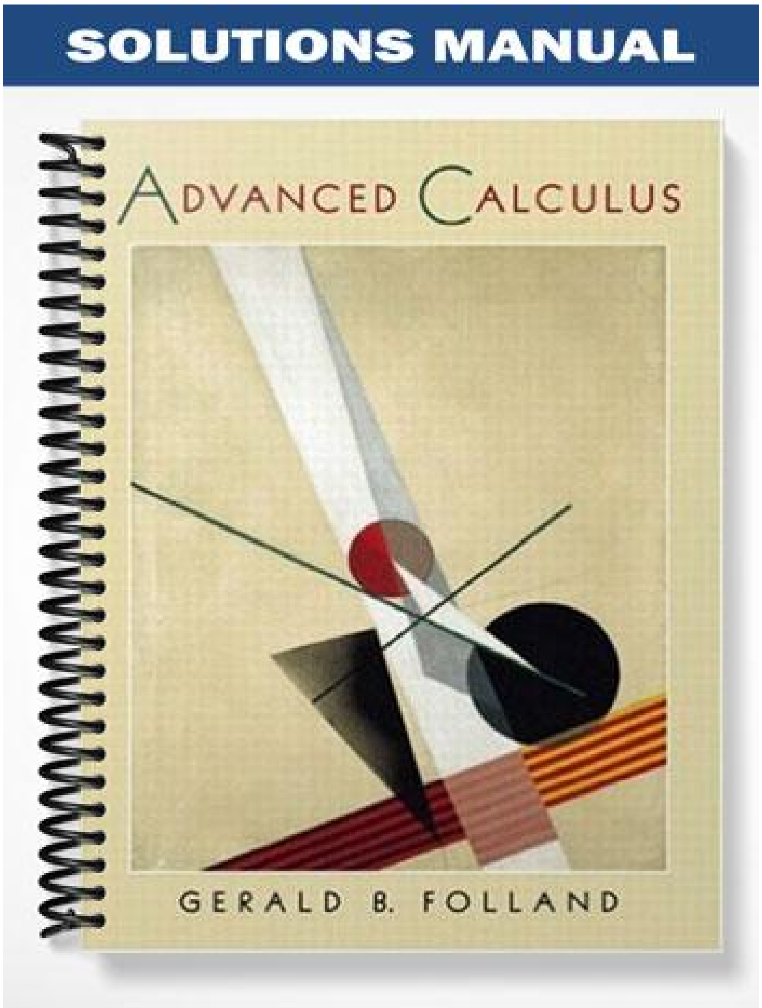
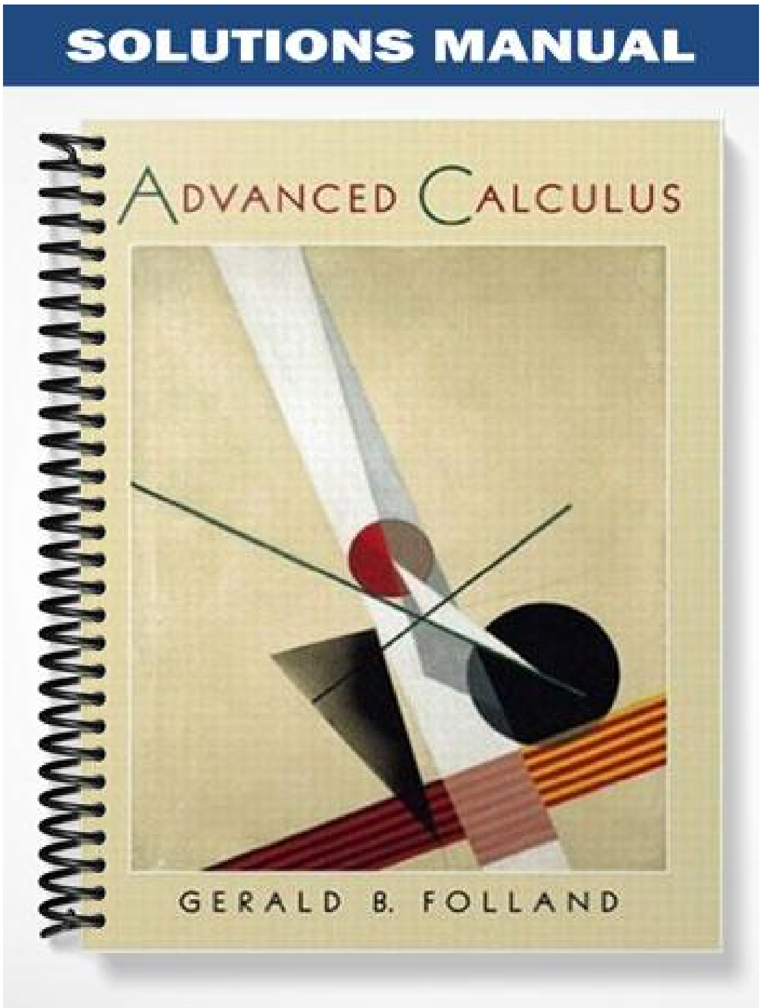

