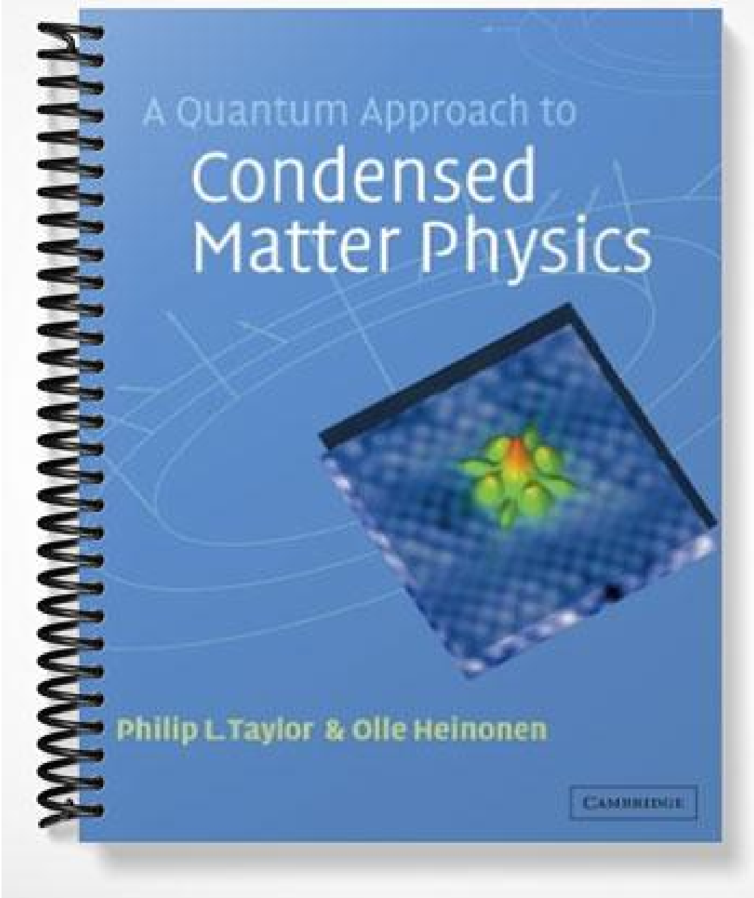
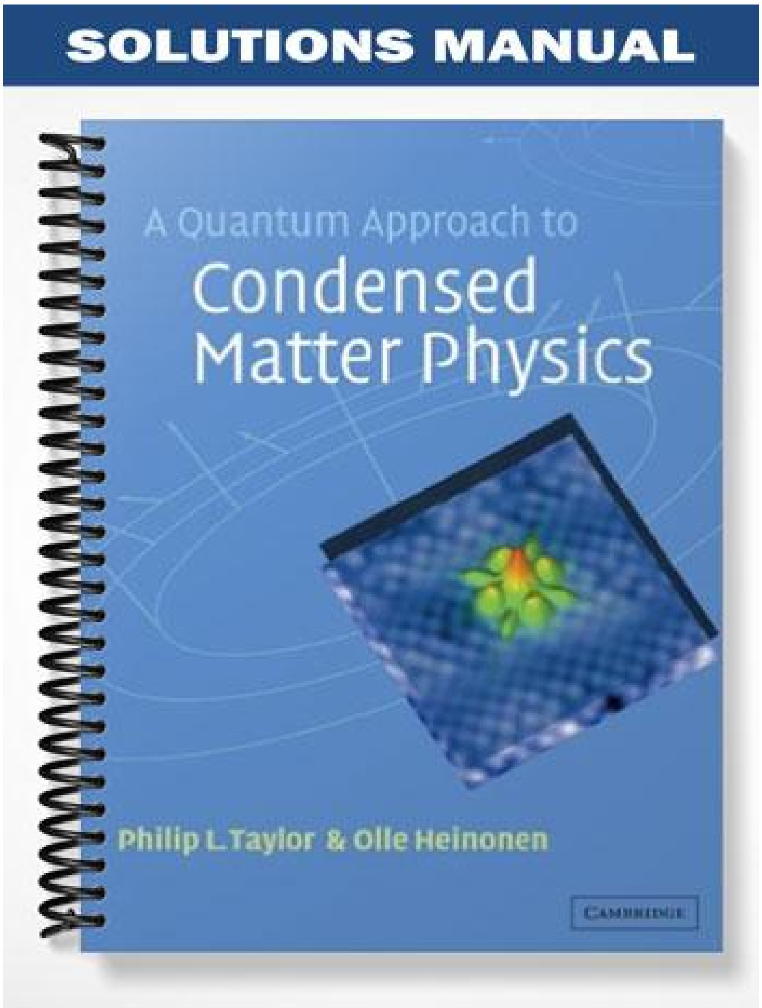
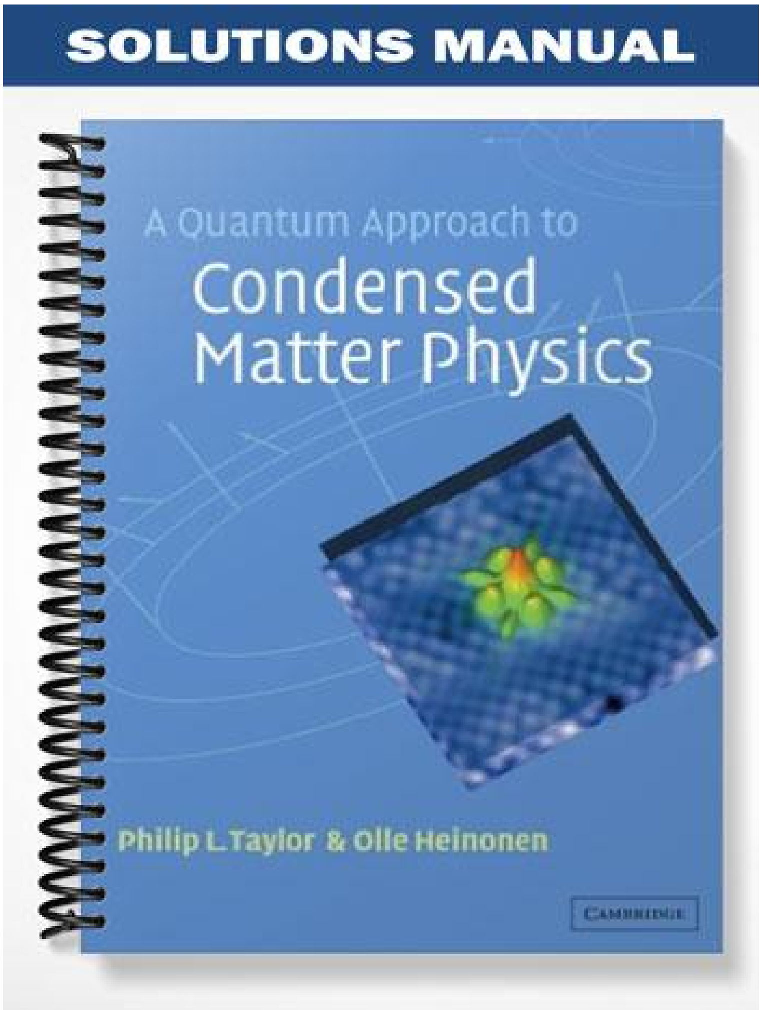
SOLUTIONS MANUAL




