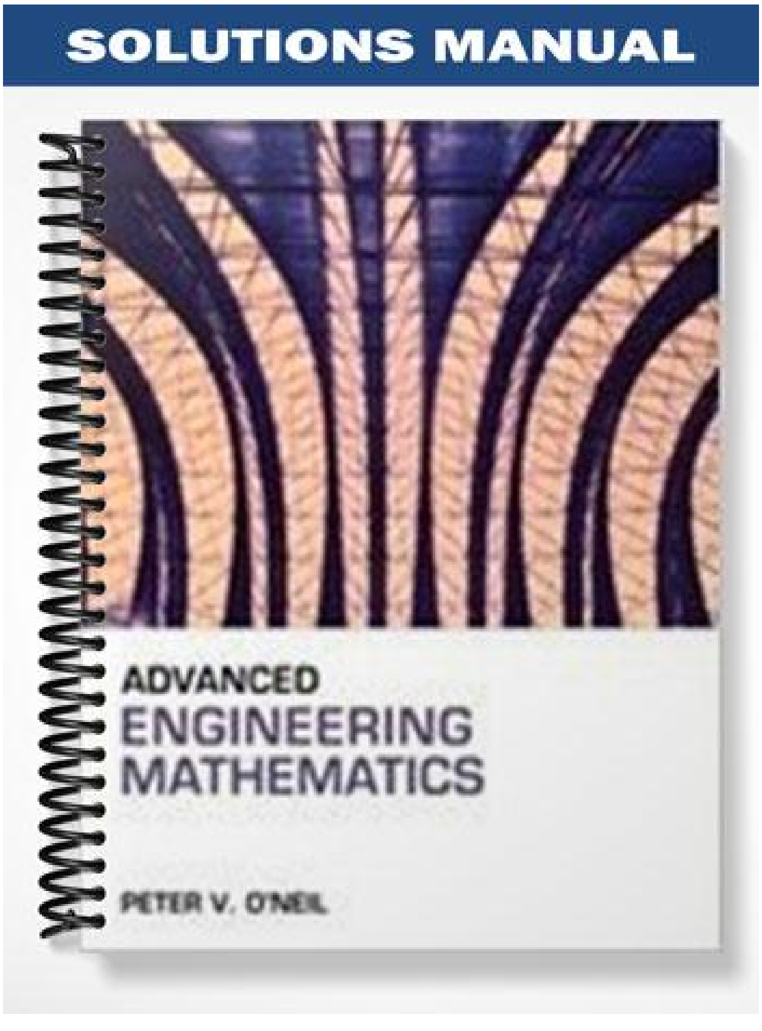
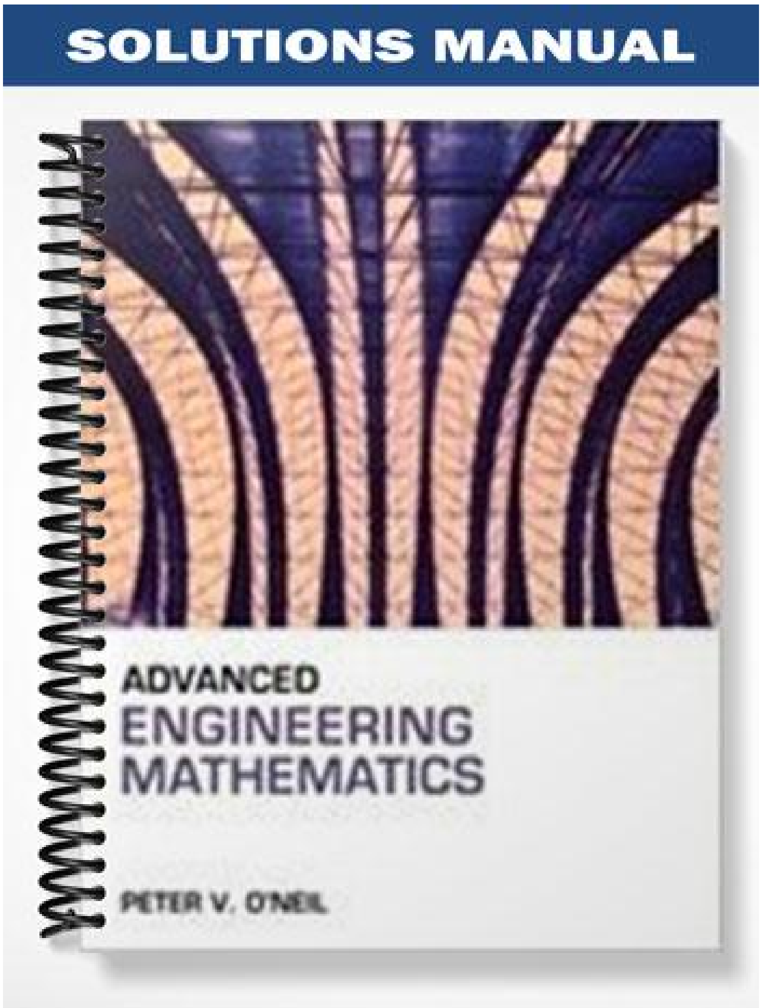

SOLUTIONS MANUAL
ADVANCED
ENGINEERING
MATHEMATICS
PETER V. O'NEIL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Chapter Two - Second Order Differential Equations

Section 2.2 Theory of Solutions

In Problems 1 - 6 direct substitution of y_1 and y_2 verifies that each is a solution of the given differential equation.

1. (b) $W = \begin{vmatrix} \cosh(2x) & \sinh(2x) \\ 2\sinh(2x) & 2\cosh(2x) \end{vmatrix} = 2[\cosh^2(2x) - \sinh^2(2x)] = 2;$
(c) $y = c_1\cosh(2x) + c_2\sinh(2x);$
(d) $y = \cosh(2x)$

2. (b) $W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} = 3;$
(c) $y = c_1\cos(3x) + c_2\sin(3x);$
(d) $y = -\frac{1}{3}\sin(3x)$

3. (b) $W = \begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix} = -5e^{-11x};$
(c) $y = c_1e^{-3x} + c_2e^{-8x};$
(d) $y = \frac{12}{5}e^{-3x} - \frac{7}{5}e^{-8x}$

4. (b) $W = \begin{vmatrix} e^{-x}\cos(\sqrt{7}x) & e^{-x}\sin(\sqrt{7}x) \\ e^{-x}(\cos(\sqrt{7}x) + \sqrt{7}\sin(\sqrt{7}x)) & e^{-x}(-\sin(\sqrt{7}x) + \sqrt{7}\cos(\sqrt{7}x)) \end{vmatrix} = \sqrt{7}e^{-x};$
(c) $y = e^{-x}[c_1\cos(\sqrt{7}x) + c_2\sin(\sqrt{7}x)];$
(d) $y = e^{-x}[2\cos(\sqrt{7}x) + \frac{2}{\sqrt{7}}\sin(\sqrt{7}x)]$

5. (b) $W = \begin{vmatrix} x^4 & x^4\ln x \\ 4x^3 & 4x^3\ln x + x^3 \end{vmatrix} = x^7;$
(c) $y = x^4(c_1 + c_2\ln(x))$;
(d) $y = 2x^4 - 4x^4\ln(x)$

6. (b) $W = \frac{2}{\pi}\begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -(\frac{\sin(x)}{\sqrt{x}} + \frac{\cos(x)}{2x\sqrt{x}}) & (\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x\sqrt{x}}) \end{vmatrix} = \frac{2}{\pi x};$
(c) $y = c_1\sqrt{\frac{2}{\pi}}\frac{\cos(x)}{\sqrt{x}} + c_2\sqrt{\frac{2}{\pi}}\frac{\sin(x)}{\sqrt{x}};$
(d) $y = -5\sqrt{\pi}\frac{\cos(x)}{\sqrt{x}} - \frac{(16\pi + 5)}{2\sqrt{\pi}}\frac{\sin(x)}{\sqrt{x}}$

7. $W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4; y_1$ and y_2 are linearly independent solutions of the differential

equation $x^2y'' - 4xy' + 6y = 0$ or $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$. Theorem 3 applies for this equations only on intervals not containing $x = 0$, and on any such interval $W = x^4 \neq 0$.

