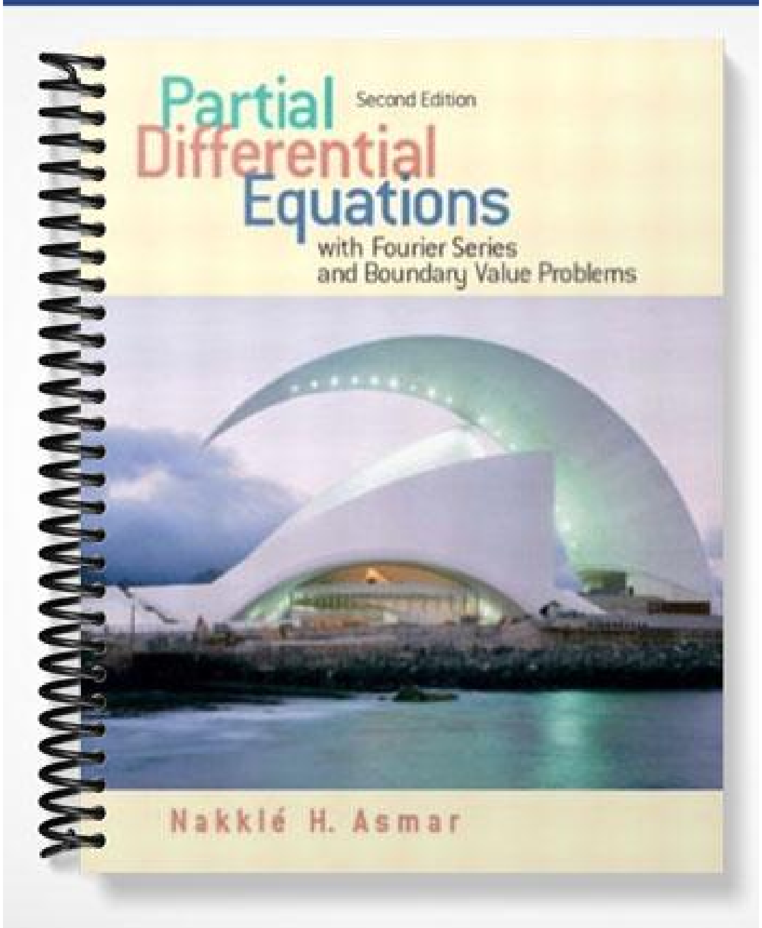
SOLUTIONS MANUAL



Instructor's Solutions Manual

PARTIAL DIFFERENTIAL EQUATIONS

with FOURIER SERIES and BOUNDARY VALUE PROBLEMS

Second Edition

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Contents

Preface

This manual contains solutions with notes and comments to problems from the textbook

Partial Differential Equations
with Fourier Series and Boundary Value Problems
Second Edition

Most solutions are supplied with complete details and can be used to supplement examples from the text. There are also many figures and numerical computions on Mathematica that can be very useful for a classroom presentation. Certain problems are followed by discussions that aim to generalize the problem under consideration.

I hope that these notes will sevre their intended purpose:

- To check the level of difficulty of assigned homework problems;
- To verify an answer or a nontrivial computation; and
- to provide worked solutions to students or graders.

As of now, only problems from Chapters 1–7 are included. Solutions to problems from the remaining chapters will be posted on my website

www.math.missouri.edu/ nakhle

as I complete them and will be included in future versions of this manual.

I would like to thank users of the first edition of my book for their valuable comments. Any comments, corrections, or suggestions from Instructors would be greatly appreciated. My e-mail address is

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Errata

The following mistakes appeared in the first printing of the second edition.

Corrections in the text and figures

p. 224, Exercise #13 is better done after Section 4.4.
p. 268, Exercise #8(b), n should be even.
p.387, Exercise#12, use $y_2 = I_0(x)$ not $y_2 = J_1(x)$.
p. 425 Figures 5 and 6: Relabel the ticks on the x-axis as $-\pi$, $-\pi/2$, $\pi/2$, π, instead of -2π, $-\pi$, π, 2π.
p. 467, l. (−3): Change reference (22) to (20).
p. 477 l. 10: $(xt) \leftrightarrow (x, t)$.
p. 477 l. 19: Change "interval" to "triangle"
p. 487, l.1: Change "is the equal" to "is equal"
p. 655, l.13: Change $\ln|\ln(x^2 + y^2)|$ to $\ln(x^2 + y^2)$.

Corrections to Answers of Odd Exercises

Section 7.2, # 7: Change i to $-i$.
Section 7.8, # 13: $f(x) = 3$ for $1 < x < 3$ not $1 < x < 2$.
Section 7.8, # 35: $\sqrt{\frac{2}{\pi}}\frac{(e^{-iw}-1)}{w}\sum_{j=1}^{3} j\sin(jw)$
Section 7.8, # 37: $i\sqrt{\frac{2}{\pi}}\frac{1}{w^3}$
Section 7.8, # 51: $\frac{3}{\sqrt{2\pi}}[\delta_1 - \delta_0]$.
Section 7.8, # 57: The given answer is the derivative of the real answer, which should be

$$\frac{1}{\sqrt{2\pi}}\Big((x+2)\big(\mathcal{U}_{-2}-\mathcal{U}_0\big)+(-x+2)\big(\mathcal{U}_0-\mathcal{U}_1\big)+\big(\mathcal{U}_1-\mathcal{U}_3\big)+(-x+4)\big(\mathcal{U}_3-\mathcal{U}_4\big)\Big)$$

Section 7.8, # 59: The given answer is the derivative of the real answer, which should be

$$\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\Big((x+3)\big(\mathcal{U}_{-3}-\mathcal{U}_{-2}\big)+(2x+5)\big(\mathcal{U}_{-2}-\mathcal{U}_{-1}\big)+(x+4)\big(\mathcal{U}_{-1}-\mathcal{U}_0\big)$$
$$+(-x+4)\big(\mathcal{U}_0-\mathcal{U}_1\big)+(-2x+5)\big(\mathcal{U}_1-\mathcal{U}_2\big)+(-x+3)\big(\mathcal{U}_2-\mathcal{U}_3\big)\Big)$$

Section 7.10, # 9: $\frac{1}{2}\big[t\sin(x+t)+\frac{1}{2}\cos(x+t)-\frac{1}{2}\cos(x-t)\big]$.

Any suggestion or correction would be greatly appreciated. Please send them to my e-mail address

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Solutions to Exercises 1.1

1. If u_1 and u_2 are solutions of (1), then

$$\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}=0 \quad \text{and} \quad \frac{\partial u_2}{\partial t}+\frac{\partial u_2}{\partial x}=0.$$

Since taking derivatives is a linear operation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(c_1u_1+c_2u_2)+\frac{\partial}{\partial x}(c_1u_1+c_2u_2) &= c_1\frac{\partial u_1}{\partial t}+c_2\frac{\partial u_2}{\partial t}+c_1\frac{\partial u_1}{\partial x}+c_2\frac{\partial u_2}{\partial x}\\ &= c_1\overbrace{\left(\frac{\partial u_1}{\partial t}+\frac{\partial u_1}{\partial x}\right)}^{=0}+c_2\overbrace{\left(\frac{\partial u_2}{\partial t}++\frac{\partial u_2}{\partial x}\right)}^{=0}=0,\end{aligned}$$

showing that $c_1u_1+c_2u_2$ is a solution of (1).

2. (a) General solution of (1): $u(x,t)=f(x-t)$. On the x-axis ($t=0$): $u(x,0)=xe^{-x^2}=f(x-0)=f(x)$. So $u(x,t)=f(x-t)=(x-t)e^{-(x-t)^2}$.

3. (a) General solution of (1): $u(x,t)=f(x-t)$. On the t-axis ($x=0$): $u(0,t)=t=f(0-t)=f(-t)$. Hence $f(t)=-t$ and so $u(x,t)=f(x-t)=-(x-t)=t-x$.

4. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

So (1) becomes

$$(a+b)\frac{\partial u}{\partial \alpha}+(c+d)\frac{\partial u}{\partial \beta}=0.$$

Let $a=1,\ b=1,\ c=1,\ d=-1$. Then

$$2\frac{\partial u}{\partial \alpha}=0 \quad\Rightarrow\quad \frac{\partial u}{\partial \alpha}=0,$$

which implies that u is a function of β only. Hence $u=f(\beta)=f(x-t)$.

5. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

Recalling the equation, we obtain

$$\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=0 \quad\Rightarrow\quad (b-a)\frac{\partial u}{\partial \alpha}+(d-c)\frac{\partial u}{\partial \beta}=0.$$

Let $a=1,\ b=2,\ c=1,\ d=1$. Then

$$\frac{\partial u}{\partial \alpha}=0 \quad\Rightarrow\quad u=f(\beta) \quad\Rightarrow\quad u(x,t)=f(x+t),$$

where f is an arbitrary differentiable function (of one variable).

6. The solution is very similar to Exercise 5. Let $\alpha=ax+bt$, $\beta=cx+dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x}=a\frac{\partial u}{\partial \alpha}+c\frac{\partial u}{\partial \beta}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t}+\frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t}=b\frac{\partial u}{\partial \alpha}+d\frac{\partial u}{\partial \beta}.\end{aligned}$$

Recalling the equation, we obtain

$$2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad (2b + 3a)\frac{\partial u}{\partial \alpha} + (2d + 3c)\frac{\partial u}{\partial \beta} = 0.$$

Let $a = 0,\ b = 1,\ c = 2,\ d = -3$. Then the equation becomes

$$2\frac{\partial u}{\partial \alpha} = 0 \quad \Rightarrow \quad u = f(\beta) \quad \Rightarrow \quad u(x,\, t) = f(cx + dt) = f(2x - 3t),$$

where f is an arbitrary differentiable function (of one variable).

7. Let $\alpha = ax + bt$, $\beta = cx + dt$, then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}.
\end{aligned}$$

The equation becomes

$$(b - 2a)\frac{\partial u}{\partial \alpha} + (d - 2c)\frac{\partial u}{\partial \beta} = 2.$$

Let $a = 1,\ b = 2,\ c = -1,\ d = 0$. Then

$$2\frac{\partial u}{\partial \beta} = 2 \quad \Rightarrow \quad \frac{\partial u}{\partial \beta} = 1.$$

Solving this ordinary differential equation in β, we get $u = \beta + f(\alpha)$ or $u(x,\, t) = -x + f(x + 2t)$.

8. Let $\alpha = c_1 x + d_1 t$, $\beta = c_2 x + d_2 t$, then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= c_1\frac{\partial u}{\partial \alpha} + c_2\frac{\partial u}{\partial \beta} \\
\frac{\partial u}{\partial t} &= d_1\frac{\partial u}{\partial \alpha} + d_2\frac{\partial u}{\partial \beta}.
\end{aligned}$$

The equation becomes

$$(ad_1 + bc_1)\frac{\partial u}{\partial \alpha} + (ad_2 + bc_2)\frac{\partial u}{\partial \beta} = u.$$

Let $c_1 = a,\ d_1 = -b,\ c_2 = 0,\ d_2 = \frac{1}{a}\ (a \neq 0)$. Then

$$\frac{\partial u}{\partial \beta} = u \quad \Rightarrow \quad u = f(\alpha)e^{\beta}.$$

Hence $u(x,\, t) = f(ax - bt)e^{\frac{t}{a}}$.

9. (a) The general solution in Exercise 5 is $u(x,\, t) = f(x + t)$. When $t = 0$, we get $u(x,\, 0) = f(x) = 1/(x^2 + 1)$. Thus

$$u(x,\, t) = f(x + t) = \frac{1}{(x + t)^2 + 1}.$$

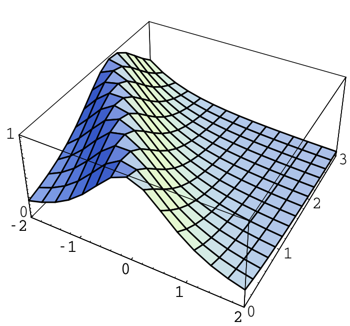


Figure for Exercise 9(b).

(c) As t increases, the wave $f(x) = \frac{1}{1+x^2}$ moves to the left.

10. (a) The directional derivative is zero along the vector $(a,\, b)$.
(b) Put the equation in the form $\frac{\partial u}{\partial x} + \frac{b}{a}\frac{\partial u}{\partial y} = 0\ (a \neq 0)$. The characteristic curves are obtained by solving

$$\frac{dy}{dx} = \frac{b}{a} \quad \Rightarrow \quad y = \frac{b}{a}x + C \quad \Rightarrow \quad y - \frac{b}{a}x = C.$$

Let $\phi(x, y) = y - \frac{b}{a}x$. The characteristic curves are the level curves of ϕ.
(c) The solution of $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$ is of the form $u(x, y) = f(\phi(x, y)) = f(y - \frac{b}{a}x)$, where f is a differentiable function of one variable.
11. The characteristic curves are obtained by solving

$$\frac{dy}{dx} = x^2 \quad \Rightarrow \quad y = \frac{1}{3}x^3 + C \quad \Rightarrow \quad y - \frac{1}{3}x^3 = C.$$

Let $\phi(x, y) = y - \frac{1}{3}x^3$. The characteristic curves are the level curves of ϕ. The solution of is of the form $u(x, y) = f(\phi(x, y)) = f(y - \frac{1}{3}x^3)$, where f is a differentiable function of one variable.

12. We follow the method of characteristic curves. Let's find the characteristic curves. For $x \neq 0$,

$$\tfrac{\partial u}{\partial x} + \tfrac{y}{x}\,\tfrac{\partial u}{\partial y} = 0;$$

$$\tfrac{dy}{dx} = \tfrac{y}{x} \Rightarrow y = Cx,$$

where we have used Theorem 1, Appendix A.1, to solve the last differential equation. Hence the characteristic curves are $\frac{y}{x} = C$ and the solution of the partial differential equation is $u(x, y) = f\left(\frac{y}{x}\right)$. To verify the solution, we use the chain rule and get $u_x = -\frac{y}{x^2}f'\left(\frac{y}{x}\right)$ and $u_y = \frac{1}{x}f'\left(\frac{y}{x}\right)$. Thus $xu_x + yu_y = 0$, as desired.

13. To find the characteristic curves, solve $\frac{dy}{dx} = \sin x$. Hence $y = -\cos x + C$ or $y + \cos x = C$. Thus the solution of the partial differential equation is $u(x, y) = f(y + \cos x)$. To verify the solution, we use the chain rule and get $u_x = -\sin x f'(y + \cos x)$ and $u_y = f'(y + \cos x)$. Thus $u_x + \sin x u_y = 0$, as desired.
14. Put the equation in the form $\frac{\partial u}{\partial x} + xe^{-x^2}\frac{\partial u}{\partial y} = 0$. The characteristic curves are obtained by solving

$$\frac{dy}{dx} = xe^{-x^2} \quad \Rightarrow \quad y = -\frac{1}{2}e^{-x^2} + C \quad \Rightarrow \quad y + \frac{1}{2}e^{-x^2} = C.$$

Let $\phi(x, y) = y + \frac{1}{2}e^{-x^2}$. The characteristic curves are the level curves of ϕ. The solution is of the form $u(x, y) = f(\phi(x, y)) = f(y + \frac{1}{2}e^{-x^2})$, where f is a differentiable function of one variable.

Exercises 1.2

1. We have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

So

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 v}{\partial t \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} = -\frac{\partial^2 u}{\partial x^2}.$$

Assuming that $\frac{\partial^2 v}{\partial t \partial x} = \frac{\partial^2 v}{\partial x \partial t}$, it follows that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, which is the one dimensional wave equation with $c = 1$. A similar argument shows that v is a solution of the one dimensional wave equation.

2. (a) For the wave equation in u, the appropriate initial conditions are $u(x, 0) = f(x)$, as given, and $u_t(x, 0) = -v_x(x, 0) = h'(x)$. (b) For the wave equation in v, the appropriate initial conditions are $v(x, 0) = h(x)$, as given, and $v_t(x, 0) = -u_x(x, 0) = f'(x)$.

3. $u_{xx} = F''(x+ct) + G''(x+ct)$, $u_{tt} = c^2 F''(x+ct) + c^2 G(x-ct)$. So $u_{tt} = c^u_{xx}$, which is the wave equation.

4. (a) Using the chain rule in two dimensions:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\
&= \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2} \\
&= \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + \frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} - c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2} \\
&= c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2}.
\end{aligned}$$

Substituting into the wave equation, it follows that

$$c^2\frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2\frac{\partial^2 u}{\partial \beta \partial \alpha} + c^2\frac{\partial^2 u}{\partial \beta^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

(b) The last equation says that $\frac{\partial u}{\partial \beta}$ is constant in α. So

$$\frac{\partial u}{\partial \beta} = g(\beta)$$

where g is an arbitrary differentiable function.

(c) Integrating the equation in (b) with respect to β, we find that $u = G(\beta) + F(\alpha)$, where G is an antiderivative of g and F is a function of α only.

(d) Thus $u(x, t) = F(x + ct) + G(x - ct)$, which is the solution in Exercise 3.