SOLUTIONS MANUAL
A Quantum Approach to
Condensed
Matter Physics
Philip L. Taylor & Olle Heinonen
CAMBRIDGE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Problem 1.1 Diatomic molecule

First, transform to center-of-mass and relative coordinates. With $\vec{r} = \vec{r}_1 - \vec{r}_2$ and the reduced mass $\mu = m/2$, the Schrödinger equation becomes

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{4}K(r-d_0)^2\right]\psi = \mathcal{E}\psi$$

Spherical symmetry allows separation of coordinates, so write $\psi = \frac{1}{r}R(r)\,Y_{\ell m}(\theta,\phi)$. The radial component is then

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2(\ell+1)\ell}{2\mu r^2} + \frac{1}{4}K(r-d_0)^2\right]R(r) = \mathcal{E}\,R(r)$$

Put $K \equiv 2\mu\omega_0^2$; $\frac{\hbar^2(\ell+1)\ell}{2\mu d_0^4} \equiv \gamma_\ell K$; $r - d_0 \equiv \rho$

Expand to second order in ρ and drop terms in γ_ℓ^2 since $\gamma_1 \ll 1$

$$\left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{1}{2}\mu\omega_0^2\left[(1+12\gamma_\ell)(\rho - 4\gamma_\ell d_0)^2 + 4\gamma_\ell d_0^2\right]\right\}R = \mathcal{E}R$$

The energy levels of this simple harmonic oscillator are

$$\mathcal{E}_{\ell n} = (n+\tfrac{1}{2})\hbar\omega_0(1+6\gamma_\ell) + \gamma_\ell K d_0^2 + O(\gamma^2)$$

So $\quad \mathcal{E}_{00} = \frac{1}{2}\hbar\sqrt{\frac{K}{m}} \qquad \mathcal{E}_{01} = \frac{3}{2}\hbar\sqrt{\frac{K}{m}}$

$$\mathcal{E}_{10} = \frac{1}{2}\hbar\sqrt{\frac{K}{m}}(1+6\gamma_1) + K d_0^2\gamma_1$$

$$\mathcal{E}_{11} = \frac{3}{2}\hbar\sqrt{\frac{K}{m}}(1+6\gamma_1) + K d_0^2\gamma_1$$

Then $\quad (\mathcal{E}_{11} - \mathcal{E}_{00}) - (\mathcal{E}_{01} - \mathcal{E}_{00}) - (\mathcal{E}_{10} - \mathcal{E}_{00})$

$$= 6\hbar\sqrt{\frac{K}{m}}\gamma_1 = \boxed{\frac{12\hbar^3}{d_0^4\sqrt{Km^3}}}$$

Solution to 1.2

Incoming wave is $E_0 e^{i(kx-\omega_0 t)}$ with $\omega_0 = kc$.