8. Clearly y_1 and y_2 are linearly independent on $[-1, 1]$ since $y_1(x) \neq ky_2(x)$. The differential equation can be written $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$, so Theorem 2.3 applies only on intervals not containing $x = 0$.

9. $y'' - y' - 2y = 0$ has solution $y_1 = e^{-x}, y_2 = e^{2x}$, but $y_1y_2 = e^x$ is not a solution.

10. Theorem 2.2 applies only to linear equations and $yy'' + 2y' - (y')^2 = 0$ is non-linear.

11. At a relative extremum of a differentiable function y, we have $y'(x_0) = 0$. Thus $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix} = 0$, and by Theorem 2.3, y_1 and y_2 are linearly dependent.

12. By Theorem 1, $y'' + p(x)y' + q(x)y = 0, y(x_0) = 0, y'(x_0) = 0$ has a unique solution, which is clearly $y(x) \equiv 0$. If $\phi'(x_0) = 0$, then $\phi(x) \equiv 0$ which contradicts the fact that ϕ is non-zero. Hence $\phi'(x_0) \neq 0$.

13. We have $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$, and by Theorem 2.3, y_1 and y_2 are linearly dependent.

Section 2.3 Reduction of Order

In problems 1 - 10 we put $y_2(x) = u(x)y_1(x)$, derive the equation satisfied by u, give its solution for $u(x)$, and give the general solution of the second order equation.

1. $u''\cos(2x) - 4\sin(2x)u' = 0; u(x) = \tan(2x); y = c_1\cos(2x) + c_2\sin(2x)$
2. $u'' + 6u' = 0; u(x) = e^{-6x}; y = c_1e^{3x} + c_2e^{-3x}$
3. $u'' = 0; u(x) = x; y = c_1e^{5x} + c_2xe^{5x}$
4. $xu'' + u' = 0; u = \ln(x); y = c_1x^4 + c_2x^4\ln(x)$.
5. $xu'' + u' = 0; u = \ln(x); y = c_1x^2 + c_2x^2\ln(x)$
6. $(2x^3 + x)u'' + 2u' = 0;\ u = 2x - \frac{1}{x};\ y = c_1x + c_2(2x^2 - 1)$.
7. $xu'' + 7u' = 0; u = x^{-6}; y = c_1x^4 + c_2x^{-2}$
8. $xu'' + \frac{2}{1+x^2}u' = 0$ which can be written as $\left[u'(\frac{x^2}{1+x^2})\right]' = 0$. Thus $u' = 1 + \frac{1}{x^2}$ and $u = \frac{x^2-1}{x}$ and $y = c_1(x^2 - 1) + c_2x$
9. $x^{-1/2}\cos(x)u'' - 2x^{-1/2}\sin(x)u' = 0; u(x) = \tan(x); y = c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\sin(x)}{\sqrt{x}}\right)$
10. $(2x^3 + 3x^2 + x)u'' + (6x^2 + 6x + 2)u' = 0$ which gives $u' = \frac{2x+1}{x^2(x+1)^2}$ and $u = \frac{1}{x(x+1)}$ and the general solution is $y = c_1x + c_2\left(\frac{1}{x+1}\right)$

11. $y = c_1e^{-ax} + c_2xe^{-ax}$

12. (a) With $u = y'$ we get $xu' - u = 2$, first order linear with solution $u = -\frac{2}{x} + c_1x$. Integrate to get $y = \int \left(-\frac{2}{x} + c_1x\right) dx = -2\ln|x| + \hat{c}_1x^2 + c_2$

(b) $xu' + 2u = x$ gives $u = \frac{x^2}{4} + \frac{c_1}{x^2}$ and then $y = \frac{x^2}{6} + \frac{\hat{c}_1}{x} + c_2$.

(c) $1 - u = 4u'$ gives $u = c_1e^{-x/4} + 1$ and then $y = x + \hat{c}_1e^{-x/4} + c_2$

(d) $u' + u^2 = 0$ gives $u = \frac{1}{(x + c_1)}$ and then $y = \ln|x + c_1| + c_2$

(e) $u' = 1 + u^2$ gives $u = \tan(x + c_1)$ and then $y = \ln|\sec(x + c_1)| + c_2$

13. (a) $yu\frac{du}{dy} + 3u^2 = 0$ is separable as $\frac{du}{u} = -\frac{3dy}{y}$. Integration gives $\ln|u| = -3\ln|y| + c$ or $uy^3 = A$. Thus $y^3dy = Adx$ and $\frac{y^4}{4} = Ax + B$ or $y^4 = c_1x + c_2$

(b) $(y - 1)e^y = c_1x + c_2$ or $y = c_3$

(c) $y = \frac{c_1e^{c_1x}}{c_2 - e^{c_1x}}$ or $y = \frac{1}{c_3 - x}$

(d) $y = \ln|\sec(x + c_1)| + c_2$

(e) $y = \ln|c_1x + c_2|$

14. With $y = uy_1$ we get $y'' + Ay' + By = \left[u'' - Au' + \frac{A^2}{4}u + A\left(u' - \frac{A}{2}u\right) + Bu\right]e^{-Ax/2} = u''e^{-Ax/2} = 0$ iff $u'' = 0$. Thus $u = c_1 + c_2x$and $y = c_1e^{-Ax/2} + c_2xe^{-Ax/2}$.

15. With $y = uy_1$ we get $y'' + \frac{A}{x}y' + \frac{B}{x^2}y =$
$\left[u''x^2 + (1 - A)xu' - \left(\frac{1-A}{2}\right)\left(\frac{1+A}{2}\right)u + A\left(xu' + \frac{(1-A)}{2}u\right) + Bu\right]x^{-(3+A)/2} = [xu'' + u']x^{(1+A)/2} = 0$ iff $xu'' + u' = 0$. Thus $u = c_1 + c_2\ln(x)$and $y = c_1x^{(1-A)/2} + c_2x^{(1-A)/2}\ln(x)$.

Section 2.4 The Constant Coefficient Homogeneous Linear Equation

1. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$ which has roots $\lambda = -2$ and $\lambda = 3$; thus the general solution is $y = c_1e^{-2x} + c_2e^{3x}$

2. The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$ which has roots $\lambda = 1 + 3i$ and $\lambda = 1 - 3i$; thus the general solution is $y = e^x[c_1\cos(3x) + c_2\sin(3x)]$

3. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$ which has repeated roots $\lambda = -3$ and $\lambda = -3$; thus the general solution is $y = e^{-3x}[c_1 + c_2x]$

4. The characteristic equation is $\lambda^2 - 3\lambda = 0$ which has roots $\lambda = 0$ and $\lambda = 3$; thus the general solution is $y = c_1 + c_2e^{3x}$

5. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$ which has roots $\lambda = -5 + i$ and $\lambda = -5 - i$; thus the general solution is $y = e^{-5x}[c_1\cos(x) + c_2\sin(x)]$

6. The characteristic equation is $\lambda^2 + 6\lambda - 40 = 0$ which has roots $\lambda = -10$ and $\lambda = 4$; thus the general solution is $y = c_1e^{-10x} + c_2e^{4x}$

7. The characteristic equation is $\lambda^2+3\lambda+18=0$ which has roots $\lambda=-\frac{3}{2}+i\frac{3\sqrt{7}}{2}$ and $\lambda=-\frac{3}{2}-i\frac{3\sqrt{7}}{2}$; thus the general solution is $y=e^{-\frac{3}{2}x}\left[c_1\cos\left(\frac{3\sqrt{7}}{2}x\right)+c_2\sin\left(\frac{3\sqrt{7}}{2}x\right)\right]$