SOLUTIONS MANUAL
ADVANCED CALCULUS
GERALD B. FOLLAND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Instructor's Solution Manual for
ADVANCED CALCULUS

Gerald B. Folland

Contents

NOTE: Users of *Advanced Calculus* should be aware of the web site

`www.math.washington.edu/~folland/Homepage/index.html`

where a list of corrections to the book can be found. In particular, some errors in the exercises and in the answers in the back of the book were discovered in the course of preparing this solution manual. The solutions given here pertain to the *corrected* exercises.

Chapter 1

Setting the Stage

1.1 Euclidean Spaces and Vectors

1. $|\mathbf{x}| = \sqrt{3^2 + (-1)^2 + (-1)^2 + 1^2} = 2\sqrt{3}$, $|\mathbf{y}| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$, $\mathbf{x} \cdot \mathbf{y} = 3(-2) + (-1)2 + (-1)1 + 1 \cdot 0 = -9$, $\theta = \arccos(-9/3 \cdot 2\sqrt{3}) = \arccos(-\sqrt{3}\,/2) = 5\pi/6$.

2. $|\mathbf{x} \pm \mathbf{y}|^2 = (\mathbf{x} \pm \mathbf{y}) \cdot (\mathbf{x} \pm \mathbf{y}) = |\mathbf{x}|^2 \pm 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2$. Taking the plus sign gives (a); adding these identities with the plus and minus signs gives (b).

3. $|\mathbf{x}_1 + \cdots + \mathbf{x}_k|^2 = \sum_{j=1}^{k} |\mathbf{x}_j|^2 + 2\sum_{1 \le i < j \le k} \mathbf{x}_i \cdot \mathbf{x}_j$. The Pythagorean theorem follows immediately.

4. With $f(t) = |\mathbf{a} - t\mathbf{b}|^2$ as in the proof, equality holds precisely when the minimum value of $f(t)$ is 0, that is, when $\mathbf{a} = t\mathbf{b}$ for some $t \in \mathbb{R}$. Thus equality holds in Cauchy's inequality precisely when $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

5. The triangle inequality is an equality precisely when $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$, that is, when the angle from $\mathbf{a}$ to $\mathbf{b}$ is 0, or when $\mathbf{a}$ is a positive scalar multiple of $\mathbf{b}$ or vice versa.

6. $|\mathbf{a}| = |(\mathbf{a} - \mathbf{b}) + \mathbf{b}| \le |\mathbf{a} - \mathbf{b}| + |\mathbf{b}|$, so $|\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a} - \mathbf{b}|$. Likewise, $|\mathbf{b}| - |\mathbf{a}| \le |\mathbf{a} - \mathbf{b}|$.

