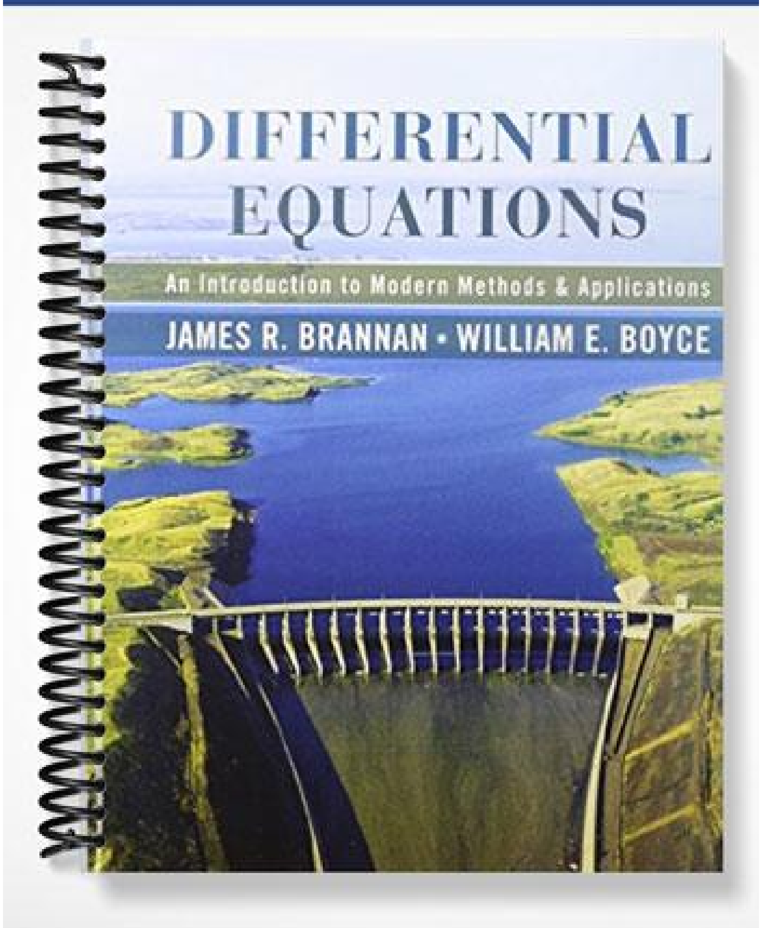
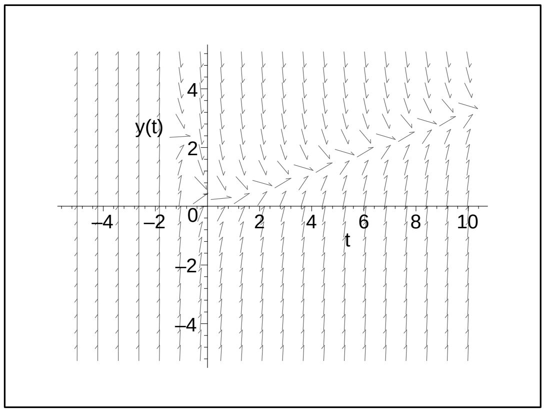
SOLUTIONS MANUAL



Chapter 2
Section 2.1

1.

(a)



(b) All solutions seem to converge to an increasing function as $t \to \infty$.

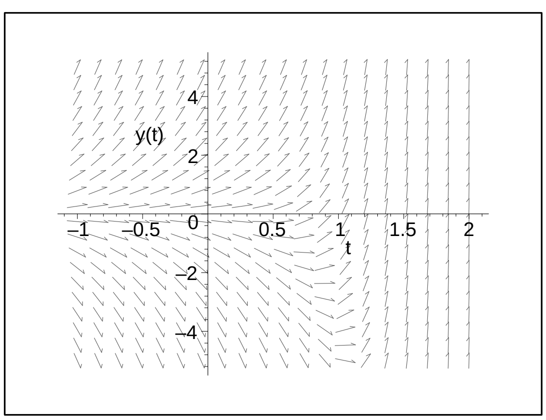
(c) The integrating factor is $\mu(t) = e^{3t}$. Then

$$
\begin{aligned}
&e^{3t}y' + 3e^{3t}y = e^{3t}(t + e^{-2t}) \implies (e^{3t}y)' = te^{3t} + e^{t} \\
&\implies e^{3t}y = \int (te^{3t} + e^{t})\,dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^{t} + c \\
&\implies y = \frac{t}{3} - \frac{1}{9} + e^{-2t} + ce^{-3t}.
\end{aligned}
$$

We conclude that y is asymptotic to $t/3 - 1/9$ as $t \to \infty$.

2.

(a)



(b) All slopes eventually become positive, so all solutions will eventually increase without bound.

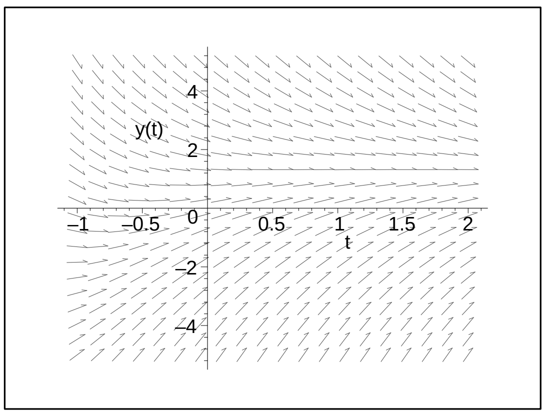
(c) The integrating factor is $\mu(t) = e^{-2t}$. Then

$$\begin{aligned} e^{-2t}y' - 2e^{-2t}y = e^{-2t}(t^2e^{2t}) &\implies (e^{-2t}y)' = t^2 \\ \implies e^{-2t}y = \int t^2\,dt = \frac{t^3}{3} + c& \\ \implies y = \frac{t^3}{3}e^{2t} + ce^{2t}.& \end{aligned}$$

We conclude that y increases exponentially as $t \to \infty$.

3.

(a)



(b) All solutions appear to converge to the function $y(t) = 1$.

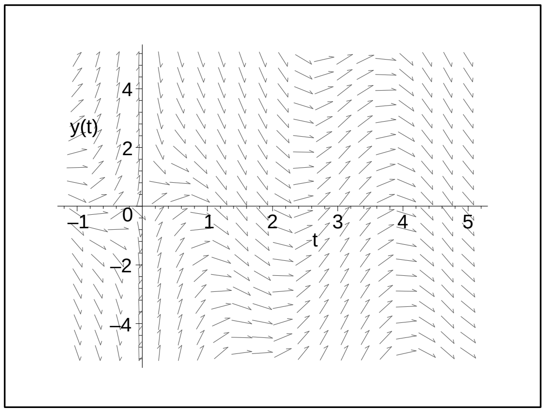
(c) The integrating factor is $\mu(t) = e^t$. Therefore,

$$\begin{aligned} e^ty' + e^ty = t + e^t &\implies (e^ty)' = t + e^t \\ \implies e^ty = \int (t + e^t)\,dt = \frac{t^2}{2} + e^t + c& \\ \implies y = \frac{t^2}{2}e^{-t} + 1 + ce^{-t}.& \end{aligned}$$

Therefore, we conclude that $y \to 1$ as $t \to \infty$.

4.

(a)



(b) The solutions eventually become oscillatory.

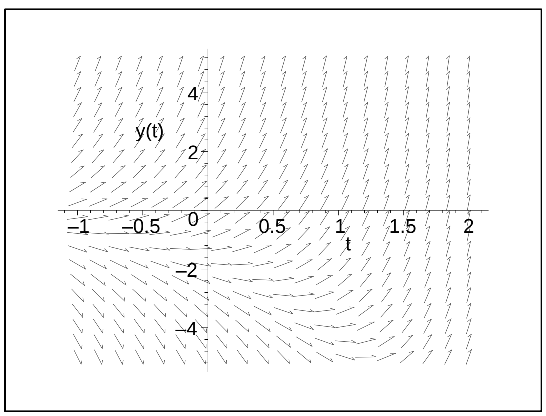
(c) The integrating factor is $\mu(t) = t$. Therefore,

$$ty' + y = 3t\cos(2t) \implies (ty)' = 3t\cos(2t)$$
$$\implies ty = \int 3t\cos(2t)\,dt = \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t) + c$$
$$\implies y = +\frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}.$$

We conclude that y is asymptotic to $(3\sin 2t)/2$ as $t \to \infty$.

5.

(a)



(b) All slopes eventually become positive so all solutions eventually increase without bound.

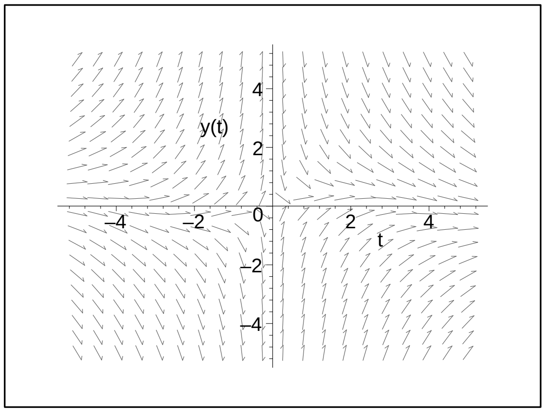
(c) The integrating factor is $\mu(t) = e^{-2t}$. Therefore,

$$e^{-2t}y' - 2e^{-2t}y = 3e^{-t} \implies (e^{-2t}y)' = 3e^{-t}$$
$$\implies e^{-2t}y = \int 3e^{-t}\,dt = -3e^{-t} + c$$
$$\implies y = -3e^{t} + ce^{2t}.$$

We conclude that y increases exponentially as $t \to \infty$.

6.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

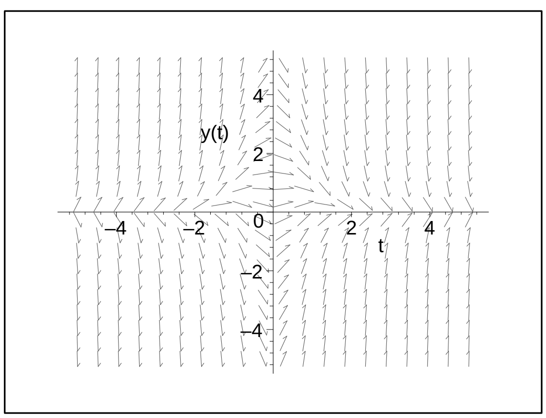
(c) The integrating factor is $\mu(t) = t^2$. Therefore,

$$t^2y' + 2ty = t\sin(t) \implies (t^2y)' = t\sin(t)$$
$$\implies t^2y = \int t\sin(t)\,dt = \sin(t) - t\cos(t) + c$$
$$\implies y = \frac{\sin t - t\cos t + c}{t^2}.$$

We conclude that $y \to 0$ as $t \to \infty$.

7.

(a)

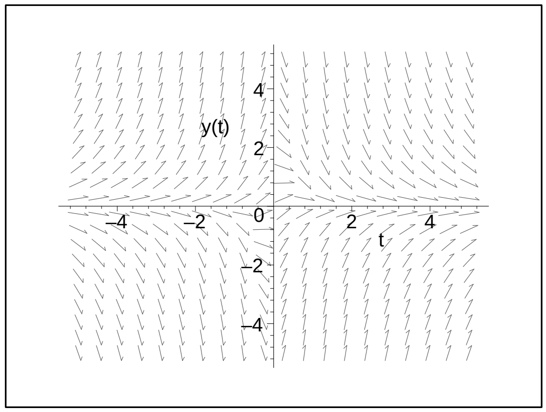


(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$. Therefore, using the techniques shown above, we see that $y(t) = t^2 e^{-t^2} + ce^{-t^2}$. We conclude that $y \to 0$ as $t \to \infty$.

8.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

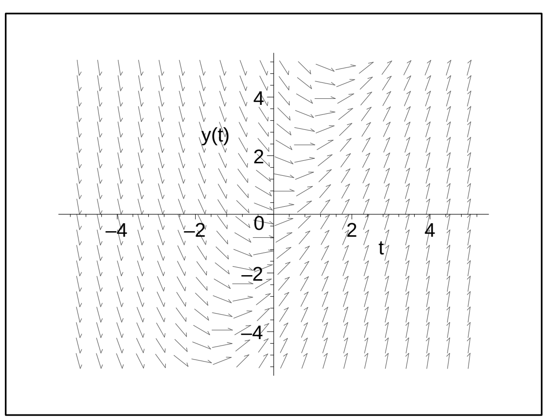
(c) The integrating factor is $\mu(t) = (1+t^2)^2$. Then

$$\begin{aligned}
(1+t^2)^2 y' + 4t(1+t^2)y &= \frac{1}{1+t^2} \\
\implies ((1+t^2)^2 y) &= \int \frac{1}{1+t^2}\,dt \\
\implies y &= (\tan^{-1}(t) + c)/(1+t^2)^2.
\end{aligned}$$

We conclude that $y \to 0$ as $t \to \infty$.

9.

(a)



(b) All slopes eventually become positive. Therefore, all solutions will increase without bound.

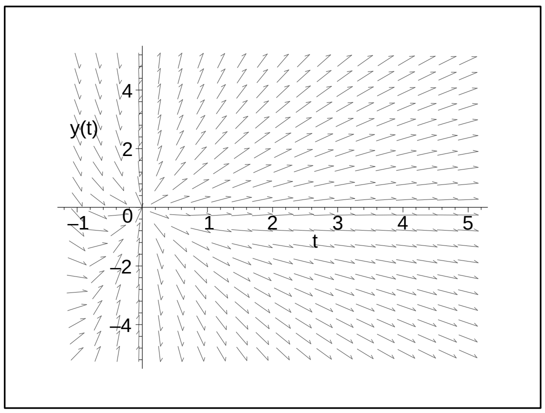
(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore,

$$2e^{t/2}y' + e^{t/2}y = 3te^{t/2} \qquad \Longrightarrow \ 2e^{t/2}y = \int 3te^{t/2}\,dt = 6te^{t/2} - 12e^{t/2} + c$$
$$\Longrightarrow \ y = 3t - 6 + ce^{-t/2}.$$

We conclude that $y \to 3t - 6$ as $t \to \infty$.

10.

(a)



(b) For $y > 0$, the slopes are all positive, and, therefore, the corresponding solutions increase without bound. For $y < 0$ almost all solutions have negative slope and therefore decrease without bound.

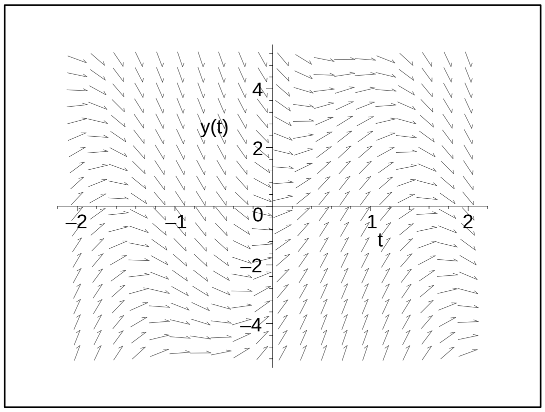
(c) By dividing the equation by t, we see that the integrating factor is $\mu(t) = 1/t$. Therefore,

$$y'/t - y/t^2 = te^{-t} \ \Longrightarrow \ (y/t)' = te^{-t}$$
$$\Longrightarrow \ \frac{y}{t} = \int te^{-t}\,dt = -te^{-t} - e^{-t} + c$$
$$\Longrightarrow \ y = -t^2e^{-t} - te^{-t} + ct.$$

We conclude that $y \to \infty$ if $c > 0$, $y \to -\infty$ if $c < 0$ and $y \to 0$ if $c = 0$.