8. The characteristic equation is $\lambda^2+16\lambda+64=0$ which has repeated roots $\lambda=-8$ and $\lambda=-8$; thus the general solution is $y=e^{-8x}[c_1+c_2x]$

9. The characteristic equation is $\lambda^2-14\lambda+49=0$ which has repeated roots $\lambda=7$ and $\lambda=7$; thus the general solution is $y=e^{7x}[c_1+c_2x]$

10. The characteristic equation is $\lambda^2-6\lambda+7=0$ which has roots $\lambda=3+i\sqrt{2}$ and $\lambda=3-i\sqrt{2}$; thus the general solution is $y=e^{3x}[c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}]$

11. The characteristic equation is $\lambda^2+4\lambda+9=0$ which has roots $\lambda=-2+i\sqrt{5}$ and $\lambda=-2-i\sqrt{5}$; thus the general solution is $y=e^{-2x}[c_1\cos(\sqrt{5}x)+c_2\sin(\sqrt{5}x)]$

12. The characteristic equation is $\lambda^2+5\lambda=0$ which has roots $\lambda=0$ and $\lambda=-5$; thus the general solution is $y=c_1+c_2e^{-5x}$

13. $y=5-2e^{-3x}$

14. $y=4e^{x}+2e^{-3x}$

15. $y=0$ for all x

16. $y=e^{2x}[3-x]$

17. $y=\frac{9}{7}e^{3(x-2)}+\frac{5}{7}e^{-4(x-2)}$

18. $y=\frac{\sqrt{6}}{4}e^{x}[e^{\sqrt{6}x}-e^{-\sqrt{6}x}]$

19. $y=e^{(x-1)}[29-17x]$

20. $y=-4(5-\sqrt{23})e^{5(x-2)/2}\sin\left(\frac{\sqrt{23}}{2}(x-2)\right)$

21. $y=e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right)+\frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$

22. (a) $\phi=e^{ax}[c_1+c_2x]$; (b) $\phi_\epsilon=e^{ax}[c_1e^{\epsilon x}+c_2e^{-\epsilon x}]$; (c) $\lim_{\epsilon\to 0}\phi_\epsilon(x)=e^{ax}[c_1+c_2]\neq\phi(x)$ in general.

23. (a) $\psi=e^{ax}[c+(d-ac)x]$

(b) $\psi_\epsilon=e^{ax}\left[\frac{(d-ac+\epsilon c)e^{\epsilon x}+(ac-d+\epsilon c)e^{-\epsilon x}}{2\epsilon}\right]$

(c) $\lim_{\epsilon\to 0}\psi_\epsilon(x)=\frac{e^{ax}}{2}\lim_{\epsilon\to 0}[x(d-ac+\epsilon c)e^{\epsilon x}-x(ac-d+\epsilon c)e^{-\epsilon x}+c(e^{\epsilon x}+e^{-\epsilon x})]=e^{ax}[c+(d-ac)x]=\psi(x)$, by L'Hopital's rule.

24. The characteristic equation has roots $\lambda_1=-\frac{A}{2}+\frac{1}{2}\sqrt{A^2-4B}$ and $\lambda_2=-\frac{A}{2}-\frac{1}{2}\sqrt{A^2-4B}$. With $B>0$ we have $A^2-4B<A^2$, so λ_1 and λ_2 are either both negative real numbers or are complex conjugates with negative real part. In the first case $\phi(x)=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$ and $\lim_{x\to+\infty}\phi(x)=0$.

In the second case

$$\phi(x) = e^{-\frac{A}{2}x}\left[c_1\cos\left(\frac{1}{2}\sqrt{4B-A^2}x\right) + c_2\sin\left(\frac{1}{2}\sqrt{4B-A^2}x\right)\right]$$

so

$$|\phi(x)| \le e^{-\frac{A}{2}}\sqrt{c_1^2+c_2^2} \text{ and } \lim_{x\to\infty}|\phi(x)| = 0, \text{hence } \lim_{x\to\infty}\phi(x) = 0.$$

Section 2.5 Euler's Equation

1. $y = c_1x^2 + c_2x^{-3}$
2. $y = x^{-1}[c_1 + c_2\ln(x)]$
3. $y = c_1\cos[2\ln(x)] + c_2\sin[2\ln(x)]$
4. $y = c_1x^2 + c_2x^{-2}$
5. $y = c_1x^4 + c_2x^{-4}$
6. $y = x^{-2}\{c_1\cos[3\ln(x)] + c_2\sin[3\ln(x)]\}$
7. $y = c_1x^{-2} + c_2x^{-3}$
8. $y = x^2\{c_1\cos[7\ln(x)] + c_2\sin[7\ln(x)]\}$
9. $y = x^{-12}[c_1 + c_2\ln(x)]$
10. $y = c_1x^7 + c_2x^5$
11. $y = x^{3/2}\left\{c_1\cos\left[\frac{\sqrt{39}}{2}\ln(x)\right] + c_2\sin\left[\frac{\sqrt{39}}{2}\ln(x)\right]\right\}$
12. $y = x^{1/2}\left\{c_1\cos\left[\frac{\sqrt{15}}{2}\ln(x)\right] + c_2\sin\left[\frac{\sqrt{15}}{2}\ln(x)\right]\right\}$
13. $y = x^{-2}\{3\cos[4\ln(-x)] - 2\sin[4\ln(-x)]\}$
14. $y = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}$
15. $y = -3 + 2x^2$
16. $y = x^2[4 - 3\ln(x)]$
17. $y = -x^{-3}\cos[2\ln(-x)]$
18. $y = \frac{7}{2}\left(\frac{x}{2}\right)^{-1} - \frac{5}{2}\left(\frac{x}{2}\right)$
19. $y = -4x^{-12}[1 + 12\ln(x)]$
20. $y = 3x^6 - 2x^4$
21. $y = \frac{11}{4}x^2 + \frac{17}{4}x^{-2}$
22. The transformation $x = e^t$ transforms the Euler equation $x^2y'' + Axy' + By = 0$ into $Y'' + (A-1)Y' + BY = 0$. Let $\lambda_1$and λ_2 designate the characteristic roots of this constant coefficient equation. Suppose on the other hand we substitute x^r directly into $x^2y''+Axy'+By = 0$ to get $r(r-1)x^r + Arx^r + Bx^r = [r^2 + (A-1)r + B]x^r = 0$. Then r must satisfy $r^2 + (A-1)r + B = 0$ and the values of r are exactly $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Thus both the transformation method, $x = e^t$, and direct substitution of x^r lead to the same solutions.

Section 2.6 The Nonhomogeneous Equation

1. By variation of parameters with $y_1 = \cos(x), y_2 = \sin(x)$, and $f(x) = \tan(x)$ we get $u' = \dfrac{-\sin^2(x)}{\cos(x)} = \cos(x) - \sec(x), v' = \sin(x)$. Thus $u(x) = \sin(x) - \ln|\sec(x) + \tan(x)| + c_1$ and $v(x) = -\cos(x)+c_2$ and the general solution is $y = c_1\cos(x)+c_2\sin(x)-\cos(x)\ln|\sec(x)+\tan(x)|$.

2. $y = c_1e^x + c_2e^{3x} + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3)$

3. $y = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\ln[\cos(3x)]\cos(3x)$ by variation of parameters.