Atom is at $x = \sum_q a_q \sin(\omega_q t + \phi_q)$

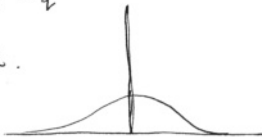
So $$E = E_0 e^{ik\sum_q a_q \sin(\omega_q t+\phi_q) - i\omega_0 t}$$
$$= E_0 e^{-i\omega_0 t}\Big[1 + \frac{i\omega_0}{c}\sum_q a_q \sin(\omega_q t+\phi_q) - \frac{\omega_0^2}{2c^2}\sum_{q,q'} a_q a_{q'} \sin(\omega_q t+\phi_q)\sin(\omega_{q'} t+\phi_{q'})\Big]$$

with higher terms neglected if $kx \ll 1$. In the sum over q and q', the phases ϕ_q are random, and so these terms average to zero unless $q = q'$, in which case $\sum a_q a_{q'} \sin(\omega_q t+\phi_q)\sin(\omega_{q'} t+\phi_{q'})$ becomes

$$\sum_q a_q^2 \langle \sin^2(\omega_q t+\phi_q)\rangle = \frac{1}{2}\sum_q a_q^2 .$$

Thus $$E(t) = e^{-i\omega_0 t}\Big(1 - \frac{\omega_0^2}{4c^2}\sum_q a_q^2\Big)$$
$$\pm \sum_q e^{-i(\omega_0 \pm \omega_q)t \pm i\phi_q} \frac{\omega_0}{2c} a_q$$

Intensity is square of amplitude.
We have an unshifted peak of height
$1 - 2\frac{\omega_0^2}{4c^2}\sum a_q^2 \simeq \exp -\Big(\frac{\omega_0^2}{2c^2}\sum a_q^2\Big)$



and side bands of intensity $\propto \frac{\omega_0^2}{4c^2}|a_q|^2$.

Note: $\exp -\Big(\frac{\omega_0^2}{4c^2}\sum a_q^2\Big)$ is known as the <u>Debye-Waller factor</u>.

Solution to 1.3

(i) Assume longitudinal wave, $\vec{q}$ in z direction

$$\sum_l \vec{E}_l = \sum_l E_{lz}\,\hat{z} = \sum_l Ze\,y_0\,e^{iql\cos\theta}(3\cos^2\theta-1)\,l^{-3}\,\hat{z}$$

For small q replace sum by integral

$$\sum_l E_{lz} \to \frac{Zey_0}{\underset{\text{vol. of unit cell}}{\Omega}} \iint e^{iql\cos\theta}(3\cos^2\theta-1)\frac{2\pi}{l}\,dl\,\sin\theta\,d\theta$$

Put $I \equiv \int e^{iql\cos\theta}\sin\theta\,d\theta = \frac{2\sin ql}{ql} = \frac{2\sin x}{x}$, say.

Then $$\sum E = -\frac{2\pi Zey_0}{\Omega}\int_{q\epsilon}^{qR}\left(I + 3\frac{d^2I}{dx^2}\right)\frac{dx}{x}$$

$$= -\frac{2\pi Zey_0}{\Omega}\int\left[\frac{2\sin x}{x} + 3\left(-\frac{2\sin x}{x} - \frac{4\cos x}{x^2} + \frac{4\sin x}{x^3}\right)\right]\frac{dx}{x}$$

$$= -\frac{2\pi Zey_0}{\Omega}\int 4\left[-\frac{\sin x}{x^2} - \frac{3\cos x}{x^3} + \frac{3\sin x}{x^4}\right]dx$$

$$= \frac{8\pi Zey_0}{\Omega}\left[\frac{\sin x - x\cos x}{x^3}\right]_{q\epsilon}^{qR}$$

(Alternative way: say $e^{i\vec{q}\cdot\vec{r}} \approx \sum(2l+1)i^l j_l(qr)P_l(\cos\theta)$
and $3\cos^2\theta - 1 \propto P_2(\cos\theta)$.
Thus $\sum E \propto \int r^{-3}j_2(qr)\,r^2\,dr \propto \left[\frac{j_1(qr)}{qr}\right]$)

So if $q\epsilon \to 0$ and $qR \to \infty$

$$\sum E_l = \frac{8\pi Zey_0}{\Omega}\left(\frac{x - x^3/6 + \dots - x + x^3/2 - \dots}{x^3}\right)_0^\infty$$