5. (a) We have $u(x, t) = F(x + ct) + G(x - ct)$. To determine F and G, we use the initial data:

$$u(x, 0) = \frac{1}{1 + x^2} \quad \Rightarrow \quad F(x) + G(x) = \frac{1}{1 + x^2}; \quad (1)$$

$$\begin{aligned}\frac{\partial u}{\partial t}(x,0)=0 \quad &\Rightarrow \quad cF'(x)-cG'(x)=0\\ &\Rightarrow \quad F'(x)=G'(x) \quad \Rightarrow \quad F(x)=G(x)+C, \qquad (2)\end{aligned}$$

where C is an arbitrary constant. Plugging this into (1), we find

$$2G(x)+C=\frac{1}{1+x^2} \quad \Rightarrow \quad G(x)=\frac{1}{2}\left[\frac{1}{1+x^2}-C\right];$$

and from (2)

$$F(x)=\frac{1}{2}\left[\frac{1}{1+x^2}+C\right].$$

Hence

$$u(x,\,t)=F(x+ct)+G(x-ct)=\frac{1}{2}\left[\frac{1}{1+(x+ct)^2}+\frac{1}{1+(x-ct)^2}\right].$$

6. We have $u(x,\,t)=F(x+ct)+G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,\,0)=e^{-x^2} \quad \Rightarrow \quad F(x)+G(x)=e^{-x^2}; \quad (1)$$

$$\begin{aligned}\frac{\partial u}{\partial t}(x,0)=0 \quad &\Rightarrow \quad cF'(x)-cG'(x)=0\\ &\Rightarrow \quad F'(x)=G'(x) \quad \Rightarrow \quad F(x)=G(x)+C, \qquad (2)\end{aligned}$$

where C is an arbitrary constant. Plugging this into (1), we find

$$2G(x)+C=e^{-x^2} \quad \Rightarrow \quad G(x)=\frac{1}{2}\left[e^{-x^2}-C\right];$$

and from (2)

$$F(x)=\frac{1}{2}\left[e^{-x^2}+C\right].$$

Hence

$$u(x,\,t)=F(x+ct)+G(x-ct)=\frac{1}{2}\left[e^{-(x+ct)^2}+e^{-(x-ct)^2}\right].$$

7. We have $u(x,\,t)=F(x+ct)+G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,\,0)=0 \quad \Rightarrow \quad F(x)+G(x)=0; \quad (1)$$

$$\begin{aligned}\frac{\partial u}{\partial t}(x,0)=-2xe^{-x^2} \quad &\Rightarrow \quad cF'(x)-cG'(x)=-2xe^{-x^2}\\ &\Rightarrow \quad cF(x)-cG(x)=\int -2xe^{-x^2}\,dx=e^{-x^2}+C\\ &\Rightarrow \quad F(x)-G(x)=\frac{e^{-x^2}}{c}+C, \qquad (2)\end{aligned}$$

where we rewrote C/c as C to denote the arbitrary constant. Adding (2) and (1), we find

$$2F(x)=\frac{e^{-x^2}}{c}+C \quad \Rightarrow \quad F(x)=\frac{1}{2c}\left[e^{-x^2}+C\right];$$

and from (1)

$$G(x)=-\frac{1}{2c}\left[e^{-x^2}+C\right].$$

Hence

$$u(x,\,t)=F(x+ct)+G(x-ct)=\frac{1}{2c}\left[e^{-(x+ct)^2}-e^{-(x-ct)^2}\right].$$

8. We have $u(x,t)=F(x+ct)+G(x-ct)$. To determine F and G, we use the initial data:

$$u(x,0)=0 \quad\Rightarrow\quad F(x)+G(x)=0 \quad\Rightarrow\quad F(x)=-G(x) \quad\Rightarrow\quad F'(x)=-G'(x); \quad (1)$$

$$\begin{aligned}\frac{\partial u}{\partial t}(x,0)=\frac{x}{(1+x^2)^2} \quad&\Rightarrow\quad cF'(x)-cG'(x)=\frac{x}{(1+x^2)^2}\\ &\Rightarrow\quad 2cF'(x)=\frac{x}{(1+x^2)^2} \quad\Rightarrow\quad F'(x)=\frac{x}{2c(1+x^2)^2}\ \text{, (from (1))}\\ &\Rightarrow\quad F(x)=\int\frac{x}{2c(1+x^2)^2}\,dx=\frac{-1}{4c(1+x^2)}+C,\end{aligned}$$

where C is an arbitrary constant. From (1),

$$G(x)=-F(x)=\frac{1}{4c(1+x^2)}-C;$$

where the C here is the same as the C in the definition of $F(x)$. So

$$u(x,t)=F(x+ct)+G(x-ct)=\frac{1}{4c}\left[\frac{-1}{(1+(x+ct))^2}+\frac{1}{(1+(x-ct))^2}\right].$$

9. As the hint suggests, we consider two separate problems: The problem in Exercise 5 and the one in Exercise 7. Let $u_1(x,t)$ denote the solution in Exercise 5 and $u_2(x,t)$ the solution in Exercise 7. It is straightforward to verify that $u=u_1+u_2$ is the desired solution. Indeed, because of the linearity of derivatives, we have $u_{tt}=(u_1)_{tt}+(u_2)_{tt}=c^2(u_1)_{xx}+c^2(u_2)_{xx}$, because u_1 and u_2 are solutions of the wave equation. But $c^2(u_1)_{xx}+c^2(u_2)_{xx}=c^2(u_1+u_2)_{xx}=u_{xx}$ and so $u_{tt}=c^2u_{xx}$, showing that u is a solution of the wave equation. Now $u(x,0)=u_1(x,0)+u_2(x,0)=1/(1+x^2)+0$, because $u_1(x,0)=1/(1+x^2)$ and $u_2(x,0)=0$. Similarly, $u_t(x,0)=-2xe^{-x^2}$; thus u is the desired solution. The explicit formula for u is

$$u(x,t)=\frac{1}{2}\left[\frac{1}{1+(x+ct)^2}+\frac{1}{1+(x-ct)^2}\right]+\frac{1}{2c}\left[e^{-(x+ct)^2}-e^{-(x-ct)^2}\right].$$

10. Reasoning as in the previous exercise, we find the solution to be $u=u_1+u_2$, wehre u_1 is the solution in Exercise 6 and u_2 is the solution in Exercise 8. Thus,

$$u(x,t)=\frac{1}{2}\left[e^{-(x+ct)^2}+e^{-(x-ct)^2}\right]+\frac{1}{4c}\left[\frac{-1}{(1+(x+ct))^2}+\frac{1}{(1+(x-ct))^2}\right].$$

11. We have

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(-L\frac{\partial I}{\partial t}-RI\right)=-L\frac{\partial^2 I}{\partial x\partial t}-R\frac{\partial I}{\partial x};$$

$$\frac{\partial I}{\partial x}=-C\frac{\partial V}{\partial t}-GV \quad\Rightarrow\quad \frac{\partial V}{\partial t}=\frac{-1}{C}\left[\frac{\partial I}{\partial x}+GV\right];$$

so

$$\frac{\partial^2 V}{\partial t^2}=\frac{-1}{C}\left[\frac{\partial^2 I}{\partial t\partial x}+G\frac{\partial V}{\partial t}\right].$$

To check that V verifies (1), we start with the right side

$$\begin{aligned}&LC\frac{\partial^2 V}{\partial t^2}+(RC+LG)\frac{\partial V}{\partial t}+RGV\\ &=\ LC\frac{-1}{C}\left[\frac{\partial^2 I}{\partial t\partial x}+G\frac{\partial V}{\partial t}\right]+(RC+LG)\frac{-1}{C}\left[\frac{\partial I}{\partial x}+GV\right]+RGV\\ &=\ -L\frac{\partial^2 I}{\partial t\partial x}-R\frac{\partial I}{\partial x}-\frac{LG}{C}\overbrace{\left[C\frac{\partial V}{\partial t}+\frac{\partial I}{\partial x}+GV\right]}^{=)}\\ &=\ -L\frac{\partial^2 I}{\partial t\partial x}-R\frac{\partial I}{\partial x}=\frac{\partial^2 V}{\partial x^2},\end{aligned}$$

which shows that V satisfies (1). To show that I satisfies (1), you can proceed as we did for V or you can note that the equations that relate I and V are interchanged if we interchange L and C, and R and G. However, (1) remains unchanged if we interchange L and C, and R and G. So I satisfies (1) if and only if V satisfies (1).

12. The function being graphed is the solution (2) with $c = L = 1$:

$$u(x, t) = \sin \pi x \cos \pi t.$$

In the second frame, $t = 1/4$, and so $u(x, t) = \sin \pi x \cos \pi/4 = \frac{\sqrt{2}}{2} \sin \pi x$. The maximum of this function (for $0 < x < \pi$ is attained at $x = 1/2$ and is equal to $\frac{\sqrt{2}}{2}$, which is a value greater than $1/2$.

13. The function being graphed is

$$u(x, t) = \sin \pi x \cos \pi t - \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

In frames 2, 4, 6, and 8, $t = \frac{m}{4}$, where $m = 1, 3, 5$, and 7. Plugging this into $u(x, t)$, we find

$$u(x, t) = \sin \pi x \cos \frac{m\pi}{4} - \frac{1}{2} \sin 2\pi x \cos \frac{m\pi}{2} + \frac{1}{3} \sin 3\pi x \cos \frac{3m\pi}{4}.$$

For $m = 1, 3, 5$, and 7, the second term is 0, because $\cos \frac{m\pi}{2} = 0$. Hence at these times, we have, for, $m = 1, 3, 5$, and 7,

$$u(x, \frac{m}{4}) = \sin \pi x \cos \pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

To say that the graph of this function is symmetric about $x = 1/2$ is equivalent to the assertion that, for $0 < x < 1/2$, $u(1/2 + x, \frac{m}{4}) = u(1/2 - x, \frac{m}{4})$. Does this equality hold? Let's check:

$$\begin{aligned} u(1/2 + x, \frac{m}{4}) &= \sin \pi(x + 1/2) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(x + 1/2) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}, \end{aligned}$$

where we have used the identities $\sin \pi(x + 1/2) = \cos \pi x$ and $\sin 2\pi(x + 1/2) = -\cos 3\pi x$. Similalry,

$$\begin{aligned} u(1/2 - x, \frac{m}{4}) &= \sin \pi(1/2 - x) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(1/2 - x) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}. \end{aligned}$$

So $u(1/2 + x, \frac{m}{4}) = u(1/2 - x, \frac{m}{4})$, as expected.

14. Note that the condition $u(0, t) = u(1, t)$ holds in all frames. It states that the ends of the string are held fixed at all time. It is the other condition on the first derivative that is specific to the frames in question.

The function being graphed is

$$u(x, t) = \sin \pi x \cos \pi t - \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

As we just stated, the equality $u(0, t) = u(1, t)$ holds for all t. Now

$$\frac{\partial}{\partial x} u(x, t) = \pi \cos \pi x \cos \pi t - \pi \cos 2\pi x \cos 2\pi t + \pi \cos 3\pi x \cos 3\pi t.$$

To say that $\frac{\partial}{\partial x} u(0, t) = 0$ and $\frac{\partial}{\partial x} u(1, t)$ means that the slope of the graph (as a function of x) is zero at $x = 0$ and $x = 1$. This is a little difficult to see in the frames 2, 4, 6, and 8 in Figure 8, but follows by plugging $t = 1/4$, $3/4$, $5/4$ and $7/4$ and $x = 0$ or 1 in the derivative. For example, when $x = 0$ and $t = 1/4$, we obtain:

$$\frac{\partial}{\partial x}u(0,\,1/4) = \pi\cos\pi/4 - \pi\cos\pi/2 + \pi\cos 3\pi/4 = \pi\cos\pi/4 + \pi\cos 3\pi/4 = 0.$$

15. Since the initial velocity is 0, from (10), we have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = f(x) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}$ implies that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sin\frac{2\pi x}{L}.$$

The equation is satisfied with the choice $b_1 = 0$, $b_2 = 1$, and all other b_n's are zero. This yields the solution

$$u(x,\,t) = \sin\frac{2\pi x}{L}\cos\frac{2c\pi t}{L}.$$

Note that the condition $u_t(x,\,0) = 0$ is also satisfied.

16. Since the initial velocity is 0, from (10), we have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = f(x) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}$ implies that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}.$$

Clearly, this equation is satisfied with the choice $b_1 = \frac{1}{2}$, $b_3 = \frac{1}{4}$, and all other b_n's are zero. This yields the solution

$$u(x,\,t) = \frac{1}{2}\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3c\pi t}{L}.$$

Note that the condition $u_t(x,\,0) = 0$ is also satisfied.

17. Same reasoning as in the previous exercise, we find the solution

$$u(x,\,t) = \frac{1}{2}\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} + \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3c\pi t}{L} + \frac{2}{5}\sin\frac{7\pi x}{L}\cos\frac{7c\pi t}{L}.$$

18. Since the initial displacement is 0, we use the functions following (1):

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = 0$ is clearly satisfied. To satisfy the second intial condition, we proceed as follows:

$$\begin{aligned}
g(x) &= \sin\frac{\pi x}{L} = \frac{\partial}{\partial t}u(x,\,0)\\
&= \frac{\partial}{\partial t}\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}\\
&= \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{\partial}{\partial t}\left(\sin\frac{cn\pi t}{L}\right)\Big|_{t=0}\\
&= \left(\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right)\Big|_{t=0}\\
&= \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\frac{cn\pi}{L}.
\end{aligned}$$

Thus

$$\sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n^* \frac{cn\pi}{L}\sin\frac{n\pi x}{L}\frac{cn\pi}{L}.$$

This equality holds if we take $b_1^*\frac{c\pi}{L} = 1$ or $b_1^* = \frac{L}{c\pi}$, and all other $b_n^* = 0$. Thus

$$u(x,\,t) = \frac{L}{c\pi}\sin\frac{\pi x}{L}\sin\frac{c\pi t}{L}.$$

This solution satisfies both initial conditions.

19. Reasoning as in the previous exercise, we satrt with the solution

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}.$$

The initial condition $u(x,\,0) = 0$ is clearly satisfied. To satisfy the second intial condition, we must have

$$\begin{aligned}
\frac{1}{4}\sin\frac{3\pi x}{L} - \frac{1}{10}\sin\frac{6\pi x}{L} &= \left[\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} b_n^* \sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}\right)\right]_{t=0}\\
&= \sum_{n=1}^{\infty} \frac{cn\pi}{L} b_n^* \sin\frac{n\pi x}{L}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{4} &= \frac{3c\pi}{L}b_3^* \quad\Rightarrow\quad b_3^* = \frac{L}{12c\pi};\\
-\frac{1}{10} &= \frac{6c\pi}{L}b_6^* \quad\Rightarrow\quad b_6^* = -\frac{L}{60c\pi};
\end{aligned}$$

and all other b_n^8 are 0. Thus

$$u(x,\,t) = \frac{L}{12c\pi}\sin\frac{3\pi x}{L}\sin\frac{3c\pi t}{L} - \frac{L}{60c\pi}\sin\frac{6\pi x}{L}\sin\frac{6c\pi t}{L}.$$

20. Write the initial condition as $u(x,\,0) = \frac{1}{4}\sin\frac{4\pi x}{L}$, then proceed as in Exercises 15 or 16 and you will get

$$u(x,\,t) = \frac{1}{4}\sin\frac{4\pi x}{L}\cos\frac{4c\pi t}{L}.$$

21. (a) We have to show that $u(\frac{1}{2},\,t)$ is a constant for all $t > 0$. With $c = L = 1$, we have

$$u(x,\,t) = \sin 2\pi x\cos 2\pi t \Rightarrow u(1/2,\,t) = \sin\pi\cos 2\pi t = 0 \quad \text{for all } t > 0.$$

(b) One way for $x = 1/3$ not to move is to have $u(x, t) = \sin 3\pi x \cos 3\pi t$. This is the solution that corresponds to the initial condition $u(x, 0) = \sin 3\pi x$ and $\frac{\partial u}{\partial t}(x, 0) = 0$. For this solution, we also have that $x = 2/3$ does not move for all t.