4. First write $2\sin^2(x) = 1 - \cos(2x)$ and get $y = c_1e^{3x} + c_2e^{-x} - \frac{1}{3} + \frac{7}{65}\cos(2x) + \frac{4}{65}\sin(2x)$

5. $y = c_1e^x + c_2e^{2x} - e^{2x}\cos(e^{-x})$

6. First write $8\sin^2(4x) = 4\cos(8x) - 4$ to get $y = c_1e^{3x}+c_2e^{2x}+\frac{2}{3}+\frac{58}{1241}\cos(8x)+\frac{40}{1241}\sin(8x)$

7. $y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4$

8. $y = c_1e^{3x} + c_2e^{-2x} - 2e^{2x}$

9. $y = e^x[c_1\cos(3x) + c_2\sin(3x)] + 2x^2 + x - 1$

10. $y = e^{2x}[c_1\cos(x) + c_2\sin(x)] + 21e^{2x}$

11. $y = c_1e^{2x} + c_2e^{4x} + e^x$

12. $y = e^{-3x}[c_1 + c_2x] + \frac{1}{2}\sin(3x)$

13. $y = c_1e^x + c_2e^{2x} + 3\cos(x) + \sin(x)$

14. $y = c_1 + c_2e^{-4x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{2}{3}e^{3x}$

15. $y = e^{2x}[c_1\cos(3x) + c_2\sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}$

16. $y = e^x[c_1 + c_2x] + 3x + 6 + \frac{3}{2}\cos(3x) - 2\sin(3x)$

17. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-x} + \frac{1}{3}xe^{2x}$

18. By variation of parameters $y = c_1x^2 + c_2x^{-6} - \frac{1}{12}\ln(x) - \frac{1}{36}$

19. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{6}x - \frac{1}{36}$

20. First write $2\sinh^2(x) = \cosh(2x) - 1$ and use undetermined coefficients to get $y = c_1e^{4x} + c_2e^{-3x} - \frac{2}{15}\cosh(2x) + \frac{1}{30}\sinh(2x) - \frac{1}{12}$

21. By variation of parameters $y = c_1x^2 + c_2x^4 + x$

22. By variation of parameters $y = x^{-1}[c_1 + c_2\ln(x)] + 2x^{-1}\ln^2(x)$

23. By variation of parameters $y = c_1\cos[2\ln(x)] + c_2\sin[2\ln(x)] - \frac{1}{4}\cos[2\ln(x)]\ln(x)$

24. By variation of parameters $y = c_1x^2 + c_2x^{-3} + \frac{1}{5}x^2\ln(x) + \frac{1}{3}$

25. By undetermined coefficients $y = c_1e^{2x} + c_2xe^{2x} + e^{3x} - \frac{1}{4}$

25. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-x} - \frac{1}{2}x - \frac{1}{4}$

27. $y = \frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x$

28. $y = 3 + 2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x$

29. $y = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$

30. $y = \frac{1}{5} + e^{3x} - \frac{1}{5}e^{2x}[\cos(x) + 3\sin(x)]$

31. $y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}$

32. $y = e^{3x} - xe^{3x} + 2x^2e^{3x}$

33. $y = -\frac{17}{4}e^{2x} + \frac{55}{13}e^{3x} + \frac{1}{52}\cos(2x) - \frac{5}{52}\sin(2x)$

34. The general solution is given by $y = e^{x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1$, but this can also be written in the form $y = e^{x/2}\left[d_1\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1$ to facilitate fitting the initial conditions specified at $x = 1$. We get $y(1) = e^{1/2}d_1 + 1 = 4$, and $y'(1) = \frac{1}{2}e^{1/2}d_1 + \frac{\sqrt{3}}{2}e^{1/2}d_2 = -2$. We find $d_1 = 3e^{-1/2}$and $d_2 = -\frac{7}{\sqrt{3}}e^{-1/2}$.
The general solution can be written in the form

$$y = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

35. The general solution is given by $y = e^{4x}[c_1e^{\sqrt{14}x} + c_2e^{-\sqrt{14}x}] + \frac{e^{-x}}{11}$. By the results of Problem 27, Section 2.4 and properties of the hyperbolic functions this solution can be written $y = e^{4x}[A\cosh(\sqrt{14}(x+1) + \frac{B}{\sqrt{14}}\sinh(\sqrt{14}(x+1))] + \frac{e^{-x}}{11}$. This form will greatly facilitate fitting the initial conditions specified at $x = -1$. We get $y(-1) = Ae^{-4} + \frac{e}{11} = 5$ and $y'(-1) = 4Ae^{-4} + Be^{-4} - \frac{e}{11} = 2$. Solving for A and B gives the general solution

$$y = \frac{e^{4(x+1)}}{11}[(55-e)\cosh(\sqrt{14}(x+1))$$

$$+\frac{(5e-198)}{\sqrt{14}}\sinh(\sqrt{14}(x+1))] + \frac{e^{-x}}{11}.$$

36. $y = \frac{1}{100}[e^{-3x}(108 - 270x) - 8\cos(x) - 6\sin(x)]$.

37. $y = 4e^{-x} - \sin^2(x) - 2$

38. $y = 4\cos(x) + 4\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$

39. $y = 2x^3 + x^{-2} - 2x^2$

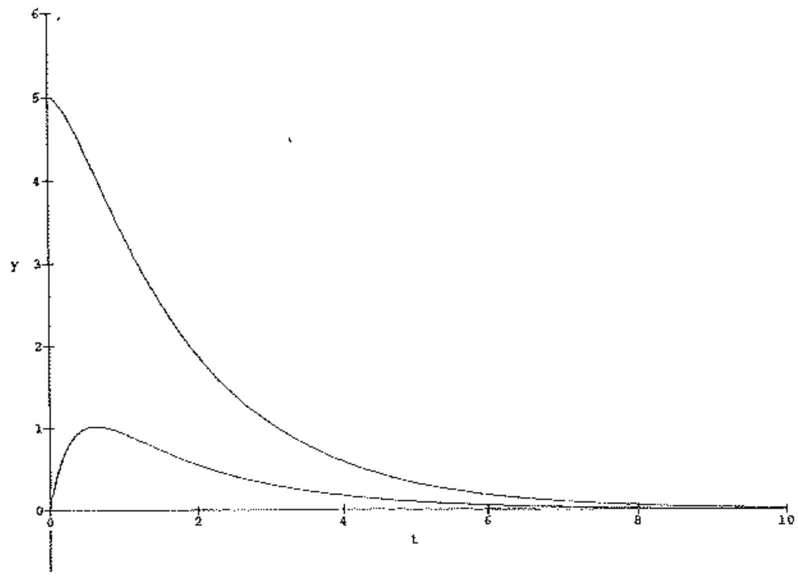
40. $y = 3x^{-3} + 2x^{-3}\ln(x) + 3\ln(x) - 2$

41. $y = x - x^2 + 3\cos[\ln(x)] + \sin[\ln(x)]$

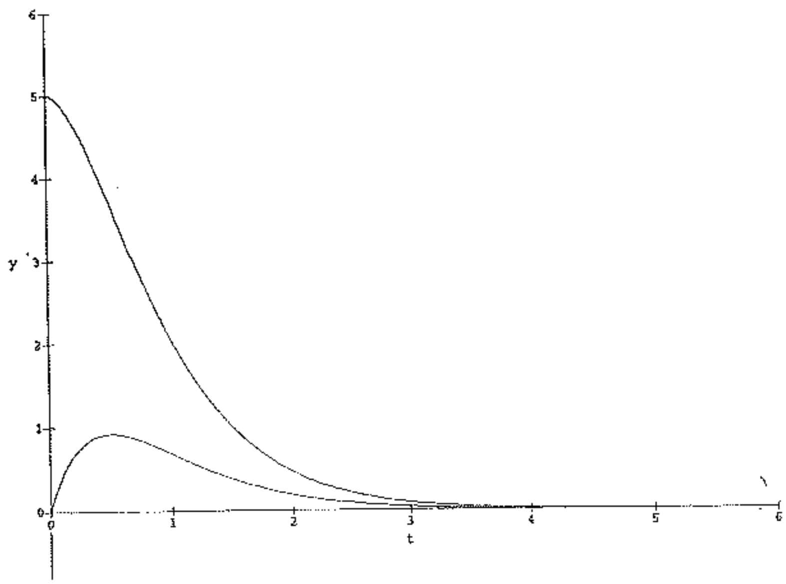
42. $y = (e^2 - 2)x^2 + \left(\frac{5}{4} - e^2\right)x^3 + x^2e^x$