11.

(a)



(b) The solution appears to be oscillatory.

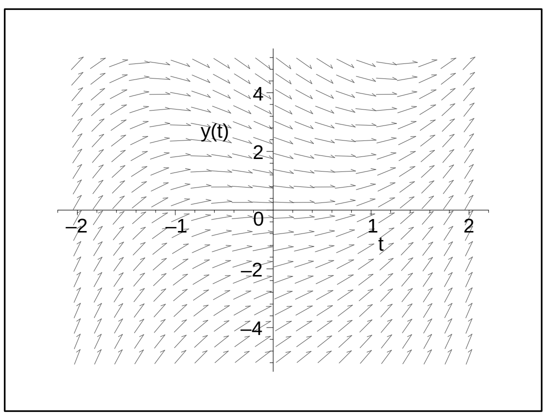
(c) The integrating factor is $\mu(t) = e^t$. Therefore,

$$e^t y' + e^t y = 5e^t \sin(2t) \implies (e^t y)' = 5e^t \sin(2t)$$
$$\implies e^t y = \int 5e^t \sin(2t)\, dt = -2e^t \cos(2t) + e^t \sin(2t) + c \quad \implies y = -2\cos(2t) + \sin(2t) + ce^{-t}.$$

We conclude that $y \to \sin(2t) - 2\cos(2t)$ as $t \to \infty$.

12.

(a)



(b) All slopes are eventually positive. Therefore, all solutions increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/}$. Therefore,

$$2e^{t/2}y' + e^{t/2}y = 3t^2 e^{t/2} \implies (2e^{t/2}y)' = 3t^2 e^{t/2}$$
$$\implies 2e^{t/2}y = \int 3t^2 e^{t/2}\, dt = 6t^2 e^{t/2} - 24te^{t/2} + 48e^{t/2} + c$$
$$\implies y = 3t^2 - 12t + 24 + ce^{-t/2}.$$

We conclude that y is asymptotic to $3t^2 - 12t + 24$ as $t \to \infty$.

13. The integrating factor is $\mu(t) = e^{-t}$. Therefore,

$$(e^{-t}y)' = 2te^t \implies y = e^t \int 2te^t\, dt = 2te^{2t} - 2e^{2t} + ce^t.$$

The initial condition $y(0) = 1$ implies $-2 + c = 1$. Therefore, $c = 3$ and $y = 3e^t + 2(t-1)e^{2t}$

14. The integrating factor is $\mu(t) = e^{2t}$. Therefore,

$$(e^{2t}y)' = t \implies y = e^{-2t} \int t\, dt = \frac{t^2}{2}e^{-2t} + ce^{-2t}.$$

The initial condition $y(1) = 0$ implies $e^{-2t}/2 + ce^{-2t} = 0$. Therefore, $c = -1/2$, and $y = (t^2 - 1)e^{-2t}/2$

15. Dividing the equation by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = t^3 - t^2 + t \implies y = t^{-2} \int (t^3 - t^2 + t)\, dt = \left(\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}\right).$$

The initial condition $y(1) = 1/2$ implies $c = 1/12$, and $y = (3t^4 - 4t^3 + 6t^2 + 1)/12t^2$.

16. The integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = \cos(t) \implies y = t^{-2} \int \cos(t)\, dt = t^{-2}(\sin(t) + c).$$

The initial condition $y(\pi) = 0$ implies $c = 0$ and $y = (\sin t)/t^2$

17. The integrating factor is $\mu(t) = e^{-2t}$. Therefore,

$$(e^{-2t}y)' = 1 \implies y = e^{2t} \int 1\, dt = e^{2t}(t + c).$$

The initial condition $y(0) = 2$ implies $c = 2$ and $y = (t+2)e^{2t}$.

18. After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = 1 \implies y = t^{-2} \int t\sin(t)\, dt = t^{-2}(\sin(t) - t\cos(t) + c).$$

The initial condition $y(\pi/2) = 1$ implies $c = (\pi^2/4) - 1$ and $y = t^{-2}[(\pi^2/4) - 1 - t\cos t + \sin t]$.

19. After dividing by t^3, we see that the integrating factor is $\mu(t) = t^4$. Therefore,

$$(t^4y)' = te^{-t} \implies y = t^{-4} \int te^{-t}\, dt = t^{-4}(-te^{-t} - e^{-t} + c).$$

The initial condition $y(-1) = 0$ implies $c = 0$ and $y = -(1+t)e^{-t}/t^4, \quad t \neq 0$

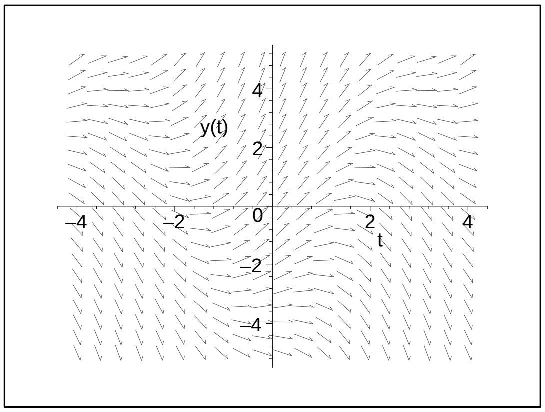
20. After dividing by t, we see that the integrating factor is $\mu(t) = te^t$. Therefore,

$$(te^ty)' = te^t \implies y = t^{-1}e^{-t} \int te^t\, dt = t^{-1}e^{-t}(te^t - e^t + c) = t^{-1}(t - 1 + ce^{-t}).$$

The initial condition $y(\ln 2) = 1$ implies $c = 2$ and $y = (t - 1 + 2e^{-t})/t, \quad t \neq 0$

21.

(a)



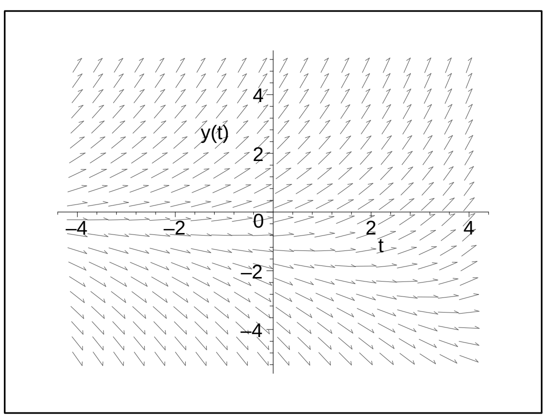
The solutions appear to diverge from an oscillatory solution. It appears that $a_0 \approx -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, the solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/2}$. From this, we conclude that the general solution is $y(t) = (8\sin(t) - 4\cos(t))/5 + ce^{t/2}$. The solution will be sinusoidal as long as $c = 0$. The initial condition for the sinusoidal behavior is $y(0) = (8\sin(0) - 4\cos(0))/5 = -4/5$. Therefore, $a_0 = -4/5$.

(c) y oscillates for $a = a_0$

22.

(a)



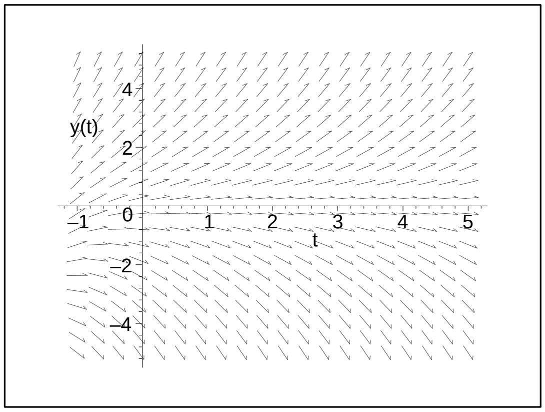
All solutions eventually increase or decrease without bound. The value a_0 appears to be approximately $a_0 = -3$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution is $y(t) = -3e^{t/3} + ce^{t/2}$. The initial condition $y(0) = a$ implies $y = -3e^{t/3} + (a+3)e^{t/2}$. The solution will behave like $(a+3)e^{t/2}$. Therefore, $a_0 = -3$.

(c) $y \to -\infty$ for $a = a_0$

23.

(a)



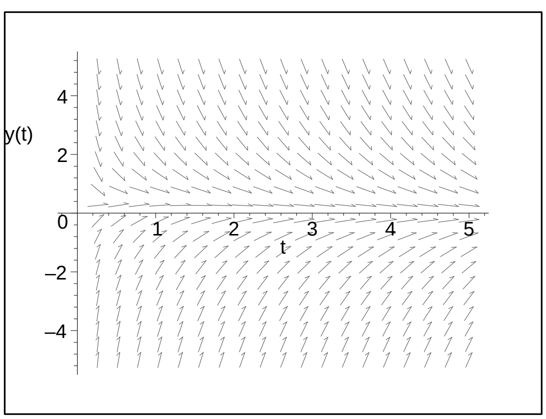
Solutions eventually increase or decrease without bound, depending on the initial value a_0. It appears that $a_0 \approx -1/8$.

(b) Dividing the equation by 3, we see that the integrating factor is $\mu(t) = e^{-2t/3}$. Therefore, the solution is $y = [(2 + a(3\pi + 4))e^{2t/3} - 2e^{-\pi t/2}]/(3\pi + 4)$. The solution will eventually behave like $(2 + a(3\pi + 4))e^{2t/3}/(3\pi + 4)$. Therefore, $a_0 = -2/(3\pi + 4)$.

(c) $y \to 0$ for $a = a_0$

24.

(a)



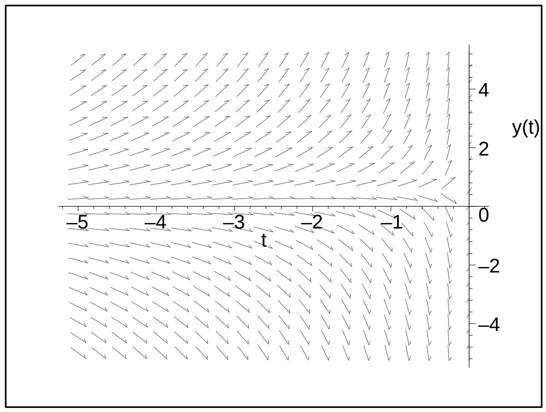
It appears that $a_0 \approx .4$. As $t \to 0$, solutions increase without bound if $y > a_0$ and decrease without bound if $y < a_0$.

(b) The integrating factor is $\mu(t) = te^t$. The general solution is $y = te^{-t} + ce^{-t}/t$. The initial condition $y(1) = a$ implies $y = te^{-t} + (ea - 1)e^{-t}/t$. As $t \to 0$, the solution will behave like $(ea - 1)e^{-t}/t$. From this, we see that $a_0 = 1/e$.

(c) $y \to 0$ as $t \to 0$ for $a = a_0$

25.

(a)



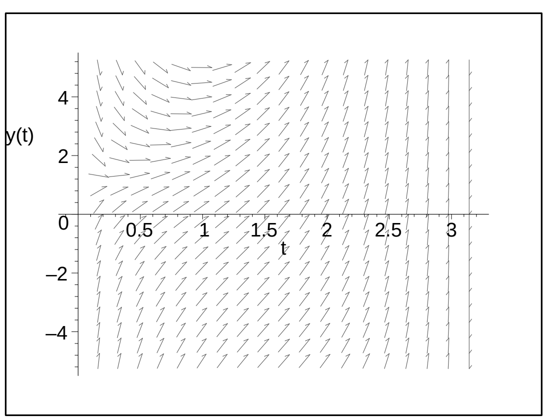
It appears that $a_0 \approx .4$. That is, as $t \to 0$, for $y(-\pi/2) > a_0$, solutions will increase without bound, while solutions will decrease without bound for $y(-\pi/2) < a_0$.

(b) After dividing by t, we see that the integrating factor is t^2, and the solution is $y = -\cos t/t^2 + \pi^2 a/4t^2$. Since $\lim_{t \to 0} \cos(t) = 1$, solutions will increase without bound if $a > 4/\pi^2$ and decrease without bound if $a < 4/\pi^2$. Therefore, $a_0 = 4/\pi^2$.

(c) For $a_0 = 4/\pi^2$, $y = (1 - \cos(t))/t^2 \to 1/2$ as $t \to 0$.

26.

(a)



It appears that $a_0 \approx 2$. For $y(1) > a_0$, the solution will increase without bound as $t \to 0$, while the solution will decrease without bound if $y(t) < a_0$.

(b) After dividing by $\sin(t)$, we see that the integrating factor is $\mu(t) = \sin(t)$. As a result, we see that the solution is given by $y = (e^t + c)\sin(t)$. Applying our initial condition, we see that our solution is $y = (e^t - e + a\sin 1)/\sin t$. The solution will increase if $1 - e + a\sin 1 > 0$ and decrease if $1 - e + a\sin 1 < 0$. Therefore, we conclude that $a_0 = (e-1)/\sin 1$

(c) If $a_0 = (e-1)\sin(1)$, then $y = (e^t - 1)/\sin(t)$. As $t \to 0$, $y \to 1$.

27. The integrating factor is $\mu(t) = e^{t/2}$. Therefore, the general solution is $y(t) = [4\cos(t) + 8\sin(t)]/5 + ce^{-t/2}$. Using our initial condition, we have $y(t) = [4\cos(t) + 8\sin(t) - 9e^{t/2}]/5$. Differentiating, we have

$$y' = [-4\sin(t) + 8\cos(t) + 4.5e^{-t/2}]/5$$
$$y'' = [-4\cos(t) - 8\sin(t) - 2.25e^{t/2}]/5.$$

Setting $y' = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point is a local maximum. The coordinates of the point are $(1.3643, .82008)$.

28. The integrating factor is $\mu(t) = e^{2t/3}$. The general solution of the differential equation is $y(t) = (21 - 6t)/8 + ce^{-2t/3}$. Using the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Therefore, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = \frac{3}{2}\ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = \frac{3}{2} + \frac{9}{4}\ln 3 - \frac{9}{8}\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9e^{4/3})/8 = -1.643$.

29.

(a) The integrating factor is $\mu(t) = e^{t/4}$. The general solution is

$$y(t) = 12 + [8\cos(2t) + 64\sin(2t)]/65 + ce^{-t/4}.$$

Applying the initial condition $y(0) = 0$, we arrive at the specific solution

$$y(t) = 12 + [8\cos(2t) + 64\sin(2t) - 788e^{-t/4}]/65.$$

For large values of t, the solution oscillates about the line $y = 12$.

(b) To find the value of t for which the solution first intersects the line $y = 12$, we need to solve the equation $8\cos(2t) + 64\sin(2t) - 788e^{-t/4} = 0$. The time t is approximately 10.519.

30. The integrating factor is $\mu(t) = e^{-t}$. The general solution is $y(t) = -1 - \frac{3}{2}\cos(t) - \frac{3}{2}\sin(t) + ce^t$. In order for the solution to remain finite as $t \to \infty$, we need $c = 0$. Therefore, y_0 must satisfy $y_0 = -1 - 3/2 = -5/2$.