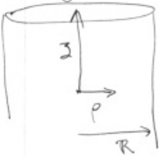
$$= \frac{8\pi Zey_0}{\Omega} \times \frac{1}{3}$$

If $M\ddot{y}_l = Ze\sum E_l$, then $\omega^2 = \frac{2}{3}\times\frac{4\pi(Ze)^2}{M\Omega} = \boxed{\frac{2}{3}\omega_p^2}$

<u>soln. to 1.3 (cont.)</u>

but if $qR \to 0$, then $\omega^2 = 0$

(ii) Cylinder $$\Sigma E = \frac{2\pi Z e y_0}{\Omega} \iint e^{iqz} \left[\frac{3z^2}{(\rho^2+z^2)^{5/2}} - \frac{1}{(\rho^2+z^2)^{3/2}} \right] \rho \, d\rho \, dz$$



$$= \frac{2\pi Z e y_0}{\Omega} \int e^{iqz} dz \left[\frac{\rho^2}{(\rho^2+z^2)^{3/2}} \right]_\epsilon^R$$

For lower limit $q\epsilon \to 0$, and integral is easy. For upper limit, put $z = R\tan\theta$
$dz = R\sec^2\theta \, d\theta$

$$\int e^{iqz} dz \frac{R^2}{(R^2+z^2)^{3/2}} = \int e^{iqR\tan\theta} \frac{R^3 \sec^3\theta \, d\theta}{R^3 \sec^3\theta}$$

$$= \int e^{iqR\tan\theta} \cos\theta \, d\theta.$$ Now the question is, what are the limits of θ?

If $z_{max} \gg R$, then limits of θ are $\pm \pi/2$, and then integral is 2 if $qR \to 0$ and 0 if $qR \to \infty$.

If $R \gg z_{max}$, limits of θ are small, and integral vanishes.

So what's going on? Explanation lies in <u>depolarizing factors</u>.

(i) Sphere, $qR \to 0$

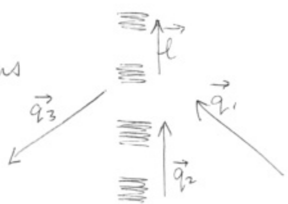
$E = 0$ inside hole

Sphere $qR \to \infty$

Outer surface cancels
$E \neq 0$ at center
as inner surface acts.

(ii) Cylinder
$z_{max} \gg R$
$qR \to 0$
$E = 0$

$R \gg z_{max}$
$E \neq 0$



Problem 1. 4. Phonon interactions

Assume a linear dispersion relation, so that $\omega = v |\vec{q}|$.

Then incoming wave has amplitude $\propto \exp(i\vec{q}_1 \cdot \vec{r} - i\omega_1 t)$ and outgoing wave $\propto \exp(i\vec{q}_3 \cdot \vec{r} - i\omega_3 t)$.

The diffraction grating is moving with velocity $\vec{v}$, so transform to coordinates moving with the grating. Then put $\vec{r}' \equiv \vec{r} - \vec{v}t$

$$\phi_{\text{incoming}} \propto \exp i[\vec{q}_1 \cdot (\vec{r}' + \vec{v}t) - \omega_1 t]$$

$$\phi_{\text{outgoing}} \propto \exp i[\vec{q}_3 \cdot (\vec{r}' + \vec{v}t) - \omega_3 t]$$

At points spaced a distance $\vec{\ell}$ along the grating, the phases must be matched, so

$$i[\vec{q}_1 \cdot (\vec{\ell} + \vec{v}t) - \omega_1 t] = i[\vec{q}_3 \cdot (\vec{\ell} + \vec{v}t) - \omega_3 t] + 2\pi i n \quad [n \text{ integer}]$$

So (A) $\vec{q}_1 \cdot \vec{v} - \omega_1 = \vec{q}_3 \cdot \vec{v} - \omega_3$

and (B) $\vec{q}_1 \cdot \vec{\ell} = \vec{q}_3 \cdot \vec{\ell} - 2\pi$ for first-order diffraction

Now $\vec{v} = \dfrac{\omega_2 \vec{q}_2}{q_2^2}$ and $\vec{\ell} = \dfrac{2\pi \vec{q}_2}{q_2^2}$, so from (B)

$$\vec{q}_1 \cdot \vec{q}_2 = \vec{q}_3 \cdot \vec{q}_2 - \vec{q}_2 \cdot \vec{q}_2 \quad \text{and by symmetry also}$$

$$\vec{q}_2 \cdot \vec{q}_1 = \vec{q}_3 \cdot \vec{q}_1 - \vec{q}_1 \cdot \vec{q}_1$$

Try $\vec{q}_3 = \vec{q}_1 + \vec{q}_2 + \vec{k}$. Then $\vec{k} \cdot \vec{q}_2 = 0$ and $\vec{k} \cdot \vec{q}_1 = 0$ so $\vec{k} = 0$.