Section 2.7 Applications

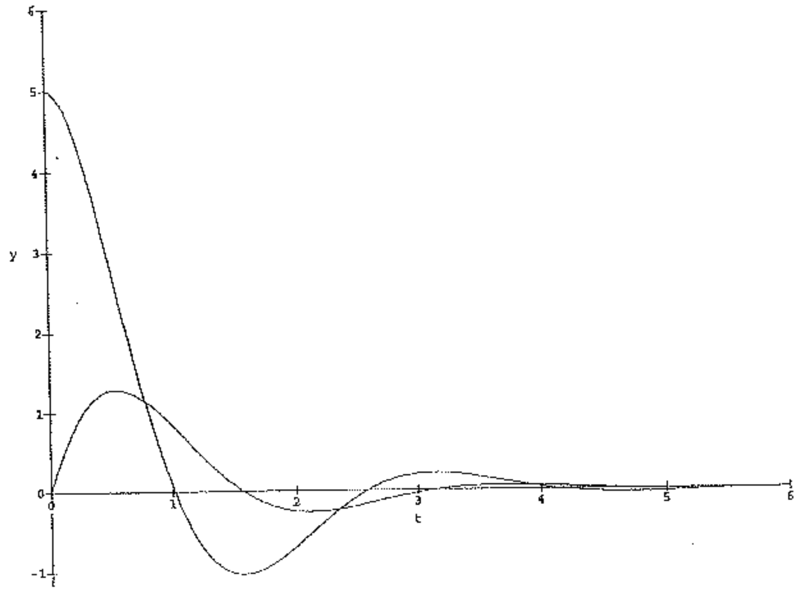
1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is $y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)]$; with initial conditions $y(0) = 0, y'(0) = 5$ is $y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t)$.



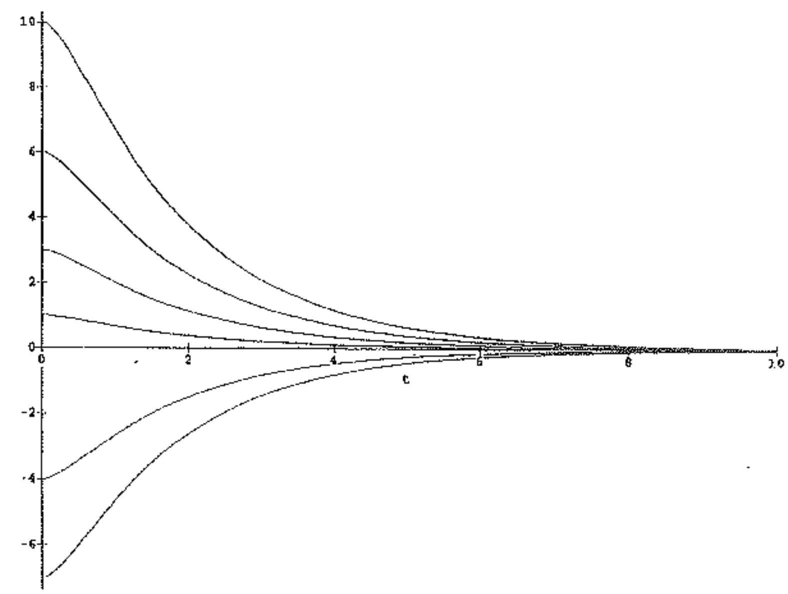
2. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is $y_1(t) = 5e^{-2t}(1 + 2t)$; with initial conditions $y(0) = 0, y'(0) = 5$ is $y_2(t) = 5te^{-2t}$.



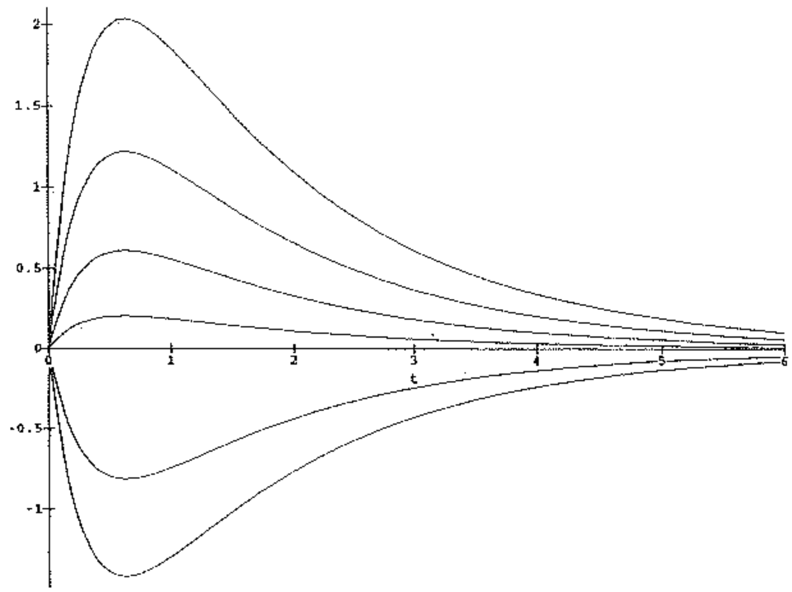
3. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is $y_1(t) = \frac{5}{2}e^{-t}[2\cos(2t) + \sin(2t)]$; with initial conditions $y(0) = 0, y'(0) = 5$ is $y_2(t) = \frac{5}{2}e^{-t}\sin(2t)$.



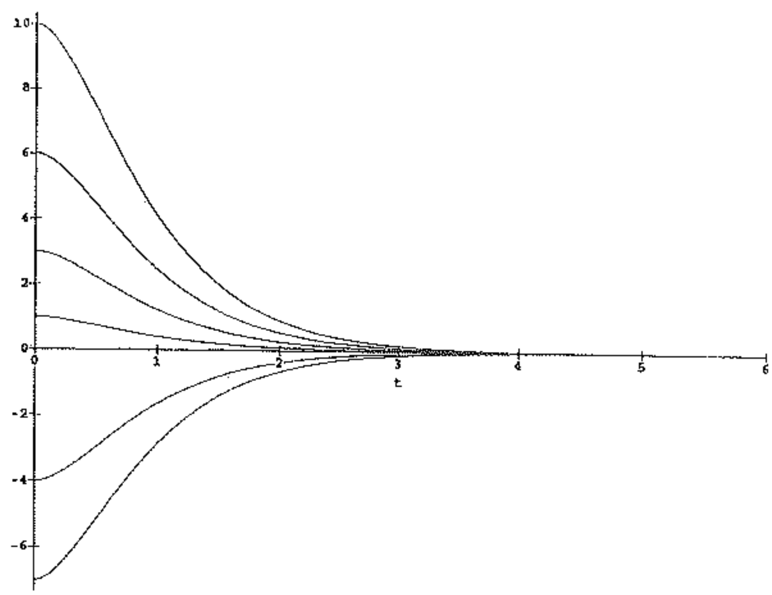
4. The solution is $y(t) = Ae^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)]$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



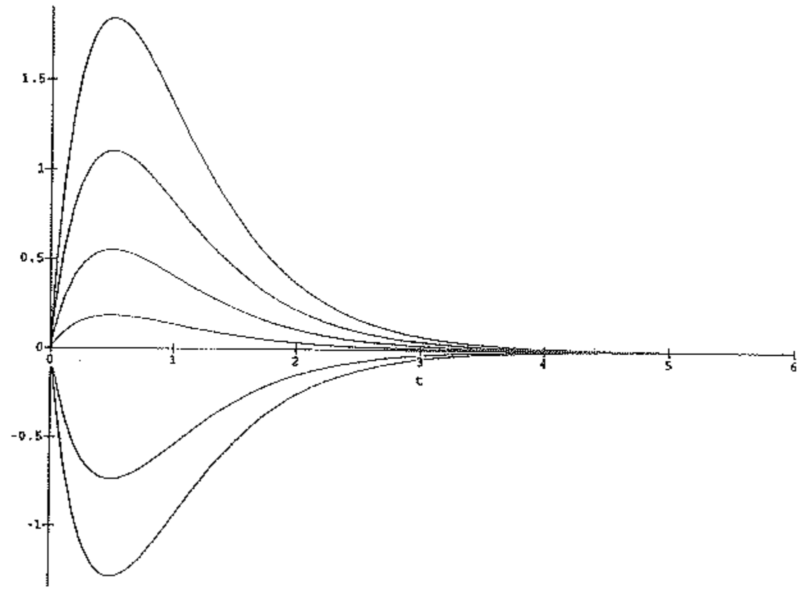
5. The solution is $y(t) = \dfrac{A}{\sqrt{2}} e^{-2t} \sinh(\sqrt{2}t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