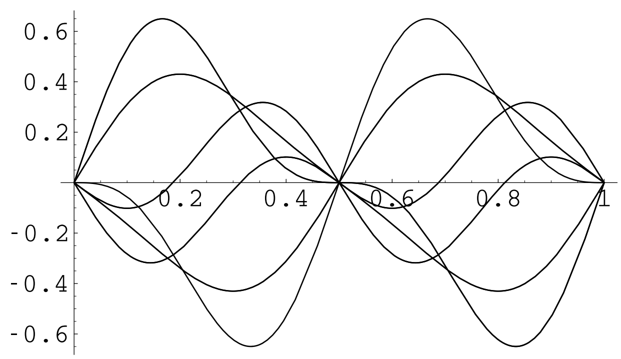
22. (a) Reasoning as in Exercise 17, we find the solution to be

$$u(x, t) = \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{4} \sin 4\pi x \cos 4\pi t.$$

(b) We used Mathematica to plot the shape of the string at times $t = 0$ to $t = 1$ by increments of .1. The string returns to some of its previous shapes. For example, when $t = .1$ and when $t = .9$, the string has the same shape.

```
u[x_, t_] = 1 / 2 Sin[2 Pi x] Cos[2 Pi t] + 1 / 4 Sin[4 Pi x] Cos[4 Pi t]
tt = Table[u[x, t], {t, 0, 1, .1}];
Plot[Evaluate[tt], {x, 0, 1}]
```

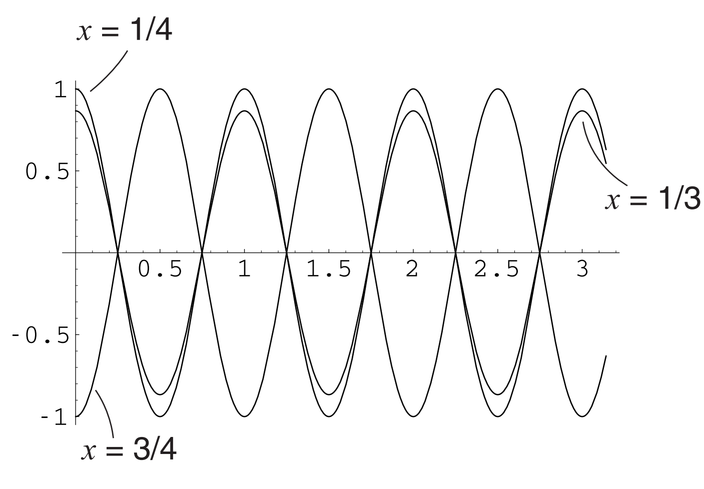
$$\frac{1}{2} \text{Cos}[2\,\pi\,\text{t}]\ \text{Sin}[2\,\pi\,\text{x}] + \frac{1}{4} \text{Cos}[4\,\pi\,\text{t}]\ \text{Sin}[4\,\pi\,\text{x}]$$



The point $x = 1/2$ does not move. This is clear: If we put $x = 1/2$ in the solution, we obtain $u(1/2, t) = 0$ for all t.

23. The solution is $u(x, t) = \sin 2\pi x \cos 2\pi t$. The motions of the points $x = 1/4, 1/3,$ and $3/4$ are illustrated by the following graphs. Note that the point $x = 1/2$ does not move, so the graph that describes its motion is identically 0.

```
u[x_, t_] = Sin[2 Pi x] Cos[2 Pi t]
Plot[{u[1 / 4, t], u[1 / 3, t], u[3 / 4, t]}, {t, 0, Pi}]
```



In each case, we have a cosine wave, namely $u(x_0, t) = \sin 2\pi x_0 \cos 2\pi t$, scaled by a factor $\sin 2\pi x_0$.

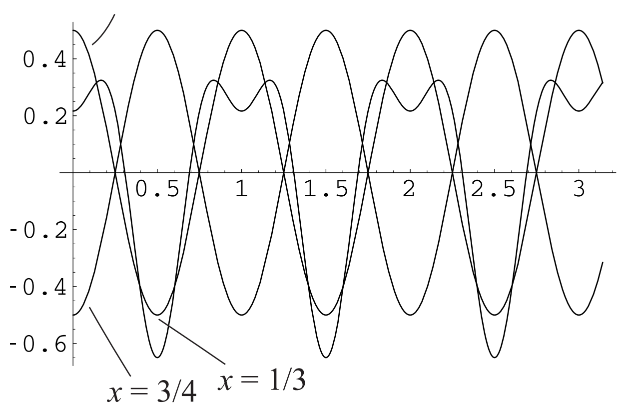
24. The solution in Exercise 22 is

$$u(x, t) = \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{4} \sin 4\pi x \cos 4\pi t.$$

The motions of the points $x = 1/4$, $1/3$, $1/2$, and $3/4$ are illustrated by the following graphs. As in the previous exercise, the point $x = 1/2$ does not move, so the graph that describes its motion is identically 0.

```
u[x_, t_] = 1 / 2 Sin[2 Pi x] Cos[2 Pi t] + 1 / 4 Sin[4 Pi x] Cos[4 Pi t]
Plot[{u[1 / 4, t], u[1 / 3, t], u[3 / 4, t]}, {t, 0, Pi}]
```

$\frac{1}{2}$ Cos[2 π t] Sin[2 π x] + $\frac{1}{4}$ Cos[4 π t] Sin[4 π x]



Unlike the previous exercises, here the motion of a point is not always a cosine wave. In fact, it is the sum of two scaled cosine waves: $u(x_0, t) = \frac{1}{2} \sin 2\pi x_0 \cos 2\pi t + \frac{1}{4} \sin 4\pi x_0 \cos 4\pi t$.

25. The solution (2) is

$$u(x, t) = \sin \frac{\pi x}{L} \cos \frac{\pi c t}{L}.$$

Its initial conditions at time $t_0 = \frac{3L}{2c}$ are

$$u(x, \frac{3L}{2c}) = \sin \frac{\pi x}{L} \cos \left(\frac{\pi c}{L} \cdot \frac{3L}{2c} \right) = \sin \frac{\pi x}{L} \cos \frac{3\pi}{2} = 0;$$

and

$$\frac{\partial u}{\partial t}(x, \frac{3L}{2c}) = -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \left(\frac{\pi c}{L} \cdot \frac{3L}{2c} \right) = -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \frac{3\pi}{2} = \frac{\pi c}{L} \sin \frac{\pi x}{L}.$$

26. We have

$$\begin{aligned} u(x, t + (3L)/(2c)) &= \sin \frac{\pi x}{L} \cos \frac{\pi c}{L} \cdot (t + \frac{3L}{2c}) \\ &= \sin \frac{\pi x}{L} \cos \left(\frac{\pi c t}{L} + \frac{3\pi}{2} \right) = \sin \frac{\pi x}{L} \sin \frac{\pi c t}{L}, \end{aligned}$$

where we used the indentity $\cos(a + \frac{3\pi}{2}) = \sin a$. Thus $u(x, t + (3L)/(2c))$ is equal to the solution given by (7). Call the latter solution $v(x, t)$. We know that $v(x, t)$ represents the motion of a string that starts with initial shape $f(x) = 0$ and intial velocity $g(x) = \frac{\pi c}{L} \sin \frac{\pi x}{L}$. Now, appealing to Exercise 25, we have that at time $t_0 = (3L)/(2c)$, the shape of the solution u is $u(x, (3L)/(2c)) = 0$ and its velocity is

$\frac{\partial u}{\partial t}(x, \frac{3L}{2c}) = \frac{\pi c}{L} \sin \frac{\pi x}{L}$. Thus the subsequent motion of the string, u, at time $t + t_0$ is identical to the motion of a string, v, starting at time $t = 0$, whenever v has initial shape $u(x, t_0)$ and the initial velocity $\frac{\partial u}{\partial t}(x, t_0)$.

27. (a) The equation is equivalent to

$$-\frac{1}{r}\frac{\partial u}{\partial t} - \frac{\kappa}{r}\frac{\partial u}{\partial x} = u.$$

The solution of this equation follows from Exercise 8, Section 1.1, by taking $a = -\frac{1}{r}$ and $b = -\frac{\kappa}{r}$. Thus

$$u(x, t) = f(-\frac{1}{r}x + \frac{\kappa}{r}t)e^{-rt}.$$

Note that this equivalent to

$$u(x, t) = f(x - \kappa t)e^{-rt},$$

by replacing the function $x \mapsto f(x)$ in the first formula by $x \mapsto f(-rx)$. This is acceptable because f is arbitrary.
(b) The number of particles at time $t \geq 0$ is given by $\int_{-\infty}^{\infty} u(x, t)\, dx$. We have $M = \int_{-\infty}^{\infty} u(x, 0)\, dx$. But $u(x, 0) = f(x)$, so $M = \int_{-\infty}^{\infty} f(x)\, dx$. For $t > 0$, the number of particles is

$$\begin{aligned} \int_{-\infty}^{\infty} u(x, t)\, dx &= \int_{-\infty}^{\infty} f(x - \kappa t)e^{-rt}\, dx \\ &= e^{-rt}\int_{-\infty}^{\infty} f(x - \kappa t)\, dx = e^{-rt}\int_{-\infty}^{\infty} f(x)\, dx = Me^{-rt}, \end{aligned}$$

where, in evaluating the integral $\int_{-\infty}^{\infty} f(x - \kappa t)\, dx$, we used the change of variables $x \leftrightarrow x - \kappa t$, and then used $M = \int_{-\infty}^{\infty} f(x)\, dx$.

Solutions to Exercises 2.1

1. (a) $\cos x$ has period 2π. (b) $\cos \pi x$ has period $T = \frac{2\pi}{\pi} = 2$. (c) $\cos \frac{2}{3}x$ has period $T = \frac{2\pi}{2/3} = 3\pi$. (d) $\cos x$ has period 2π, $\cos 2x$ has period π, 2π, 3π,:. A common period of $\cos x$ and $\cos 2x$ is 2π. So $\cos x + \cos 2x$ has period 2π.

2. (a) $\sin 7\pi x$ has period $T = \frac{2\pi}{7\pi} = 2/7$. (b) $\sin n\pi x$ has period $T = \frac{2\pi}{n\pi} = \frac{2}{n}$. Since any integer multiple of T is also a period, we see that 2 is also a period of $\sin n\pi x$. (c) $\cos mx$ has period $T = \frac{2\pi}{m}$. Since any integer multiple of T is also a period, we see that 2π is also a period of $\cos mx$. (d) $\sin x$ has period 2π, $\cos x$ has period 2π; $\cos x + \sin x$ so has period 2π. (e) Write $\sin^2 2x = \frac{1}{2} - \frac{\cos 4x}{2}$. The function $\cos 4x$ has period $T = \frac{2\pi}{4} = \frac{\pi}{2}$. So $\sin^2 2x$ has period $\frac{\pi}{2}$.

3. (a) The period is $T = 1$, so it suffices to describe f on an interval of length 1. From the graph, we have

$$f(x) = \begin{cases} 0 & \text{if } -\frac{1}{2} \le x < 0, \\ 1 & \text{if } 0 \le x < \frac{1}{2}. \end{cases}$$

For all other x, we have $f(x+1) = f(x)$.
(b) f is continuous for all $x \neq \frac{k}{2}$, where k is an integer. At the half-integers, $x = \frac{2k+1}{2}$, using the graph, we see that $\lim_{h\to x^+} f(h) = 0$ and $\lim_{h\to x^-} f(h) = 1$. At the integers, $x = k$, from the graph, we see that $\lim_{h\to x^+} f(h) = 1$ and $\lim_{h\to x^-} f(h) = 0$. The function is piecewise continuous.
(c) Since the function is piecewise constant, we have that $f'(x) = 0$ at all $x \neq \frac{k}{2}$, where k is an integer. It follows that $f'(x+) = 0$ and $f'(x-) = 0$ (Despite the fact that the derivative does not exist at these points; the left and right limits exist and are equal.)

4. The period is $T = 4$, so it suffices to describe f on an interval of length 4. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -2 \le x \le 0, \\ -x+1 & \text{if } 0 < x < 2. \end{cases}$$

For all other x, we have $f(x+4) = f(x)$. (b) The function is continuous at all x. (c) (c) The function is differentiable for all $x \neq 2k$, where k is an integer. Note that f' is also 4-periodic. We have

$$f'(x) = \begin{cases} 1 & \text{if } -2 < x \le 0, \\ -1 & \text{if } 0 < x < 2. \end{cases}$$

For all other $x \neq 2k$, we have $f(x+4) = f(x)$. If $x = 0, \pm 4, \pm 8, \ldots$, we have $f'(x+) = 1$ and $f'(x-) = -1$. If $x = \pm 2, \pm 6, \pm 10, \ldots$, we have $f'(x+) = -1$ and $f'(x-) = 1$.

5. This is the special case $p = \pi$ of Exercise 6(b).

6. (a) A common period is $2p$. (b) The orthogonality relations are

$$\int_{-p}^{p} \cos\frac{m\pi x}{p} \cos\frac{n\pi x}{p}\, dx = 0 \quad \text{if } m \neq n,\ m,\, n = 0, 1, 2, \ldots;$$

$$\int_{-p}^{p} \sin\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\, dx = 0 \quad \text{if } m \neq n,\ m,\, n = 1, 2, \ldots;$$

$$\int_{-p}^{p} \cos\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\, dx = 0 \quad \text{for all } m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots.$$

These formulas are established by using various addition formulas for the cosine and sine. For example, to prove the first one, if $m \neq n$, then

$$\begin{aligned}
&\int_{-p}^{p} \cos\frac{m\pi x}{p} \cos\frac{n\pi x}{p}\, dx \\
&= \frac{1}{2}\int_{-p}^{p} \left[\cos\frac{(m+n)\pi x}{p} + \cos\frac{(m-n)\pi x}{p}\right] dx \\
&= \frac{1}{2}\left[\frac{p}{m+n)\pi} \sin\frac{(m+n)\pi x}{p} + \frac{p}{m-n)\pi} \sin\frac{(m-n)\pi x}{p}\right]\Bigg|_{-p}^{p} = 0.
\end{aligned}$$

7. Suppose that Show that $f_1, f_2, \ldots, f_n, \ldots$ are T-periodic functions. This means that $f_j(x+T) = f(x)$ for all x and $j = 1, 2, \ldots, n$. Let $s_n(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$. Then

$$\begin{aligned} s_n(x+T) &= = a_1 f_1(x+T) + a_2 f_2(x+T) + \cdots + a_n f_n(x+T) \\ &= a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x) = s_n(x); \end{aligned}$$

which means that s_n is T-periodic. In general, if $s(x) = \sum_{j=1}^{\infty} a_j f_j(x)$ is a series that converges for all x, where each f_j is T-periodic, then

$$s(x+T) = \sum_{j=1}^{\infty} a_j f_j(x+T) = \sum_{j=1}^{\infty} a_j f_j(x) = s(x);$$

and so $s(x)$ is T-periodic.

8. (a) Since $|\cos x| \leq 1$ and $|\cos \pi x| \leq 1$, the equation $\cos x + \cos \pi x = 2$ holds if and only if $\cos x = 1$ and $\cos \pi x = 1$. Now $\cos x = 1$ implies that $x = 2k\pi$, k and integer, and $\cos \pi x = 1$ implies that $x = 2m$, m and integer. So $\cos x + \cos \pi x = 2$ implies that $2m = 2k\pi$, which implies that $k = m = 0$ (because π is irrational). So the only solution is $x = 0$. (b) Since $f(x) = \cos x + \cos \pi x = 2$ takes on the value 2 only at $x = 0$, it is not periodic.

9. (a) Suppose that f and g are T-periodic. Then $f(x+T) \cdot g(x+T) = f(x) \cdot g(x)$, and so $f \cdot g$ is T periodic. Similarly,

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)},$$

and so f/g is T periodic.
(b) Suppose that f is T-periodic and let $h(x) = f(x/a)$. Then

$$\begin{aligned} h(x+aT) &= f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a} + T\right) \\ &= f\left(\frac{x}{a}\right) \quad \text{(because } f \text{ is } T\text{-periodic)} \\ &= h(x). \end{aligned}$$

Thus h has period aT. Replacing a by $1/a$, we find that the function $f(ax)$ has period T/a.
(c) Suppose that f is T-periodic. Then $g(f(x+T)) = g(f(x))$, and so $g(f(x))$ is also T-periodic.

10. (a) $\sin x$ has period 2π, so $\sin 2x$ has period $2\pi/2 = \pi$ (by Exercise 9(b)).

(b) $\cos \frac{1}{2}x$ has period 4π and $\sin 2x$ has period π (or any integer multiple of it). So a common period is 4π. Thus $\cos \frac{1}{2}x + 3\sin 2x$ had period 4π (by Exercise 7)
(c) We can write $\frac{1}{2+\sin x} = g(f(x))$, where $g(x) = 1/x$ and $f(x) = 2 + \sin x$. Since f is 2π-periodic, it follows that $\frac{1}{2+\sin x}$ is 2π-periodic, by Exercise 9(c).
(d) Same as part (c). Here we let $f(x) = \cos x$ and $g(x) = e^x$. Then $e^{\cos x}$ is 2π-periodic.

11. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} \sin x\,dx = 2.$$

12. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} \cos x\,dx = 0.$$

13.

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi/2} 1\,dx = \pi/2.$$

14. Using Theorem 1,

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} x^2\,dx = \frac{\pi^3}{3}.$$

15. Let $F(x) = \int_a^x f(t)\,dt$. If F is 2π-periodic, then $F(x) = F(x+2\pi)$. But

$$F(x+2\pi) = \int_a^{x+2\pi} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+2\pi} f(t)\,dt = F(x) + \int_x^{x+2\pi} f(t)\,dt.$$

Since $F(x) = F(x+2\pi)$, we conclude that

$$\int_x^{x+2\pi} f(t)\,dt = 0.$$

Applying Theorem 1, we find that

$$\int_x^{x+2\pi} f(t)\,dt = \int_0^{2\pi} f(t)\,dt = 0.$$

The above steps are reversible. That is,

$$\begin{aligned}
\int_0^{2\pi} f(t)\,dt = 0 \quad &\Rightarrow \quad \int_x^{x+2\pi} f(t)\,dt = 0 \\
&\Rightarrow \quad \int_a^x f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+2\pi} f(t)\,dt = \int_a^{x+2\pi} f(t)\,dt \\
&\Rightarrow \quad F(x) = F(x+2\pi);
\end{aligned}$$

and so F is 2π-periodic.

