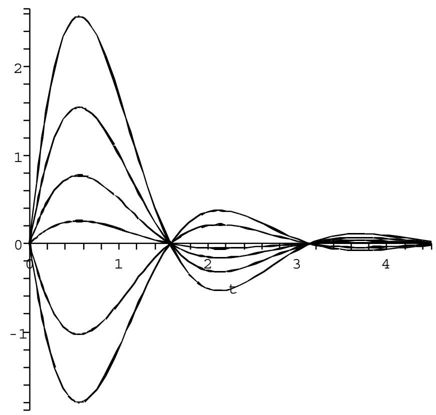

Figure 2.9: Solutions to Problem 9, Section 2.4.

The solution in this underdamped problem is

$$y(t) = e^{-5t}[3\cos(2t) + 7\sin(2t)].$$

If the condition on $y'(0)$ is $y'(0) = A$, this solution is

$$y(t) = e^{-5t}\left[3\cos(2t) + \left(\frac{A+15}{2}\right)\sin(2t)\right].$$

Graphs of this solution are shown in Figure 2.10 for $A = -1, -2, -4, 7, -12$ cm/sec (recall that down is the positive direction).

11. For overdamped motion the displacement is given by

$$y(t) = e^{-\alpha t}(A + Be^{\beta t}),$$

where α is the smaller of the roots of the characteristic equation and is positive, and β equals the larger root minus the smaller root. The factor $A + Be^{\beta t}$ can be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by the initial conditions. In fact, if $y_0 = y(0)$ and $v_0 = y'(0)$, we have

$$A + B = y_0 \text{ and } -\alpha(A + B) + \beta B = v_0.$$

We find from these that

$$-\frac{A}{B} = 1 - \frac{\beta y_0}{v_0 + \alpha y_0}.$$

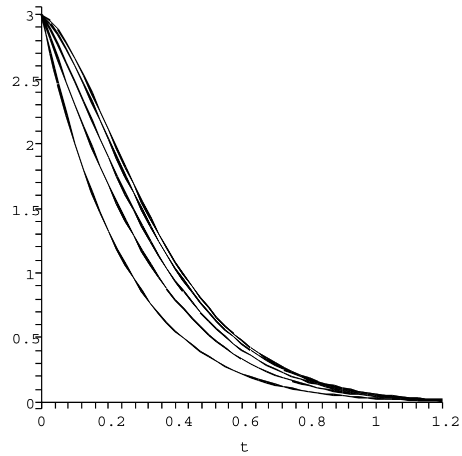

Figure 2.10: Solutions to Problem 10, Section 2.4.

No condition on only y_0 will ensure that $-A/B \leq 1$. If we also specify that $v_0 > -\alpha y_0$, we ensure that the overdamped bob will never pass through the equilibrium point.

12. For critically damped motion the displacement has the form

$$y(t) = e^{-\alpha t}(A + Bt),$$

with $\alpha > 0$ and A and B determined by the initial conditions. From the linear factor, the bob can pass through the equilibrium at most once, and will do this for some $t > 0$ if and only if $B \neq 0$ and $AB < 0$. Now note that $y_0 = A$ and $v_0 = y'(0) = -\alpha A + B$. Thus to ensure that the bob never passes through equilibrium we need $AB > 0$, which becomes the condition $(v_0 + \alpha y_0)y_0 > 0$. No condition on y_0 alone can ensure this. We would also need to specify $v_0 > -\alpha y_0$, and this will ensure that the critically damped bob never passes through the equilibrium point.

13. For underdamped motion, the solution has the appearance

$$y(t) = e^{-ct/2m}[c_1 \cos(\sqrt{4km - c^2}t/2m) + c_2 \sin(\sqrt{4km - c^2}t/2m)]$$

having frequency

$$\omega = \frac{\sqrt{4km - c^2}}{2m}.$$

Thus increasing c decreases the frequency of the the motion, and decreasing c increases the frequency.

