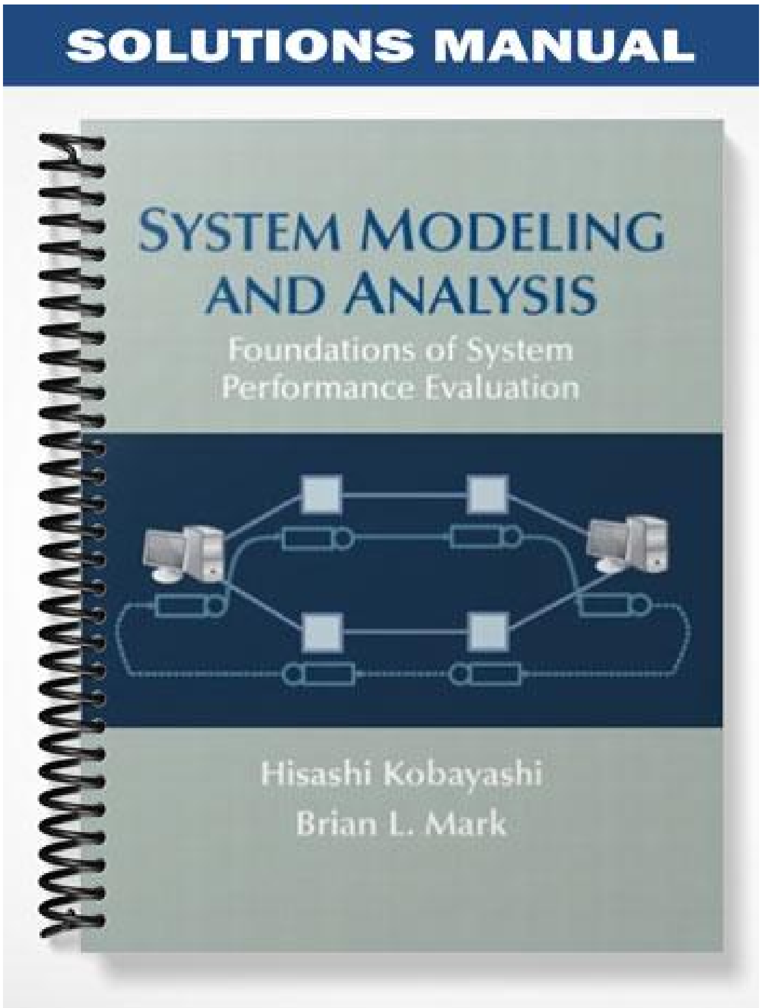
SOLUTIONS MANUAL
SYSTEM MODELING
AND ANALYSIS
Foundations of System
Performance Evaluation
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System Modeling and Analysis:

Foundations of System Performance Evaluation

Solution Manual (Chapters 2-13)

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CHAPTER 2

Basic Queueing Models

2.2 LITTLE'S FORMULA AND ITS GENERALIZATION

2.2-1. Little's formula.

- **(a)** False : Little's formula holds for arbitrary arrival processes.
- **(b)** False : For the same reason as part (a).
- **(c)** True: Little's formula holds for arbitrary work-conserving disciplines.
- **(d)** True: For the same reason as part (a).

2.2-2. Little's formula for multiple type jobs. Little's law can be be generalized to

$$\overline{Q}_r = \lambda_r E[W_r], \tag{2.1}$$

where $\overline{Q}_r$ is the mean number of type r jobs in queue and $E[W_r]$ is the mean waiting time for type r jobs in queue, for $r = 1, \ldots, R$. The average queue size for a type r job is given by (2.1) for the FCFS queue discipline or any other work-conserving discipline.

2.2-3 Distributions seen by arrivals and departures. In the interval $[0, T]$, for each arrival which causes $Q(t)$ to increase from n to $n+1$ ($n = 0, 1, \cdots$), there must be a corresponding departure that causes $Q(t)$ to decrease from $n+1$ to n (since $Q(0) = Q(T) = 0$). This implies that the average queue size seen by an arrival is the same as that seen by a departure in the interval $[0, T]$.

BIRTH-AND-DEATH PROCESSES

2.3-1 Superposition of Poisson processes.