16. We have

$$F(x+T) = \int_a^{x+T} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt.$$

So F is T periodic if and only if

$$\begin{aligned}
F(x+T) = F(x) \quad &\Leftrightarrow \quad \int_a^x f(t)\,dt + \int_x^{x+T} f(t)\,dt = \int_a^x f(t)\,dt \\
&\Leftrightarrow \quad \int_x^{x+T} f(t)\,dt = 0 \\
&\Leftrightarrow \quad \int_a^{a+T} f(t)\,dt = 0,
\end{aligned}$$

where the last assertion follows from Theorem 1.

17. By Exercise 16, F is 2 periodic, because $\int_0^2 f(t)\,dt = 0$ (this is clear from the graph of f). So it is enough to describe F on any interval of length 2. For $0 < x < 2$, we have

$$F(x) = \int_0^x (1-t)\,dt = t - \frac{t^2}{2}\Big|_0^x = x - \frac{x^2}{2}.$$

For all other x, $F(x+2) = F(x)$. (b) The graph of F over the interval [0, 2] consists of the arch of a parabola looking down, with zeros at 0 and 2. Since F is 2-periodic, the graph is repeated over and over.

18. (a) We have

$$\begin{aligned}\int_{nT}^{(n+1)T} f(x)\,dx &= \int_0^T f(s+nT)\,ds \quad (\text{let } x = s+nT,\ dx = ds)\\ &= \int_0^T f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_0^T f(x)\,dx.\end{aligned}$$

(b)

$$\begin{aligned}\int_{(n+1)T}^{a+T} f(x)\,dx &= \int_{nT}^{a} f(s+T)\,ds \quad (\text{let } x = s+T,\ dx = ds)\\ &= \int_{nT}^{a} f(s)\,ds \quad (\text{because } f \text{ is } T\text{-periodic})\\ &= \int_{nT}^{a} f(x)\,dx.\end{aligned}$$

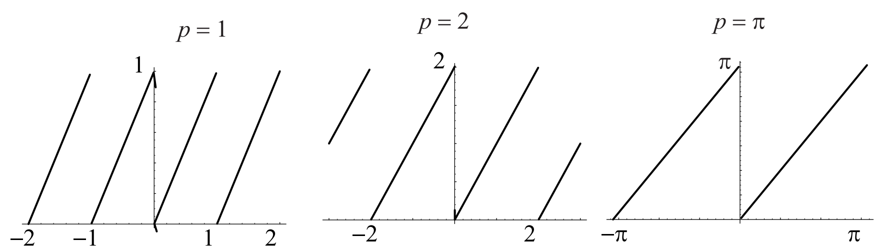
(c)

$$\begin{aligned}\int_a^{a+T} f(x)\,dx &= \int_a^{(n+1)T} f(x)\,dx + \int_{(n+1)T}^{a+T} f(x)\,dx\\ &= \int_a^{(n+1)T} f(x)\,dx + \int_{nT}^{a} f(x)\,dx \quad (\text{by (b)})\\ &= \int_{nT}^{(n+1)T} f(x)\,dx = \int_0^T f(x)\,dx \quad (\text{by (a)}).\end{aligned}$$

19. (a) The plots are shown in the following figures.

The Mathematica command to plot the graphs is the following:

```
p=Pi
Plot[x - p Floor[x/p], {x, -Pi,  Pi}]
```

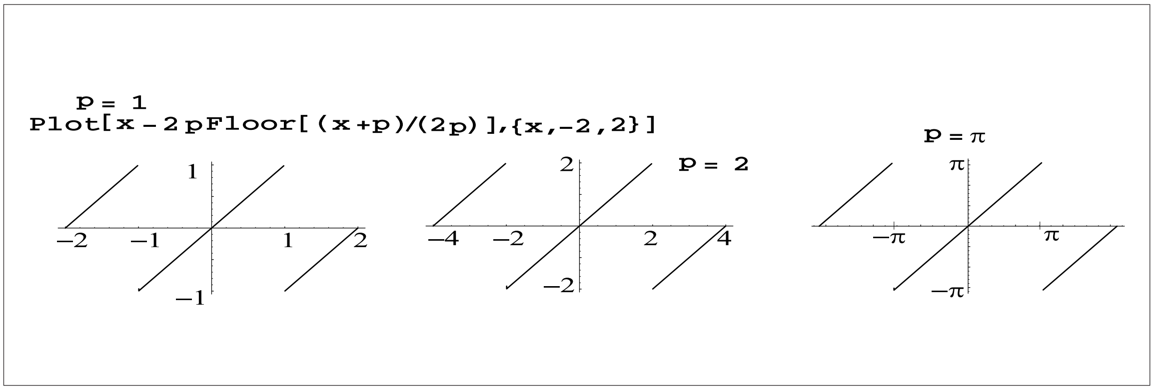


(b) Let us show that $f(x) = x - p\left[\frac{x}{p}\right]$ is p-periodic.

$$\begin{aligned}f(x+p) &= x+p-p\left[\frac{x+p}{p}\right] = x+p-p\left[\frac{x}{p}+1\right] = x+p-p\left(\left[\frac{x}{p}\right]+1\right)\\ &= x-p\left[\frac{x}{p}\right] = f(x).\end{aligned}$$

From the graphs it is clear that $f(x) = x$ for all $0 < x < p$. To see this from the formula, use the fact that $[t] = 0$ if $0 \le t < 1$. So, if $0 \le x < p$, we have $0 \le \frac{x}{p} < 1$, so $\left[\frac{x}{p}\right] = 0$, and hence $f(x) = x$.

20. (a) Plot of the function $f(x) = x - 2p\left[\frac{x+p}{2p}\right]$ for $p = 1,\ 2,$ and π.

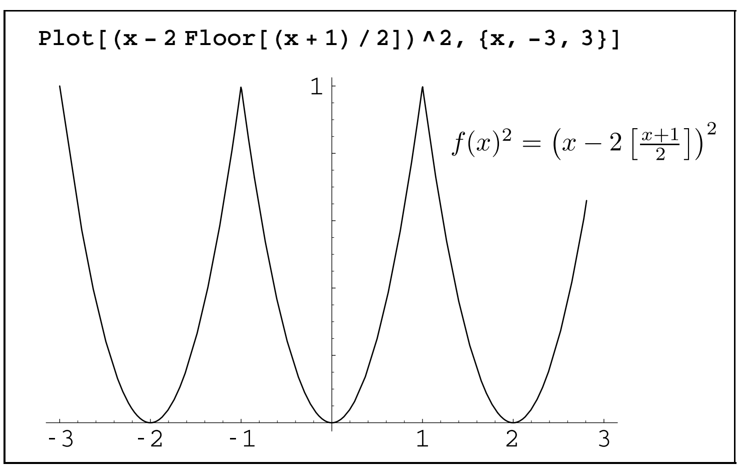


(b)

$$\begin{aligned} f(x+2p) &= (x+2p) - 2p\left[\frac{(x+2p)+p}{2p}\right] = (x+2p) - 2p\left[\frac{x+p}{2p}+1\right] \\ &= (x+2p) - 2p\left(\left[\frac{x+p}{2p}\right]+1\right) = x - 2p\left[\frac{x+p}{2p}\right] = f(x). \end{aligned}$$

So f is $2p$-periodic. For $-p < x < p$, we have $0 < \frac{x+p}{2p} < 1$, hence $\left[\frac{x+p}{2p}\right] = 0$, and so $f(x) = x - 2p\left[\frac{x+p}{2p}\right] = x$.

21. (a) With $p = 1$, the function f becomes $f(x) = x - 2\left[\frac{x+1}{2}\right]$, and its graph is the first one in the group shown in Exercise 20. The function is 2-periodic and is equal to x on the interval $-1 < x < 1$. By Exercise 9(c), the function $g(x) = h(f(x)$ is 2-periodic for any function h; in particular, taking $h(x) = x^2$, we see that $g(x) = f(x)^2$ is 2-periodic. (b) $g(x) = x^2$ on the interval $-1 < x < 1$, because $f(x) = x$ on that interval. (c) Here is a graph of $g(x) = f(x)^2 = \left(x - 2\left[\frac{x+1}{2}\right]\right)^2$, for all x.

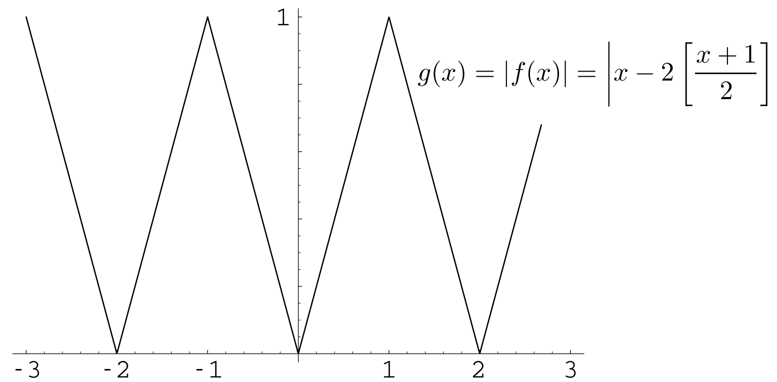


22. (a) As in Exercise 21, the function $f(x) = x - 2\left[\frac{x+1}{2}\right]$ is 2-periodic and is equal to x on the interval $-1 < x < 1$. So, by Exercise 9(c), the function

$$g(x) = |f(x)| = \left|x - 2\left[\frac{x+1}{2}\right]\right|$$

is 2-periodic and is clearly equal to $|x|$ for all $-1 < x < 1$. Its graph is a triangular wave as shown in (b).

```
Plot[Abs[x - 2 Floor[(x + 1) / 2]], {x, -3, 3}]
```



(c) To obtain a triangular wave of arbitrary period $2p$, we use the $2p$-periodic function

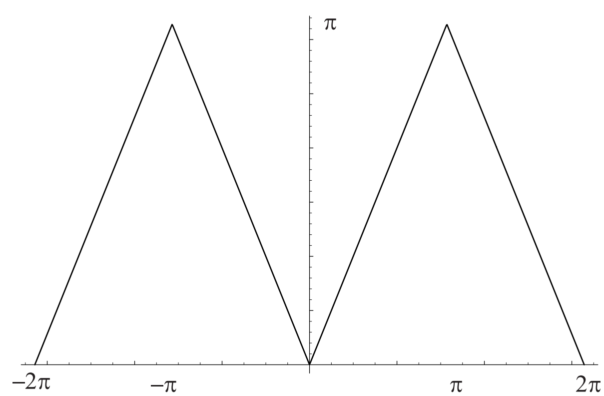
$$f(x) = x - 2p\left[\frac{x+p}{2p}\right],$$

which is equal to x on the interval $-p < x < p$. Thus,

$$g(x) = \left| x - 2p\left[\frac{x+p}{2p}\right]\right|$$

is a $2p$-periodic triangular wave, which equal to $|x|$ in the interval $-p < x < p$. The following graph illustrates this function with $p = \pi$.

```
p = Pi
Plot[Abs[x - 2 p Floor[(x + p) / (2 p)]], {x, -2 Pi, 2 Pi}]
```



23. (a) Since $f(x + 2p) = f(x)$, it follows that $g(f(x + 2p)) = g(f(x))$ and so $g(f(x))$ is $2p$-periodic. For $-p < x < p$, $f(x) = x$ and so $g(f(x)) = g(x)$.
(b) The function $e^{g(x)}$, with $p = 1$, is the 2-periodic extension of the function which equals e^x on the interval $-1 < x < 1$. Its graph is shown in Figure 1, Section 2.6 (with $a = 1$).

24. Let f_1, f_2, $\dots$, f_n be the continuous components of f on the interval $[0, T]$, as described prior to the exercise. Since each f_j is continuous on a closed and bounded interval, it is bounded: That is, there exists $M > 0$ such that $|f_j(x)| \le M$ for all x in the domain of f_j. Let M denote the maximum value of M_j for $j = 1, 2, \dots, n$. Then $|f(x)| \le M$ for all x in $[0, T]$ and so f is bounded.

25. We have

$$\begin{aligned} |F(a+h)-F(a)| &= \left| \int_0^a f(x)\,dx - \int_0^{a+h} f(x)\,dx \right| \\ &= \left| \int_a^{a+h} f(x)\,dx \right| \leq M \cdot h, \end{aligned}$$

where M is a bound for $|f(x)|$, which exists by the previous exercise. (In deriving the last inequality, we used the following property of integrals:

$$\left| \int_a^b f(x)\,dx \right| \leq (b-a) \cdot M,$$

which is clear if you interpret the integral as an area.) As $h \to 0$, $M \cdot h \to 0$ and so $|F(a+h)-F(a)| \to 0$, showing that $F(a+h) \to F(a)$, showing that F is continuous at a.
(b) If f is continuous and $F(a) = \int_0^a f(x)\,dx$, the fundamental theorem of calculus implies that $F'(a) = f(a)$. If f is only piecewise continuous and a_0 is a point of continuity of f, let (x_{j-1}, x_j) denote the subinterval on which f is continuous and a_0 is in (x_{j-1}, x_j). Recall that $f = f_j$ on that subinterval, where f_j is a continuous component of f. For a in (x_{j-1}, x_j), consider the functions $F(a) = \int_0^a f(x)\,dx$ and $G(a) = \int_{x_{j-1}}^a f_j(x)\,dx$. Note that $F(a) = G(a) + \int_0^{x_{j-1}} f(x)\,dx = G(a) + c$. Since f_j is continuous on (x_{j-1}, x_j), the fundamental theorem of calculus implies that $G'(a) = f_j(a) = f(a)$. Hence $F'(a) = f(a)$, since F differs from G by a constant.

26. We have

$$F(a) = \int_0^{a+T} f(x)\,dx - \int_0^a f(x)\,dx.$$

By the previous exercise, F is a sum of two continuous and piecewise smooth functions. (The first term is a translate of $\int_0^a f(x)\,dx$ by T, and so it is continuous and piecewise smooth.) Thus F is continuous and piecewise smooth. Since each term is differentiable the points of continuity of f, we conclude that F is also differentiable at the points of continuity of f.
(b) By Exercise 25, we have, at the points where f is continuous, $F'(a) = f(a+T) - f(a) = 0$, because f is periodic with period T. Thus F is piecewise constant.
(c) A piecewise constant function that is continuous is constant (just take left and right limits at points of discontinuity.) So F is constant.

27. (a) The function $\sin \frac{1}{x}$ does not have a right or left limit as $x \to 0$, and so it is not piecewise continuous. (To be piecewise continuous, the left and right limits must exist.) The reason is that $1/x$ tends to $+\infty$ as $x \to 0^+$ and so $\sin 1/x$ oscillates between $+1$ and -1. Similarly, as $x \to 0^-$, $\sin 1/x$ oscillates between $+1$ and -1. See the graph.
(b) The function $f(x) = x \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0. The reason for this is that $\sin 1/x$ is bounded by 1, so, as $x \to 0$, $x \sin 1/x \to 0$, by the squeeze theorem. The function, however, is not piecewise smooth. To see this, let us compute its derivative. For $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

As $x \to 0^+$, $1/x \to +\infty$, and so $\sin 1/x$ oscillates between $+1$ and -1, while $\frac{1}{x} \cos \frac{1}{x}$ oscillates between $+\infty$ and $-\infty$. Consequently, $f'(x)$ has no right limit at 0. Similarly, it fails to have a left limit at 0. Hence f is not piecewise smooth. (Recall that to be piecewise smooth the left and right limits of the derivative have to exist.)
(c) The function $f(x) = x^2 \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0, as in case (b).