14. For critical damping,

$$y(t) = e^{-ct/2m}(A + Bt).$$

For the maximum displacement at time t^* we need $y'(t^*) = 0$. This gives us

$$t^* = \frac{2mB - cA}{Bc}.$$

Now $y(0) = A$ and $y'(0) = B - Ac/2m$. Since we are given that $y(0) = y'(0) \neq 0$, we find that

$$t^* = \frac{4m^2}{2mc + c^2}$$

and this is independent of $y(0)$. The maximum displacement is

$$y(t^*) = \frac{y(0)}{c}(2m + c)e^{-2m/(2m+c)}.$$

15. The general solution of the overdamped problem

$$y'' + 6y' + 2y = 4\cos(3t)$$

is

$$y(t) = e^{-3t}[c_1\cosh(\sqrt{7}t) + c_2\sinh(\sqrt{7}t)] - \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us

$$c_1 = \frac{2266}{373} \text{ and } c_2 = \frac{6582}{373\sqrt{7}}.$$

Now the solution is

$$y_a(t) = \frac{1}{373}[e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us $c_1 = 28/373$ and $c_2 = 2106/373$ and the unique solution

$$y_b(t) = \frac{1}{373}[e^{-3t}[29\cosh(\sqrt{7}t) + \tfrac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)].$$

These solutions are graphed in Figure 2.11.

16. The general solution of the critically damped problem

$$y'' + 4y' + 4y = 4\cos(3t)$$

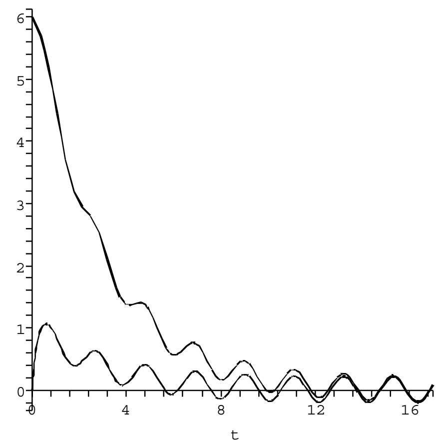

Figure 2.11: Solutions to Problem 15, Section 2.4.

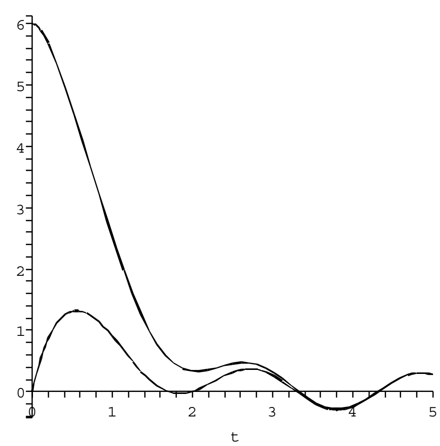

Figure 2.12: Solutions to Problem 16, Section 2.4.

is

$$y(t) = e^{-2t}[c_1 + c_2 t] - \frac{20}{169}\cos(3t) + \frac{48}{169}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give us the unique solution

$$y_a(t) = \frac{1}{169}[e^{-2t}[1034 + 1924t] - 20\cos(3t) + 48\sin(3t)].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give us the unique solution

$$y_b(t) = \frac{1}{169}[e^{-2t}[20 + 910t] - 20\cos(3t) + 48\sin(3t)].$$

These solutions are graphed in Figure 2.12.

17. The general solution of the underdamped problem

$$y''(t) + y' + 3y = 4\cos(3t)$$

is

$$y(t) = e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{11}t}{2}\right) + c_2\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t).$$

(a) The initial conditions $y(0) = 6, y'(0) = 0$ yield the unique solution

$$y_a(t) = \frac{1}{15}\left[e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ yield the unique solution

$$y_b(t) = \frac{1}{15}\left[e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right].$$

These solutions are graphed in Figure 2.13.

2.5 Euler's Equation

In Problems 1 - 3, details are given with the solution. Solutions for Problems 4 through 10, just the general solution is given. All solutions are for $x > 0$.

1. Let $x = e^t$ to obtain

$$Y'' + Y' - 6Y = 0$$

which we can read directly from the original differential equation without further calculation. Then

$$Y(t) = c_1 e^{2t} + c_2 e^{-3t}.$$

In terms of x,

$$y(x) = c_1 e^{2\ln(x)} + c_2 e^{-3\ln(x)} = c_1 x^2 + c_2 x^{-3}.$$

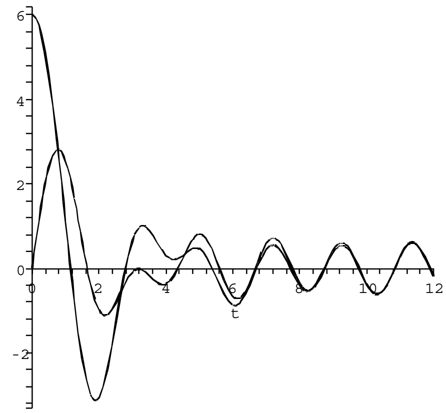

Figure 2.13: Solutions to Problem 17, Section 2.4.