(a) We have

$$\begin{aligned} P[Y \geq y] &= P[X_1 \geq y, X_2 \geq y, \ldots, X_m \geq y] \\ &= P[X_1 \geq y] \cdot P[X_2 \geq y] \cdots P[X_m \geq y] \\ &= e^{-\lambda_1 y} \cdot e^{-\lambda_2 y} \cdots e^{-\lambda_m y} = e^{-\sum_{i=1}^m \lambda_i y} = e^{-\lambda y}, \end{aligned}$$

where $\lambda \triangleq \sum_{i=1}^m \lambda_i$. Therefore,

$$F_Y(y) = 1 - e^{-\lambda y}, \quad y \geq 0,$$

so Y is exponentially distributed with parameter λ.

(b) Let X_j denote the inter-arrival time of the jth arrival stream. Since the arrival streams are independent, so to are the random variables X_j, $j = 1, \ldots, m$. Furthermore, since the jth arrival stream is Poisson, X_j is an exponentially distributed random variable with parameter λ_j. The inter-arrival time of the aggregate stream is given by

$$Y = \min\{X_1, \ldots, X_m\}.$$

From the result of part (a), we can conclude that Y is exponentially distributed with parameter $\lambda = \sum_{i=1}^{m} \lambda_i$. Therefore, the aggregate stream is a Poisson process with rate λ.

2.3-2. **Consistency check of the Poisson process.**

(a) Let I_h denote a small interval of length h. From (2.3-2) we have

$$\begin{aligned}
P[\text{no arrival in } I_h] &= P[N(h) = 0] = e^{-\lambda h} \\
&= 1 - \lambda h + \frac{(\lambda h)2}{2!} - \frac{(\lambda h)3}{3!} + \frac{(\lambda h)4}{4!} + \cdots \\
&= 1 - \lambda h + o(h), \\
P[1 \text{ arrival in } I_h] &= P[N(h) = 1] = \lambda h e^{-\lambda h} \\
&= \lambda h(1 - \lambda h + o(h)) = \lambda h + o(h), \\
P[\geq 2 \text{ arrivals in } I_h] &= \sum_{j=2}^{\infty} P[N(h) = j] = \sum_{j=2}^{\infty} \frac{(\lambda h)^j}{j!} \cdot e^{-\lambda h} = o(h).
\end{aligned}$$

(b) Let X_1 denote the time of the first arrival after the time origin (say $t = 0$) and X_2 denote the inter-arrival time between the first arrival and the second arrival. The RVs X_1 and X_2 are both exponentially distributed with parameter λ and have a common cdf:

$$F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

The event of no arrival in the interval I_h is equivalent to the event $\{X_1 > h\}$. Therefore,

$$P[\text{no arrival in } I_h] = P[X_1 > h] = e^{-\lambda h} = 1 - \lambda h + o(h).$$

The event of two or more arrivals in the interval I_h is equivalent to the event $\{X_1 + X_2 \leq h\}$. Let $Y = X_1 + X_2$. Then,

$$P[\geq 2 \text{ arrivals in } I_h] = P[Y \leq h] = F_Y(h). \tag{2.2}$$

There are several ways of determining the cdf $F_Y(y)$. Since X_1 and X_2 are independent, the pdf of Y is given by

$$f_Y(y) = f_{X_1}(y) \circledast f_{X_2}(y).$$

One can further show that in general, the cdf of Y is given by

$$F_Y(y) = F_{X_1}(y) \circledast f_{X_2}(y) = f_{X_1}(y) \circledast F_{X_2}(y). \tag{2.3}$$

Therefore,

$$\begin{aligned} F_Y(y) &= F_X(y) \circledast f_X(y) = \int_0^y (1 - e^{-\lambda x}) \cdot \lambda e^{-\lambda(y-x)} dx \\ &= \lambda e^{-\lambda y} \int_0^y (e^{\lambda x} - 1) dx = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}. \end{aligned}$$

Returning to (2.2), we obtain

$$\begin{aligned} P[\text{2 or more arrivals in } I_h] &= F_Y(h) = 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} \\ &= 1 - (1 - \lambda h + o(h)) - (\lambda h + o(h)) = o(h). \end{aligned}$$