31. The integrating factor is $\mu(t) = e^{-3t/2}$ and the general solution of the equation is $y(t) = -2t - 4/3 - 4e^t + ce^{3t/2}$. The initial condition implies $y(t) = -2t - 4/3 - 4e^t + (y_0 + 16/3)e^{3t/2}$. The solution will behave like $(y_0 + 16/3)e^{3t/2}$ (for $y_0 \neq -16/3$). For $y_0 > -16/3$, the solutions will increase without bound, while for $y_0 < -16/3$, the solutions will decrease without bound. If $y_0 = -16/3$, the solution will decrease without bound as the solution will be $-2t - 4/3 - 4e^t$.

32. By equation (41), we see that the general solution is given by

$$y = e^{-t^2/4} \int_0^t e^{s^2/4}\, ds + ce^{-t^2/4}.$$

Applying L'Hospital's rule,

$$\lim_{t\to\infty} \frac{\int_0^t e^{s^2/4}\, ds}{e^{t^2/4}} = \lim_{t\to\infty} \frac{e^{t^2/4}}{(t/2)e^{t^2/4}} = 0.$$

Therefore, $y \to 0$ as $t \to \infty$.

33. The integrating factor is $\mu(t) = e^{at}$. First consider the case $a \neq \lambda$. Multiplying the equation by e^{at}, we have

$$(e^{at}y)' = be^{(a-\lambda)t} \implies y = e^{-at} \int be^{(a-\lambda)t} = e^{-at}\left(\frac{b}{a-\lambda}e^{(a-\lambda)t} + c\right) = \frac{b}{a-\lambda}e^{-\lambda t} + ce^{-at}.$$

Since a, λ are assumed to be positive, we see that $y \to 0$ as $t \to \infty$. Now if $a = \lambda$ above, then we have

$$(e^{at}y)' = b \implies y = e^{-at}(bt + c)$$

and similarly $y \to 0$ as $t \to \infty$.

34. We notice that $y(t) = ce^{-t} + 3$ approaches 3 as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 3$. Then $f' + f = 0$ and $g' + g = 3$. Therefore, $y(t) = ce^{-t} + 3$ satisfies the equation $y' + y = 3$. That is, the equation $y' + y = 3$ has the desired properties.

35. We notice that $y(t) = ce^{-t} + 3 - t$ approaches $3 - t$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 3 - t$. Then $f' + f = 0$ and $g' + g = -1 + 3 - t = -2 - t$. Therefore, $y(t) = ce^{-t} + 3 - t$ satisfies the equation $y' + y = -2 - t$. That is, the equation $y' + y = -2 - t$ has the desired properties.

36. We notice that $y(t) = ce^{-t} + 2t - 5$ approaches $2t - 5$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 2t - 5$. Then $f' + f = 0$ and $g' + g = 2 + 2t - 5 = 2t - 3$. Therefore, $y(t) = ce^{-t} + 2t - 5-$ satisfies the equation $y' + y = 2t - 3$. That is, the equation $y' + y = 2t - 3$ has the desired properties.

37. We notice that $y(t) = ce^{-t} + 4 - t^2$ approaches $4 - t^2$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 4 - t^2$. Then $f' + f = 0$ and

$g' + g = -2t + 4 - t^2 = 4 - 2t - t^2$. Therefore, $y(t) = ce^{-t} + 2t - 5-$ satisfies the equation $y' + y = 4 - 2t - t^2$. That is, the equation $y' + y = 4 - 2t - t^2$ has the desired properties.

38. Multiplying the equation by $e^{a(t-t_0)}$, we have

$$\begin{aligned}
&e^{a(t-t_0)}y' + ae^{a(t-t_0)}y = e^{a(t-t_0)}g(t) \\
\implies &(e^{a(t-t_0)}y)' = e^{a(t-t_0)}g(t) \\
\implies &y(t) = \int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds + e^{-a(t-t_0)}y_0.
\end{aligned}$$

Assuming $g(t) \to g_0$ as $t \to \infty$,

$$\int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds \to \int_{t_0}^{t} e^{-a(t-s)}g_0\,ds = \frac{g_0}{a} - \frac{e^{-a(t-t_0)}}{a}g_0 \to \frac{g_0}{a} \quad \text{as } t \to \infty$$

For an example, let $g(t) = e^{-t} + 1$. Assume $a \neq 1$. By undetermined coefficients, we look for a solution of the form $y = ce^{-at} + Ae^{-t} + B$. Substituting a function of this form into the differential equation leads to the equation

$$[-A + aA]e^{-t} + aB = e^{-t} + 1 \implies -A + aA = 1 \text{ and } aB = 1.$$

Therefore, $A = 1/(a-1)$, $B = 1/a$ and $y = ce^{-at} + \frac{1}{a-1}e^{-t} + 1/a$. The initial condition $y(0) = y_0$ implies $y(t) = (y_0 - \frac{1}{a-1} - \frac{1}{a})e^{-at} + \frac{1}{a-1}e^{-t} + 1/a \to 1/a$ as $t \to \infty$.

39.

(a) The integrating factor is $e^{\int p(t)\,dt}$. Multiplying by the integrating factor, we have

$$e^{\int p(t)\,dt}y' + e^{\int p(t)\,dt}p(t)y = 0.$$

Therefore,

$$\left(e^{\int p(t)\,dt}y\right)' = 0$$

which implies

$$y(t) = Ae^{-\int p(t)\,dt}$$

is the general solution.

(b) Let $y = A(t)e^{-\int p(t)\,dt}$. Then in order for y to satisfy the desired equation, we need

$$A'(t)e^{-\int p(t)\,dt} - A(t)p(t)e^{-\int p(t)\,dt} + A(t)p(t)e^{-\int p(t)\,dt} = g(t).$$

That is, we need

$$A'(t) = g(t)e^{\int p(t)\,dt}.$$

(c) From equation (iv), we see that

$$A(t) = \int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C.$$

Therefore,

$$y(t) = e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right).$$

40. Here, $p(t) = -2$ and $g(t) = t^2 e^{2t}$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) \\ &= e^{\int 2\,dt}\left(\int_0^t \tau^2 e^{2\tau} e^{\int -2\,d\tau}\,d\tau + C\right) \\ &= e^{2t}\left(\int_0^t \tau^2\,d\tau + C\right) \\ &= e^{2t}\left(\frac{t^3}{3} + c\right). \end{aligned}$$

41. Here, $p(t) = 1/t$ and $g(t) = 3\cos(2t)$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) \\ &= e^{-\int \frac{1}{t}\,dt}\left(\int_0^t 3\cos(2\tau)e^{\int \frac{1}{\tau}\,d\tau}\,d\tau + C\right) \\ &= \frac{1}{t}\left(\int_0^t 3\tau\cos(2\tau)\,d\tau + C\right) \\ &= \frac{1}{t}\left(\frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t) + C\right). \end{aligned}$$

42. Here, $p(t) = 2/t$ and $g(t) = \sin(t)/t$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) \\ &= e^{-\int \frac{2}{t}\,dt}\left(\int_0^t \frac{\sin(\tau)}{\tau}e^{\int \frac{2}{\tau}\,d\tau}\,d\tau + C\right) \\ &= \frac{1}{t^2}\left(\int_0^t \frac{\sin(\tau)^2}{\tau}\,d\tau + C\right) \\ &= \frac{1}{t^2}\left(\int_0^t \tau\sin(\tau)\,d\tau + C\right) \\ &= \frac{1}{t^2}\left(\sin(t) - t\cos(t) + C\right). \end{aligned}$$

43. Here, $p(t) = 1/2$ and $g(t) = 3t^2/2$. The general solution is given by

$$\begin{aligned}
y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right)\\
&= e^{-\int \frac{1}{2}\,dt}\left(\int_0^t \frac{3t^2}{2}e^{\int \frac{1}{2}\,d\tau}\,d\tau + C\right)\\
&= e^{-t/2}\left(\int_0^t \frac{3\tau^2}{2}e^{\tau/2}\,d\tau + C\right)\\
&= e^{-t/2}\left(3t^2e^{t/2} - 12te^{t/2} + 24e^{t/2} + C\right)\\
&= et^2 - 12t + 24 + ce^{-t/2}.
\end{aligned}$$

Section 2.2

1. Rewriting as $ydy = x^2dx$, then integrating both sides, we have $y^2/2 = x^3/3 + C$, or $3y^2 - 2x^3 = c; \quad y \neq 0$

2. Rewriting as $ydy = [x^2/(1+x^3)]dx$, then integrating both sides, we have $y^2/2 = \ln|1 + x^3|/3 + C$, or $3y^2 - 2\ln|1+x^3| = c; \quad x \neq -1, y \neq 0$

3. Rewriting as $y^{-2}dy = -\sin(x)dx$, then integrating both sides, we have $-y^{-1} = \cos(x)+C$, or $y^{-1} + \cos x = c$ if $y \neq 0$;. Also, we have $y = 0$ everywhere

4. Rewriting as $(3+2y)dy = (3x^2-1)dx$, then integrating both sides, we have $3y + y^2 - x^3 + x + C$ as long as $y \neq -3/2$.

5. Rewriting as $\sec^2(2y)dy = \cos^2(x)dx$, then integrating both sides, we have $\tan(2y)/2 = x/2 + \sin(2x)/4 + C$, or $2\tan 2y - 2x - \sin 2x = C$ as long as $\cos 2y \neq 0$. Also, if $y = \pm(2n+1)\pi/4$ for any integer n, then $y' = 0 = \cos(2y)$

6. Rewriting as $(1-y^2)^{-1/2}dy = dx/x$, then integrating both sides, we have $\sin^{-1}(y) = \ln|x| + C$. Therefore, $y = \sin[\ln|x| + c]$ as long as $x \neq 0$ and $|y| < 1$;. We also notice that if $y = \pm 1$, then $xy' = 0 = (1-y^2)^{1/2}$ is a solution.

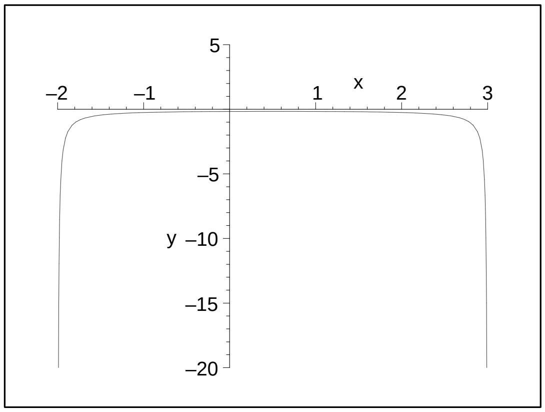
7. Rewriting as $(y + e^y)dy = (x - e^{-x})dx$, then integrating both sides, we have $y^2/2 + e^y = x^2/2 + e^{-x} + C$, or $y^2 - x^2 + 2(e^y - e^{-x}) = C$ as long as $y + e^y \neq 0$.

8. Rewriting as $(1+y^2)dy = x^2dx$, then integrating both sides, we have $y + y^3/3 = x^3/3 + C$, or $3y + y^3 - x^3 = c$;.

9.

(a) Rewriting as $y^{-2}dy = (1-2x)dx$, then integrating both sides, we have $-y^{-1} = x - x^2 + C$. The initial condition, $y(0) = -1/6$ implies $C = 6$. Therefore, $y = 1/(x^2 - x - 6)$.

(b)

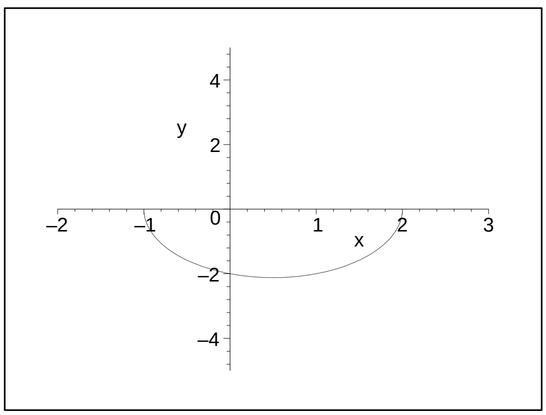


(c) $-2 < x < 3$

10.

(a) Rewriting as $ydy = (1-2x)dx$, then integrating both sides, we have $y^2/2 = x - x^2 + C$. Therefore, $y = \pm\sqrt{2x - 2x^2 + 4}$. The initial condition, $y(1) = -2$ implies $C = 2$ and $y = -\sqrt{2x - 2x^2 + 4}$.

(b)

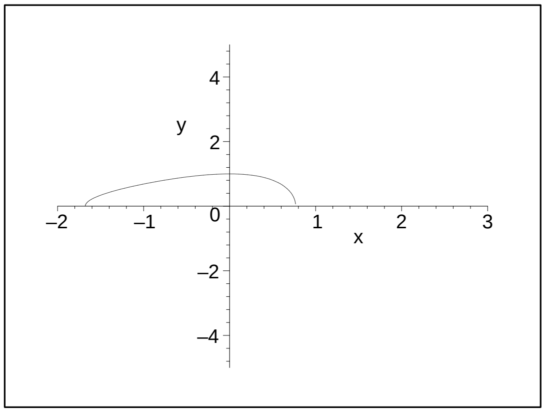


(c) $-1 < x < 2$

11.

(a) Rewriting as $xe^x dx = -ydy$, then integrating both sides, we have $xe^x - e^x = -y^2/2 + C$. The initial condition, $y(0) = 1$ implies $C = -1/2$. Therefore, $y = [2(1-x)e^x - 1]^{1/2}$.

(b)

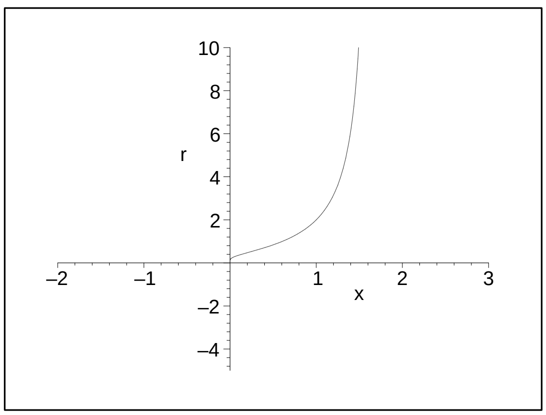


(c) $-1.68 < x < 0.77$ approximately

12.

(a) Rewriting as $r^{-2}dr = \theta^{-1}d\theta$, then integrating both sides, we have $-r^{-1} = \ln\theta + C$. The initial condition, $r(1) = 2$ implies $C = -1/2$. Therefore, $r = 2/(1 - 2\ln\theta)$.

(b)

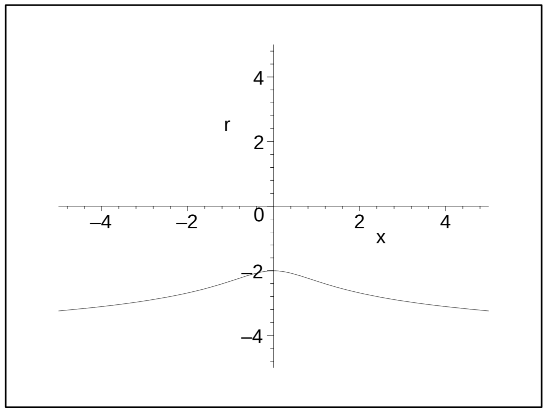


(c) $0 < \theta < \sqrt{e}$

13.