From (A) $\dfrac{\vec{q}_1 \cdot \vec{q}_2}{q_2^2}\omega_2 - \omega_1 = \dfrac{(\vec{q}_1 + \vec{q}_2) \cdot \vec{q}_2}{q_2^2}\omega_2 - \omega_3$

and so $\boxed{\omega_3 = \omega_1 + \omega_2}$ when $\boxed{\vec{q}_3 = \vec{q}_1 + \vec{q}_2}$

This is the classical analog of energy and momentum conservation for phonons.

Problem 1.5 Anharmonic chain

Equations of motion are $M\ddot{y}_n = 4g\left[(y_{n+1}-y_n)^3-(y_n-y_{n-1})^3\right]$

Put $r_n \equiv y_n - y_{n-1}$. Then $M\ddot{r}_n = 4g\left[r_{n+1}^3 - 2r_n^3 + r_{n-1}^3\right]$

Call lattice spacing $= a$. For a traveling wave $r_n = A f(na - vt)$.

Here A is an amplitude and f is dimensionless

Now $\ddot{r}_n = \dfrac{Av^2}{a^2}\dfrac{\partial^2 f}{\partial n^2}$ and $\dfrac{MAv^2}{a^2}\dfrac{\partial^2 f}{\partial n^2} = 4gA^3\left(f_{n+1}^3 - 2f_n^3 + f_{n-1}^3\right)$

Put $h \equiv \sqrt{\dfrac{4g A^2 a^2}{Mv^2}}\, f$. Then $\dfrac{\partial^2 h_n}{\partial n^2} = h_{n+1}^3 - 2h_n^3 + h_{n-1}^3$

This equation has no parameters. Suppose it has a solution $\tilde{h}_n$. Then $r_n = \sqrt{\dfrac{Mv^2}{4ga^2}}\,\tilde{h}$, and

$(r_n)_{max} = \dfrac{v}{2a}\sqrt{\dfrac{M}{g}}\,(\tilde{h})_{max}$ so $\boxed{v \propto (r_n)_{max}}$

(A simpler solution comes from dimensional analysis

Velocity v must be proportional to the lattice spacing, so try $v = \text{constant} \times a \times g^\alpha M^\beta (r_{max})^\gamma$

$$\left[\frac{\text{length}}{\text{time}}\right] = [\text{length}] \times \left[\frac{\text{mass}}{\text{length}^2\,\text{time}^2}\right]^\alpha \times [\text{mass}]^\beta \times [\text{length}]^\gamma$$

$\alpha = \frac{1}{2}$; $\beta = -\frac{1}{2}$; $\gamma = 1$ so $\boxed{v \propto r_{max}}$)

Problem 1.6 Suppose original Fermi radius k_F is reduced to $k_\downarrow$ for down-spin electrons and increased to $k_\uparrow$ for up-spin electrons. By conservation of the number of electrons $k_\uparrow^3 + k_\downarrow^3 = 2k_F^3$

so we can put $k_\uparrow^3 = (1+\alpha)k_F^3$ and $k_\downarrow^3 = (1-\alpha)k_F^3$

Magnetic field shifts the Fermi energies to make

$\frac{\hbar^2}{2m}\left(k_\uparrow^2 - k_\downarrow^2\right) = 2\mu_B H$ so $\mathcal{E}_F\left[(1+\alpha)^{2/3} - (1-\alpha)^{2/3}\right] = 2\mu_B H$

For small α, $\left(1+\frac{2}{3}\alpha - \frac{1}{9}\alpha^2\ldots\right) - \left(1-\frac{2}{3}\alpha - \frac{1}{9}\alpha^2\ldots\right) = \frac{2\mu_B H}{\mathcal{E}_F}$

$$\text{so } \alpha \simeq \frac{3\mu_B H}{2\mathcal{E}_F}$$

We were given the change in total kinetic energy. Now total kinetic energy is proportional to k_F^5, as energy $\propto k_F^2$ and number of electrons $\propto k_F^3$.

Thus $\frac{k_\uparrow^5 + k_\downarrow^5}{2k_F^5} = 1 + 5\times 10^{-8}$

$\frac{1}{2}\left[(1+\alpha)^{5/3} + (1-\alpha)^{5/3}\right] \simeq 1 + \frac{5/3 \times 2/3}{2}\alpha^2 = 1 + \frac{5}{9}\alpha^2$

So $\alpha^2 = 9\times 10^{-8}$ and $\alpha = 3\times 10^{-4}$

Hence $\boxed{H = 2\times 10^{-4}\,\frac{\mathcal{E}_F}{\mu_B}}$

1.7 <u>Solution</u> $\mathcal{E}_{kinetic} \propto N_\uparrow k_\uparrow^2 + N_\downarrow k_\downarrow^2 \propto k_\uparrow^5 + k_\downarrow^5$

$\mathcal{E}_{int} \propto N_\uparrow^{4/3} + N_\downarrow^{4/3} \propto k_\uparrow^4 + k_\downarrow^4$.