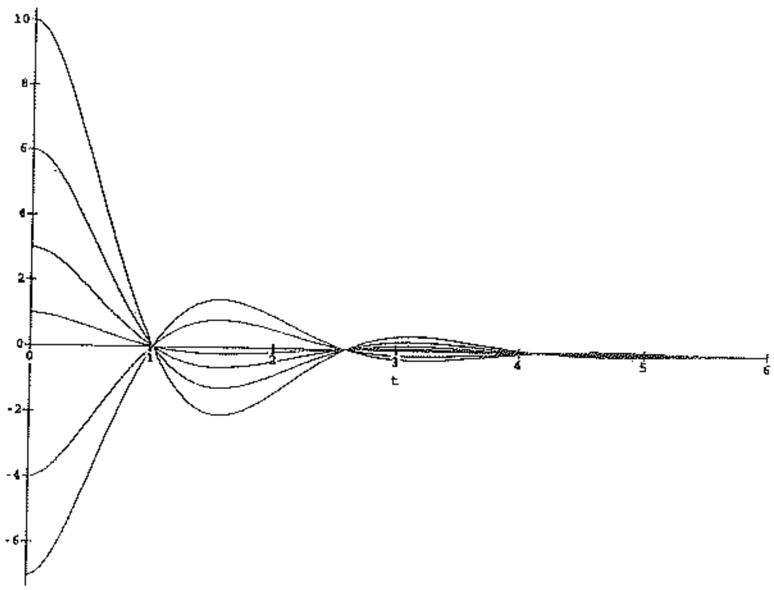
6. The solution is $y(t) = Ae^{-2t}(1+2t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



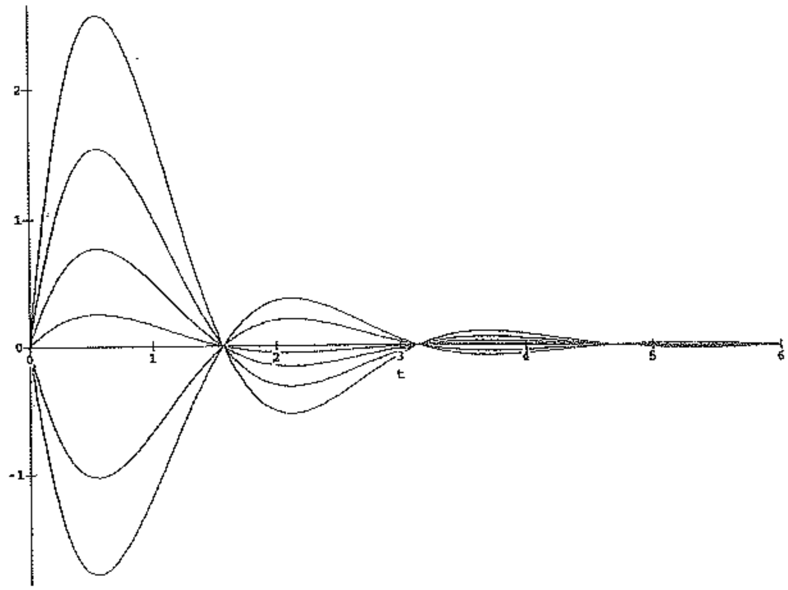
7. The solution is $y(t) = Ate^{-2t}$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



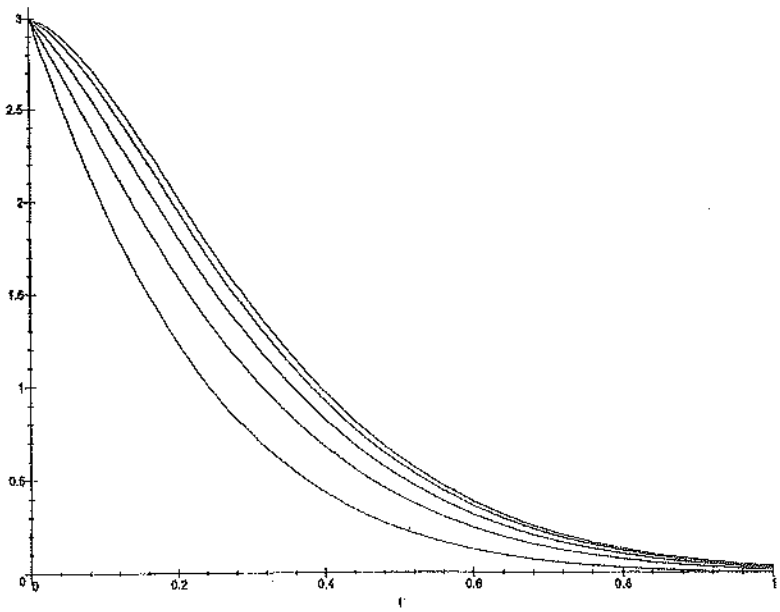
8. The solution is $y(t) = \dfrac{A}{2}e^{-t}[2\cos(2t)+\sin(2t)]$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



9. The solution is $y(t) = \dfrac{A}{2}e^{-t}\sin(2t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.

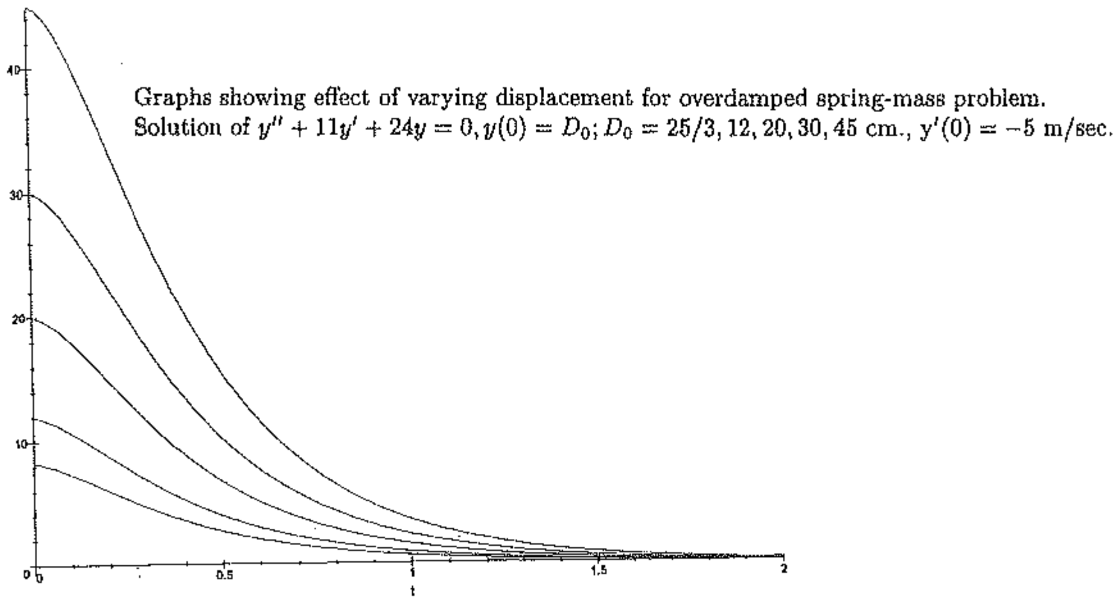


10. From Newton's second law we have $y'' = \sum \text{forces} = -29y - 10y'$, so the motion is described by the solution of $y''+10y'+29y = 0, y(0) = 3, y'(0) = -1$. The solution of this underdamped case is $y(t) = e^{-5t}[3\cos(2t) + 7\sin(2t)] = \sqrt{58}e^{-5t}\cos(2t - \phi)$, where $\phi = \tan^{-1}(7/3)$. Comparative graphs are shown below for $y(0) = 3$ cm. $y'(0) = -1, -2, -4, -7, -12$ cm./sec. (recall down is the positive direction).