Also, as in (b), the function is not piecewise smooth. To see this, let us compute its derivative. For $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

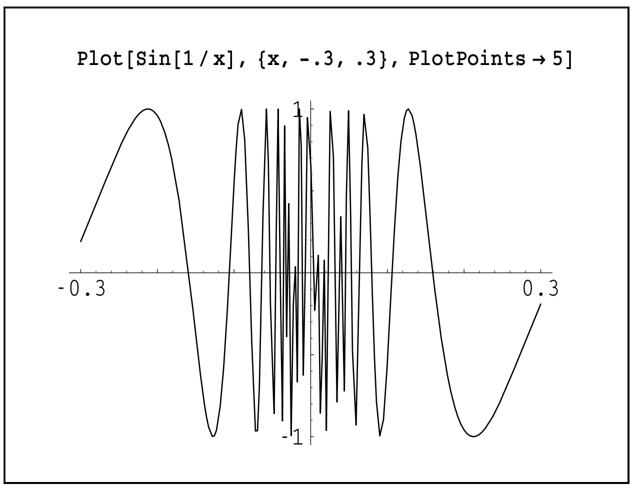
As $x \to 0^+$, $1/x \to +\infty$, and so $2x \sin 1/x \to 0$, while $\cos \frac{1}{x}$ oscillates between $+1$ and -1. Hence, as $x \to 0^+$, $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ oscillates between $+1$ and -1, and so $f'(x)$ has no right limit at 0. Similarly, it fails to have a left limit at 0. Hence f is not piecewise smooth. (d) The function $f(x) = x^3 \sin \frac{1}{x}$ and $f(0) = 0$ is continuous at 0, as in case (b). It is also smooth. We only need to check the derivative at $x = 0$. We have

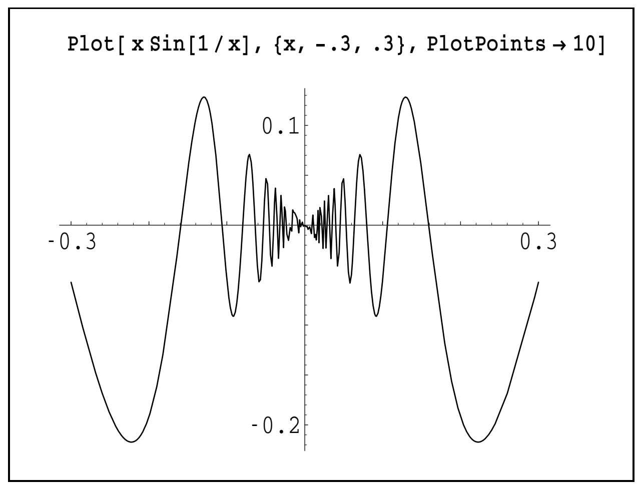
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h}}{h} \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0.$$

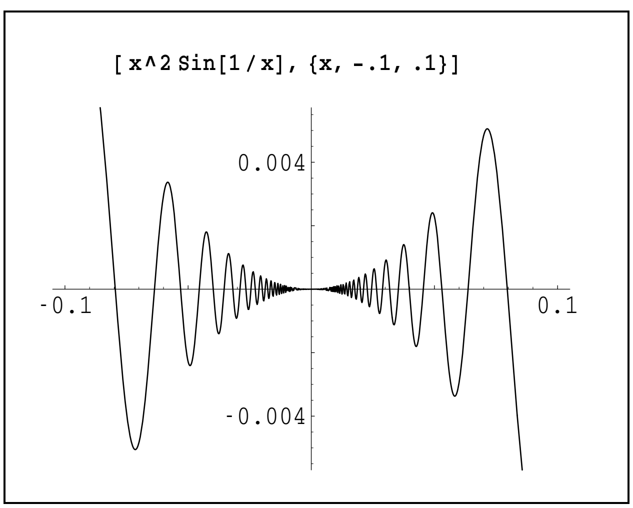
For $x \neq 0$, we have

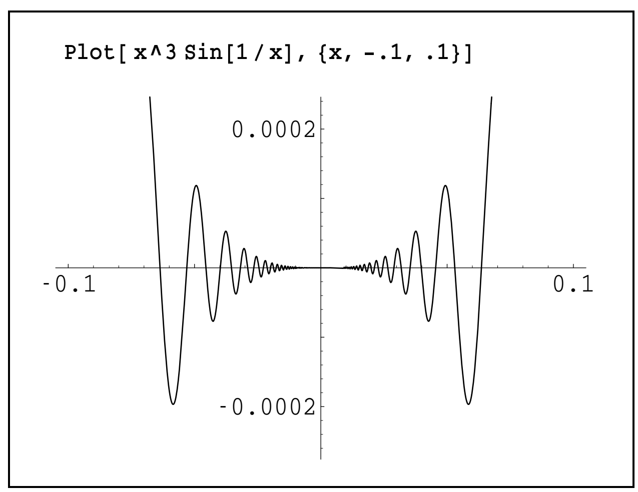
$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}.$$

Since $f'(x) \to 0 = f'(0)$ as $x \to 0$, we conclude that f' exists and is continuous for all x.





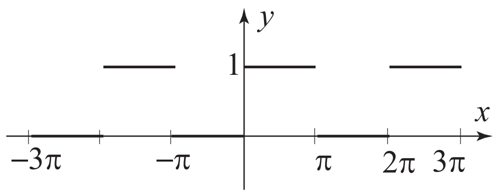




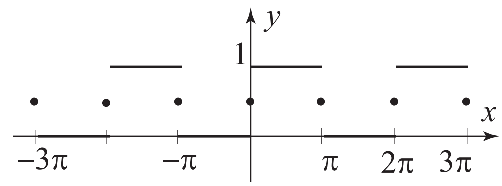
Solutions to Exercises 2.2

1. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is $\frac{1}{2}$.

Function

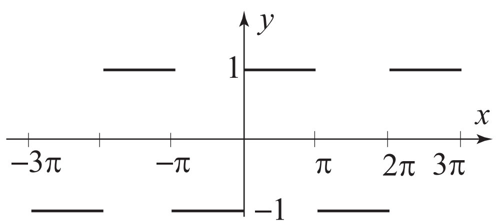


Fourier series

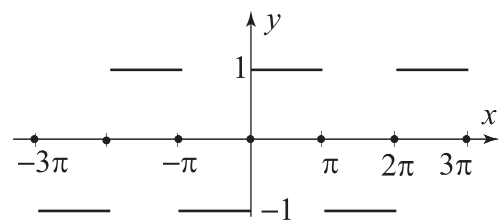


2. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is 0 in this case.

Function

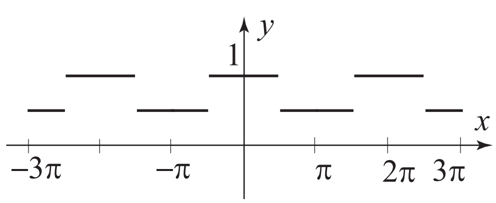


Fourier series

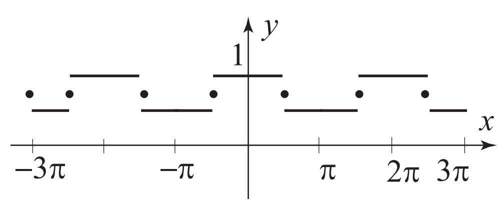


3. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is 3/4 in this case.

Function

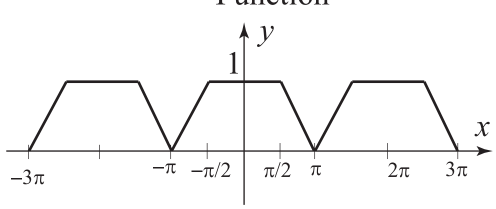


Fourier series

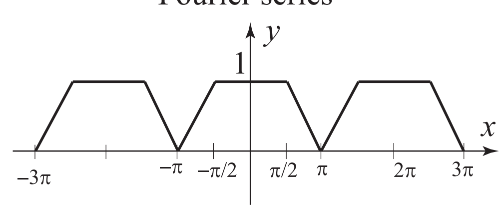


4. Since the function is continuous and piecewise smooth, it is equal to its Fourier series.

Function



Fourier series



5. We compute the Fourier coefficients using he Euler formulas. Let us first note that since $f(x) = |x|$ is an even function on the interval $-\pi < x < \pi$, the product $f(x) \sin nx$ is an odd function. So

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{|x| \sin nx}^{\text{odd function}} \, dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. For the other coefficients, we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= = \frac{1}{2\pi}\int_{-\pi}^{0} (-x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big|_0^{\pi} = \frac{\pi}{2}. \end{aligned}$$

In computing a_n $(n \geq 1)$, we will need the formula

$$\int x\cos ax\,dx = \frac{\cos(a\,x)}{a^2} + \frac{x\,\sin(a\,x)}{a} + C \quad (a \neq 0),$$

which can be derived using integration by parts. We have, for $n \geq 1$,

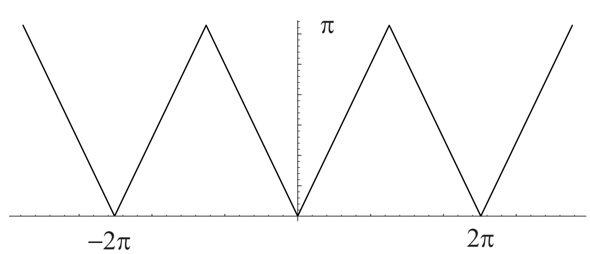
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\big[(-1)^n - 1\big] \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is

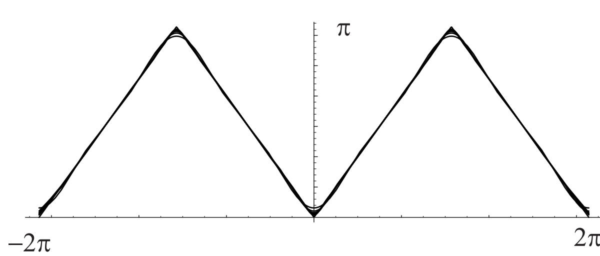
$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos(2k+1)x\,.$$

```
In[2]:=  s[n_, x_] := Pi / 2 - 4 / Pi Sum[1 / (2 k + 1) ^2 Cos[(2 k + 1) x], {k, 0, n}]

In[25]:= partialsums = Table[s[n, x], {n, 1, 7}];
         f[x_] = x - 2 Pi Floor[(x + Pi) / (2 Pi)]
         g[x_] = Abs[f[x]]
         Plot[g[x], {x, -3 Pi, 3 Pi}]
         Plot[Evaluate[{g[x], partialsums}], {x, -2 Pi, 2 Pi}]
```



The function g(x) = | x | and its periodic extension



Partial sums of the Fourier series. Since we are summing over the odd integers, when n = 7, we are actually summing the 15th partial sum.

6. We compute the Fourier coefficients using he Euler formulas. Let us first note that $f(x)$ is an odd function on the interval $-\pi < x < \pi$, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. A similar argument shows that $a_n = 0$ for all n. This leaves b_n. We have, for $n \geq 1$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi/2}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi/2}\sin nx\,dx \\
&= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]\Big|_0^{\pi/2} \\
&= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right]
\end{aligned}$$

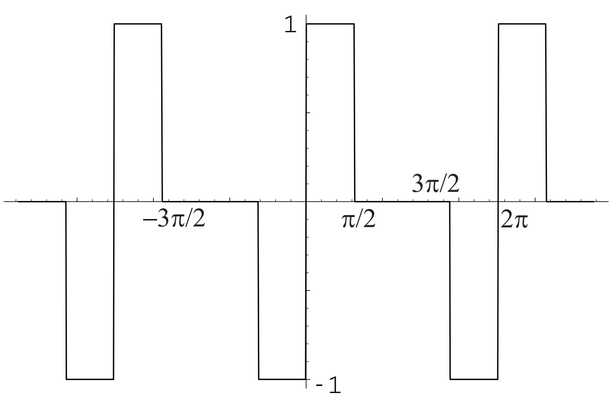
Thus the Fourier series is

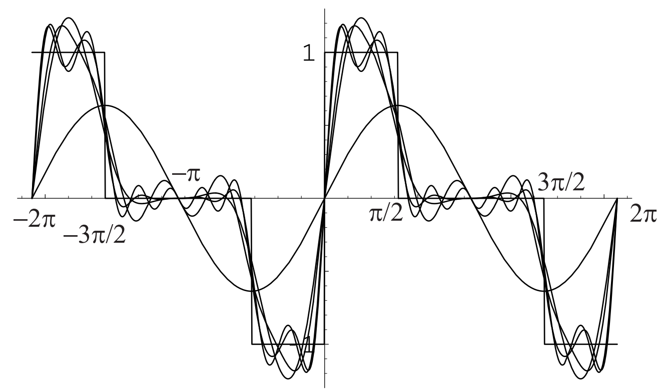
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin nx.$$

(b) The new part here is defining the periodic function. We willuse the same construction as in the previous exercise. We will use the Which command to define a function by several formulas over the interval - 2 Pi to 2 Pi. We then use the construction of Exrcises 20 and 22 of Section 2.1 to define the periodic extension.

```
s[n_, x_] := 2 / Pi Sum[(1 - Cos[ k Pi / 2]) / k Sin[k x], {k, 1, n}]

partialsums = Table[s[n, x], {n, 1, 10, 2}];
f[x_] = Which[x < -Pi / 2, 0, -Pi / 2 < x < 0, -1, 0 < x < Pi / 2, 1, x > Pi / 2, 0]
g[x_] = x - 2 Pi Floor[(x + Pi) / ( 2 Pi)]
h[x_] = f[g[x]]
Plot[h[x], {x, -3 Pi, 3 Pi}]
Plot[Evaluate[{h[x], partialsums}], {x, -2 Pi, 2 Pi}]
```





The partial sums of the Fourier series converge to the function at all points of continuity. At the points of discontinuity, we observe a Gibbs phenomenon. At the points of discontinuity, the Fourier series is converging to 0 or $\pm 1/2$. depending on the location of the point of discontinuity: For all $x = 2k\pi$, the Fourier series converges to 0; and for all $x = (2k+1)\pi/2$, the Fourier series converges to $(-1)^k/2$.

7. f is even, so all the b_n's are zero. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = \frac{2}{\pi}.$$

We will need the trigonometric identity

$$\sin a \cos b = \frac{1}{2}\big(\sin(a-b) + \sin(a+b)\big).$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\big(\sin(1-n)x + \sin(1+n)x\big)\,dx \\
&= \frac{1}{\pi}\left[\frac{-1}{1-n}\cos(1-n)x - \frac{1}{(1+n)}\cos(1+n)x\right]\Big|_0^{\pi} \quad (\text{if } n \neq 1) \\
&= \frac{1}{\pi}\left[\frac{-1}{1-n}(-1)^{1-n} - \frac{1}{(1+n)}(-1)^{1+n} + \frac{1}{1-n} + \frac{1}{1+n}\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4}{\pi(1-n^2)} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

If $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\,dx = 0.$$

Thus, the Fourier series is : $\displaystyle |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1}\cos 2kx\,.$

8. You can compute the Fourier series directly as we did in Exercise 7, or you can use Exercise 7 and note that $|\cos x| = |\sin(x + \frac{\pi}{2})|$. This identity can be verified by comparing the graphs of the functions or as follows:

$$\sin(x + \frac{\pi}{2}) = \sin x \underbrace{\cos \frac{\pi}{2}}^{=0} + \cos x \underbrace{\sin\frac{\pi}{2}}^{=1} = \cos x.$$

So

$$\begin{aligned}
|\cos x| &= |\sin(x + \frac{\pi}{2})| \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1}\cos\Big(2k(x + \frac{\pi}{2})\Big) \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2 - 1}\cos 2kx,
\end{aligned}$$

where we have used

$$\cos\Big(2k(x + \frac{\pi}{2})\Big) = \cos 2kx\,\cos k\pi - \sin 2kx\,\sin k\pi = (-1)^k \cos 2kx.$$

9. Just some hints:
(1) f is even, so all the b_n's are zero.
(2)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

(3) Establish the identity

$$\int x^2 \cos(ax)\,dx = \frac{2\,x\,\cos(a\,x)}{a^2} + \frac{\left(-2 + a^2\,x^2\right)\,\sin(a\,x)}{a^3} + C \qquad (a \neq 0),$$

using integration by parts.

10. The function $f(x) = 1 - \sin x + 3\cos 2x$ is already given by its own Fourier series. If you try to compute the Fourier coefficients by using the Euler formulas, you will find that all a_ns and b_n's are 0 except $a_0 = 1$, $a_2 = 3$, and $b_1 = -1$. This is because of the orthogonality of the trigonometric system. Let us illustrate this by computing the b_n's ($n \geq 1$):

$$\begin{aligned}
b_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - \sin x + 3\cos 2x)\sin nx\,dx \\
&= \frac{1}{2\pi}\overbrace{\int_{-\pi}^{\pi}\sin nx\,dx}^{=0 \text{ for all } n} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x \sin nx\,dx + \frac{3}{2\pi}\overbrace{\int_{-\pi}^{\pi}\cos 2x \sin nx\,dx}^{=0 \text{ for all } n} \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x \sin nx\,dx \\
&= \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which shows that $b_1 = -1$, while all other b_ns are 0.

11. We have $f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $g(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$. Both functions are given by their Fourier series.