Thus $\mathcal{E}_{total} = a(k_\uparrow^5 + k_\downarrow^5) + b(k_\uparrow^4 + k_\downarrow^4)$

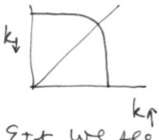
Minimize this subject to the constraint $k_\uparrow^3 + k_\downarrow^3 = 2k_F^3$ Ⓐ

Lagrange undetermined multiplier λ for constraint gives

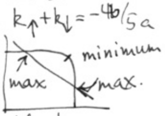
$5a k_\uparrow^4 + 4b k_\uparrow^3 - 3\lambda k_\uparrow^2 = 0$; $5a k_\downarrow^4 + 4b k_\downarrow^3 - 3\lambda k_\downarrow^2 = 0$

Eliminate λ to find $5a(k_\uparrow^2 - k_\downarrow^2) + 4b(k_\uparrow - k_\downarrow) = 0$

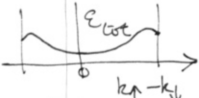
Hence $\boxed{k_\downarrow = k_\uparrow}$ or $\boxed{k_\uparrow + k_\downarrow = -4b/5a}$.



Plot a graph of the constraint Ⓐ.
Question is: Where on this line is $\mathcal{E}_{tot}$ a minimum? If b is positive, then by inspection of the expression for $\mathcal{E}_{tot}$ we see that $k_\uparrow = k_\downarrow$ is a minimum, so magnetism is unstable. This remains true until line $k_\uparrow + k_\downarrow = -4b/5a$ intersects the constraint curve. Then →



But if there is a maximum on this curve, then magnetism can be metastable, since variation of $\mathcal{E}_{tot}$ with $k_\uparrow$ can show that there is an energy barrier separating magnetized from unmagnetized state. This intersection occurs when $k_\downarrow = 0$ $k_\uparrow = \boxed{2^{1/3} k_F = -4b/5a}$. As b becomes more negative, we reach the point where $\mathcal{E}_{tot}(k_\uparrow = k_\downarrow) = \mathcal{E}_{tot}(k_\uparrow = 0)$. This occurs



when $2ak_F^5 + 2bk_F^4 = a\,2^{5/3}k_F^5 + b\,2^{4/3}k_F^4$, from which

$\boxed{b = -(2^{1/3}+1)\,a\,k_F}$ Beyond this point the magnetized state is stable.

Now $\mathcal{E}_{kin}(k_\uparrow = k_\downarrow) = \frac{3}{5}N\mathcal{E}_F$, so $a = \frac{3}{5}N\mathcal{E}_F/2k_F^5$.

$\mathcal{E}_{int}(k_\uparrow = k_\downarrow) = 2K(N/2)^{4/3}$, so $b = K(N/2)^{4/3}/k_F^4$

So stable if $K < -(2^{1/3}+1)2^{1/3}\cdot 3\mathcal{E}_F/5N^{1/3}$ i.e. $K < -1.708\,\mathcal{E}_F/N^{1/3}$

unstable if $K > -3\mathcal{E}_F/2^{4/3}N^{1/3}$ i.e. $K > -1.191\,\mathcal{E}_F/N^{1/3}$

metastable in-between.

Problem 1.8 Antiferromagnetic magnons.

Analysis similar to that of section 1.4 leads to substitutions

$$\vec{\mu}^{(a)} = \vec{\mu}_0^{(a)} + \vec{\mu}_1^{(a)} e^{i(\omega t + \vec{k}\cdot\vec{\ell})} \; ; \; \vec{\mu}_1^{(b)} = \vec{\mu}_0^{(b)} + \vec{\mu}_1^{(b)} e^{i(\omega t + \vec{k}\cdot\vec{\ell})}$$

Linearized equations of motion are

$$i\omega \vec{\mu}_1^{(a)} = C \sum_{\text{neighbors}} \left(\vec{\mu}_0^{(b)} \times \vec{\mu}_1^{(a)} + \vec{\mu}_1^{(b)} \times \vec{\mu}_0^{(a)} e^{i\vec{k}\cdot(\vec{\ell}'-\vec{\ell})} \right)$$

$$i\omega \vec{\mu}_1^{(b)} = C \sum \left(\vec{\mu}_0^{(a)} \times \vec{\mu}_1^{(b)} + \vec{\mu}_1^{(a)} \times \vec{\mu}_0^{(b)} e^{i\vec{k}\cdot(\ell'-\ell)} \right)$$

Now put $\sum e^{i\vec{k}\cdot(\vec{\ell}'-\vec{\ell})} \equiv z f(\vec{k})$ with z = co-ordination number

Then
$$(Cz\mu_0 - \omega)\mu_1^{(a)} + Cz\mu_0 f(\vec{k})\, \mu_1^{(b)} = 0$$
$$-Cz\mu_0 f(\vec{k})\, \mu_1^{(a)} - (Cz\mu_0 + \omega)\mu_1^{(b)} = 0$$