Graphs showing effect of varying initial velocity for underdamped spring-mass problem.
Solution of $y'' + 10y' + 29y = 0, y(0) = 3, y'(0) = V_0; V_0 = -1, -2, -4, -7, -12$ cm/sec.

11. The motion is described by the solution of $y'' + 11y' + 24y = 0; y(0) = \dfrac{1}{12}$ m, $y'(0) = -5$ m/sec. The displacement is $y(t) = \dfrac{1}{60}[57e^{-8t} - 52e^{-3t}]$. Comparative graphs are shown below for $y(0) = \frac{25}{3}, 12, 20, 30, 45$ cm; $y'(0) = -5$ m/sec.



12. Since one pound $\doteq 4.45$ Newton and one inch $= 2.54$ cm we calculate the spring modulus $k = 4$ lbs/in $= \dfrac{4(4.45)}{.0254}$ Newton/m ≈ 700 Newton/m. The equation in the mks system will be $7y'' + 700y = 0; y(0) = 0, y'(0) = -4$ with solution $y = -\dfrac{2}{5}\sin(10t)$ meters.

13. For overdamped motion, the displacement is given by $y(t) = e^{-\alpha t}(A + Be^{\beta t})$ where $\alpha > 0$ ($-\alpha$ is the smaller characteristic root) and $\beta > 0$ is the difference $\beta =$ (larger root - smaller root). The factor $A + Be^{\beta t}$ could be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by given initial conditions, in fact if $y_0 = y(0)$ and $v_0 = y'(0)$ we have $A + B = y_0$ and $-\alpha(A + B) + \beta B = v_0$. With a bit of algebra we find $-\dfrac{A}{B} = 1 - \dfrac{\beta y_0}{v_0 + \alpha y_0}$. To ensure that $-\dfrac{A}{B} \leq 1$ we see that no condition on only y_0 will be sufficient. If we also specify that $v_0 > -\alpha y_0$, this will ensure that the overdamped bob never passes through the equilibrium point.

14. For critically damped motion, the displacement is given by $y(t) = e^{-\alpha t}(A + Bt)$ with $\alpha > 0$ and A and B determined by the initial conditions. From the linear factor we see that the bob could pass through equilibrium at most once, and will for some $t > 0$ if and only if $B \neq 0$ and $AB < 0$. Now note that $y_0 = y(0) = A$ and $v_0 = y'(0) = -\alpha A + B$. Thus to ensure that the bob never passes through equilibrium we need $AB > 0$, which becomes $(v_0 + \alpha y_0)y_0 > 0$. No condition on $y_0 = y(0)$ alone can ensure this. We would also need to specify $v_0 > -\alpha y_0$, and this will ensure that the critically damped bob never passes through the equilibrium point.

15. For underdamped motion we have $y(t) = e^{-ct/2m}[c_1 \cos(\sqrt{4km - c^2}t/2m) + c_2 \sin(\sqrt{4km - c_2}t/2m)]$ which has frequency $\omega = \dfrac{\sqrt{4km - c^2}}{2m}$. Thus increasing c decreases the frequency and decreasing c increases the frequency.

16. For critical damping $y(t) = e^{-ct/2m}(A + Bt)$. For maximum displacement we need $y'(t^*) = 0$ from which we find $t^* = \dfrac{2mB - cA}{Bc}$. Now $y(0) = A, y'(0) = -\dfrac{Ac}{2m} + B$ and since we are given $y(0) = y'(0) \neq 0$ we find $t^* = \dfrac{4m^2}{2mc + c^2}$, independent of $y(0)$. The maximum displacement is $y(t^*) = \dfrac{y(0)}{c}(2m + c)e^{-(\frac{2m}{2m+c})}$.

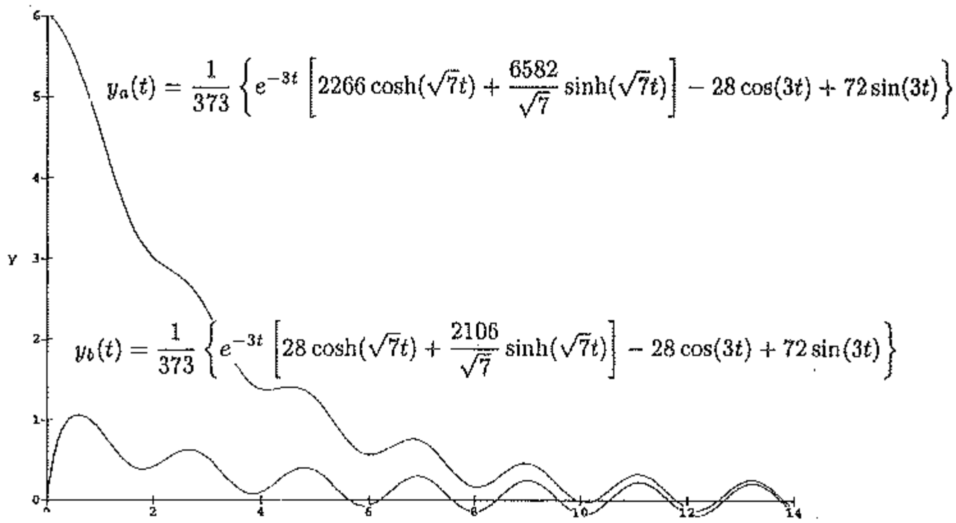
17. In the case of undamped motion, $y = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, therefore $y'' = -\omega^2 y$. When $y = d, y'' = a$, so $\omega^2 = -\dfrac{a}{d}$, or $\omega = \sqrt{-\dfrac{a}{d}}$. The period of the motion is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{-\dfrac{d}{a}}$. Note that d and a (as scalars) will have opposite signs.

18. The period of the original system with mass m_1 is $p = 2\pi\sqrt{\frac{m_1}{k}}$. The new system with mass $(m_1 + m_2)$ will have period $p_2 = 2\pi\sqrt{\frac{m_1+m_2}{k}} = 2\pi\sqrt{\frac{m_1}{k}}\sqrt{1 + \frac{m_2}{m_1}} = p\sqrt{1 + \frac{m_2}{m_1}}$.