(a) Rewriting as $ydy = 2x/(1+x^2)dx$, then integrating both sides, we have $y^2/2 = \ln(1+x^2) + C$. The initial condition, $y(0) = -2$ implies $C = 2$. Therefore, $y = -[2\ln(1+x^2)+4]^{1/2}$.

(b)

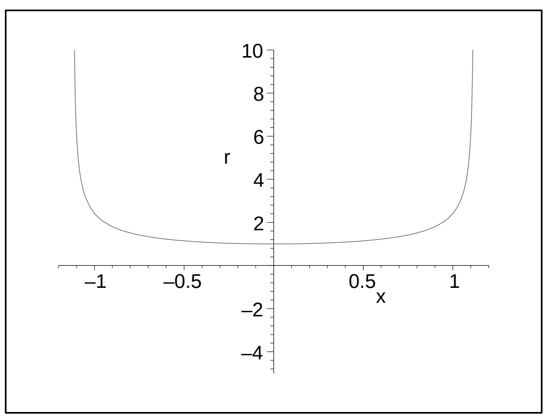


(c) $-\infty < x < \infty$

14.

(a) Rewriting as $y^{-3}dy = x(1+x^2)^{-1/2}dx$, then integrating both sides, we have $-y^{-2}/2 = \sqrt{1+x^2} + C$. The initial condition, $y(0) = 1$ implies $C = -3/2$. Therefore, $y = \left[3 - 2\sqrt{1+x^2}\right]^{-1/2}$.

(b)

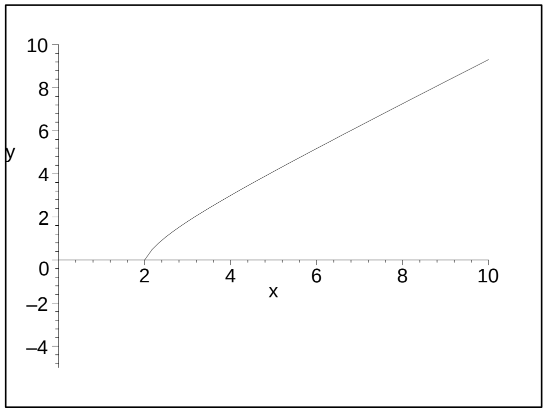


(c) $|x| < \dfrac{1}{2}\sqrt{5}$

15.

(a) Rewriting as $(1+2y)dy = 2xdx$, then integrating both sides, we have $y + y^2 = x^2 + C$. The initial condition, $y(2) = 0$ implies $C = -4$. Therefore, $y^2 + y = x^2 - 4$. Completing the square, we have $(y+1/2)^2 = x^2 - 15/4$, and, therefore, $y = -\dfrac{1}{2} + \dfrac{1}{2}\sqrt{4x^2 - 15}$.

(b)

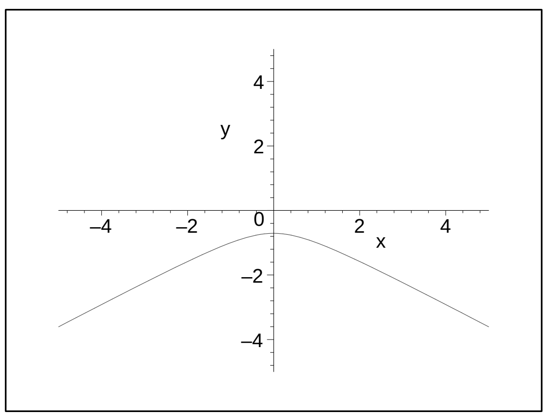


(c) $x > \dfrac{1}{2}\sqrt{15}$

16.

(a) Rewriting as $4y^3 dy = x(x^2+1)dx$, then integrating both sides, we have $y^4 = (x^2+1)^2/4 + C$. The initial condition, $y(0) = -1/\sqrt{2}$ implies $C = 0$. Therefore, $y = -\sqrt{(x^2+1)/2}$.

(b)

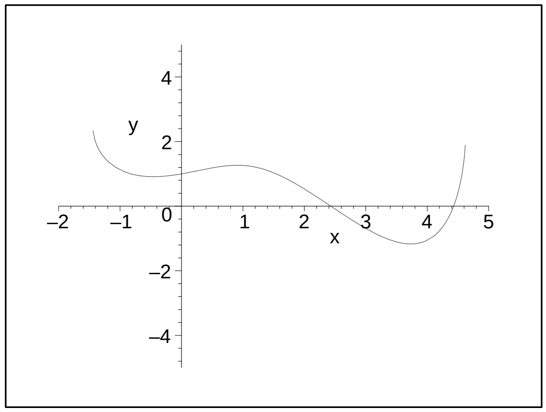


(c) $-\infty < x < \infty$

17.

(a) Rewriting as $(2y-5)dy = (3x^2 - e^x)dx$, then integrating both sides, we have $y^2 - 5y = x^3 - e^x + C$. The initial condition, $y(0) = 1$ implies $C = -3$. Completing the square, we have $(y - 5/2)^2 = x^3 - e^x + 13/4$. Therefore, $y = 5/2 - \sqrt{x^3 - e^x + 13/4}$.

(b)

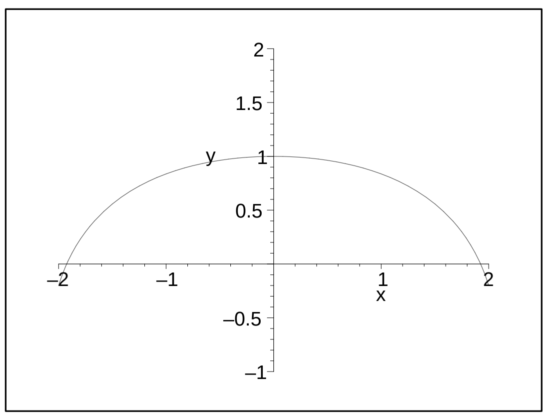


(c) $-1.4445 < x < 4.6297$ approximately

18.

(a) Rewriting as $(3+4y)dy = (e^{-x} - e^{x})dx$, then integrating both sides, we have $3y + 2y^2 = -(e^x + e^{-x}) + C$. The initial condition, $y(0) = 1$ implies $C = 7$. Completing the square, we have $(y + 3/4)^2 = -(e^x + e^{-x})/2 + 65/16$. Therefore, $y = -\frac{3}{4} + \frac{1}{4}\sqrt{65 - 8e^x - 8e^{-x}}$.

(b)

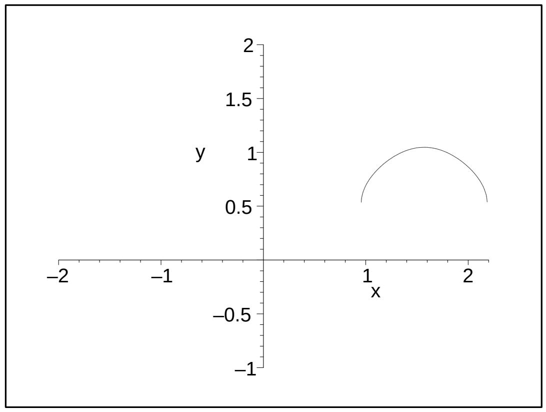


(c) $|x| < 2.0794$ approximately

19.

(a) Rewriting as $\cos(3y)dy = -\sin(2x)dx$, then integrating both sides, we have $\sin(3y)/3 = \cos(2x)/2 + C$. The initial condition, $y(\pi/2) = \pi/3$ implies $C = 1/2$. Therefore, $y = [\pi - \arcsin(3\cos^2 x)]/3$.

(b)

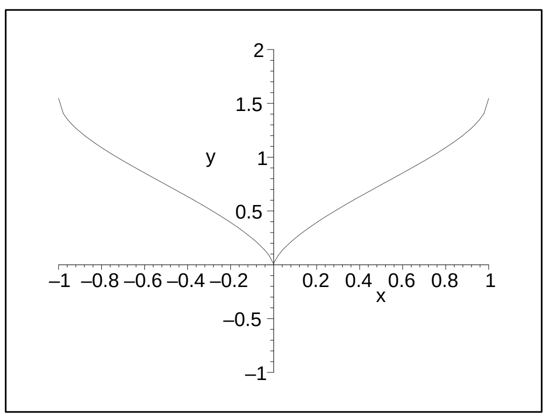


(c) $|x - \pi/2| < 0.6155$ approximately

20.

(a) Rewriting as $y^2 dy = \arcsin(x)/\sqrt{1-x^2}dx$, then integrating both sides, we have $y^3/3 = (\arcsin(x))^2/2 + C$. The initial condition, $y(0) = 1/$ implies $C = 0$. Therefore, $y = \left[\frac{3}{2}(\arcsin x)^2\right]^{1/3}$.

(b)



(c) $-1 < x < 1$

21. Rewriting the equation as $(3y^2 - 6y)dy = (1 + 3x^2)dx$ and integrating both sides, we have $y^3 - 3y^2 = x + x^3 + C$. The initial condition, $y(0) = 1$ implies $c = -2$. Therefore, $y^3 - 3y^2 - x - x^3 + 2 = 0$. When $3y^2 - 6y = 0$, the integral curve will have a vertical tangent. In particular, when $y = 0, 2$. From our solution, we see that $y = 0$ implies $x = 1$ and $y = 2$ implies $x = -1$. Therefore, the solution is defined for $-1 < x < 1$.

22. Rewriting the equation as $(3y^2 - 4)dy = 3x^2 dx$ and integrating both sides, we have $y^3 - 4y = x^3 + C$. The initial condition $y(1) = 0$ implies $C = -1$. Therefore, $y^3 - 4y - x^3 = -1$. When $3y^2 - 4 = 0$, the integral curve will have a vertical tangent. In particular, when $y = \pm 2/\sqrt{3}$. At these values for y, we have $x = -1.276, 1.598$. Therefore, the solution is defined for $-1.276 < x < 1.598$

23. Rewriting the equation as $y^{-2}dy = (2+x)dx$ and integrating both sides, we have $-y^{-1} = 2x + x^2/2 + C$. The initial condition $y(0) = 1$ implies $C = -1$. Therefore, $y = -1/(x^2/2 + 2x - 1)$. To find where the function attains it minimum value, we look where $y' = 0$. We see that $y' = 0$ implies $y = 0$ or $x = -2$. But, as seen by the solution formula, y is never zero. Further, it can be verified that $y''(-2) > 0$, and, therefore, the function attains a minimum at $x = -2$.

24. Rewriting the equation as $(3+2y)dy = (2-e^x)dx$ and integrating both sides, we have $3y + y^2 = 2x - e^x + C$. By the initial condition $y(0) = 0$, we have $C = 1$. Completing the square, it follows that $y = -3/2 + \sqrt{2x - e^x + 13/4}$. The solution is defined if $2x - e^x + 13/4 \geq 0$, that is, $-1.5 \leq x \leq 2$ (approximately). In that interval, $y = 0$ for $x = \ln 2$. It can be verified that $y''(\ln 2) < 0$, and, therefore, the function attains its maximum value at $x = \ln 2$.

25. Rewriting the equation as $(3+2y)dy = 2\cos(2x)dx$ and integrating both sides, we have $3y + y^2 = \sin(2x) + C$. By the initial condition $y(0) = -1$, we have $C = -2$. Completing the square, it follows that $y = -3/2 + \sqrt{\sin(2x) + 1/4}$. The solution is defined for $\sin(2x) + 1/4 \geq 0$. That is, $-0.126 \leq x \leq 1.44$. To find where the solution attains its maximum value, we need to check where $y' = 0$. We see that $y' = 0$ when $2\cos(2x) = 0$. In the interval of definition above, that occurs when $2x = \pi/2$, or $x = \pi/4$.

26. Rewriting this equation as $(1+y^2)^{-1}dy = 2(1+x)dx$ and integrating both sides, we have $\tan^{-1}(y) = 2x + x^2 + C$. The initial condition implies $C = 0$. Therefore, the solution is $y = \tan(x^2 + 2x)$. The solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. We note that $2x + x^2 \geq -1$. Further, $2x + x^2 = \pi/2$ for $x = -2.6$ and 0.6. Therefore, the solution is valid in the interval $-2.6 < x < 0.6$. We see that $y' = 0$ when $x = -1$. Furthermore, it can be verified that $y''(x) > 0$ for all x in the interval of definition. Therefore, y attains a global minimum at $x = -1$.

27.

(a) First, we rewrite the equation as $dy/[y(4-y)] = t\,dt/3$. Then, using partial fractions, we write

$$\frac{1/4}{y}\,dy + \frac{1/4}{4-y}\,dy = \frac{t}{3}\,dt.$$

Integrating both sides, we have

$$\begin{aligned} \frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y| &= \frac{t^2}{6} + C \\ \implies \ln\left|\frac{y}{y-4}\right| &= \frac{2}{3}t^2 + C \\ \implies \left|\frac{y}{y-4}\right| &= Ce^{2t^2/3}. \end{aligned}$$

From the equation, we see that $y_0 = 0 \implies C = 0 \implies y(t) = 0$ for all t. Otherwise, $y(t) > 0$ for all t or $y(t) < 0$ for all t. Therefore, if $y_0 > 0$ and $|y/(y-4)| = Ce^{2t^2/3} \to \infty$, we must have $y \to 4$. On the other hand, if $y_0 < 0$, then $y \to -\infty$ as $t \to \infty$. (In particular, $y \to -\infty$ in finite time.)

(b) For $y_0 = 0.5$, we want to find the time T when the solution first reaches the value 3.98. Using the fact that $|y/(y-4)| = Ce^{2t^2/3}$ combined with the initial condition, we have $C = 1/7$. From this equation, we now need to find T such that $|3.98/.02| = e^{2T^2/3}/7$. Solving this equation, we have $T = 3.29527$.

28.

(a) Rewriting the equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$ and integrating both sides, we have $\ln|y| - \ln|y-4| = 4t - 4\ln|1+t| + C$. Therefore, $|y/(y-4)| = Ce^{4t}/(1+t)^4 \to \infty$ as $t \to \infty$ which implies $y \to 4$.

(b) The initial condition $y(0) = 2$ implies $C = 1$. Therefore, $y/(y-4) = -e^{4t}/(1+t)^4$. Now we need to find T such that $3.99/-.01 = -e^{4T}/(1+T)^4$. Solving this equation, we have $T = 2.84367$.

(c) Using our results from part (b), we note that $y/(y-4) = y_0/(y_0-4)e^{4t}/(1+t)^4$. We want to find the range of initial values y_0 such that $3.99 < y < 4.01$ at time $t = 2$. Substituting $t = 2$ into the equation above, we have $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Since the function $y/(y-4)$ is monotone, we need only find the values y_0 satisfying $y_0/(y_0-4) = -399(3/e^2)^4$ and $y_0/(y_0-4) = 401(3/e^2)^4$. The solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$. Therefore, we need $3.6622 < y_0 < 4.4042$.