12. The function is clearly odd and so, as in Exercise 6, all the a_ns are 0. Also,

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 x - x^3)\sin nx\,dx.$$

To compute this integral, we use integration by parts, as follows:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \overbrace{(\pi^2 x - x^3)}^{u}\overbrace{\sin nx}^{v'}\,dx \\
&= \frac{2}{\pi}(\pi^2 x - x^3)\frac{-\cos nx}{n}\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}(\pi^2 - 3x^2)\frac{-\cos nx}{n}\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} \overbrace{(\pi^2 - 3x^2)}^{u}\frac{\overbrace{\cos nx}^{v'}}{n}\,dx \\
&= \frac{2}{\pi}(\pi^2 - 3x^2)\frac{\sin nx}{n^2}\Big|_0^{\pi} - \frac{12}{\pi}\int_0^{\pi} x\frac{\sin nx}{n^2}\,dx \\
&= \frac{12}{\pi}x\frac{(-\cos nx)}{n^3}\Big|_0^{\pi} + \frac{12}{\pi}\overbrace{\int_0^{\pi}\frac{\cos nx}{n^3}\,dx}^{=0} \\
&= 12\frac{(-\cos n\pi)}{n^3} = 12\frac{(-1)^{n+1}}{n^3}
\end{aligned}$$

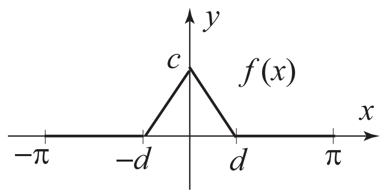
Thus the Fourier series is

$$12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx.$$

13. You can compute directly as we did in Example 1, or you can use the result of Example 1 as follows. Rename the function in Example 1 $g(x)$. By comparing graphs, note that $f(x) = -2g(x+\pi)$. Now using the Fourier series of $g(x)$ from Example, we get

$$f(x) = -2\sum_{n=1}^{\infty} \frac{\sin n(\pi + x)}{n} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

14. The function is even. Its graph is as follows:



Since $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, by interpreting the integral as an area, we see that a_0 is $1/(2\pi)$ times the area under the graph of $f(x)$, above the x-axis, from $x=-d$ to $x=d$. Thus $a_0 = \frac{1}{2\pi}cd$. To compute a_n, we use integration by parts:

$$\begin{aligned} a_n &= \frac{2c}{d\pi}\int_0^d \overbrace{(x-d)}^{u}\,\overbrace{\cos nx}^{v'}\,dx \\ &= \frac{2c}{d\pi}(x-d)\frac{\sin nx}{n}\Big|_0^d - \frac{2}{\pi}\int_0^d \frac{\sin nx}{n}\,dx \\ &= \frac{2c}{d\pi}\frac{-\cos nx}{n^2}\Big|_0^d = \frac{2c}{d\pi n^2}\left(1-\cos(nd)\right). \end{aligned}$$

Using the identity $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$, we find that

$$b_n = \frac{4c}{d\pi n^2}\sin^2\frac{nd}{2}.$$

Thus the Fourier series is

$$\frac{cd}{2\pi} + \frac{4c}{d\pi}\sum_{n=1}^{\infty}\frac{\sin^2\frac{nd}{2}}{n^2}\cos nx.$$

15. f is even, so all the b_n's are zero. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} e^{-x}\,dx = -\frac{e^{-x}}{\pi}\Big|_0^{\pi} = \frac{1-e^{-\pi}}{\pi}.$$

We will need the integral identity

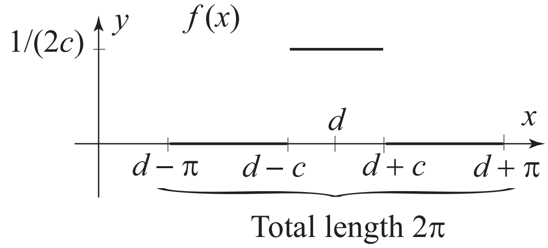
$$\int e^{ax}\cos(bx)\,dx = \frac{a\,e^{a\,x}\,\cos(b\,x)}{a^2+b^2} + \frac{b\,e^{a\,x}\,\sin(b\,x)}{a^2+b^2} + C \quad (a^2+b^2\neq 0),$$

which can be established by using integration by parts; alternatively, see Exercise 17, Section 2.6. We have, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{n}{n^2+1}e^{-x}\sin nx - \frac{1}{n^2+1}e^{-x}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi(n^2+1)}\left[-e^{-\pi}(-1)^n+1\right] = \frac{2(1-(-1)^n e^{-\pi})}{\pi(n^2+1)}. \end{aligned}$$

Thus the Fourier series is $\displaystyle \frac{e^{\pi}-1}{\pi e^{\pi}} + \frac{2}{\pi e^{\pi}}\sum_{n=1}^{\infty}\frac{1}{n^2+1}(e^{\pi}-(-1)^n)\cos nx\,.$

16. In general, the function is neither even nor odd. Its graph is the following:



As in Exercise 14, we interpret a_0 as a scaled area and obtain $a_0 = \frac{1}{2\pi}(2c)/(2c) = 1/(2\pi)$. For the other coefficients, we will compute the integrals over the interval $(d-\pi, d+\pi)$, whose length is 2π. This is possible by Theorem 1, Section 2.1, because the integrands are 2π-periodic. We have

$$\begin{aligned} a_n &= \frac{1}{2c\pi}\int_{d-c}^{d+c}\cos nx\,dx \\ &= \frac{1}{2cn\pi}\sin nx\Big|_{d-c}^{d+c} = \frac{1}{2cn\pi}\left(\sin n(d+c) - \sin n(d-c)\right) \\ &= \frac{1}{cn\pi}\cos(nd)\sin(nc). \end{aligned}$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{2c\pi}\int_{d-c}^{d+c}\sin nx\,dx \\ &= -\frac{1}{2cn\pi}\cos nx\Big|_{d-c}^{d+c} = \frac{1}{2cn\pi}\left(\cos n(d-c) - \cos n(d+c)\right) \\ &= \frac{1}{cn\pi}\sin(nd)\sin(nc). \end{aligned}$$

Thus the Fourier series is

$$\frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left(\frac{\cos(nd)\sin(nc)}{n}\cos nx + \frac{\sin(nd)\sin(nc)}{n}\sin nx\right).$$

Another way of writing the Fourier series is as follows:

$$\begin{aligned} &\frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(nc)}{n}\left(\cos(nd)\cos nx + \sin(nd)\sin nx\right)\right\} \\ &= \frac{1}{2\pi} + \frac{1}{c\pi}\sum_{n=1}^{\infty}\left\{\frac{\sin(nc)}{n}\cos n(x-d)\right\}. \end{aligned}$$

17. Setting $x = \pi$ in the Fourier series expansion in Exercise 9 and using the fact that the Fourier series converges for all x to $f(x)$, we obtain

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

where we have used $\cos n\pi = (-1)^n$. Simplifying, we find

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

18. Take $x = \frac{\pi}{2}$ in the Fourier series of Exercise 13, note that $\sin(n\frac{\pi}{2}) = 0$ if $n = 2k$ and $\sin(n\frac{\pi}{2}) = (-1)^k$ if $n = 2k+1$, and get

$$\frac{\pi}{2} = 2\sum_{k=0}^{\infty}\frac{(-1)^{2k+2}(-1)^k}{2k+1} = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1},$$

and the desired series follows upon dividing by 2.

19. (a) Let $f(x)$ denote the function in Exercise 1 and $w(x)$ the function in Example 5. Comparing these functions, we find that $f(x) = \frac{1}{\pi}w(x)$. Now using the Fourier series of w, we find

$$f(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + 2\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}\right] = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}.$$

(b) Let $g(x)$ denote the function in Exercise 2 and $f(x)$ the function in (a). Comparing these functions, we find that $g(x) = 2f(x) - 1$. Now using the Fourier series of f, we find

$$g(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}.$$

(c) Let $k(x)$ denote the function in Figure 13, and let $f(x)$ be as in (a). Comparing these functions, we find that $k(x) = f\big(x + \frac{\pi}{2}\big)$. Now using the Fourier series of f, we get

$$\begin{aligned}
k(x) &= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)(x+\frac{\pi}{2})}{2k+1} \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\Big(\sin[(2k+1)x]\overbrace{\cos[(2k+1)\frac{\pi}{2}]}^{=0} + \cos[(2k+1)x]\overbrace{\sin[(2k+1)\frac{\pi}{2}]}^{=(-1)^k}\Big) \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos[(2k+1)x].
\end{aligned}$$

(d) Let $v(x)$ denote the function in Exercise 3, and let $k(x)$ be as in (c). Comparing these functions, we find that $v(x) = \frac{1}{2}\big(k(x) + 1\big)$. Now using the Fourier series of k, we get

$$v(x) = \frac{3}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos[(2k+1)x].$$

20. (a) Let $f(x)$ denote the function in Figure 14 and let $|\sin x|$. By comparing graphs, we see that $|\sin x| + \sin x = 2f(x)$. So

$$f(x) = \frac{1}{2}\left(|\sin x| + \sin x\right).$$

Now the Fourier series of $\sin x$ is $\sin x$ and the Fourier series of $|\sin x|$ is computed in Exercise 7. Combining these two series, we obtain

$$f(x) = \frac{1}{2}\sin x + \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k)^2 - 1}\cos 2kx.$$

In particular, $f(x)$ has only one nonzero b_n; namely, $b_1 = \frac{1}{2}$. All other b_ns are 0.
(b) Let $g(x)$ denote the function in Figure 15. Then $g(x) = f(x + \frac{\pi}{2})$. Hence, using

$\sin(a + \frac{\pi}{2}) = \cos a$, we obtain

$$\begin{aligned} g(x) &= f\big(x + \frac{\pi}{2}\big) \\ &= \frac{1}{2}\sin\big(x + \frac{\pi}{2}\big) + \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k)^2-1}\cos\Big[2k\big(x + \frac{\pi}{2}\big)\Big] \\ &= \frac{1}{\pi} + \frac{1}{2}\cos x \\ &\quad -\frac{2}{\pi}\sum_{k=1}^{\infty}\Big[\frac{1}{(2k)^2-1}\cos(2kx)\overbrace{\cos(2k\frac{\pi}{2})}^{(-1)^k} - \sin(2kx)\overbrace{\sin(2k\frac{\pi}{2})}^{=0}\Big] \\ &= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{k=1}^{\infty}\Big[\frac{(-1)^k}{(2k)^2-1}\cos(2kx)\Big]. \end{aligned}$$

21. (a) Interpreting the integral as an area (see Exercise 16), we have

$$a_0 = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{8}.$$

To compute a_n, we first determine the equation of the function for $\frac{\pi}{2} < x < \pi$. From Figure 16, we see that $f(x) = \frac{2}{\pi}(\pi - x)$ if $\frac{\pi}{2} < x < \pi$. Hence, for $n \geq 1$,

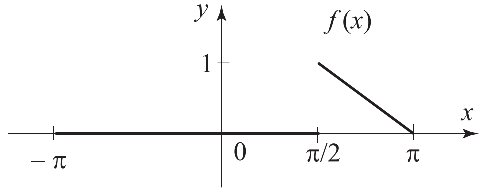
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\cos nx}^{v'}\,dx \\ &= \frac{2}{\pi^2}(\pi - x)\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi} + \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\sin nx}{n}\,dx \\ &= \frac{2}{\pi^2}\left[\frac{-\pi}{2n}\sin\frac{n\pi}{2}\right] - \frac{2}{\pi^2 n^2}\cos nx\Big|_{\pi/2}^{\pi} \\ &= -\frac{2}{\pi^2}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]. \end{aligned}$$

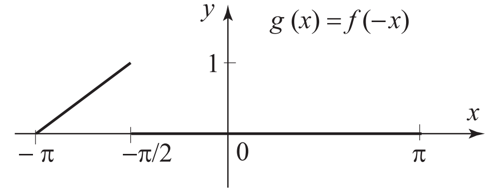
Also,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi}\frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\sin nx}^{v'}\,dx \\ &= -\frac{2}{\pi^2}(\pi - x)\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} - \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,dx \\ &= \frac{2}{\pi^2}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]. \end{aligned}$$

Thus the Fourier series representation of f is

$$\begin{aligned} f(x) &= \frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Big\{-\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\cos nx \\ &\qquad + \left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin nx\Big\}. \end{aligned}$$





(b) Let $g(x) = f(-x)$. By performing a change of variables $x \leftrightarrow -x$ in the Fourier series of f, we obtain (see also Exercise 24 for related details) Thus the Fourier series representation of f is

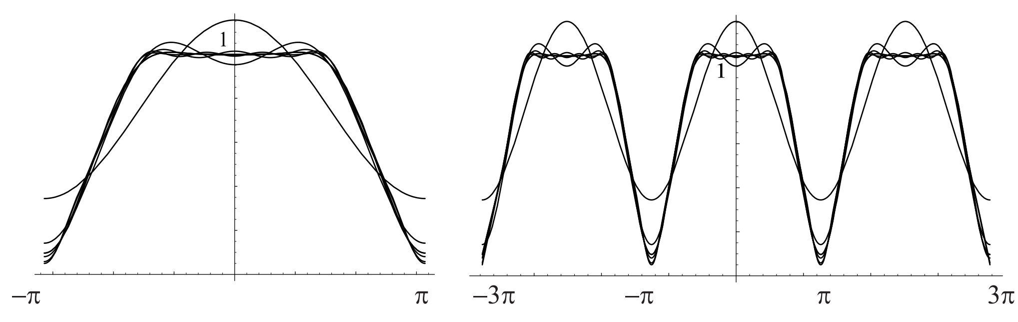
$$\begin{aligned} g(x) &= \frac{1}{8} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left\{ -\left[\frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos \frac{n\pi}{2} \right] \cos nx \right. \\ &\qquad \left. - \left[\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \sin nx \right\}. \end{aligned}$$

22. By comparing graphs, we see that the function in Exercise 4 (call it $k(x)$) is the sum of the three functions in Exercises 19(c) and 21(a) and (b) (call them h, f, and g, respectively). Thus, adding and simplifying, we obtain

$$\begin{aligned} k(x) &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n} \cos nx \\ &\quad + \frac{1}{8} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left\{ -\left[\frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos \frac{n\pi}{2} \right] \cos nx \right. \\ &\qquad \left. + \left[\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \sin nx \right\} \\ &\quad + \frac{1}{8} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left\{ -\left[\frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos \frac{n\pi}{2} \right] \cos nx \right. \\ &\qquad \left. - \left[\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \sin nx \right\} \\ &= \frac{3}{4} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left\{ \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos \frac{n\pi}{2} \right] \cos nx \right\}. \end{aligned}$$

We illustrate the convergence of the Fourier series in the following figure.

```
s[n_, x_] := 3 / 4 - 4 / Pi^2 Sum[((-1)^k - Cos[k Pi / 2]) Cos[k x] / k^2, {k, 1, n}]
partialsums = Table[s[n, x], {n, 1, 7}];
Plot[Evaluate[partialsums], {x, - Pi, Pi}]
Plot[Evaluate[partialsums], {x, - 3 Pi, 3 Pi}]
```



23. This exercise is straightforward and follows from the fact that the integral is linear.

24. (a) This part follows by making appropriate changes of variables. We have

$$\begin{aligned} a(g,0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-x)\,dx \\ &= \frac{1}{2\pi} \int_{\pi}^{-\pi} f(x)\,(-1)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = a(f,\,0). \end{aligned}$$

Similarly,

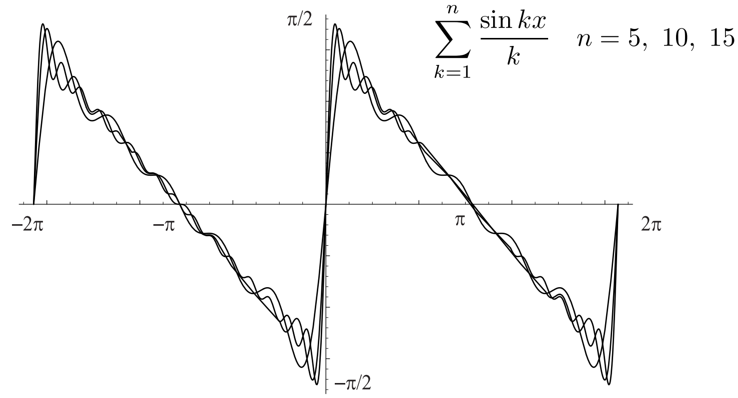
$$\begin{aligned}
a(g,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos(-nx)\,(-1)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a(f,\,n); \\
b(g,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(-x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\sin(-nx)\,(-1)dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -b(f,\,n).
\end{aligned}$$

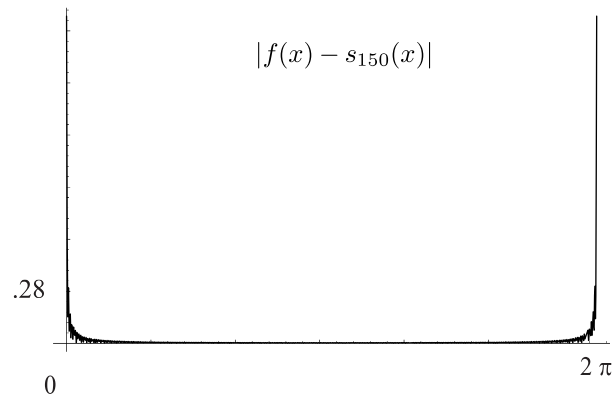
(b) As in part (a),

$$\begin{aligned}
a(h,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\alpha)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a(f,\,0);\ \text{(by Theorem 1, Section 2.1)} \\
a(h,\,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\alpha)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos(n(x+\alpha))\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos nx\,dx - \frac{\sin n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin nx\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - \frac{\sin n\alpha}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&\qquad \text{(by Theorem 1, Section 2.1)} \\
&= a(f,\,n)\cos n\alpha - b(f,\,n)\sin n\alpha; \\
b(h,\,n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\alpha)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin(n(x+\alpha))\,dx \\
&= \frac{\cos n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\sin nx\,dx + \frac{\sin n\alpha}{\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x)\cos nx\,dx \\
&= b(f,\,n)\cos n\alpha + a(f,\,n)\sin n\alpha.
\end{aligned}$$

25. For (a) and (b), see plots.

(c) We have $s_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$. So $s_n(0) = 0$ and $s_n(2\pi) = 0$ for all n. Also, $\lim_{x\to 0^+} f(x) = \frac{\pi}{2}$, so the difference between $s_n(x)$ and $f(x)$ is equal to $\pi/24$ at $x = 0$. But even we look near $x = 0$, where the Fourier series converges to $f(x)$, the difference $|s_n(x) - f(x)|$ remains larger than a positive number, that is about .28 and does not get smaller no matter how large n. In the figure, we plot $|f(x) - s_{150}(x)|$. As you can see, this difference is 0 everywhere on the interval $(0,\,2\pi)$, except near the points 0 and 2π, where this difference is approximately .28. The precise analysis of this phenomenon is done in the following exercise.