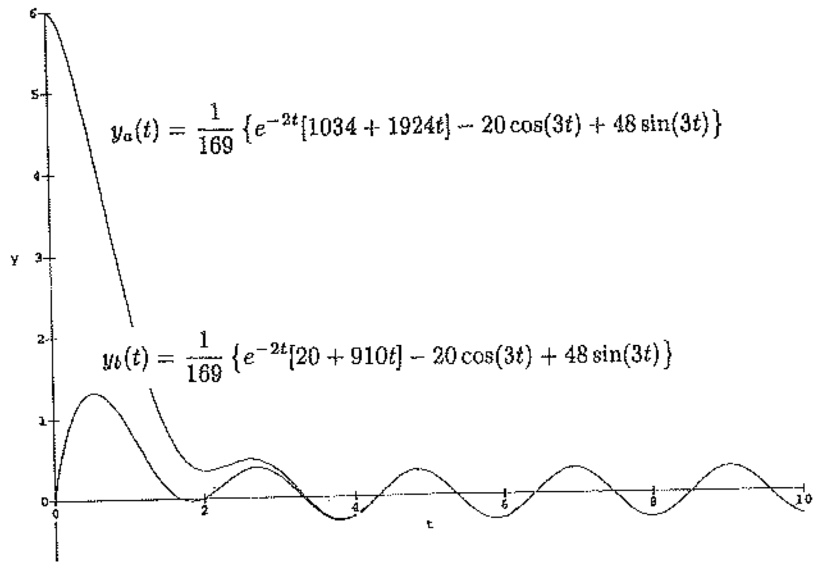
19. With $\omega \neq \omega_0$ the solution of $y'' + \omega_0^2 y = \dfrac{A}{m}\cos(\omega t); y(0) = y'(0) = 0$ is $y(t) = \dfrac{A}{m}\left[\dfrac{\cos(\omega_0 t) - \cos(\omega t)}{(\omega^2 - \omega_0^2)}\right]$. Letting $\omega \to \omega_0$ and using L'Hopitals' rule we find $\lim\limits_{\omega \to \omega_0} y(t) = \dfrac{A}{2m\omega_0} t \sin(\omega_0 t)$ which is the solution of $y'' + \omega_0^2 y = \dfrac{A}{m}\cos(\omega_0 t); y(0) = y'(0) = 0$.

20. The spring constant is $k = 16 \cdot \dfrac{11}{8} = 22$ pounds/ft, mass m$= \dfrac{16}{32} = \dfrac{1}{2}$ slug so the equation of motion is $\dfrac{1}{2}y'' + 2y' + 22y = 4\cos(\omega t)$. The general solution of this equation is $y(t) = e^{-2t}[A\cos(2\sqrt{10}t) + B\sin(2\sqrt{10}t)] + \dfrac{8}{(44 - \omega^2)^2 + 16\omega^2}\{(44 - \omega^2)\cos(\omega t) + 4\omega \sin(\omega t)\}$. As $t \to +\infty$, the exponential term dies out and the steady state solution can be written $y_{ss} = \dfrac{8}{\sqrt{(44 - \omega^2)^2 + 16\omega^2}}\cos(\omega t + \delta)$. The amplitude is maximized when ω is chosen to minimize the radicand $(44 - \omega^2)^2 + 16\omega^2 = (\omega^2 - 36)^2 + 640$. From this form we see we should choose $\omega = 6$ to get maximum amplitude of $\dfrac{8}{\sqrt{640}} = \dfrac{1}{\sqrt{10}}$ feet.

21. The general solution of the overdamped problem $y'' + 6y' + 2y = 4\cos(3t)$ can be written as $y(t) = e^{-3t}[c_1 \cosh(\sqrt{7}t) + c_2 \sinh(\sqrt{7}t)] - \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t)$.
(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{2266}{373}$ and $c_2 = \frac{6582}{373\sqrt{7}}$ and unique solution $y_a(t) = \frac{1}{373}\{e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)\}$
(b) The initial conditions $y(0) = 0, y'(0) = 6$ give $c_1 = \frac{28}{373}$ and $c_2 = \frac{2106}{373}$ and unique solution $y_b(t) = \frac{1}{373}\{e^{-3t}[28\cosh(\sqrt{7}t) + \frac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)\}$.
(c) These two solutions are graphed on the axes below.



22. The general solution of the critically damped problem $y'' + 4y' + 4y = 4\cos(3t)$ is given by $y(t) = e^{-2t}[c_1 + c_2 t] - \dfrac{20}{169}\cos(3t) + \dfrac{48}{169}\sin(3t)$.
(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{1034}{169}$ and $c_2 = \frac{1924}{169}$ and unique solution $y_a(t) = \dfrac{1}{169}\{e^{-2t}[1034 + 1924t] - 20\cos(3t) + 48\sin(3t)\}$
(b) The initial conditions $y(0) = 0$ and $y'(0) = 6$ give $c_1 = \frac{20}{169}$ and $c_2 = \frac{910}{169}$ and unique solution $y_b(t) = \dfrac{1}{169}\{e^{-2t}[20 + 910t] - 20\cos(3t) + 48\sin(3t)\}$.
(c) These two solutions are graphed on the axes below.

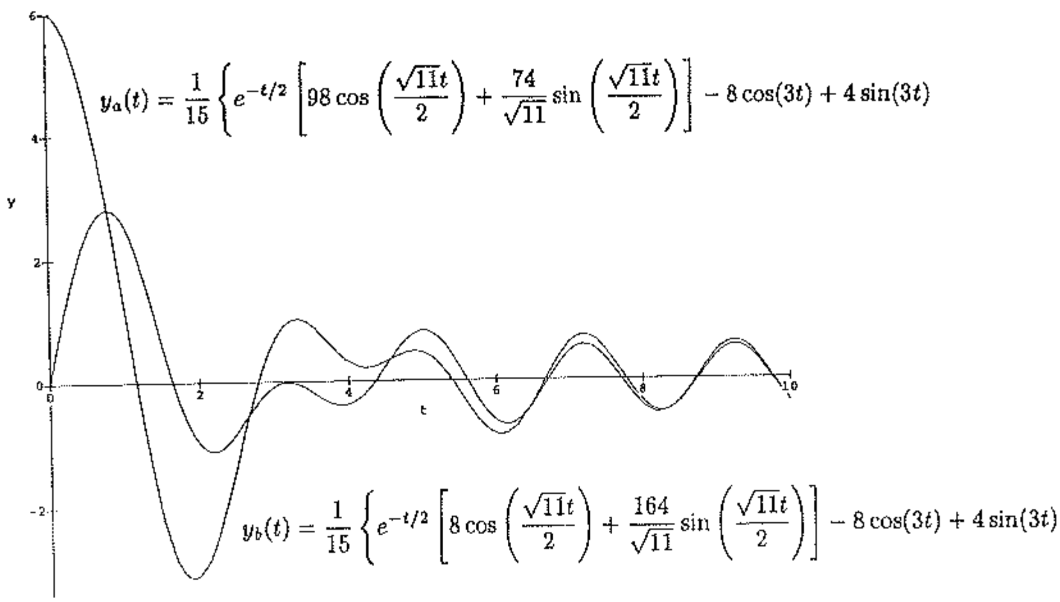


23. The general solution of the underdamped problem $y'' + y' + 3y = 4\cos(3t)$ is given by $y(t) = e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{11}t}{2}\right) + c_2\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t)$.

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{98}{15}$ and $c_2 = \frac{74}{15\sqrt{11}}$ and unique solution $y_a(t) = \frac{1}{15}\left\{e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right\}$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give $c_1 = \frac{8}{15}$ and $c_2 = \frac{164}{15\sqrt{11}}$ and unique solution $y_b(t) = \frac{1}{15}\left\{e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right\}$

(c) These two solutions are graphed on the axes below.



In Problems 24 through 27 the RLC circuit driven by the potential $E(t)$ is modeled by the differential equation $Lq'' + Rq' + \frac{1}{C}q = E(t)$ for charge q. Since current $i = q'$ this can be written $Li' + Ri + \frac{1}{C}q = E(t)$ and by differentiation we get the second order equation $Li'' + Ri' + \frac{1}{C}i = E'(t)$. With $q(0) = i(0) = 0$ we get $i'(0+) = \frac{E(0+)}{L}$ as the second initial condition for the current problem. In the answers below, some terms with exceedingly small coefficients ($\approx 10^{-6}$) have been dropped so initial conditions may not be satisfied exactly.

24. $i(t) = -0.005027e^{-0.8337t} + (0.001003t + 0.005027)e^{-t}$

25. $i(t) = -0.000938e^{-0.0625t} + 0.018000e^{-3333.27t} - 0.000862\cos(20t) + 0.299998\sin(20t)$

26. $i(t) = 0.0007695e^{-0.1334t} - 0.0007704e^{-t}$

27. $i(t) = -0.000511e^{-0.3176t} + 0.001633e^{-t} - 0.001121e^{-t}\cos(6t) = 0.000044e^{-t}\sin(6t)$