29. We can rewrite the equation as

$$\left(\frac{cy+d}{ay+b}\right) dy = dx \implies \frac{cy}{ay+b} + \frac{d}{ay+b} dy = dx \implies \frac{c}{a} - \frac{bc}{a^2y+ab} + \frac{d}{ay+b} dy = dx.$$

Then integrating both sides, we have

$$\frac{c}{a}y - \frac{bc}{a^2}\ln|a^2y+ab| + \frac{d}{a}\ln|ay+b| = x + C.$$

Simplifying, we have

$$\begin{aligned} \frac{c}{a}y - \frac{bc}{a^2}\ln|a| - \frac{bc}{a^2}\ln|ay+b| + \frac{d}{a}\ln|ay+b| &= x + C \\ \implies \frac{c}{a}y + \left(\frac{ad-bc}{a^2}\right)\ln|ay+b| &= x + C. \end{aligned}$$

Note, in this calculation, since $\frac{bc}{a^2}\ln|a|$ is just a constant, we included it with the arbitrary constant C. This solution will exist as long as $a \neq 0$ and $ay + b \neq 0$.

30.

(a) Factoring an x out of each term in the numerator and denominator of the right-hand side, we have

$$\frac{dy}{dx} = \frac{x((y/x)-4)}{x(1-(y/x))} = \frac{(y/x)-4}{1-(y/x)},$$

as claimed.

(b) Letting $v = y/x$, we have $y = xv$, which implies that $dy/dx = v + x \cdot dv/dx$.

(c) Therefore,

$$v + x \cdot \frac{dv}{dx} = \frac{v-4}{1-v}$$

which implies that

$$x \cdot \frac{dv}{dx} = \frac{v - 4 - v(1-v)}{(1-v)} = \frac{v^2-4}{1-v}.$$

(d) To solve the equation above, we rewrite as

$$\frac{1-v}{v^2-4} dv = \frac{dx}{x}.$$

Integrating both sides of this equation, we have

$$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C.$$

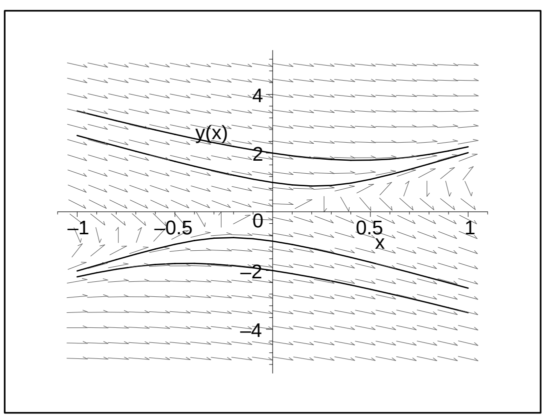
Applying the exponential function to both sides of the equation, we have

$$|v-2|^{-1/4}|v+2|^{-3/4} = C|x|.$$

(e) Replacing v with y/x, we have

$$\left|\frac{y}{x} - 2\right|^{-1/4}\left|\frac{y}{x} + 2\right|^{-3/4} = C|x| \implies |x||y-2x|^{-1/4}|y+2x|^{-3/4} = C|x| \implies |y+2x|^3|y-2x| = C.$$

(f)



31.

(a)

$$\frac{dy}{dx} = 1 + (y/x) + (y/x)^2.$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

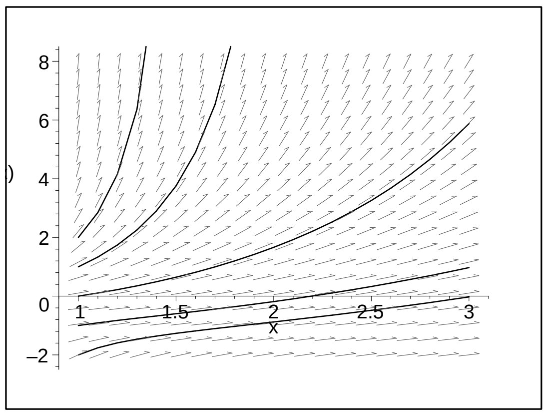
$$v + x\frac{dv}{dx} = 1 + v + v^2 \implies x\frac{dv}{dx} = 1 + v^2.$$

This equation can be rewritten as

$$\frac{dv}{1+v^2} = \frac{dx}{x}$$

which has solution $\arctan(v) = \ln|x| + c$. Rewriting back in terms of y, we have $\arctan(y/x) - \ln|x| = c$.

(c)



32.

(a)

$$\frac{dy}{dx} = (y/x)^{-1} + \frac{3}{2}(y/x).$$

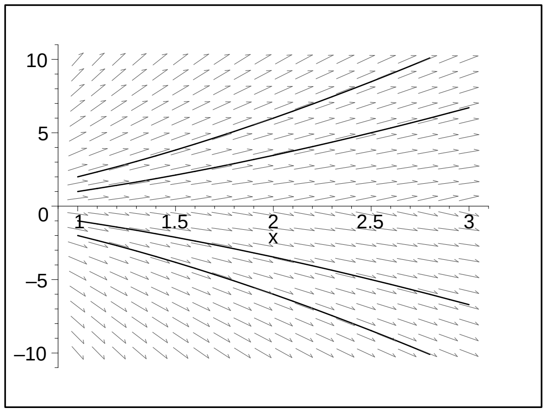
Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x^2v^2}{2x^2v} \implies \frac{dv}{dx} = \frac{1+v^2}{2xv}.$$

The solution of this separable equation is $v^2 + 1 = cx$. Rewriting back in terms of y, we have $x^2 + y^2 - cx^3 = 0$.

(c)



33.

(a)

$$\frac{dy}{dx} = \frac{4(y/x) - 3}{2 - (y/x)}.$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

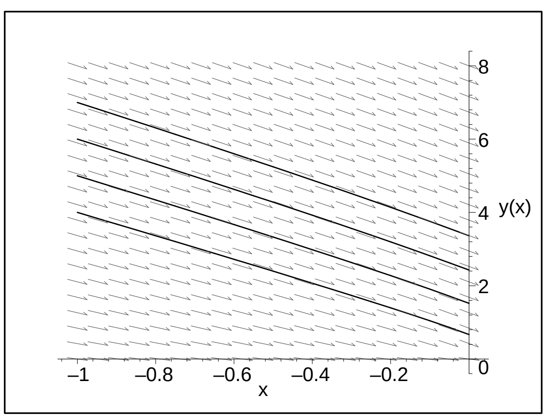
$$v + x\frac{dv}{dx} = \frac{4v - 3}{2 - v} \implies x\frac{dv}{dx} = \frac{v^2 + 2v - 3}{2 - v}.$$

This equation can be rewritten as

$$\frac{2 - v}{v^2 + 2v - 3} dv = \frac{dx}{x}.$$

Integrating both sides and simplifying, we arrive at the solution $|v + 3|^{-5/4}|v - 1|^{1/4} = |x| + c$. Rewriting back in terms of y, we have $|y - x| = c|y + 3x|^5$. We also have the solution $y = -3x$.

(c)



34.

(a)

$$\frac{dy}{dx} = -2 - \frac{y}{x}\left[2 + \frac{y}{x}\right]^{-1}.$$

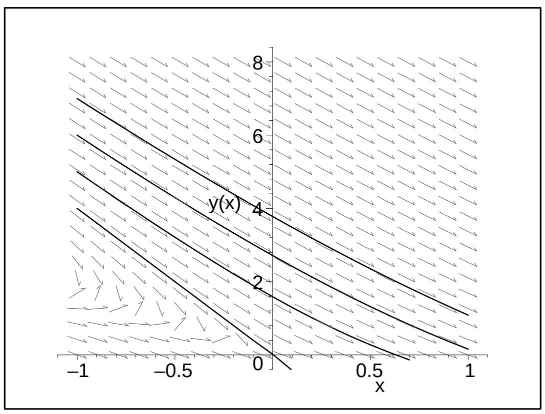
Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

$$v + x\frac{dv}{dx} = -2 - \frac{v}{2+v} \implies \frac{dv}{dx} = -\frac{v^2+5v+4}{x(2+v)}.$$

This equation is separable with solution $(v+4)^2|v+1| = C/x^3$. Rewriting back in terms of y, we have $|y+x|(y+4x)^2 = c$.

(c)



35.

(a)

$$\frac{dy}{dx} = \frac{1+3(y/x)}{1-(y/x)}.$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

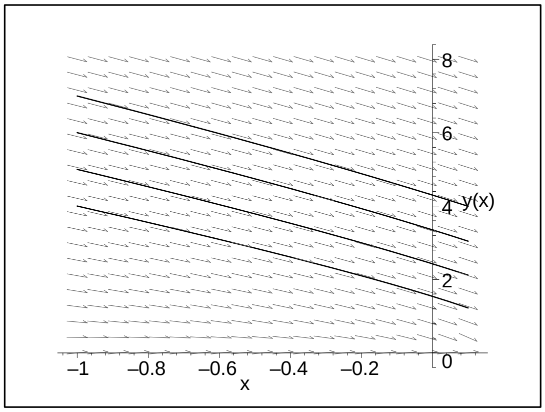
$$v + x\frac{dv}{dx} = \frac{1+3v}{1-v} \implies x\frac{dv}{dx} = \frac{v^2+2v+1}{1-v}.$$

This equation can be rewritten as

$$\frac{1-v}{v^2+2v+1}dv = \frac{dx}{x}$$

which has solution $-\frac{2}{v+1} - \ln|v+1| = \ln|x| + c$. Rewriting back in terms of y, we have $2x/(x+y) + \ln|x+y| = c$. We also have the solution $y = -x$.

(c)



36.

(a)

$$\frac{dy}{dx} = 1 + 3(y/x) + (y/x)^2.$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

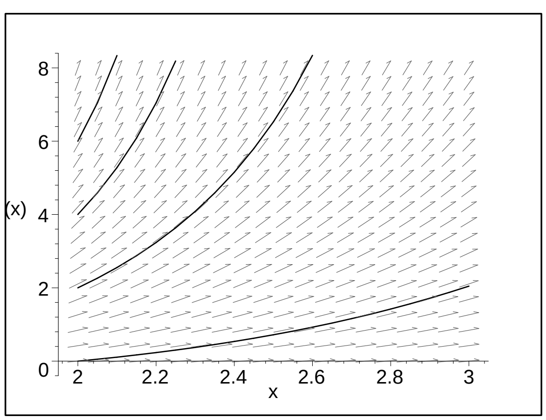
$$v + x\frac{dv}{dx} = 1 + 3v + v^2 \implies x\frac{dv}{dx} = 1 + 2v + v^2.$$

This equation can be rewritten as

$$\frac{dv}{1 + 2v + v^2} = \frac{dx}{x}$$

which has solution $-1/(v+1) = \ln|x| + c$. Rewriting back in terms of y, we have $x/(x+y) + \ln|x| = c$. We also have the solution $y = -x$.

(c)



37.

(a)

$$\frac{dy}{dx} = \frac{1}{2}(y/x)^{-1} - \frac{3}{2}(y/x).$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

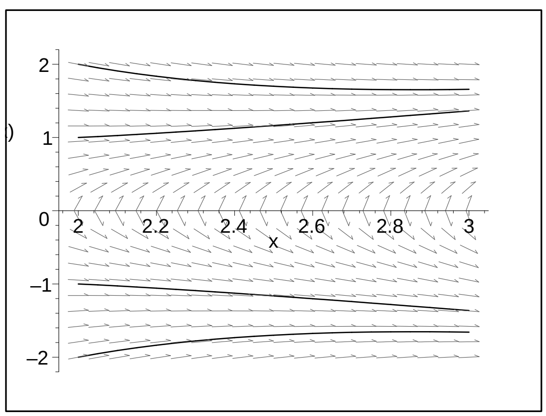
$$v + x\frac{dv}{dx} = 1 + \frac{1}{2v} - \frac{3}{2}v \implies x\frac{dv}{dx} = \frac{1-5v^2}{2v}.$$

This equation can be rewritten as

$$\frac{2v}{1-5v^2}dv = \frac{dx}{x}$$

which has solution $-\frac{1}{5}\ln|1-5v^2| = \ln|x| + c$. Applying the exponential function, we arrive at the solution $1 - 5v^2 = c/|x|^5$. Rewriting back in terms of y, we have $|x|^3(x^2 - 5y^2) = c$

(c)



38.

(a)

$$\frac{dy}{dx} = \frac{3}{2}(y/x) - \frac{1}{2}(y/x)^{-1}.$$

Therefore, the equation is homogeneous.

(b) The substitution $v = y/x$ results in the equation

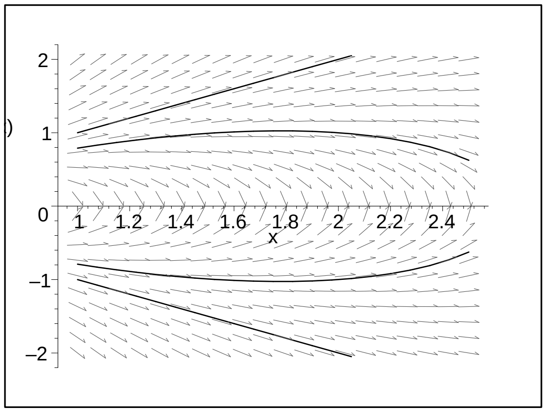
$$v + x\frac{dv}{dx} = \frac{3}{2}v - \frac{1}{2}v^{-1} \implies x\frac{dv}{dx} = \frac{v^2-1}{2v}.$$

This equation can be rewritten as

$$\frac{2v}{v^2-1}dv = \frac{dx}{x}$$

which has solution $\ln|v^2-1| = \ln|x| + c$. Applying the exponential function, we have $v^2 - 1 = C|x|$. Rewriting back in terms of y, we have $c|x|^3 = (y^2 - x^2)$

(c)



Section 2.3

1. Let $Q(t)$ be the quantity of dye in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out}.$$

Here, fresh water is flowing in. Therefore, no dye is coming in. The dye is flowing out at the rate of $(Q/200)g/l \cdot 2l/min = Q/100$ l/min. Therefore,

$$\frac{dQ}{dt} = -\frac{Q}{100}.$$

The solution of this equation is $Q(t) = Ce^{-t/100}$. Since $Q(0) = 200$ grams, $C = 200$. We need to find the time T when the amount of dye present is 1% of what it is initially. That is, we need to find the time T when $Q(T) = 2$ grams. Solving the equation $2 = 200e^{-t/100}$, we conclude that $T = 100\ln(100)$ minutes.

2. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out}.$$

Here, water containing γ g/liter of salt is flowing in at a rate of 2 liters/minute. The salt is flowing out at the rate of $(Q/120)g/l \cdot 2l/min = Q/60$ l/min. Therefore,

$$\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}.$$

The solution of this equation is $Q(t) = 120\gamma + Ce^{-t/60}$. Since $Q(0) = 0$ grams, $C = -120\gamma$. Therefore, $Q(t) = 120\gamma[1 - e^{-t/60}]$. As $t \to \infty$, $Q(t) \to 120\gamma$.

3. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing 1/2 lb/gallon of salt is flowing in at a rate of 2 gal/minute. The salt is flowing out at the rate of $(Q/100)lb/gal \cdot 2gal/min = Q/50$ gal/min. Therefore,

$$\frac{dQ}{dt} = 1 - \frac{Q}{50}.$$

The solution of this equation is $Q(t) = 50 + Ce^{-t/50}$. Since $Q(0) = 0$ grams, $C = -50$. Therefore, $Q(t) = 50[1 - e^{-t/50}]$ for $0 \leq t \leq 10$ minutes. After 10 minutes, the amount of salt in the tank is $Q(10) = 50[1 - e^{-1/5}] \approx 9.06$ lbs. Starting at that time (and resetting the time variable), the new equation for dQ/dt is given by

$$\frac{dQ}{dt} = -\frac{Q}{50},$$

since fresh water is being added. The solution of this equation is $Q(t) = Ce^{-t/50}$. Since we are now starting with 9.06 lbs of salt, $Q(0) = 9.06 = C$. Therefore, $Q(t) = 9.06e^{-t/50}$. After 10 minutes, $Q(10) = 9.06e^{-1/5} \cong 7.42$ lbs.

4. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing 1 lb/gallon of salt is flowing in at a rate of 3 gal/minute. The salt is flowing out at the rate of $(Q/(200+t))lb/gal \cdot 2gal/min = 2Q/(200+t)$ lb/min. Therefore,

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200+t}.$$

This is a linear equation with integrating factor $\mu(t) = (200 + t)^2$. The solution of this equation is $Q(t) = 200 + t + C(200 + t)^{-2}$. Since $Q(0) = 100$ lbs, $C = -4,000,000$. Therefore, $Q(t) = 200 + t - 4,000,000/(200 + t)^2$. Since the tank has a net gain of 1 gallon of water every minute, the tank will reach its capacity after 300 minutes. When $t = 300$, we see that $Q(300) = 484$ lbs. Therefore, the concentration of salt when it is on the point of overflowing is $121/125$ lbs/gallon. The concentration of salt is given by $Q(t)/(200+t)$ (since t gallons of water are added every t minutes). Using the equation for Q above, we see that if the tank had infinite capacity, the concentration would approach 1 as $t \to \infty$.

5.

(a) Let $Q(t)$ be the quantity of salt in the tank. We know that

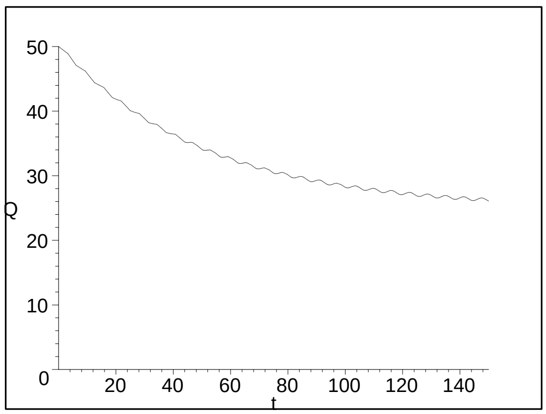
$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing $\frac{1}{4}\left(1+\frac{1}{2}\sin t\right)$ oz/gallon of salt is flowing in at a rate of 2 gal/minute. The salt is flowing out at the rate of $Q/100 oz/gal \cdot 2gal/min = Q/50$ oz/min. Therefore,

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}.$$

This is a linear equation with integrating factor $\mu(t) = e^{t/50}$. The solution of this equation is $Q(t) = (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501 + C$. The initial condition, $Q(0) = 50$ oz implies $C = 25$. Therefore, $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501$.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of amplitude 1/4 about a level of 25 oz.

6.

(a) Using the Principle of Conservation of Energy, we know that the kinetic energy of a particle after it has fallen from a height h is equal to its potential energy at a height t. Therefore, $mv^2/2 = mgh$. Solving this equation for v, we have $v = \sqrt{2gh}$.

(b) The volumetric outflow rate is (outflow cross-sectional area) $\times$ (outflow velocity): $\alpha a\sqrt{2gh}$. The volume of water in the tank is

$$V(h) = \int_0^h A(u)\,du$$

where $A(u)$ is the cross-sectional area of the tank at height u. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = A(h)\frac{dh}{dt}.$$

Therefore,

$$\frac{dV}{dt} = A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

(c) The cross-sectional area of the cylinder is $A(h) = \pi(1)^2 = \pi$. The outflow cross-sectional area is $a = \pi(.1)^2 = .01\pi$. From part (a), we take $\alpha = 0.6$ for water. Then by part (b), we have

$$\pi\frac{dh}{dt} = -0.006\pi\sqrt{2gh}.$$

This is a separable equation with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, we have $h(t) = 0.0001764t^2 - 0.046t + 3$. Then $h(t) = 0$ implies $t \approx 130.4$ seconds.

7.

(a) The equation for S is

$$\frac{dS}{dt} = rS$$

with an initial condition $S(0) = S_0$. The solution of the equation is $S(t) = S_0e^{rt}$. We want to find the time T such that $S(T) = 2S_0$. Our equation becomes $2S_0 = S_0e^{rT}$. Dividing by S_0 and applying the logarithmic function to our equation, we have $rT = \ln(2)$. That is, $T = \ln(2)/r$.

(b) If $r = .07$, then $T = \ln(2)/.07 \cong 9.90$ years.

(c) By part (a), we also know that $r = \ln(2)T$ where T is the doubling time. If we want the investment to double in $T = 8$ years, then we need $r = \ln(2)/8 \cong 8.66\%$.

8.

(a) The equation for S is given by

$$\frac{dS}{dt} = rS + k.$$

This is a linear equation with solution $S(t) = \dfrac{k}{r}[e^{rt} - 1]$.

(b) Using the function in part (a), we need to find k so that $S(40) = 1,000,000$ assuming $r = .075$. That is, we need to solve

$$1,000,000 = \frac{k}{.075}[e^{.075 \cdot 40} - 1].$$

The solution of this equation is $k \cong 3930$.

(c) Now we assume that $k = 2000$ and want to find r. Our equation becomes

$$1,000,000 = \frac{2000}{r}[e^{40r} - 1].$$

The solution of this equation is approximately 9.77%.

9.

(a) Let $S(t)$ be the balance due on the loan at time t. To determine the maximum amount the buyer can afford to borrow, we will assume that the buyer will pay \$800 per month. Then

$$\frac{dS}{dt} = .09S - 12(800).$$

The solution is given by equation (18), $S(t) = S_0 e^{.09t} - 106,667(e^{.09t} - 1)$. If the term of the mortgage is 20 years, then $S(20) = 0$. Therefore, $0 = S_0 e^{.09(20)} - 106,667(e^{.09(20)} - 1)$ which implies $S_0 \approx \$89,035$.

(b) Since the homeowner pays \$800 per month for 20 years, he ends up paying a total of \$192,000 for the house. Since the house loan was \$89,035, the rest of the amount was interest payments. Therefore, the amount of interest was approximately \$102,965.

10.

(a) Let $S(t)$ be the balance due on the loan at time t. Taking into account that t is measured in years, we rewrite the monthly payment as $800(1 + t/10)$ where t is now in years. The equation for S is given by

$$\frac{dS}{dt} = .09S - 12(800)(1 + t/10).$$

This is a linear equation. Its solution is $S(t) = 225185 + 10667t + ce^{.09t}$. The initial condition $S(0) = 100,000$ implies $c = -125185$. Therefore, the particular solution is $S(t) = 225185 + 10667t - 125185e^{.09t}$. To find when the loan will be paid, we just need to solve $S(t) = 0$. Solving this equation, we conclude that the loan will be paid off in 11.28 years (135.36 months).

(b) From part (a), we know the general solution is given by $S(t) = 225185 + 10667t + ce^{.09t}$. Now we want to find c such that $S(20) = 0$. The solution of this equation is $c = -72486$. Therefore, the solution of the equation will be $S(t) = 225185 + 10667 - 72846e^{.09t}$. Therefore, $S(0) = 225185 - 72846 = 152699$.

11.

(a) If S_0 is the initial balance, then the balance after one month is

$$\begin{aligned} S_1 &= \text{ initial balance + interest - monthly payment} \\ &= S_0 + rS_0 - k. \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &= S_1 + rS_1 - k \\ &= (1 + r)S_1 - k. \end{aligned}$$

In general,

$$S_n = (1 + r)S_{n-1} - k.$$

(b) $R = 1 + r$ implies $S_n = RS_{n-1} - k$. Therefore,

$$\begin{aligned}
&S_1 = RS_0 - k \\
&S_2 = RS_1 - k = R[RS_0 - k] - k = R^2 S_0 - (R+1)k \\
&S_3 = RS_2 - k = R[R^2 S_0 - (R+1)k] - k = R^3 S_0 - (R^2 + R + 1)k.
\end{aligned}$$

(c) We check the base case, $n = 1$. We see that

$$S_1 = RS_0 - k = RS_0 - \left(\frac{R-1}{R-1}\right) k,$$

which implies that that the condition is satisfied for $n = 1$. We assume that

$$S_n = R^n S_0 - \frac{R^n - 1}{R - 1} k$$

to show that

$$S_{n+1} = R^{n+1} S_0 - \frac{R^{n+1} - 1}{R - 1} k.$$

We see that

$$\begin{aligned}
S_{n+1} &= RS_n - k \\
&= R\left[R^n S_0 - \frac{R^n - 1}{R - 1} k\right] - k \\
&= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R - 1}\right) k - k \\
&= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R - 1}\right) k - \left(\frac{R - 1}{R - 1}\right) k \\
&= R^{n+1} S_0 - \left(\frac{R^{n+1} - R + R - 1}{R - 1}\right) k \\
&= R^{n+1} S_0 - \left(\frac{R^{n+1} - 1}{R - 1}\right) k.
\end{aligned}$$

(d) We are assuming that $S_0 = 20,000$ and $r = .08/12$. We need to find k such that $S_{48} = 0$. Our equation becomes

$$S_{48} = R^{48} S_0 - \left(\frac{R^{48} - 1}{R - 1}\right) k = 0.$$

Therefore,

$$\left(\frac{(1 + .08/12)^{48} - 1}{.08/12}\right) k = \left(1 + \frac{.08}{12}\right)^{48} \cdot 20,000,$$

which implies $k \approx 488.26$, which is very close to the result in example 2.

12.

(a) The general solution is $Q(t) = Q_0 e^{-rt}$. If the half-life is 5730, then $Q_0/2 = Q_0 e^{-5730r}$ implies $-5730r = \ln(1/2)$. Therefore, $r = 1.2097 \times 10^{-4}$ per year.

(b) Therefore, $Q(t) = Q_0 e^{-1.2097 \times 10^{-4} t}$.

(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097 \times 10^{-4} T}$. Solving for T, we have $T = 13,304.65$ years.

13. Let $P(t)$ be the population of mosquitoes at any time t, measured in days. Then

$$\frac{dP}{dt} = rP - 20,000.$$

The solution of this linear equation is $P(t) = P_0 e^{rt} - \frac{20,000}{r}(e^{rt} - 1)$. In the absence of predators, the equation is $dP_1/dt = rP_1$. The solution of this equation is $P_1(t) = P_0 e^{rt}$. Since the population doubles after 7 days, we see that $2P_0 = P_0 e^{7r}$. Therefore, $r = \ln(2)/7 = .09902$ per day. Therefore, the population of mosquitoes at any time t is given by $P(t) = 200,000e^{.099t} - 201,997(e^{.099t} - 1) = 201,997 - 1997e^{.099t}$.

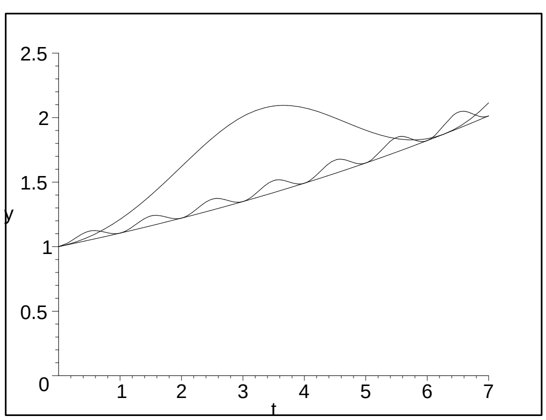
14.

(a) The solution of this separable equation is given by $y(t) = \exp[2/10 + t/10 - 2\cos(t)/10]$. The doubling-time is found by solving the equation $2 = \exp[2/10 + t/10 - 2\cos(t)/10]$. The solution of this equation is given by $\tau \approx 2.9632$.

(b) The differential equation will be $dy/dt = y/10$ with solution $y(t) = y(0)e^{t/10}$. The doubling time is found by setting $y(t) = 2y(0)$. In this case, the doubling time is $\tau \approx 6.9315$.

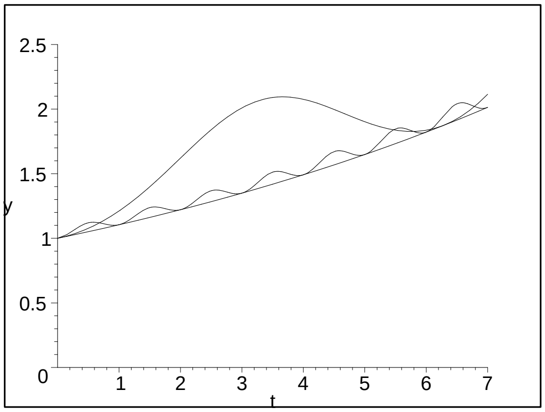
(c) Consider the differential equation $dy/dt = (0.5 + \sin(2\pi t))y/5$. This equation is separable with solution $y(t) = \exp[(1 + \pi t - \cos(2\pi t))/(10\pi)]$. The doubling time is found by setting $y(t) = 2$. The solution is given by 6.9167.

(d)



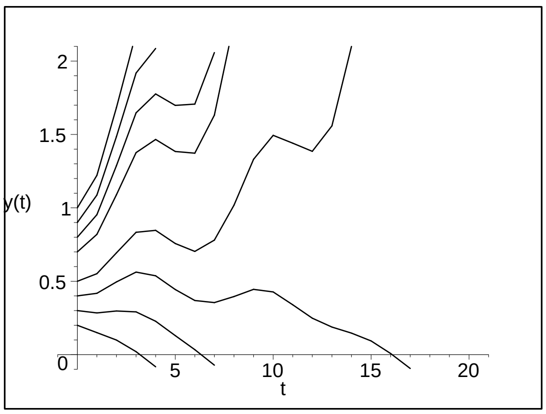
15.

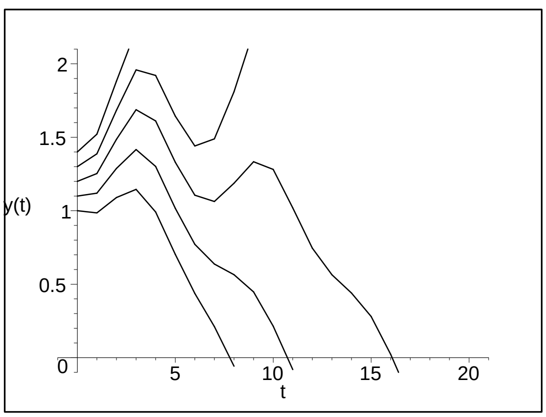
(a)



(b) Based on the graph, we estimate that $y_c \approx 0.83$.

(c) We sketch the graphs below for $k = 1/10$ and $k = 3/10$, respectively. Based on these graphs, we estimate that $y_c(1/10) \approx .41$ and $y_c(3/10) \approx 1.24$.





(d) From our results from above, we conclude that y_c is a linear function of k.

16. Let $T(t)$ be the temperature of the coffee at time t. The governing equation is given by

$$\frac{dT}{dt} = -k(T - 70).$$

This is a linear equation with solution $T(t) = 70 + ce^{-kt}$. The initial condition $T(0) = 200$ implies $c = 130$. Therefore, $T(t) = 70 + 130e^{-kt}$. Using the fact that $T(1) = 190$, we see that $190 = 70 + 130e^{-k}$ which implies $k = -\ln(12/13) \cong .08$ per minute. To find when the temperature reaches 150 degrees, we just need to solve $T(t) = 70 + 130e^{-0.08t} = 150$. The solution of this equation is $t = -\ln(80/130)/.08 \cong 6.07$ minutes.