26. (a) We have

$$s_N\left(\frac{\pi}{N}\right) = \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n} = \sum_{n=1}^{N} \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}}.$$

(b) Consider the Riemann integral

$$I = \int_0^{\pi} \frac{\sin x}{x}\, dx,$$

where the integrand $f(x) = \frac{\sin x}{x}$ may be considered as a continuous function on the interval $[0,\, \pi]$ if we define $f(0) = 0$. This is because $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Since the definite integral of a continuous function on a closed interval is the limit of Riemann sums, it follows that the integral I is the limit of Riemann sums corresponding to the function $f(x)$ on the interval $[0,\, \pi]$. We now describe these Riemann sums. Let $\Delta x = \frac{\pi}{N}$ and partition the interval $[0,\, \pi]$ into N subintervals of equal length Δx. Form the Nth Riemann sum by evaluating the function $f(x)$ at the right endpoint of each subinterval. We have N of these endpoints and they are $\frac{n\pi}{N}$, for $n = 1,\, 2,\, \ldots,\, N$. Thus,

$$\begin{aligned} I &= \lim_{N\to\infty} \sum_{n=1}^{N} f\left(\frac{n\pi}{N}\right) \Delta x \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n} \\ &= \lim_{N\to\infty} s_N\left(\frac{\pi}{N}\right). \end{aligned}$$

Hence

$$\lim_{N\to\infty} s_N\left(\frac{\pi}{N}\right) = \int_0^{\pi} \frac{\sin x}{x}\, dx,$$

as desired.
(c) From the Taylor series

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty < x < \infty),$$

we obtain, for all $x \neq 0$,

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}, \quad (-\infty < x < \infty,\ x \neq 0).$$

The right side is a power series that converges for all $x \neq 0$. It is also convergent for $x = 0$ and its value for $x = 0$ is 1. Since the left side is also equal to 1 (in the limit), we conclude that

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}, \quad (-\infty < x < \infty).$$

(d) A power series can be integrated term-by-term within its radius of convergence. Thus

$$\begin{aligned}
\int_0^{\pi} \frac{\sin x}{x}\,dx &= \int_0^{\pi}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\right)dx \\
&= \sum_{n=0}^{\infty}\int_0^{\pi}\left((-1)^n \frac{x^{2n}}{(2n+1)!}\right)dx \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{x^{2n+1}}{2n+1}\Big|_0^{\pi} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1}.
\end{aligned}$$

We have thus expressed the integral I as an alternating series. Using a property of an alternating series with decreasing coefficients, we know that I is greater than any partial sum that is obtained by leaving out terms starting with a negative term; and I is smaller than any partial sum that is obtained by leaving out terms starting with a positive term. So if $I = \sum_{n=0}^{\infty}(-1)^n a_n$, where a_n are positive and decreasing to 0, then $I > \sum_{n=0}^{N}(-1)^n a_n$ if N is even and $I < \sum_{n=0}^{N}(-1)^n a_n$ if N is odd. So

$$\sum_{n=0}^{5}\frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1} < I < \sum_{n=0}^{4}\frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2n+1}.$$

More explicitely,

$$\pi - \frac{1}{3!}\frac{\pi^3}{3} + \frac{1}{5!}\frac{\pi^5}{5} - \frac{1}{7!}\frac{\pi^7}{7} + + \frac{1}{9!}\frac{\pi^9}{9} - - \frac{1}{11!}\frac{\pi^{11}}{11} < I < \pi - \frac{1}{3!}\frac{\pi^3}{3} + \frac{1}{5!}\frac{\pi^5}{5} - \frac{1}{7!}\frac{\pi^7}{7} + + \frac{1}{9!}\frac{\pi^9}{9}.$$

With the help of a calculator, we find that

$$1.8519 < I < 1.85257,$$

which is slightly better than what is in the text.
(e) We have

$$\left|f\left(\frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = \left|\frac{1}{2}(\pi - \frac{\pi}{N}) - s_N\left(\frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\frac{1}{2}(\pi - \frac{\pi}{N}) \to \frac{\pi}{2}$ and $s_N\left(\frac{\pi}{N}\right) \to I$ so

$$\lim_{N\to\infty}\left|f\left(\frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = |\frac{\pi}{2} - I|.$$

Using part (f), we find that this limit is between $1.86 - \frac{\pi}{2} \approx .2892$ and $1.85 - \frac{\pi}{2} \approx .2792$.

(f) The fact that there is a hump on the graph of $s_n(x)$ does not contradict the convergence theorem. This hump is moving toward the endpoints of the interval. So if you fix $0 < x < 2\pi$, the hump will eventually move away from x (toward the endpoints) and the partial sums will converge at x.

27. The graph of the sawtooth function is symmetric with respect to the point $\pi/2$ on the interval $0, \pi)$; that is, we have $f(x) = -f(2\pi - x)$. The same is true for the partial sums of the Fourier series. So we expect an overshoots of the partial sums near π of the same magnitude as the overshoots near 0. More precisely, since $s_N(x) = \sum_{n=1}^N \frac{\sin nx}{n}$, it follows that

$$s_N\left(2\pi - \frac{\pi}{N}\right) = \sum_{n=1}^N \frac{\sin\left(n(2\pi - \frac{\pi}{N})\right)}{n} = \sum_{n=1}^N \frac{\sin\left(n(-\frac{\pi}{N})\right)}{n} = -\sum_{n=1}^N \frac{\sin(n\frac{\pi}{N})}{n}.$$

So, by Exercise 26(b), we have

$$\lim_{N\to\infty} s_N\left(2\pi - \frac{\pi}{N}\right) = \lim_{N\to\infty} -s_N\left(\frac{\pi}{N}\right) = -\int_0^\pi \frac{\sin x}{x}\,dx.$$

Similarly,

$$\left|f\left(2\pi - \frac{\pi}{N}\right) - s_N\left(2\pi - \frac{\pi}{N}\right)\right| = \left|\frac{1}{2}(-\pi + \frac{\pi}{N}) + s_N\left(\frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\frac{1}{2}(-\pi + \frac{\pi}{N}) \to -\frac{\pi}{2}$ and $s_N\left(\frac{\pi}{N}\right) \to I$ so

$$\lim_{N\to\infty} \left|f\left(2\pi - \frac{\pi}{N}\right) - s_N\left(2\pi - \frac{\pi}{N}\right)\right| = |-\frac{\pi}{2} + I| \approx .27.$$

The overshoot occurs at $2\pi - \frac{\pi}{N+1} = \frac{(2N+1)\pi}{N+1}$ (using the result from Exercise 26(f)).

28. This exercise is very much like Exercise 26. We outline the details.
(a) We have

$$s_N\left(\pi - \frac{\pi}{N}\right) = 2\sum_{n=1}^N (-1)^{n+1} \frac{\sin\left(n\pi - \frac{n\pi}{N}\right)}{n} = 2\sum_{n=1}^N \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}},$$

where we have used the identity $\sin(n\pi - \alpha) = -\cos n\pi \sin\alpha = (-1)^{n+1}\sin\alpha$.
(b) As in Exercise 26(b), we have Thus,

$$\lim_{N\to\infty} s_N\left(\pi - \frac{\pi}{N}\right) = 2\lim_{N\to\infty} \sum_{n=1}^N \frac{\pi}{N} \cdot \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}} \quad = \quad 2\int_0^\pi \frac{\sin x}{x}\,dx,$$

as desired.
(c) We have

$$\left|f\left(\pi - \frac{\pi}{N}\right) - s_N\left(\pi - \frac{\pi}{N}\right)\right| = \left|\pi - \frac{\pi}{N} - s_N\left(\pi - \frac{\pi}{N}\right)\right|.$$

As $N \to \infty$, we have $\pi - \frac{\pi}{N} \to \pi$ and $s_N\left(\pi - \frac{\pi}{N}\right) \to 2I$ so

$$\lim_{N\to\infty} \left|f\left(\pi - \frac{\pi}{N}\right) - s_N\left(\frac{\pi}{N}\right)\right| = |\pi - 2I|.$$

Using Exercise 26(f), we find that this limit is between $2(1.86) - \pi \approx .578$ and $2(1.85) - \pi \approx .559$.

Solutions to Exercises 2.3

1. (a) and (b) Since f is odd, all the a_n's are zero and

$$\begin{aligned} b_n &= \frac{2}{p}\int_0^p \sin\frac{n\pi}{p}\,dx \\ &= \frac{-2}{n\pi}\cos\frac{n\pi}{p}\Big|_0^\pi = \frac{-2}{n\pi}\big[(-1)^n - 1\big] \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus the Fourier series is $\displaystyle \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\frac{(2k+1)\pi}{p}x$. At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph.

2. (a) and (b). The function is odd. Using the Fourier series from Exercise 13, Section 2.2, we have

$$t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt \qquad \text{for } -\pi < t < \pi.$$

Let $t = \frac{\pi x}{p}$, then

$$\frac{\pi x}{p} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{p} \qquad \text{for } -\pi < \frac{\pi x}{p} < \pi.$$

Hence

$$x = \frac{2p}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi}{p}x) \quad \text{for } -p < x < p;$$

which yields the desired Fourier series. At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph.

3. (a) and (b) The function is even so all the b_n's are zero,

$$a_0 = \frac{1}{p}\int_0^p a\big[(1-(\frac{x}{p})^2\big]\,dx = \frac{a}{p}(x - \frac{1}{3p^2}x^3)\Big|_0^p = \frac{2}{3}a;$$

and with the help of the integral formula from Exercise 9, Section 2.2, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2a}{p}\int_0^p (1-\frac{x^2}{p^2})\cos\frac{n\pi x}{p}\,dx = -\frac{2a}{p^3}\int_0^p x^2\cos\frac{n\pi x}{p}\,dx \\ &= -\frac{2a}{p^3}\left[2x\frac{p^2}{(n\pi)^2}\cos\frac{n\pi x}{p} + \frac{p^3}{(n\pi)^3}(-2+\frac{(n\pi)^2}{p^2})x^2\sin\frac{n\pi x}{p}\right]\Big|_0^p \\ &= -\frac{4a(-1)^n}{n^2\pi^2}. \end{aligned}$$

Thus the Fourier series is $\displaystyle \frac{2}{3}a + 4a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n\pi)^2}\cos(\frac{n\pi}{p}x)$. Note that the function is continuous for all x.

4. (a) and (b) The function is even. It is also continuous for all x. Using the Fourier series from Exercise 9, Section 2.2, we have

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt \qquad \text{for } -\pi < t < \pi.$$

Let $t = \frac{\pi x}{p}$, then

$$(\frac{\pi x}{p})^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\frac{\pi x}{p} \qquad \text{for } -\pi < \frac{\pi x}{p} < \pi.$$

Hence

$$x^2 = \frac{p^2}{3} + \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\frac{\pi x}{p} \qquad \text{for } -p < x < p,$$

which yields the desired Fourier series.

5. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x = 0$ to $x = p$, we see that $a_0 = 0$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^p -\left(\frac{2c}{p}\right)(x - p/2)\cos\frac{n\pi}{p}x\,dx = -\frac{4c}{p^2}\int_0^p \overbrace{(x-p/2)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= -\frac{4c}{p^2}\left[\overbrace{\frac{p}{n\pi}(x-p/2)\sin\frac{n\pi}{p}x\Big|_{x=0}^{p}}^{=0} - \frac{p}{n\pi}\int_0^p \sin\frac{n\pi}{p}x\,dx\right] \\
&= -\frac{4c}{p^2}\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_{x=0}^{p} = \frac{4c}{n^2\pi^2}(1-\cos n\pi) \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{8c}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{8c}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos\left[(2k+1)\frac{\pi}{p}x\right]}{(2k+1)^2}.$$

6. (a) and (b) The function is even. It is not continuous at $\pm d + 2kp$. At these points, the Fourier series converges to $c/2$. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x = 0$ to $x = p$, we see that $a_0 = cd/p$. We have

$$\begin{aligned}
a_n &= \frac{2c}{p}\int_0^d \cos\frac{n\pi}{p}x\,dx \\
&= \frac{2c}{n\pi}\sin\frac{n\pi}{p}x\Big|_{x=0}^{d} = \frac{2c}{n\pi}\sin\frac{n\pi d}{p}.
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{cd}{p} + \frac{2c}{\pi}\sum_{k=0}^{\infty} \frac{\sin\frac{n\pi d}{p}}{n}\cos(\frac{n\pi}{p}x).$$

7. The function in this exercise is similar to the one in Example 3. Start with the Fourier series in Example 3, multiply it by $1/c$, then change $2p \leftrightarrow p$ (this is not a change of variables, we are merely changing the notation for the period from $2p$ to p) and you will get the desired Fourier series

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n\pi x}{p}\right)}{n}.$$

The function is odd and has discontinuities at $x = \pm p + 2kp$. At these points, the Fourier series converges to 0.

8. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x=0$ to $x=d$, we see that $a_0=(cd)/(2p)$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^d -\left(\frac{c}{d}\right)(x-d)\cos\frac{n\pi}{p}x\,dx = -\frac{2c}{dp}\int_0^d \overbrace{(x-d)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= -\frac{2c}{dp}\left[\overbrace{\frac{p}{n\pi}(x-d)\sin\frac{n\pi}{p}x\Big|_{x=0}^{d}}^{=0} - \frac{p}{n\pi}\int_0^d \sin\frac{n\pi}{p}x\,dx\right] \\
&= -\frac{2c}{dp}\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_{x=0}^{d} = \frac{2cp}{dn^2\pi^2}\left(1-\cos\frac{n\pi d}{p}\right).
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{cd}{2p} + \frac{2cp}{d\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi d}{p}}{n^2}\cos\left(n\frac{\pi}{p}x\right).$$

9. The function is even; so all the b_n's are 0,

$$a_0 = \frac{1}{p}\int_0^p e^{-cx}\,dx = -\frac{1}{cp}e^{-cx}\Big|_0^p = \frac{1-e^{-cp}}{cp};$$

and with the help of the integral formula from Exercise 15, Section 2.2, for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^p e^{-cx}\cos\frac{n\pi x}{p}\,dx \\
&= \frac{2}{p}\frac{1}{n^2\pi^2+p^2c^2}\left[n\pi p e^{-cx}\sin\frac{n\pi x}{p} - p^2 c e^{-cx}\cos\frac{n\pi x}{p}\right]\Big|_0^p \\
&= \frac{2pc}{n^2\pi^2+p^2c^2}\left[1-(-1)^n e^{-cp}\right].
\end{aligned}$$

Thus the Fourier series is

$$\frac{1}{pc}(1-e^{-cp}) + 2cp\sum_{n=1}^{\infty}\frac{1}{c^2p^2+(n\pi)^2}(1-e^{-cp}(-1)^n)\cos(\frac{n\pi}{p}x).$$

10. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x=0$ to $x=p$, we see that $a_0=(1/p)(c+(p-c)/2)=(p+c)/(2p)$. Now, using integration by parts, we obtain

$$\begin{aligned}
a_n &= \frac{2}{p}\int_0^c \cos\frac{n\pi}{p}x\,dx + \frac{2}{p}\int_c^p -\left(\frac{1}{p-c}\right)(x-p)\cos\frac{n\pi}{p}x\,dx \\
&= \frac{2}{p}\frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^c - \frac{2}{p(p-c)}\int_c^p \overbrace{(x-p)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\
&= \frac{2}{n\pi}\sin\frac{n\pi c}{p} - \frac{2}{p(p-c)}\left[\frac{p}{n\pi}(x-p)\sin\frac{n\pi}{p}x\Big|_c^p - \frac{p}{n\pi}\int_c^p \sin\frac{n\pi}{p}x\,dx\right] \\
&= \frac{2}{n\pi}\sin\frac{n\pi c}{p} - \frac{2}{p(p-c)}\left[-\frac{p}{n\pi}(c-p)\sin(\frac{n\pi}{p}c) + \frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_c^p\right] \\
&= -\frac{2}{p(p-c)}\frac{p^2}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi c}{p}\right] \\
&= \frac{2p}{\pi^2(c-p)n^2}\left[(-1)^n - \cos\frac{n\pi c}{p}\right].
\end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{p+c}{2p} + \frac{2p}{\pi^2(c-p)} \sum_{n=1}^{\infty} \frac{(-1)^n - \cos\frac{n\pi c}{p}}{n^2} \cos\left(\frac{n\pi}{p}x\right).$$

11. We note that the function $f(x) = x\sin x$ $(-\pi < x < \pi)$ is the product of $\sin x$ with a familiar function, namely, the 2π-periodic extension of x $(-\pi < x < \pi)$. We can compute the Fourier coefficients of $f(x)$ directly or we can try to relate them to the Fourier coefficients of $g(x) = x$. In fact, we have the following useful fact.