7. (a) If $\mathbf{a} \cdot \mathbf{b} = 0$ then $\mathbf{a} \perp \mathbf{b}$, so $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$; hence if also $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

(b) If $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ then $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = 0$ and $(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0}$, so by (a), either $\mathbf{a} - \mathbf{b} = \mathbf{0}$ or $\mathbf{c} = \mathbf{0}$; the latter possibility is excluded.

(c) We always have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. If $\mathbf{a}$ and $\mathbf{b}$ are proportional, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ too. If not, then $\mathbf{a} \times \mathbf{b}$ is a nonzero vector perpendicular to $\mathbf{a}$, so $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) \ne \mathbf{0}$.

8. This follows from the definitions by a simple calculation.

1.2 Subsets of Euclidean Space

1. (a)–(d): See the answers in the back of the text.

(e) $S^{\text{int}} = \varnothing$ and $\partial S = \overline{S} = S \cup \{(y, 0) : -1 \le y \le 1\}$.

(f) $S^{\text{int}} = S \setminus \{(0, 0)\}$, $\overline{S} = \{(x, y) : x^2 + y^2 \le 1\}$, and ∂S is the union of the unit circle and the line segment $[-1, 0] \times \{0\}$.

(g) $S^{\text{int}} = \varnothing$ and $\partial S = \overline{S} = [0, 1] \times [0, 1]$.

2. If $\mathbf{x} \in S^{\text{int}}$, there is a ball $B = B(r, \mathbf{x})$ contained in S. B is open, so every point of B is an interior point of B and hence of S, so in fact $B \subset S^{\text{int}}$ and $\mathbf{x}$ is an interior point of S^{int}. Thus S^{int} is open by Proposition 1.4a. Next, $\overline{S}$ and ∂S are the complements of $(S^c)^{\text{int}}$ and $S^{\text{int}} \cup (S^c)^{\text{int}}$, respectively, so they are closed by Proposition 1.4b.

3. We use Proposition 1.4a. If $\mathbf{x} \in S_1 \cup S_2$, some ball centered at $\mathbf{x}$ is contained in either S_1 or S_2 and hence in $S_1 \cup S_2$, so $\mathbf{x}$ is an interior point of $S_1 \cup S_2$. If $\mathbf{x} \in S_1 \cap S_2$, there are balls B_1 and B_2 centered at $\mathbf{x}$ and contained in S_1 and S_2, respectively; the smaller of these balls is contained in $S_1 \cap S_2$, so again $\mathbf{x}$ is an interior point of $S_1 \cap S_2$.

4. The complements of $S_1 \cup S_2$ and $S_1 \cap S_2$ are $S_1^c \cap S_2^c$ and $S_1^c \cup S_2^c$, respectively, which are both open by Exercise 3 and Proposition 1.4b.

5. This follows from the remarks preceding Proposition 1.4: $\mathbb{R}^n$ is the disjoint union of S^{int}, ∂S, and $(S^c)^{\text{int}}$, whereas $\overline{S} = S^{\text{int}} \cup \partial S$ and $\overline{S^c} = (S^c)^{\text{int}} \cup \partial S$.

6. One example (in $\mathbb{R}^1$) is $S_j = [0,\, 1 - j^{-1}]$, for which $\bigcup_1^\infty S_j = [0, 1)$.

7. $\mathbb{R}^n$ and $\varnothing$.

8. The sets in Exercise 1a and 1f are both examples.

9. If $|\mathbf{x} - \mathbf{a}| < r$ then $|\mathbf{x}| = |(\mathbf{x} - \mathbf{a}) + \mathbf{a}| \le r + |\mathbf{a}|$. Thus, if $S \subset B(r, \mathbf{a})$ then $S \subset B(r + |\mathbf{a}|,\, \mathbf{0})$.

1.3 Limits and Continuity

1. (a) $f(0, y) = 1$ for $y > 0$ and $f(0, y) = -1$ for $y < 0$.
(b) $f(x, 0) = x^{-3} \to \infty$ as $x \to 0$.
(c) $f(t, t) = 1/8t^4 \to \infty$ as $t \to 0$.