17.

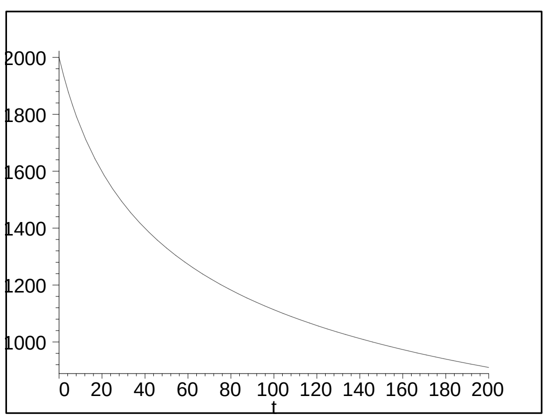
(a) The solution of this separable equation is given by

$$u^3 = \frac{u_0^3}{3\alpha u_0^3 t + 1}.$$

Since $u_0 = 2000$, the specific solution is

$$u(t) = \frac{2000}{(6t/125 + 1)^{1/3}}.$$

(b)

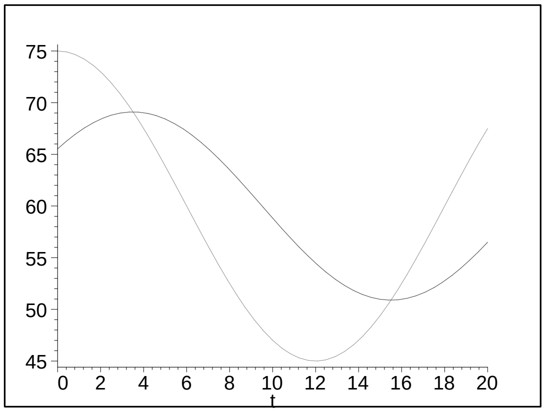


(c) We look for τ so that $u(\tau) = 600$. The solution of this equation is $t \approx 750.77$ seconds.

18.

(a) The integrating factor is $\mu(t) = e^{kt}$. Then $u = e^{-kt} \int ke^{kt}(T_0 + T_1 \cos(\omega t)) = ce^{-kt} + T_0 + kT_1(k \cos \omega t + \omega \sin \omega t)/(k^2 + \omega^2)$. Since $e^{-kt} \to 0$ as $t \to \infty$, we see that the steady state is $S(t) = T_0 + kT_1(k \cos(\omega t) + \omega \sin(\omega t))/(k^2 + \omega^2)$.

(b)



The amplitude R of the oscillatory part of S(t) is approximately 9 degrees Fahrenheit. The time lag τ between maxima is approximately 3.5 seconds.

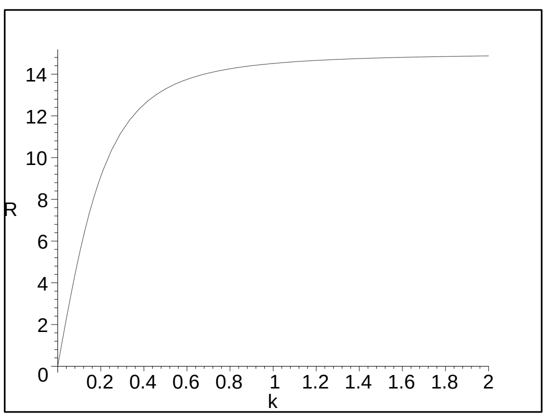
(c) From above, the oscillatory part of $S(t)$ is given by

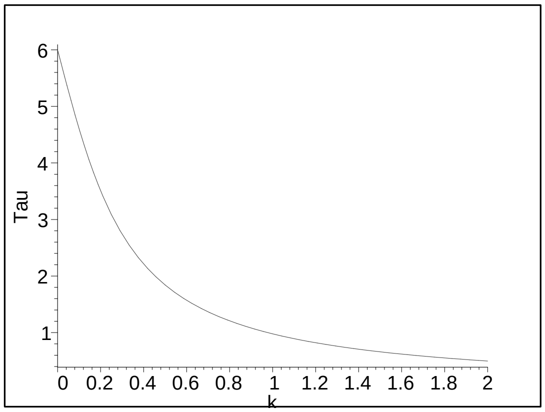
$$kT_1 \frac{k\cos(\omega t) + \omega\sin(\omega t)}{k^2 + \omega^2} = \frac{kT_1}{\sqrt{k^2 + \omega^2}}(\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau))$$

for τ such that $\cos(\omega\tau) = k/\sqrt{k^2 + \omega^2}$ and $\sin(\omega\tau) = \omega/\sqrt{k^2 + \omega^2}$. That is, $\tau = \frac{1}{\omega}\arctan(\omega/k)$. Further, letting $R = kT_1/\sqrt{k^2 + \omega^2}$, we can write the oscillatory part of $S(t)$ as

$$R[\cos(\omega t)\cos(\omega\tau) + \sin(\omega t)\sin(\omega\tau)] = R\cos(\omega(t - \tau)).$$

Below we show graphs of R and τ versus k.





19.

(a) The differential equation for Q is

$$\frac{dQ}{dt} = kr + P - \frac{Q(t)}{V}r.$$

Therefore,

$$V\frac{dc}{dt} = kr + P - c(t)r.$$

The solution of this equation is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. As $t \to \infty$, $c(t) \to k + P/r$.

(b) In this case, we will have $c(t) = c_0 e^{-rt/V}$. The reduction times are $T_{50} = \ln(2)V/r$ and $T_{10} = \ln(10)V/r$.

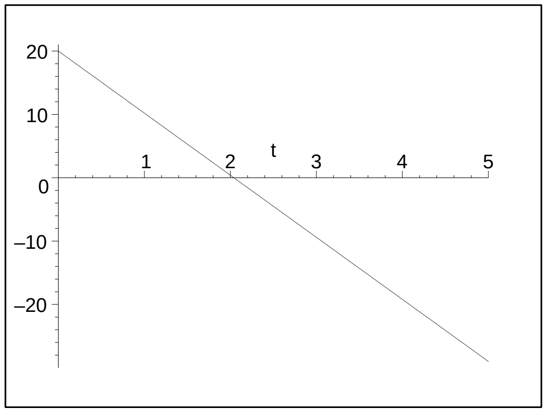
(c) Using the results from part (b), we have: Superior, $T = 431$ years; Michigan, $T = 71.4$ years; Erie, $T = 6.05$ years; Ontario, $T = 17.6$ years.

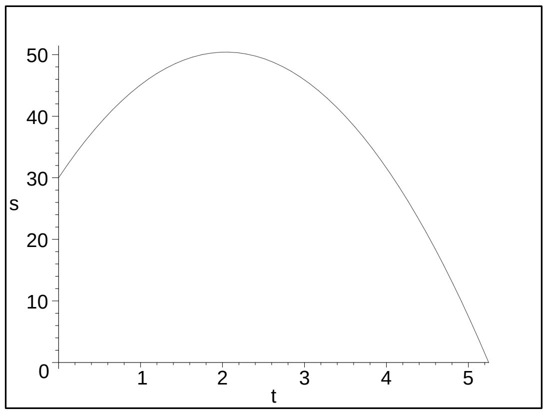
20.

(a) Assuming no air resistance, we have $dv/dt = -9.8$. Therefore, $v(t) = -9.8t + v_0 = -9.8t + 20$ and its position at time t is given by $s(t) = -4.9t^2 + 20t + 30$. When the ball reaches its max height, the velocity will be zero. We see that $v(t) = 0$ implies $t = 20/9.8 \cong 2.04$ seconds. When $t = 2.04$, we see that $s(2.04) \cong 50.4$ meters.

(b) Solving $s(t) = -4.9t^2 + 20t + 30 = 0$, we see that $t = 5.248$ seconds.

(c)



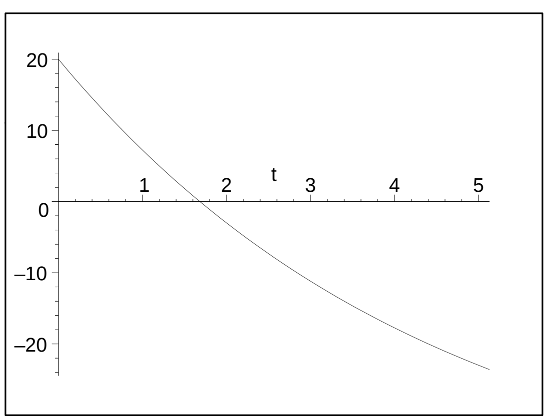


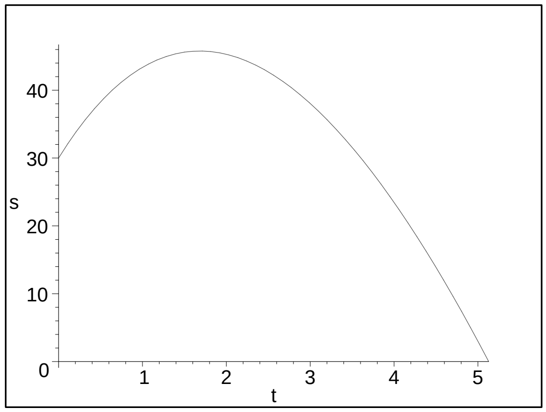
21.

(a) We have $mdv/dt = -v/30 - mg$. Given the conditions from problem 20, we see that the solution is given by $v(t) = -44.1 + 64.1e^{-t/4.5}$. The ball will reach its max height when $v(t) = 0$. This occurs at $t = 1.683$ seconds. The height of the ball is given by $s(t) = -318.45 - 44.1t - 288.45e^{-t/4.5}$. When $t = 1.683$, we have $s(1.683) = 45.78$ meters, the maximum height.

(b) The ball will hit the ground when $s(t) = 0$. This occurs when $t = 5.128$ seconds.

(c)



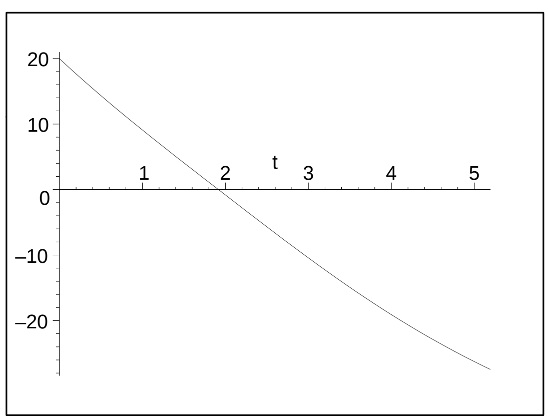


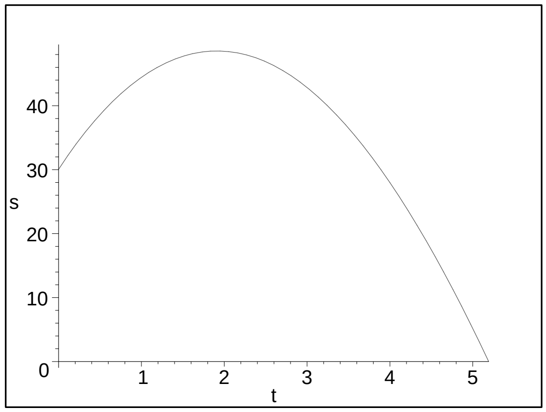
22.

(a) The equation for the upward motion is $mdv/dt = -\mu v^2 - mg$ where $\mu = 1/1325$. Using the data from exercise 20, and the fact that this equation is separable, we see its solution is given by $v(t) = 44.133\tan(.425 - .222t)$. Setting $v(t) = 0$, we see the ball will reach its max height at $t = 1.916$ seconds. Integrating $v(t)$, we see the position at time t is given by $s(t) = 198.75\ln(\cos(0.222t - 0.425)) + 48.57$. Therefore, the max height is given by $s(1.916) = 48.56$ meters.

(b) The differential equation for the downward motion is $mdv/dt = +\mu v^2 - mg$. The solution of this equation is given by $v(T) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we see that the position is given by $s(t) = 99.29\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. Setting $s(t) = 0$, we see that the ball will spend $t = 3.276$ seconds going downward before hitting the ground. Combining this time with the amount of time the ball spends going upward, 1.916 seconds, we conclude that the ball will hit the ground 5.192 seconds after being thrown upward.

(c)





23.

(a) Measure the positive direction of motion downward. Then the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75v + mg & 0 < t < 10 \\ -12v + mg & t > 10. \end{cases}$$

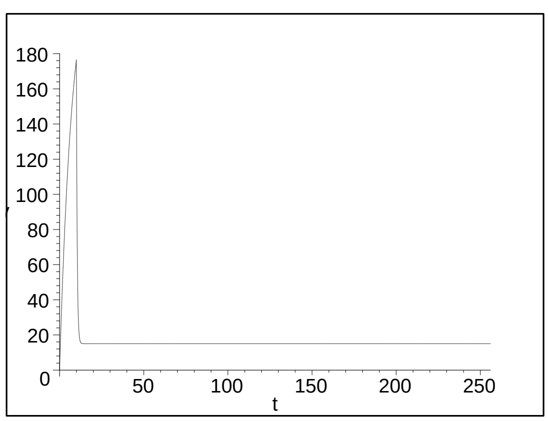
For the first 10 seconds, the equation becomes $dv/dt = -v/7.5 + 32$ which has solution $v(t) = 240(1 - e^{-t/7.5})$. Therefore, $v(10) = 176.7$ feet per second.

(b) Integrating the velocity function from part (a), we see that the height of the skydiver at time t $(0 < t < 10)$ is given by $s(t) = 240t + 1800e^{-t/7.5} - 1800$. Therefore, $s(10) = 1074.5$ feet.

(c) After the parachute opens, the equation for v is given by $dv/dt = -32v/15 + 32$ (as discussed in part (a)). We will reset t to zero. The solution of this differential equation is given by $v(t) = 15 + 161.7e^{-32t/15}$. As $t \to \infty$, $v(t) \to 15$. Therefore, the limiting velocity is $v_l = 15$ feet/second.

(d) Integrating the velocity function from part (c), we see that the height of the sky diver after falling t seconds with his parachute open is given by $s(t) = 15t - 75.8e^{-32t/15} + 1150.3$. To find how long the skydiver is in the air after the parachute opens, we find T such that $s(T) = 0$. Solving this equation, we have $T = 256.6$ seconds.

(e)



24.

(a) The equation of motion is given by $dv/dx = -\mu v$.

(b) The speed of the sled satisfies $\ln(v/v_0) = -\mu x$. Therefore, μ must satisfy $\ln(15/150) = -2000\mu$. Therefore, $\mu = \ln(10)/2000$ f^{-1}.

(c) The solution of $dv/dt = -\mu v^2$ can be expressed as $1/v - 1/v_0 = \mu t$. Using the fact that 1 mi/hour $\approx$ 1.467 feet/second, the elapsed time is $t \approx 35.56$ seconds.

25.