Finally,

$$\begin{aligned} P[\text{one arrival in } I_h] &= 1 - P[\text{no arrival in } I_h] - P[\geq 2 \text{ arrivals in } I_h] \\ &= 1 - (1 - \lambda h + o(h)) - o(h) = \lambda h + o(h). \end{aligned}$$

To prove (2.3), note that

$$\begin{aligned} F_Y(y) &= \int_0^y f_{X_1}(x) \circledast f_{X_2}(x)\, dx = \int_0^y \int_0^x f_{X_1}(x-t) f_{X_2}(t)\, dt\, dx \\ &= \int_0^y \int_t^y f_{X_1}(x-t) f_{X_2}(t)\, dx\, dt = \int_0^y \int_0^{y-t} f_{X_1}(\alpha)\, d\alpha\, f_{X_2}(t)\, dt \\ &= \int_0^y F_{X_1}(y-t) f_{X_2}(t)\, dt = F_{X_1}(y) \circledast f_{X_2}(y). \end{aligned}$$

2.3-3. **Decomposition of a Poisson process**

(a) We are given that $\{X_j\}$ is a sequence of i.i.d. random variables, exponentially distributed with parameter λ. Then for fixed n,

$$S_n = X_1 + \cdots + X_n$$

and has an Erlang-n distribution. The cdf of S_n is given by

$$\begin{aligned} F_{S_n}(x) &= P[S_n \leq x] = 1 - P[< n \text{ arrivals in an interval of length } x] \\ &= 1 - \sum_{j=0}^{n-1} P[A(x) = j] = 1 - e^{-\lambda x} \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!}. \end{aligned}$$

Therefore,

$$P[S_n > x] = \sum_{j=0}^{n-1} P[A(x) = j] = 1 - e^{-\lambda x} \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!}.$$

Hence,

$$\begin{aligned} P[S_N > x] &= \sum_{n=1}^{\infty} P[S_N > s|N = n]P[N = n] \\ &= \sum_{n=1}^{\infty} e^{-\lambda x} \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} \cdot (1-r)^{n-1} r \\ &= re^{-\lambda x} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(\lambda x)^j}{j!} \cdot (1-r)^n \\ &= re^{-\lambda x} \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{(\lambda x)^j}{j!} \cdot (1-r)^n \\ &= re^{-\lambda x} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\lambda x)^j}{j!} \cdot (1-r)^{m+j} \\ &= re^{-\lambda x} \sum_{j=0}^{\infty} \frac{[(\lambda(1-r)x]^j}{j!} \cdot \sum_{m=0}^{\infty} (1-r)^m \\ &= re^{-\lambda x} \cdot e^{\lambda(1-r)x} \left(\frac{1}{r}\right) = e^{-\lambda r x} \end{aligned}$$

which shows that S_N has an exponential distribution with parameter λr.

(b) In decomposing the Poisson stream into m substreams, each arrival is assigned independently to the kth substream with probability r_k, where $\sum_{k=1}^{m} r_k = 1$. Consider an arrival that is assigned to the kth substream. The number of subsequent arrivals of the original Poisson stream until the next arrival that is assigned to the kth substream is a random variable N_k with distribution

$$P[N_k = n] = (1 - r_k)^{n-1} r_k, \quad n = 0, 1, \cdots .$$

Therefore, the inter-arrival time between arrivals assigned to the kth substream is a random variable

$$S_{N_k} = X_1 + \cdots X_{N_k},$$

where X_i are inter-arrival times of the original Poisson process. Hence, the X_i are i.i.d. and exponentially distributed with parameter λ. By the result from part (a), S_{N_k} is exponentially distributed with parameter $r_k\lambda$. Therefore, the kth substream is Poisson with rate $r_k\lambda$.

2.3-4. **Alternate decomposition of a Poisson stream.** Let X_i represent the interarrival time between the ith and the $(i+1)$-st arrival. For substream 1, the time between the first and the second arrival is given by

$$Y = X_1 + X_2 + \cdots X_m.$$

The event $\{Y \leq y\}$ is equivalent to the event that there are fewer than m arrivals of the original Poisson stream in an interval of length y, i.e.,

$$\begin{aligned} F_Y(y) &\triangleq P[Y \leq y] = 1 - P[< m \text{ arrivals in an interval of length } y] \\ &= 1 - \sum_{j=0}^{m-1} P[A(y) = j] = 1 - \sum_{j=0}^{m-1} \frac{(\lambda y)^j}{j!} e^{-\lambda y}, \end{aligned}$$

which is an Erlang-m distribution with mean m/λ.