2. (a) Since $|xy| \le \frac{1}{2}(x^2 + y^2)$, we have $|f(x, y)| \le \frac{1}{4}(x^2 + y^2) \to 0$ as $x, y \to 0$.
(b) Since $|3x^4 - y^4| \le 3(x^4 + y^4)$, we have $|f(x, y)| \le 3|x| \to 0$ as $x, y \to 0$.

3. $f(x, y) \to y$ as $x \to 0$, so take $f(0, y) = y$.

4. $f(x, a)$ and $f(a, y)$ are continuous for $a \neq 0$ since f is continuous except at $(0, 0)$. Moreover, $f(x, 0) = f(0, y) = 0$ for all x, y, also continuous.

5. The two formulas for f agree along the curves $y = 0$ and $y = x^2$, $x \neq 0$, so f is continuous except at the origin. It is discontinuous there since $f(0, 0) = 0$ but $f(x, \frac{1}{2}x^2) = \frac{1}{2} \not\to 0$ as $x \to 0$.

6. Since $|f(x)| \le |x|$ for all x, we have $f(x) \to 0 = f(0)$ as $x \to 0$. Suppose $a \neq 0$. If a is irrational, then $f(a) = a \neq 0$, but there are points x arbitrarily close to a with $f(x) = 0$. If a is rational, then $f(a) = 0$, but there are points x arbitrarily close to a with $|f(x)| > \frac{1}{2}|a|$. In both cases f is discontinuous at a.

7. Clearly $|f(x)| \le |x|$ for all x, so f is continuous at 0. If $a \neq 0$ is rational, then $f(a) \neq 0$, but there are points x arbitrarily close to a with $f(x) = 0$; hence f is discontinuous at a. If a is irrational and δ is the distance from a to the nearest rational number with denominator $\le k$, then $|f(x)| < 1/k$ for $|x - a| < \delta$; hence f is continuous at a. (There are only finitely many rational numbers with denominator $\le k$ in any bounded interval.)

8. Given $\mathbf{a} \in \mathbb{R}^n$ and $\epsilon > 0$, let $U = B(\epsilon, \mathbf{f}(\mathbf{a}))$. Then U is open, and hence so is $V = \{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in U\}$. We have $\mathbf{a} \in V$, so there exists $\delta > 0$ such that $B(\delta, \mathbf{a}) \subset V$. But this says that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$, so $\mathbf{f}$ is continuous at $\mathbf{a}$. One can replace "open" by "closed" in the hypothesis by the reasoning of the second paragraph of the proof of Theorem 1.13.

9. The fact that since $\mathbf{f}$ is a one-to-one correspondence between the points of U and the points of V has the following consequences that we shall use: (i) If $A \subset U$, $\mathbf{f}(U \setminus A) = V \setminus \mathbf{f}(A)$. (ii) If $B \subset V$, $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in B\} = \mathbf{f}^{-1}(B)$.

Suppose $\mathbf{b} \in \mathbf{f}(\partial S)$, and let $\epsilon > 0$ be small enough so that $B(\epsilon, \mathbf{b}) \subset V$. Since $\mathbf{f}$ is continuous, $\mathbf{f}^{-1}(B(\epsilon, \mathbf{b}))$ is a neighborhood of $\mathbf{f}^{-1}(\mathbf{b})$ by Theorem 1.13 and the remarks following it. Hence it contains points in S and points not in S, and therefore $B(\epsilon, \mathbf{b})$ contains points in $\mathbf{f}(S)$ and points not in $\mathbf{f}(S)$. It follows that $\mathbf{b} \in \partial(\mathbf{f}(S))$.