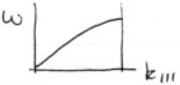
From which $\omega^2 = (Cz\mu_0)^2 (1 - f^2(\vec{k}))$

Put $Cz\mu_0 \equiv \omega_{max}$ $\qquad \boxed{\omega = \omega_{max}\sqrt{1 - f^2(k)}}$

For simple cubic lattice $f(\vec{k}) = \frac{1}{3}[\cos k_x a + \cos k_y a + \cos k_z a]$

In (111) direction $\vec{k} = k(1,1,1)$ and $f(k) = \cos ka$

$\sqrt{1-f^2} = \sin ka$



$$\frac{\mu_1^{(a)}}{\mu_1^{(b)}} = \frac{-f}{1 \pm \sqrt{1-f^2}} = -\sqrt{\frac{\omega_{max}+\omega}{\omega_{max}-\omega}} \text{ or } -\sqrt{\frac{\omega_{max}-\omega}{\omega_{max}+\omega}}$$

so for small ω, $\mu_1^{(a)} \simeq -\mu_1^{(b)}$

for large ω $\dfrac{\mu_1^{(a)}}{-\mu_1^{(b)}} \to 0 \text{ or } \infty$

ω small $\qquad$ ω large

Problem 1.9

$$\Delta\mathcal{E}_{magnetic} = -\mu_B H (N_\uparrow - N_\downarrow) = -\mu_B H \frac{N}{2}\left(\frac{k_\uparrow^3 - k_\downarrow^3}{k_F^3}\right)$$

With $k_\uparrow^3 = (1+\alpha)k_F^3$, $\Delta\mathcal{E}_{magnetic} = -N\mu_B H\alpha$

$$\mathcal{E}_{kinetic} = \frac{3}{5}N\mathcal{E}_F\left(\frac{k_\uparrow^5 + k_\downarrow^5}{2k_F^5}\right) \simeq \frac{3}{5}N\mathcal{E}_F\left(1+\frac{5}{9}\alpha^2\right)$$

Also $\mathcal{E}_F\left[(1+\alpha)^{2/3} - (1-\alpha)^{2/3}\right] = 2\mu_B H$, so $\alpha \simeq 3\mu_B H / 2\mathcal{E}_F$.

Thus $\Delta\mathcal{E}_{total} = \Delta\mathcal{E}_{kinetic} + \Delta\mathcal{E}_{magnetic}$

$$= \frac{3}{5}N\mathcal{E}_F\left[\frac{5}{9}\left(\frac{3\mu_B H}{2\mathcal{E}_F}\right)^2 - \frac{3(\mu_B H)^2}{\frac{3}{5}\mathcal{E}_F \times 2\mathcal{E}_F}\right] = \frac{3}{5}N\mathcal{E}_F\left[-\frac{5}{4}\left(\frac{\mu_B H}{\mathcal{E}_F}\right)^2\right]$$

So $-\frac{5}{4}\left(\frac{\mu_B H}{\mathcal{E}_F}\right)^2 = -5\times 10^{-8}$ and $\boxed{H = 2\times 10^{-4}\,\frac{\mathcal{E}_F}{\mu_B}}$ as before.

(Quick solution:

Total energy is of the form $ax^2 - bx$ where $x \propto N_\uparrow - N_\downarrow$.

Minimum of this expression is at $x = \frac{b}{2a}$

Then $\Delta(\text{kinetic energy}) = ax^2 = b^2/4a$

$\Delta(\text{potential energy}) = -bx = -\frac{b^2}{2a}$

So $\Delta(\text{total energy}) = -\Delta(\text{kinetic energy})$

and thus H must be the same as in problem 1.6)

Problem 1.10 Energy of a Toda soliton.

From Eq. (1.3.2) $r_n = -\frac{1}{b}\ln\left[1+\sinh^2\mu\,\mathrm{sech}^2(\mu n-\beta t)\right]$

Now $1+\sinh^2\mu\,\mathrm{sech}^2(\mu n-\beta t) = \dfrac{\cosh^2(\mu n-\beta t)+\sinh^2\mu}{\cosh^2(\mu n-\beta t)}$

$$= \frac{\cosh^2(\mu n-\beta t)(\cosh^2\mu-\sinh^2\mu)+\sinh^2\mu\left(\cosh^2(\mu n-\beta t)-\sinh^2(\mu n-\beta t)\right)}{\cosh^2(\mu n-\beta t)}$$

$$= \frac{\cosh[\mu(n+1)-\beta t]\cosh[\mu(n-1)-\beta t]}{\cosh^2(\mu n-\beta t)}$$