2.3-5. Derivation of the Poisson distribution.

(a) Equation (2.3-9) for $n = 0$ can be written as:

$$\frac{d}{dt} \ln(P_0(t)) = -\lambda,$$

which is a simple, separable first-order differential equation. Integrating both sides and solving for $P_0(t)$ yields $P_0(t) = Ke^{-\lambda t}$, where the constant K is determined by the initial condition $P_0(0) = 1$. Hence, $K = 1$ and

$$P_0(t) = e^{-\lambda t}. \tag{2.4}$$

Substituting (6.5) into (2.3-9) for $n = 1$, we obtain

$$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}. \tag{2.5}$$

Equation (2.5) is a first-order differential equation that can be reduced to a separable form by multiplying both sides by an integrating factor. More generally, let us re-write (2.5) as follows:

$$P_1'(t) + R(t)P_1(t) = Q(t), \tag{2.6}$$

where in this case, $R(t) = \lambda$ and $Q(t) = \lambda e^{-\lambda t}$. The integrating factor can be obtained by supposing that the left-hand side of (2.6) to be the derivative of a product $\phi(t)P_1(t)$, given by

$$\phi(t)P_1'(t) + \phi'(t)P_1(t). \tag{2.7}$$

Multiplying the left-hand side of (2.6) by $\phi(t)$, we have

$$\phi(t)P_1'(t) + \phi(t)R(t)P_1(t) = \phi(t)Q(t). \tag{2.8}$$

Equating the left-hand side of (2.8) with (2.7), we see that they can be made equal by choosing $\phi(t)$ such that

$$\phi'(t) = \phi(t)R(t). \tag{2.9}$$

This is a simple separable equation that has the solution

$$\phi(t) = e^{\int R(t)dt}, \tag{2.10}$$

which is the integrating factor we seek.

After multiplying (2.5) by the integrating factor $\phi(t)$, we obtain:

$$\frac{d}{dt}\left[e^{\int R(t)dt}P_1(t)\right] = Q(t)e^{\int R(t)dt}. \tag{2.11}$$

The left-hand side is an exact derivative that can be integrated directly. In particular, we have

$$\frac{d}{dt}[e^{\lambda t}P_1(t)] \quad = \quad \lambda. \tag{2.12}$$

Hence, we obtain (using the fact that $P_1(0) = 1$):

$$P_1(t) = \lambda t e^{-\lambda t}. \tag{2.13}$$

To proceed by induction, we postulate the result

$$P_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t} \tag{2.14}$$

and show that the result holds for $P_{n+1}(t)$. From (4.61), we have:

$$P'_{n+1}(t) + \lambda P_{n+1}(t) = \lambda P_n(t) = \lambda\frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

Multiplying both sides by the integrating factor $\phi(t) = e^{\lambda t}$, we have:

$$\frac{d}{dt}[e^{\lambda t}P_{n+1}(t)] = \lambda\frac{(\lambda t)^n}{n!}.$$

Integrating both sides and using the fact that $P_{n+1}(0) = 0$, we obtain the required result:

$$P_{n+1}(t) = \frac{(\lambda t)^{(n+1)}}{(n+1)!}e^{-\lambda t}.$$

Thus, by induction, we have shown the validity of (2.14) for all $n \geq 0$.