Conversely, suppose $\mathbf{b} \in \partial(\mathbf{f}(S))$, and let $\mathbf{a} = \mathbf{f}^{-1}(\mathbf{b})$; let $\epsilon > 0$ be small enough so that $B(\epsilon, \mathbf{a}) \subset U$. Since $\mathbf{f}^{-1}$ is continuous, $\mathbf{f}(B(\epsilon, \mathbf{a})) = (\mathbf{f}^{-1})^{-1}(B(\epsilon, \mathbf{a}))$ is a neighborhood of $\mathbf{b}$ by Theorem 1.13 again. Hence it contains points in $\mathbf{f}(S)$ and points not in $\mathbf{f}(S)$, and so $B(\epsilon, \mathbf{a})$ contains points in S and points not in S. It follows that $\mathbf{a} \in \partial S$ and hence $\mathbf{b} \in \mathbf{f}(\partial S)$.

1.4 Sequences

1. (a) Divide top and bottom by $\sqrt{k}$ to get $x_k = \dfrac{\sqrt{2 + k^{-1}}}{2 + k^{-1/2}} \to \dfrac{\sqrt{2}}{2}$.

(b) $|\sin k/k| \le 1/k \to 0$.

(c) Diverges since x_k is 0, $\frac{1}{2}\sqrt{3}$, and $-\frac{1}{2}\sqrt{3}$ for infinitely many k each.

2. $|x_k - 3| = 19/|k - 5| < \epsilon$ whenever $k > 5 + 19\epsilon^{-1}$.

3. $x_k = 1 \cdot \dfrac{1}{2} \cdot \dfrac{2}{3} \cdots \dfrac{k-1}{k} = \dfrac{1}{k} \to 0$.

4. If $x_k \to a$ and $y_k \to b$, then $(x_k, y_k) \to (a, b)$. By continuity of addition and multiplication (Theorem 1.10) and the sequential characterization of continuity (Theorem 1.15), the result follows.

5. If $\mathbf{f}(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{a}$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{l}| < \epsilon$ whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$. If $\mathbf{x}_k \to \mathbf{a}$, there exists K such that $|\mathbf{x}_k - \mathbf{a}| < \delta$ whenever $k > K$, and hence $|\mathbf{f}(\mathbf{x}_k) - \mathbf{l}| < \epsilon$. On the other hand, if $\mathbf{f}(\mathbf{x}) \not\to \mathbf{l}$ as $\mathbf{x} \to \mathbf{a}$, there exists $\epsilon > 0$ such that for every $\delta > 0$ there is an $\mathbf{x}$ with $0 < |\mathbf{x} - \mathbf{a}| < \delta$ but $|\mathbf{f}(\mathbf{x}) - \mathbf{l}| > \epsilon$. Let $\mathbf{x}_k$ be such a point for $\delta = 1/k$. Then $\mathbf{x}_k \to \mathbf{a}$ but $\mathbf{f}(\mathbf{x}_k) \not\to \mathbf{l}$.

6. If $\mathbf{x}_k \in S$, $\mathbf{x}_k \neq \mathbf{a}$, and $\mathbf{x}_k \to \mathbf{a}$, then the sequence $\{\mathbf{x}_k\}$ must assume infinitely many distinct values, and for $\epsilon > 0$, all but finitely many of them are in $B(\epsilon, \mathbf{a})$; thus $\mathbf{a}$ is an accumulation point of S. Conversely, if $\mathbf{a}$ is an accumulation point of S, for each positive integer k the ball $B(\mathbf{a}, 1/k)$ contains points of S other than $\mathbf{a}$; let $\mathbf{x}_k$ be one.

7. If $\mathbf{a}$ is an accumulation point of S, then $\mathbf{a} \in \overline{S}$ by Theorem 1.14 and Exercise 6. If $\mathbf{a} \notin S$ and $\mathbf{a}$ is not an accumulation point of S, there is a neighborhood of $\mathbf{a}$ that contains only finitely many points of S. If ϵ is less than the minimum distance from $\mathbf{a}$ to any of these points (which do not coincide with $\mathbf{a}$ since $\mathbf{a} \notin S$), $B(\epsilon, \mathbf{a})$ is a neighborhood of $\mathbf{a}$ that is disjoint from S, and hence $\mathbf{a} \notin \overline{S}$.