Define $A_n \equiv \dfrac{\cosh(\mu n-\beta t)}{\cosh(\mu(n-1)-\beta t)}$ so $r_n = -\frac{1}{b}\ln\left(\dfrac{A_{n+1}}{A_n}\right)$

or $r_n = \frac{1}{b}(\ln A_n - \ln A_{n+1})$ so $\sum_{-\infty}^{\infty} r_n = -\frac{1}{b}(\ln A_\infty - \ln A_{-\infty})$
$= -2\mu/b$

The energy of the soliton is

$$\mathcal{E} = \sum_{-\infty}^{\infty}\left[\frac{m}{2}\dot{y}_n^2 + \frac{a}{b}\left(e^{-br_n}-1\right)+ar_n\right]$$

Now $\dot{y}_n$ can be found from the equation $y_n - y_{-\infty} = \sum_{-\infty}^{n} r_n$

and $\sum_{-\infty}^{n} r_n = -\frac{1}{b}\left(\ln A_{n+1} - \ln A_{-\infty}\right)$

so $\dot{y}_n = -\frac{1}{b}\frac{d}{dt}\left(\ln A_{n+1}\right) = \frac{\beta}{b}\left[\tanh(\mu(n+1)-\beta t) - \tanh(\mu n-\beta t)\right]$

The energy is constant, so we can put $t=0$, and $\beta = \sqrt{\frac{ab}{m}}\sinh\mu$

$$\mathcal{E} = \sum_{-\infty}^{\infty}\frac{a}{2b}\sinh^2\mu\left\{\left(\tanh(\mu(n+1))-\tanh\mu n\right)^2 + 2\,\mathrm{sech}^2\mu n\right\} - \frac{2\mu a}{b}$$

$$= \frac{a}{2b}\sinh^2\mu\sum_{-\infty}^{\infty}\left[\tanh^2(\mu(n+1)) - \tanh^2\mu n + 2\left(1-\tanh\mu n\tanh\mu(n+1)\right)\right]$$

$$= \frac{a}{b}\sinh^2\mu\sum_{-\infty}^{\infty}\left(1-\tanh\mu n\tanh\mu(n+1)\right) - \cancel{\frac{2\mu a}{b}}\ -\frac{2\mu a}{b}$$

Prob. 1.10 continued

Now $\tanh\mu = \tanh[(n+1)\mu - n\mu] = \dfrac{\tanh(n+1)\mu - \tanh n\mu}{1 - \tanh(n+1)\mu \tanh n\mu}$

so $1 - \tanh\mu n \tanh\mu(n+1) = \coth\mu\left(\tanh(n+1)\mu - \tanh n\mu\right)$

and $\mathcal{E} = \dfrac{a}{b}\sinh^2\mu \coth\mu \sum\left(\tanh(n+1)\mu - \tanh\mu n\right) - \dfrac{2\mu a}{b}$

$= \dfrac{a}{b}\sinh\mu \cosh\mu \times 2 \quad - \dfrac{2\mu a}{b}$

$$\boxed{\mathcal{E} = \frac{2a}{b}\left(\sinh\mu \cosh\mu - \mu\right)}$$

(Thanks to Xin-Yi Wang for this formulation)

Easy version: For large μ at $t=0$, $r_0 \sim -\frac{1}{b}\ln(\sinh^2\mu)$
so $r_0 \sim -\frac{2\mu}{b}$ and $r_{\pm 1} \sim -\frac{1}{b}\ln 2 \ll r_0$

Potential energy $V \sim a r_0 + \frac{a}{b}e^{-br_0} \sim \frac{a}{b}\sinh^2\mu$.

Also $\dot{r}_0 = 0$ $\quad \dot{r}_{\pm 1} \sim \frac{\beta}{b}(\pm 1) \sim \mp\sqrt{\frac{a}{bM}}\sinh\mu$

and kinetic energy $\sim \frac{a}{b}\sinh^2\mu$ so $\boxed{\mathcal{E}_{\text{large}\,\mu} \sim \frac{2a}{b}\sinh^2\mu}$

For small μ at $t=0$ the motion is harmonic, and so potential energy = kinetic energy. We just need to calculate the kinetic energy and double it.

Now $\dot{r}_n \sim \frac{\beta}{\mu}\frac{\partial r_n}{\partial n}$ and thus $\dot{y}_n \sim \frac{\beta}{\mu} r_n$ so $\frac{1}{2}m\dot{y}_n^2 \sim \frac{m\beta^2 r_n^2}{2\mu^2}$

$\sum \frac{1}{2}m\dot{y}_n^2 \sim \frac{a}{2b}\frac{\sinh^2\mu}{\mu^2}\sum_n\left[\sinh^2\mu \operatorname{sech}^2\mu n\right]^2$

$\sim \frac{a\mu^3}{2b}\int_{-\infty}^{\infty}\operatorname{sech}^4 x\, dx \sim \frac{2a\mu^3}{3b}$ so $\boxed{\mathcal{E}_{\text{small}\,\mu} \sim \frac{a\mu^3}{b}}$