2. The differential equation transforms to

$$Y'' + 2Y' + Y = 0,$$

with general solution

$$Y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Then

$$y(x) = c_1 x^{-1} + c_2 x^{-1} \ln(x) = \frac{1}{x}(c_1 + c_2 \ln(x)).$$

3. Solve

$$Y'' + 4Y = 0$$

to obtain

$$Y(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

Then

$$y(x) = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)).$$

4. $y(x) = c_1 x^2 + c_2 x^{-2}$

5. $y(x) = c_1 x^4 + c_2 x^{-4}$

6. $y(x) = x^{-2}(c_2 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$

7. $y(x) = c_1 x^{-2} + c_2 x^{-3}$

8. $y(x) = x^2(c_1 \cos(7\ln(x)) + c_2 \sin(7\ln(x))$

9. $y(x) = x^{-12}(c_1 + c_2 \ln(x))$

10. $y(x) = c_1x^7 + c_2x^5$

11. The general solution of the differential equation is

$$y(x) = c_1x^3 + c_2x^{-7}.$$

We need

$$y(2) = 1 = c_1 2^3 + c_2 2^{-7} \text{ and } y'(2) = 0 = 3c_1 2^2 - 7c_2 2^{-8}.$$

Solve for c_1 and c_2 to obtain the solution of the initial value problem

$$y(x) = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}$$

12. The solution of the initial value problem is

$$y(x) = -3 + 2x^2$$

13. $y(x) = x^2(4 - 3\ln(x))$

14. $y(x) = -4x^{-12}(1 + 12\ln(x))$

15. $y(x) = 3x^6 - 2x^4$

16.

$$y(x) = \frac{11}{4}x^2 + \frac{17}{4}x^{-2}$$

17. The transformation $x = e^t$ transforms the Euler equation $x^2y'' + axy' + by = 0$ into

$$Y'' + (a-1)Y' + bY = 0,$$

with characteristic equation

$$\lambda^2 + (a-1)\lambda + b = 0,$$

with roots λ_1 and λ_2. If we substitute $y = x^r$ directly into Euler's equation, we obtain

$$r(r-1)x^r + arx^r + bx^r = 0,$$

or, after dividing by x^r,

$$r^2 + (a-1)r + b = 0.$$

This equation for r is the same as the quadratic equation for λ, so its roots are $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Therefore both the transformation method, and direct substitution of $y = x^r$ into Euler's equation, lead to the same solutions.

18. If $x < 0$, use the transformation $x = -e^t$, so $t = \ln(-x) = \ln|x|$. Note that

$$\frac{dt}{dx} = \frac{1}{-x}(-1) = \frac{1}{x},$$

just as in the case that $x > 0$. With $y(x) = y(-e^t) = Y(t)$, proceeding as in the text with chain rule derivatives. First

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}Y'(t)$$

and, similarly,

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dt}{dx}Y''(t) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x^2}Y''(t). \end{aligned}$$

Then,

$$x^2y''(x) = Y''(t) - Y'(t)$$

just as in the case that x is positive. Therefore Euler's equation transforms to

$$Y'' + (A-1)Y' + BY = 0,$$

and in effect we obtain the solution of Euler's equation for negative x by replacing x with $|x|$. For example, suppose we want to solve

$$x^2y'' + xy' + y = 0$$

for $x < 0$. We know that, for $x > 0$, this Euler equation transforms to

$$Y'' + Y = 0,$$

so $Y(t) = c_1\cos(t) + c_2\sin(t)$ and

$$y(x) = c_1\cos(\ln(x)) + c_2\sin(\ln(x))$$

for $x > 0$. For $x < 0$, the solution is

$$y(x) = c_1\cos(\ln(|x|)) + c_2\sin(\ln(|x|)).$$