(b) Taking the Laplace transform of the system of differential equations (4.61) and (4.62), we have:

$$sP_n^*(s) - P_n(0) \quad = \quad -\lambda P_n^*(s) + \lambda P_{n-1}^*(s), \quad n \geq 1 \tag{2.15}$$

$$sP_0^*(s) - P_0(0) \quad = \quad -\lambda P_0^*(s). \tag{2.16}$$

From (2.16) and the fact that $P_0(0) = 1$, we obtain $P_0^*(s) = \frac{1}{s+\lambda}$. Noting that $P_n(0) = 0$ in (2.15) we have, in particular for $n = 1$,

$$(s+\lambda)P_1^*(s) = \lambda P_0^*(s).$$

Therefore, $P_1^*(s) = \frac{\lambda}{(s+\lambda)2}$. Using induction, it is straightforward to show that

$$P_n^*(s) = \frac{\lambda^n}{(s+\lambda)^{n+1}} \tag{2.17}$$

Inverting (2.17), we obtain the desired result:

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where we have used the following Laplace transform properties:

$$\mathcal{L}^{-1}\{f^*(s+a)\} = f(t)e^{-at} \tag{2.18}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}. \tag{2.19}$$

2.3-6. Uniformity of Poisson arrivals.

(a) Suppose there are n arrivals in the interval $(0, T]$. The joint probability that there are i arrivals in a subinterval $(0, t]$, one arrival in $(t, t+h]$, and $n-i-1$ arrivals in $(t+h, T]$ is the product of three factors obtained from the Poisson distribution:

$$\left(\frac{(\lambda t)^i}{i!} e^{-\lambda t}\right) \left(\lambda h e^{-\lambda h}\right) \left(\frac{[\lambda(T-t-h)]^{n-i-1}}{(n-i-1)!} e^{-\lambda(T-t-h)}\right)$$
$$= \frac{\lambda^n h e^{-\lambda T}}{(n-1)!} \binom{n-1}{i} t^i (T-t-h)^{n-i-1}$$

Summing the above expression over the possible values of i, we find that the joint probability that there are n arrivals in $(t, T]$ with one arrival in $(t, t+h]$ is

$$\frac{\lambda^n h e^{-\lambda T}}{(n-1)!} \sum_{i=0}^{n} \binom{n-1}{i} t^i (T-t-h)^{n-i-1} = \frac{[\lambda(T-h)]^{n-1}}{(n-1)!} \lambda h e^{-\lambda T}, \tag{2.20}$$

where we used the binomial formula

$$\sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i} = (x+y)^k.$$

Since h is an infinitesimal interval, we rewrite (2.20) as

$$P[n \text{ arrivals in } (0,T],\ 1 \text{ arrival in } (t,t+h]] = \frac{(\lambda T)^{n-1}}{(n-1)!} \lambda h e^{-\lambda T} + o(h),$$

obtaining the conditional probability

$$P[1 \text{ arrival in } (t,t+h] | n \text{ arrivals in } (0,T]] = \frac{\frac{(\lambda T)^{n-1}}{(n-1)!} \lambda h e^{-\lambda T} + o(h)}{\frac{(\lambda T)^n}{n!} e^{-\lambda T}}$$
$$= \frac{nh}{T} + o(h).$$

(b) Since the n arrivals are independent, any one of them will fall into the interval $(t, t+h]$ with equal chance. Thus, the conditional probability that a call arrives in $(t, t+h]$, given that it is one of n arrivals in $(0, T]$ is $\frac{h}{T}$. This final expression is independent of n, hence the conditional probability is unconditional. Thus, we have proved (2.3-34).

2.3-7. Pure birth process. When $\lambda(n) = \lambda$ and $\mu(n) = 0$ for all $n \geq 0$, the differential-difference equations of the B-D process become:

$$p'_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad n = 1, 2, \cdots, \tag{2.21}$$
$$p'_0(t) = -\lambda p_0(t). \tag{2.22}$$

Using the same procedure as in Exercise 2.3-6, these equations can be solved to obtain the Poisson distribution:

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \cdots .$$

2.3-8. Time-dependent solution. When $\mu(n) = 0$ for all $n \geq 0$, but state-dependent birth rates $\lambda(n)$ are permitted, the differential-difference equations of the B-D process become:

$$p'_n(t) = -\lambda(n) p_n(t) + \lambda(n-1) p_{n-1}(t), \quad n = 1, 2, \cdots, \tag{2.23}$$
$$p'_0(t) = -\lambda p_0(t). \tag{2.24}$$

If we multiply both sides of (2.23) by the integrating factor $e^{\lambda(n)t}$, we obtain:

$$\frac{d}{dt}[e^{\lambda(n)t} p_n(t)] = \lambda(n-1) p_{n-1}(t) e^{\lambda(n)t}. \tag{2.25}$$

After integrating both sides from 0 to t and re-arranging, we obtain:

$$p_n(t) = e^{-\lambda(n)t} \left[\lambda(n-1) \int_0^t p_{n-1}(x) e^{\lambda(n)x} dx + K \right], \tag{2.26}$$

where K is a constant determined by the initial condition $p_n(0) = K$.