Suppose that $g(x)$ is an odd function and write its Fourier series representation as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where b_n is the nth Fourier coefficient of g. Let $f(x) = g(x)\sin x$. Then f is even and its nth cosine Fourier coefficients, a_n, are given by

$$a_0 = \frac{b_1}{2}, \qquad a_1 = \frac{b_2}{2}, \qquad a_n = \frac{1}{2}\left[b_{n+1} - b_{n-1}\right] \quad (n \geq 2).$$

To prove this result, proceed as follows:

$$\begin{aligned} f(x) &= \sin x \sum_{n=1}^{\infty} b_n \sin nx \\ &= \sum_{n=1}^{\infty} b_n \sin x \sin nx \\ &= \sum_{n=1}^{\infty} \frac{b_n}{2}\big[-\cos[(n+1)x] + \cos[(n-1)x]\big]. \end{aligned}$$

To write this series in a standard Fourier series form, we reindex the terms, as follows:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty}\left(-\frac{b_n}{2}\cos[(n+1)x]\right) + \sum_{n=1}^{\infty}\left(\frac{b_n}{2}\cos[(n-1)x]\right) \\ &= \sum_{n=2}^{\infty}\left(-\frac{b_{n-1}}{2}\cos nx\right) + \sum_{n=0}^{\infty}\left(\frac{b_{n+1}}{2}\cos nx\right) \\ &= \sum_{n=2}^{\infty}\left(-\frac{b_{n-1}}{2}\cos nx\right)\frac{b_1}{2} + \frac{b_2}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{b_{n+1}}{2}\cos nx\right) \\ &= \frac{b_1}{2} + \frac{b_2}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{b_{n+1}}{2} - \frac{b_{n-1}}{2}\right)\cos nx. \end{aligned}$$

This proves the desired result. To use this result, we recall the Fourier series from Exercise 2: For $-\pi < x < \pi$, $g(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$; so $b_1 = 2$, $b_2 = -1/2$ and $b_n = \frac{2(-1)^{n+1}}{n}$ for $n \geq 2$. So, $f(x) = x\sin x = \sum_{n=1}^{\infty} a_n \cos nx$, where $a_0 = 1$, $a_1 = -1/2$, and, for $n \geq 2$,

$$a_n = \frac{1}{2}\left[\frac{2(-1)^{n+2}}{n+1} - \frac{2(-1)^n}{n-1}\right] = \frac{2(-1)^{n+1}}{n^2-1}.$$

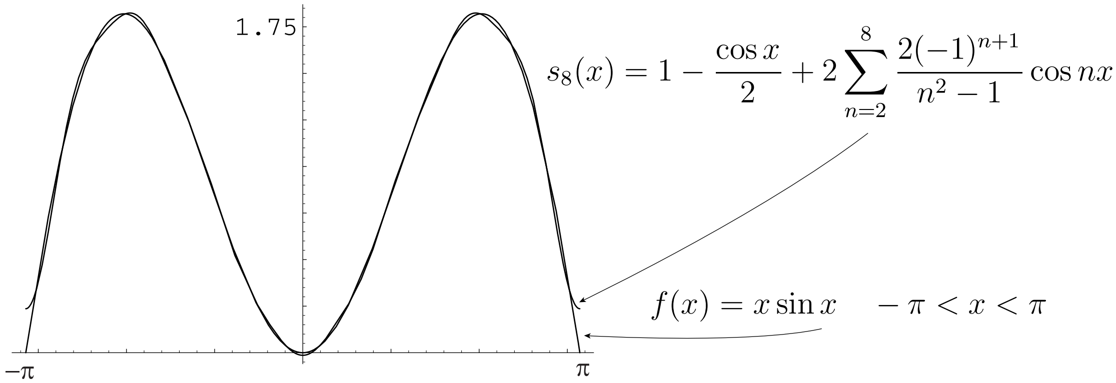
Thus, for $-\pi < x < \pi$,

$$x\sin x = 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx.$$

The convergence of the Fourier series is illustrated in the figure. Note that the partial sums converge uniformly on the entire real line. This is a consequence of the fact that the function is piecewise smooth and continuous for all x. The following is the 8th partial sum.

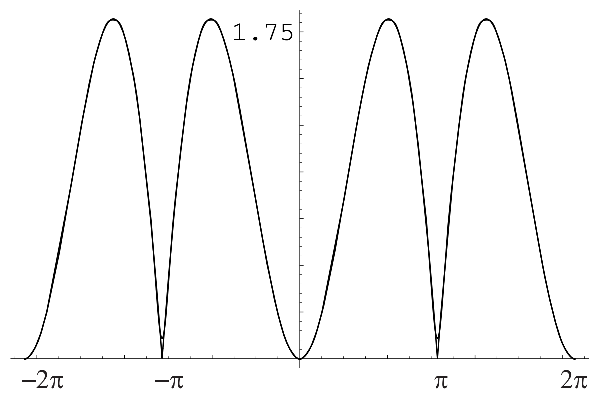
```
s[n_, x_] = 1 - Cos[x] / 2 + 2 Sum[(-1) ^ (k + 1) / (k^2 - 1) Cos[k x], {k, 2, n}];

Plot[Evaluate[{x Sin[x], s[8, x]}], {x, - Pi, Pi}]
```



To plot the function over more than one period, we can use the Floor function to extend it outside the interval $(-\pi, \pi)$. In what follows, we plot the function and the 18th partial sum of its Fourier series. The two graphs are hard to distinguish from one another.

```
f[x_] = x Sin[x];
g[x_] = x - 2 Pi Floor[(x + Pi) / ( 2 Pi)];
h[x_] = f[g[x]];
Plot[Evaluate[{h[x], s[18, x]}], {x, -2 Pi, 2 Pi}]
```



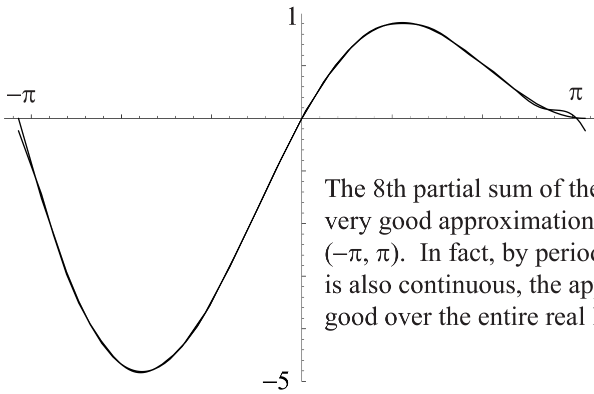
12. The Fourier series of the function $f(x) = (\pi - x)\sin x$ $(-\pi < x < \pi)$ can be obtained from that of the function in Exercise 11, as follows. Call the function in Exercise 11 $g(x)$. Then, on the interval $-\pi < x < \pi$, we have

$$\begin{aligned} f(x) &= (\pi - x)\sin x = \pi \sin -x \sin x \\ &= \pi \sin x - g(x). \end{aligned}$$

Since $\pi \sin x$ is its own Fourier series, using the Fourier series from Exercise 11, we obtain

$$\begin{aligned} f(x) &= \pi \sin x - \left(1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx\right) \\ &= \pi \sin x - 1 + \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{2(-1)^{n}}{n^2-1}\cos nx. \end{aligned}$$

```
s[n_, x_] =
  Pi Sin[x] - 1 + Cos[x] / 2 - 2 Sum[(-1) ^ (k + 1) / (k^2 - 1) Cos[k x], {k, 2, n}];
Plot[Evaluate[{(Pi - x) Sin[x], s[8, x]}], {x, - Pi, Pi}]
```



The 8th partial sum of the Fourier series provides a very good approximation of the function on the interval (−π, π). In fact, by periodicity, because the function is also continuous, the approximation is uniformly good over the entire real line.

13. Take $p = 1$ in Exercise 1, call the function in Exercise 1 $f(x)$ and the function in this exercise $g(x)$. By comparing graphs, we see that

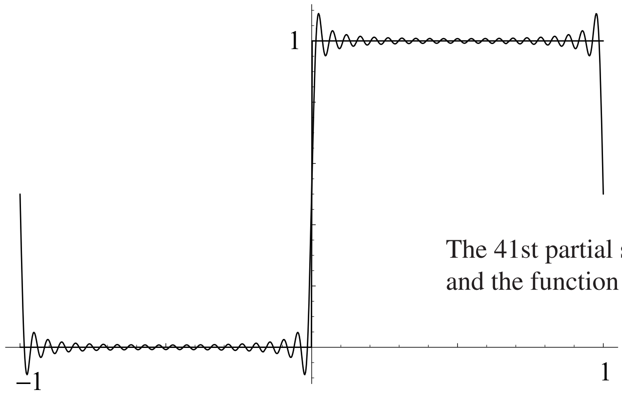
$$g(x) = \frac{1}{2}\left(1 + f(x)\right).$$

Thus the Fourier series of g is

$$\frac{1}{2}\left(1 + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)}\sin(2k+1)\pi x\right) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)}\sin(2k+1)\pi x.$$

```
f[x_] = Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 1 / 2 + 2 / Pi Sum[1 / (2 k + 1) Sin[(2 k + 1 ) Pi x], {k, 0, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]

Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
```



The 41st partial sum of the Fourier series and the function on the interval (-1, 1).

14. Call the function in Exercise 13 $g(x)$ and the function in this exercise $h(x)$. By comparing graphs of the 2-periodic extensions, we see that

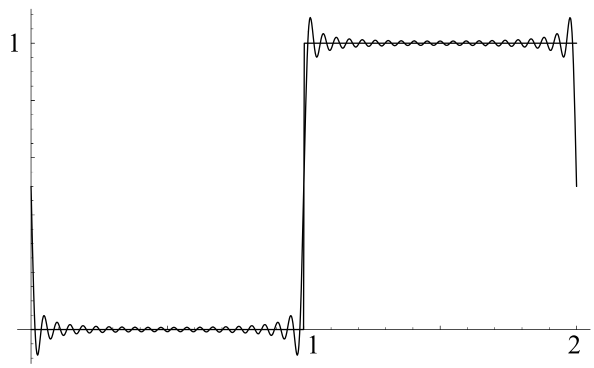
$$h(x) = g(-x).$$

Thus the Fourier series of h is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin[(2k+1)\pi(-x)] = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin[(2k+1)\pi x].$$

```
h[x_] = Which[x < 1, 0, 1 < x < 2, 1, x > 2, 0]
s[n_, x_] = 1 / 2 - 2 / Pi Sum[1 / (2 k + 1) Sin[(2 k + 1) Pi x], {k, 0, n}];
Plot[Evaluate[{h[x], s[20, x]}], {x, 0, 2}]

Which[x < 1, 0, 1 < x < 2, 1, x > 2, 0]
```



15. To match the function in Example 2, Section 2.2, take $p = a = \pi$ in Example 2 of this section. Then the Fourier series becomes

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x,$$

which is the Fourier series of Example 2, Section 2.2.

16. (a) As $d \to p$, the function becomes the constant function $f(x) = c$. Its Fourier series is itself; in particular, all the Fourier coefficients are 0, except $a_0 = c$. This is clear if we let $d \to p$ in the formulas for the Fourier coefficients, because

$$\lim_{d\to p}\frac{cd}{p} = 0 \quad \text{and} \quad \lim_{d\to p}\frac{2c}{\pi}\frac{\sin\frac{dn\pi}{p}}{n} = 0.$$

(b) If $c = p/d$ and $d \to 0$, interesting things happen. The function tends to 0 pointwise, except at $x = 0$, where it tends to ∞. Also, for any fixed d (no matter how small), the area under the graph of f and above the x-axis, from $x = -d$ to $x = d$, is $2p$. Taking $c = p/d$, we find that $a_0 = 1$ for all d and, hence, $a_0 \to 1$, as $d \to 0$. For $n \geq 1$, we have

$$\lim_{d\to 0} a_n = \lim_{d\to 0}\frac{2p}{d\pi}\frac{\sin\frac{dn\pi}{p}}{n} = \lim_{d\to 0}\frac{2p}{\pi}\frac{n\pi}{p}\frac{\cos\frac{dn\pi}{p}}{n} = 2,$$

by using l'Hospital's rule. Thus, even though the function tends to 0, its Fourier coefficients are not tending to 0. In fact, the Fourier coefficients are tending to 1 for a_0 and to 2 for all other a_n. These limiting Fourier coefficients do not correspond to any function! That is, there is no function with Fourier coefficients given by $a_0 = 1$, $a_n = 2$ and $b_n = 0$ for all $n \geq 1$. There is, however, a generalized function (a constant multiple of the Dirac delta function) that has precisely those coefficients. You may have encountered the Dirac delta function previously. You can read more about it in Sections 7.8-7.10.

17. (a) Take $x = 0$ in the Fourier series of Exercise 4 and get

$$0 = \frac{p^2}{3} - \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

(b) Take $x = p$ in the Fourier series of Exercise 4 and get

$$p^2 = \frac{p^2}{3} - \frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^2} \quad \Rightarrow \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Summing over the even and odd integers separately, we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k)^2}.$$

But $\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4}\frac{\pi^2}{6}$. So

$$\frac{\pi^2}{6} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{\pi^2}{24} \quad \Rightarrow \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

18. To derive (4), repeat the proof of Theorem 1 until you get to the equation (7). Then continue as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right)\cos nx\,dx \quad (\text{let } t = \frac{px}{\pi}) \\ &= \frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)\,dt, \end{aligned}$$

which is formula (4). Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right)\sin nx\,dx \quad (\text{let } t = \frac{px}{\pi}) \\ &= \frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)\,dt, \end{aligned}$$

which is formula (5).

19. This is very similar to the proof of Theorem 2(i). If $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$, then, for all x,

$$f(-x) = \sum_{n=1}^{\infty} b_n \sin(-\frac{n\pi}{p}x) = -\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x = -f(x),$$

and so f is odd. Conversely, suppose that f is odd. Then $f(x)\cos\frac{n\pi}{p}x$ is odd and, from (10), we have $a_n = 0$ for all n. Use (5), (9), and the fact that $f(x)\sin\frac{n\pi}{p}x$ is even to get the formulas for the coefficients in (ii).

20. (a) f_e is even because

$$f_e(-x) = \frac{f(-x) + f(--x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x).$$

f_o is odd because

$$f_o(-x) = \frac{f(-x) - f(- - x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x).$$

(b) We have

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x).$$

To show that this decomposition is unique, suppose that $f(x) = g(x) + h(x)$, where g is even and h is odd. Then $f(-x) = g(-x) + h(-x) = g(x) - h(x)$, and so

$$f(x) + f(-x) = g(x) + h(x) + g(x) - h(x) = 2g(x);$$

equivalently,

$$g(x) = \frac{f(x) + f(-x)}{2} = f_e(x).$$

By considering $f(x) - f(-x)$, we obtain that $h(x) = f_o(x)$; and hence the decomposition is unique.
(c) If $f(x)$ is $2p$-periodic, then clearly $f(-x)$ is also $2p$-periodic. So f_e and f_o are both $2p$-periodic, being linear combinations of $2p$-periodic functions.
(d) If $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p})$, then

$$f(-x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi(-x)}{p} + b_n \sin \frac{n\pi(-x)}{p}) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{p} - b_n \sin \frac{n\pi x}{p});$$

and so

$$f_e(x) = \frac{f(x) + f(-x)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p},$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}.$$

21. From the graph, we have

$$f(x) = \begin{cases} -1 - x & \text{if } -1 < x < 0, \\ 1 + x & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 - x & \text{if } -1 < x < 0, \\ -1 + x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} -x & \text{if } -1 < x < 0, \\ x & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Note that, $f_e(x) = |x|$ for $-1 < x < 1$. The Fourier series of f is the sum of the Fourier series of f_e and f_o. From Example 1 with $p = 1$,

$$f_e(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x].$$

From Exercise 1 with $p = 1$,

$$f_o(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1} \sin[(2k+1)\pi x].$$

Hence

$$f(x) = \frac{1}{2} + \frac{4}{\pi}\sum_{k=0}^{\infty}\left[-\frac{\cos[(2k+1)\pi x]}{\pi(2k+1)^2} + \frac{\sin[(2k+1)\pi x]}{2k+1}\right].$$

22. From the graph, we have

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

So

$$f(-x) = \begin{cases} 1 & \text{if } -1 < x < 0, \\ 1-x & \text{if } 0 < x < 1; \end{cases}$$

hence

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \begin{cases} \frac{x}{2}+1 & \text{if } -1 < x < 0, \\ 1-\frac{x}{2} & \text{if } 0 < x < 1, \end{cases}$$

and

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{x}{2} \quad (-1 < x < 1).$$

As expected, $f(x) = f_e(x) + f_o(x)$. Let $g(x)$ be the function in Example 2 with $p = 1$ and $a = 1/2$. Then $f_e(x) = g(x) + 1/2$. So from Example 2 with $p = 1$ and $a = 1/2$, we obtain

$$\begin{aligned} f_e(x) &= \frac{1}{2} + \frac{1}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] \\ &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x]. \end{aligned}$$

From Exercise 2 with $p = 1$,

$$f_o(x) = \frac{1}{2}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

Hence

$$\begin{aligned} f(x) &= \frac{3}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)\pi x] + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) \\ &= \frac{3}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{\pi n^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n}\sin(n\pi x). \end{aligned}$$

Let's illustrate the convergence of the Fourier series. (This is one way to check that our answer is correct.)

```
f[x_] = Which[x < -1, 0, -1 < x < 0, x + 1, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 3 / 4 + 1 / Pi Sum[(1 - (-1) ^k) / (Pi k^2) Cos[ k Pi x] +
      (-1) ^ (k + 1) / k Sin[k Pi x], {k, 1, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]
```