2.3-9. Pure death process. When $\lambda(n) = 0$ and $\mu(n) = \mu$ for all $n \geq 0$, the differential-difference equations of the B-D process become:

$$p'_{N_0}(t) = -\mu p_{N_0}(t), \tag{2.27}$$
$$p'_n(t) = -\mu p_n(t) + \mu p_{n+1}(t), \quad n = 1, \cdots, N_0 - 1 \tag{2.28}$$
$$p'_0(t) = \mu p_1(t). \tag{2.29}$$

Solving (2.27), we obtain

$$p_{N_0}(t) = e^{-\mu t}. \tag{2.30}$$

Similar to the approach in Problem 4.7, we can obtain from (2.28) and (2.29), the following result:

$$p_n(t) = \mu e^{-\mu t} \int_0^t e^{\mu x} p_{n+1}(x)dx, \quad n = 1, \cdots, N_0 - 1. \tag{2.31}$$

Applying (2.31) successively for $n = 1, 2, \cdots, N_0 - 1$, we obtain:

$$p_n(t) = \frac{(\mu t)^{N_0 - n}}{(N_0 - n)!} e^{-\mu t}, \quad n = 1, \cdots N_0.$$

For each t we have:

$$p_0(t) + \sum_{n=1}^{N_0} p_n(t) = 1.$$

Thus, we find that

$$p_0(t) = 1 - \sum_{n=1}^{N_0} \frac{(\mu t)^{N_0 - n}}{(N_0 - n)!} e^{-\mu t} = 1 - \sum_{n=0}^{N_0 - 1} \frac{(\mu t)^n}{n!} e^{-\mu t}.$$

2.3-10 The time-dependent PGF. Multiply both sides of (2.21) and (2.22) by z^n and sum from $n = 0$ to ∞ to obtain:

$$\sum_{n=0}^{\infty} p_n'(t) z^n = -\lambda \sum_{n=0}^{\infty} p_n(t) z^n + \lambda \sum_{n=1}^{\infty} p_{n-1}(t) z^n, \tag{2.32}$$

which is equivalent to

$$\frac{\partial}{\partial t} G(z,t) = -\lambda G(z,t) + \lambda z G(z,t) \tag{2.33}$$

$$= -\lambda(1 - z) G(z,t). \tag{2.34}$$

Equation (2.34) is a simple, separable first order differential equation whose solution is:

$$G(z,t) = e^{-\lambda(1-z)t} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} z^n. \tag{2.35}$$

Hence,

$$p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

2.3-11. Time-dependent solution for a certain BD process. When $\lambda_n = \lambda$ and $\mu_n = n\mu$ for all n, equations (2.3-46) become

$$p_n'(t) = -(\lambda + n\mu) p_n(t) + \lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t), \quad n = 1, 2, 3, \cdots, \tag{2.36}$$

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t). \tag{2.37}$$

Multiply both sides of (2.36) by z^n, sum from $n = 1$ to ∞ and then add (2.37) to obtain

$$\sum_{n=0}^{\infty} p_n'(t)z^n = -\lambda \sum_{n=0}^{\infty} p_n(t)z^n - \mu z \sum_{n=0}^{\infty} np_n(t)z^{n-1} + \lambda z \sum_{n=0}^{\infty} p_n(t)z^n + \mu \sum_{n=0}^{\infty} np_n(t)z^{n-1}. \tag{2.38}$$

Noting that

$$\frac{\partial}{\partial t}G(z,t) = \sum_{n=0}^{\infty} p_n'(t)z^n \text{ and } \frac{\partial}{\partial z}G(z,t) = \sum_{n=0}^{\infty} np_n(t)z^{n-1},$$

we can rewrite (2.38) as

$$\frac{\partial}{\partial t}G(z,t) = (z-1)\left[\lambda G(z,t) + \mu \frac{\partial}{\partial z}G(z,t)\right]. \tag{2.39}$$

One can easily verify by direct substitution that

$$G(z,t) = \exp\left\{\frac{\lambda}{\mu}(1 - e^{-\mu t})(z-1)\right\}. \tag{2.40}$$

is the unique solution to (2.39).

Since $p_n(t)$ is the coefficient of z^n in the power series expansion of $G(z,t)$, we have

$$p_n(t) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}G(z,t)\mid_{z=0}. \tag{2.41}$$

From (2.40), we obtain that

$$\frac{\partial}{\partial z}G(z,t) = \frac{\lambda}{\mu}(1 - e^{-\mu t})G(z,t),$$

which implies that

$$\frac{\partial^n}{\partial z^n}G(z,t) = \left[\frac{\lambda}{\mu}(1 - e^{-\mu t})\right]^n G(z,t). \tag{2.42}$$

From (2.40), (2.41), and (2.42), we obtain

$$p_n(t) = \frac{\left[\frac{\lambda}{\mu}(1 - e^{-\mu t})\right]^n}{n!}G(z,t)|_{z=0} = \frac{\left[\frac{\lambda}{\mu}(1 - e^{-\mu t})\right]^n}{n!}\exp\left\{\frac{-\lambda}{\mu}(1 - e^{-\mu t})\right\}.$$

2.4 BIRTH-AND-DEATH QUEUEING MODELS

2.4-1. Splitting a Poisson stream.

(a) As shown in Section 2.3, the k-th substream is a Poisson process with rate $p_k\lambda$, $k = 1, 2, \ldots, K$, and the K Poisson streams are statistically independent.

(b) The interarrival time T of each substream is K-stage Erlangian distributed with mean K/λ.

$$F_T(t) = 1 - e^{-\lambda t}\sum_{j=0}^{K-1}\frac{(\lambda t)^j}{j!}, \quad t \geq 0.$$

2.4-2. Erlangian distribution.

(a) The LT of an exponential random variable with mean μ is given by

$$f^*(s) = \frac{\mu}{s+\mu}.$$

Therefore, the LT of Y_i is

$$f_Y^*(s) = \frac{n\lambda}{s+n\lambda}.$$

Since $X = Y_1 + \cdots + Y_n$,

$$f_X^*(s) = [f_Y^*(s)]^n = \left(\frac{n\lambda}{s+n\lambda}\right)^n. \tag{2.43}$$

(b) The pdf of X can be obtained by inverting the LT given in (2.43). Using properties of the LT we have

$$\begin{aligned} f_X(x) &= (n\lambda)^n\mathcal{L}^{-1}\left[\frac{1}{(s+n\lambda)^n}\right] = (n\lambda)^n e^{-\lambda n x}\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] \\ &= (n\lambda)^n e^{-\lambda n x}\cdot\frac{x^{n-1}}{(n-1)!} = \frac{(n\lambda x)^n}{x(n-1)!}e^{-\lambda n x}. \end{aligned}$$

(c) The mean of X is

$$E[X] = nE[Y_i] = n\frac{1}{n\lambda} = \frac{1}{\lambda}.$$

The variance of X is

$$\text{Var}[X] = n\text{Var}[Y_i] = n\frac{1}{(n\lambda)2} = \frac{1}{n\lambda 2}.$$

2.4-3. Erlangian distribution (continued). The service completions of customers 1 through n may be considered as arrivals of a Poisson process, since the service times are exponentially distributed and i.i.d. The event that the total time to serve n customers, W, exceeds some value x is equivalent to the event that there are $n-1$ or fewer arrivals in the interval $[0,x]$, i.e.,

$$P[W > x|n] = P[n-1 \text{ or fewer arrivals in } [0,x)] = \sum_{j=0}^{n-1} \frac{(\mu x)^j}{j!} e^{-\mu x}.$$

Hence,

$$F_W(x|n) = P[W \le x|n] = 1 - \sum_{j=0}^{n-1} \frac{(\mu x)^j}{j!} e^{-\mu x}.$$

2.4-4. Balance equation of M/M/1.

(a) The detailed balance equation for the M/M/1 queue equates the probability flow rate from state $n-1$ to state n with that from state n to state $n-1$. The flow rates must be equal if the queue reaches a stable equilibrium. Therefore, the balance equations are given by

$$\mu p_n = \lambda p_{n-1}, \quad n = 1, 2, \cdots, \tag{2.44}$$

or equivalently,

$$p_n = \rho p_{n-1}, \quad n = 1, 2, \cdots, \tag{2.45}$$

where $\rho = \lambda/\mu$.

(b) Multiply both sides of (2.45) by z^n and sum from $n = 1$ to ∞ to obtain

$$\sum_{n=1}^{\infty} p_n z^n = \rho \sum_{n=1}^{\infty} p_{n-1} z^n. \tag{2.46}$$

Simplifying the above equation, we have

$$G(z) - p_0 = \rho z G(z). \tag{2.47}$$

Solving for $G(z)$, we have

$$G(z) = \frac{p_0}{1 - \rho z}.$$

Using the fact that $G(z) = 1$, we find that $p_0 = 1 - \rho$. Hence,

$$G(z) = \frac{1-\rho}{1-\rho z}.$$

(c) Expanding $G(z)$ as a power series, we find that

$$G(z) = \sum_{n=0}^{\infty} (1-\rho)\rho^n z^n.$$

Hence, we see that $p_n = (1-\rho)\rho^n$ for $n = 0, 1, \cdots$.

2.4-5. PASTA and related properties in the M/M/1 queue.

(a) Due to the uniformity property of the Poisson process, the proportion of arriving calls in the interval $(0, T)$ that find n in the system can be expressed as the following ratio:

$$a_n = \frac{\text{expected number of arrivals in } (0,T) \text{ that find } n \text{ in the system}}{\text{expected number of arriving calls in } (0,T)}.$$

The expected number of calls during $(0, T)$ that find exactly n calls in the system is $\lambda p_n T$. Thus,

$$a_n = \frac{\lambda p_n T}{\sum_{i=0}^{\infty} \lambda p_i T} = \frac{\lambda p_n T}{\lambda T} = p_n$$

(b) From Problem (2.2-1), we know that $a_n = d_n$ holds for any work-conserving queueing discipline. Hence, in this case $p_n = d_n$, i.e., the probability distribution of the number of system seen by departing customers is $\{p_n\}$.

(c) If the arrival process is state-dependent, i.e., the arrival rate $\lambda(n)$ depends on the state of the system, then

$$a_n = \frac{\lambda(n) p_n T}{\sum_{i=0}^{\infty} \lambda(i) p_i T} = \frac{\lambda(n) p_n}{\sum_{i=0}^{\infty} \lambda(i) p_i},$$

which does not equal p_n in general.

2.4-6. Derivation of the waiting time distribution. From (2.4-25) we have

$$\begin{aligned}
F_W(x) &= 1 - \rho + (1-\rho)\sum_{n=1}^{\infty} \rho^n - (1-\rho)e^{-\mu x} \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \rho^n \frac{(\mu x)^j}{j!} \\
&= 1 - (1-\rho)e^{-\mu x} \sum_{j=0}^{\infty} \sum_{n=j+1}^{\infty} \rho^n \frac{(\mu x)^j}{j!} \\
&= 1 - (1-\rho)e^{-\mu x} \sum_{j=0}^{\infty} \frac{\rho^{j+1}}{1-\rho} \frac{(\mu x)^j}{j!} = 1 - \rho e^{-\mu x} \sum_{j=0}^{\infty} \frac{(\rho \mu x)^j}{j!} \\
&= 1 - \rho e^{-\mu(1-\rho)x}.
\end{aligned}$$

2.4-7. Laplace transform method. The waiting time experienced by a call that arrives with $n \geq 1$ calls ahead of it in the system is given by:

$$W = R_1 + S_2 + \cdots + S_n, \tag{2.48}$$

where $R_1, S_2, \cdots, S_n$ are i.i.d. according to an exponential distribution of parameter μ. If there are 0 calls ahead of the arriving call, its waiting time will be 0. That is, the conditional pdf of W given $N = 0$ is given by $f_W(t|0) = \delta(t)$,