(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Solving $v(t) = 0$, we see that the mass will reach its max height $t_m = (m/k)\ln[(mg + kv_0)/mg]$ seconds after being projected upward. Integrating the velocity equation, we see that the position of the mass at this time will be given by the position equation

$$s(t) = -mgt/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{mv_0}{k}\right](1 - e^{-kt/m}).$$

Therefore, the max height reached is

$$x_m = s(t_m) = \frac{mv_0}{k} - g\left(\frac{m}{k}\right)^2 \ln\left[\frac{mg + kv_0}{mg}\right].$$

,

(b) These formulas for t_m and x_m come from the fact that for $\delta << 1$, $\ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$. This formula is just Taylor's formula.

(c) Consider the result for t_m in part (b). Multiplying the equation by $\frac{g}{v_0}$, we have

$$\frac{t_m g}{v_0} = \left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \ldots\right].$$

The units on the left, must match the units on the right. Since the units for $t_m g/v_0 = (s \cdot m/s^2)/(m/s)$, the units cancel. As a result, we can conclude that kv_0/mg is dimensionless.

26.

(a) The equation of motion is given by $mdv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$.

(b) Applying L'Hospital's rule, as $k \to 0$, we have

$$\lim_{k\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = v_0 - gt.$$

(c)

$$\lim_{m\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = 0.$$

27.

(a) The equation of motion is given by

$$m\frac{dv}{dt} = -6\pi\mu a v + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

We can rewrite this equation as

$$v' + \frac{6\pi\mu a}{m}v = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho).$$

Multiplying by the integrating factor $e^{6\pi\mu a t/m}$, we have

$$(e^{6\pi\mu a t/m}v)' = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho)e^{6\pi\mu a t/m}.$$

Integrating this equation, we have

$$\begin{aligned} v &= e^{-6\pi\mu a t/m}\left[\frac{2a^2 g(\rho' - \rho)}{9\mu}e^{6\pi\mu a t/m} + C\right] \\ &= \frac{2a^2 g(\rho' - \rho)}{9\mu} + Ce^{-6\pi\mu a t/m}. \end{aligned}$$

Therefore, we conclude that the limiting velocity is $v_L = (2a^2 g(\rho' - \rho))/9\mu$.

(b) By the equation above, we see that the force exerted on the droplet of oil is given by

$$Ee = -6\pi\mu a v + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

If $v = 0$, then solving the above equation for e, we have

$$e = \frac{4\pi a^3 g(\rho' - \rho)}{3E}.$$

28.

(a) The equation is given by $m dv/dt = -kv - mg$. The solution of this equation is $v(t) = -(mg/k)(1 - e^{-kt/m})$. Integrating, we see that the position function is given by $x(t) = -(mg/k)t + (m/k)^2 g(1 - e^{-kt/m}) + 30$. First, by setting $x(t) = 0$, we see that the ball will hit the ground $t = 3.63$ seconds after it is dropped. Then $v(3.63) = 11.58$ m/second will be the speed when the mass hits the ground.

(b) In terms of displacement, we have $mvdv/dx = -kv + mg$. This equation comes from applying the chain rule: $dv/dt = dv/dx \cdot dx/dt = vdv/dx$. The solution of this differential equation is given by

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} \ln \left| \frac{mg - kv}{mg} \right|.$$

Plugging in the given values for k, m, g, we have $x(v) = -1.25v - 15.31 \ln |0.0816v - 1|$. If $v = 10$, then $x(10) = 13.45$ meters.

(c) Using the equation for $x(v)$ above, we set $x(v) = 30$, $v = 10$, $m = 0.25$, $g = 9.8$. Then solving for k, we have $k = 0.24$.

29.

(a) The equation of motion is given by $mdv/dt = -GMm/(R + x)^2$. By the chain rule,

$$m\frac{dv}{dx} \cdot \frac{dx}{dt} = -G\frac{Mm}{(R + x)^2}.$$

Therefore,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R + x)^2}.$$

This equation is separable with solution $v^2 = 2GM(R+x)^{-1} + 2gR - 2GM/R$. Here we have used the initial condition $v_0 = \sqrt{2gR}$. From physics, we know that $g = GM/R^2$. Using this substitution, we conclude that $v(x) = \sqrt{2g}[R/\sqrt{R + x}]$.

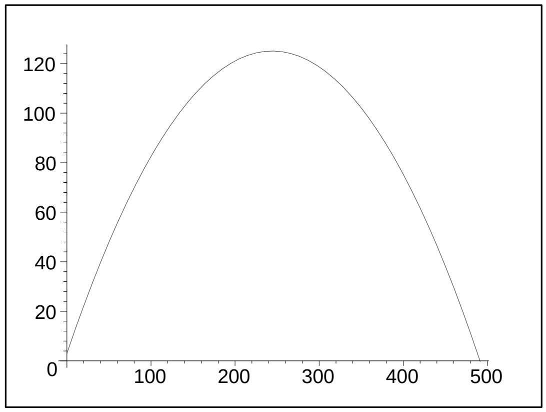
(b) By part (a), we know that $dx/dt = v(x) = \sqrt{2g}[R/\sqrt{R + x}]$. We want to solve this differential equation with the initial condition $x(0) = 0$. This equation is separable with solution $x(t) = [\frac{3}{2}(\sqrt{2g}Rt + \frac{2}{3}R^{3/2}]^{2/3} - R$. We want to find the time T such that $x(T) = 240,000$. Solving this equation, we conclude that $T \approx 50.6$ hours.

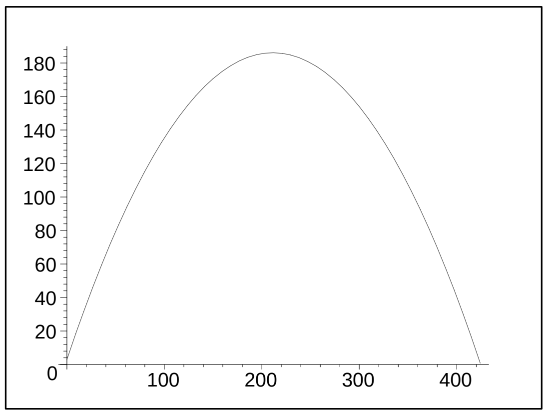
30.

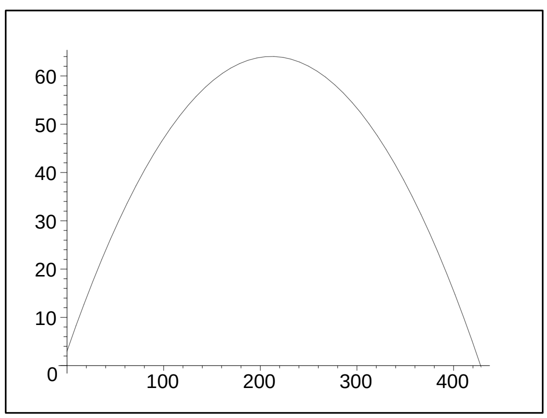
(a) $dv/dt = 0$ implies v is constant, but clearly by the initial condition $v = u \cos A$. $dw/dt = -g$ implies $w = -gt + C$, but also by the initial condition $w = -gt + u \sin A$.

(b) The equation $dx/dt = v = u \cos A$ along with the initial condition implies $x(t) = u \cos At$. The equation $dy/dt = w = -gt + u \sin A$ along with the initial condition implies $y(t) = -gt^2/2 + u \sin At + h$.

(c) Below we have plotted the trajectory of the ball in the cases $\pi/4$, $\pi/3$ and $\pi/6$ respectively.







(d) First, let T be the time it takes for the ball to travel L feet horizontally. Using the equation for x, we know that $x(T) = u\cos AT = L$ implies $T = L/u\cos A$. Then, when the ball reaches this wall, we need the height of the ball to be at least H feet. That is, we need $y(T) \geq H$. Now $y(t) = -16t^2 + u\sin At + 3$ implies we need $y(T) = -16L^2/(u^2\cos^2(A)) + L\tan A + 3 \geq H$.

(e) If $L = 350$ and $H = 10$, then our inequality becomes

$$-\frac{1,960,000}{(u^2\cos^2(A))} + 350\tan A + 3 \geq 10.$$

Now if $u = 110$, then our inequality becomes

$$-\frac{162}{\cos^2(A)} + 350\tan(A) \geq 7.$$

Solving this inequality, we conclude that $0.63 \text{ rad} \leq A \leq 0.96 \text{ rad}$.

(f) We rewrite the inequality in part (c) as

$$\cos^2(A)(350\tan A - 7) \geq \frac{1,960,000}{u^2}.$$

In order to determine the minimum value necessary, we will maximize the function on the left-hand side. Letting $f(A) = \cos^2(A)(350\tan A - 7)$, we see that $f'(A) = 350\cos(2A) + 7\sin(2A)$. Therefore, $f'(A) = 0$ implies $\tan(2A) = -50$. For $0 < A < \pi/2$, we see that this occurs at $A = 0.7954$ radians. Substituting this value for A into the inequality above, we conclude that

$$u^2 \geq 11426.24.$$

Therefore, the minimum velocity necessary is 106.89 mph and the optimal angle necessary is 0.7954 radians.

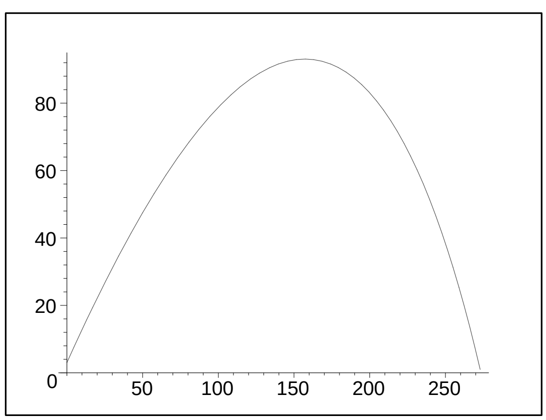
31.

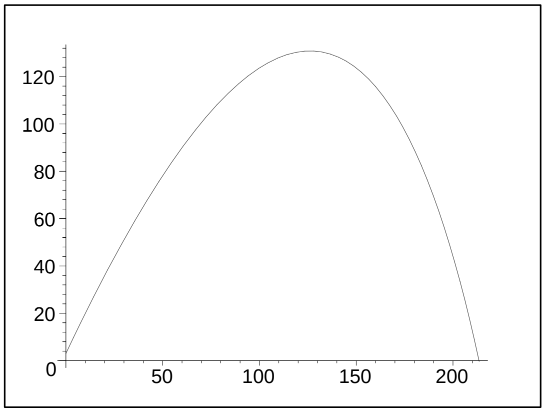
(a) The initial conditions are $v(0) = u\cos(A)$ and $w(0) = u\sin(A)$. Therefore, the solutions of the two equations are $v(t) = u\cos(A)e^{-rt}$ and $w(t) = -g/r + (u\sin(A) + g/r)e^{-rt}$.

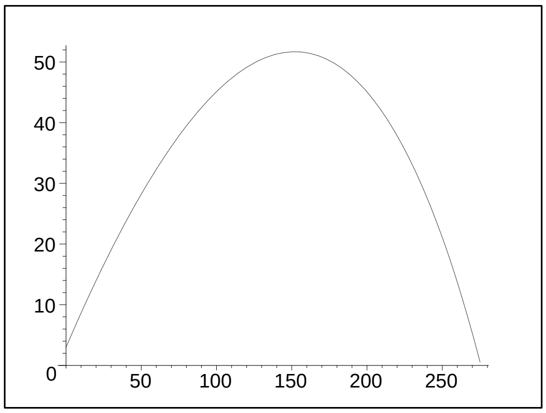
(b) Now $x(t) = \int v(t) = \frac{u}{r}\cos(A)(1 - e^{-rt})$, and

$$y(t) = \int w(t) = -\frac{gt}{r} + \frac{(g + ur\sin(A) + hr^2)}{r^2} - \left(\frac{u}{r}\sin(A) + \frac{g}{r^2}\right)e^{-rt}.$$

(c) Below we show trajectories for the cases $A = \pi/4, \pi/3$ and $\pi/6$, respectively.







(d) Let T be the time it takes the ball to go 350 feet horizontally. Then from above, we see that $e^{-T/5} = (u\cos(A) - 70)/u\cos(A)$. At the same time, the height of the ball is given by $y(T) = -160T + 267 + 125u\sin(A) - (800 + 5u\sin(A))[(u\cos(A) - 70)/u\cos(A)]$. Therefore, u and A must satisfy the inequality

$$800\ln\left[\frac{u\cos(A) - 70}{u\cos(A)}\right] + 267 + 125u\sin(A) - (800 + 5u\sin(A))\left[\frac{u\cos(A) - 70}{u\cos(A)}\right] \geq 10.$$

32.

(a) Solving equation (i), we have $y'(x) = [(k^2 - y)/y]^{1/2}$. The positive answer is chosen since y is an increasing function of x.

(b) $y = k^2\sin^2 t \implies dy/dt = 2k^2\sin t\cos t$. Substituting this into the equation in part (a), we have

$$\frac{2k^2\sin t\cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Therefore, $2k^2\sin^2 t dt = dx$.

(c) Letting $\theta = 2t$, we have $k^2\sin^2(\theta/2)d\theta = dx$. Integrating both sides, we have $x(\theta) = k^2(\theta - \sin\theta)/2$. Further, using the fact that $y = k^2\sin^2 t$, we conclude that $y = k^2\sin^2(\theta/2) = k^2(1 - \cos(\theta))/2$.

(d) From part (c), we see that $y/x = (1-\cos\theta)/(\theta-\sin\theta)$. If $x = 1$ and $y = 2$, the solution of the equation is $\theta \approx 1.401$. Substituting that value of θ into either of the equations in part (c), we conclude that $k \approx 2.193$.

Section 2.4

1. Rewriting the equation as

$$y' + \frac{\ln t}{t-3} y = \frac{2t}{t-3}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 3$.

2. Rewriting the equation as

$$y' + \frac{1}{t(t-4)} y = 0$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 4$.

3. By Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $\pi/2 < t < 3\pi/2$.

4. Rewriting the equation as

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-\infty < t < -2$.

5. Rewriting the equation as

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-2 < t < 2$.

6. Rewriting the equation as

$$y' + \frac{1}{lnt} y = \frac{\cot t}{lnt}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $1 < t < \pi$.

7. Using the fact that

$$f = \frac{t-y}{2t+5y} \implies f_y = \frac{3t-10y}{(2t+5y)^2},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied as long as $2t + 5y \neq 0$.

8. Using the fact that

$$f = (1-t^2-y^2)^{1/2} \implies f_y = -\frac{y}{(1-t^2-y^2)^{1/2}},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied as long as $t^2 + y^2 < 1$.

9. Using the fact that

$$f = \frac{ln|ty|}{1-t^2+y^2} \implies f_y = \frac{1-t^2+y^2-2y^2\ln|ty|}{y(1-t^2+y^2)^2},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied as long as $y, t \neq 0$ and $1-t^2+y^2 \neq 0$.

10. Using the fact that

$$f = (t^2+y^2)^{3/2} \implies f_y = 3y(t^2+y^2)^{1/2},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied for all $t \in \mathbb{R}$.

11. Using the fact that

$$f = \frac{1+t^2}{3y-y^2} \implies f_y = -\frac{(1+t^2)(3-2y)}{(3y-y^2)^2},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied as long as $y \neq 0, 3$.

12. Using the fact that

$$f = \frac{(\cot t)y}{1+y} \implies f_y = \frac{1}{(1+y)^2},$$

we see that the hypothesis of Theorem 2.4.2 are satisfied as long as $y \neq -1, t \neq n\pi$ for $n = 0, 1, 2 \ldots$.

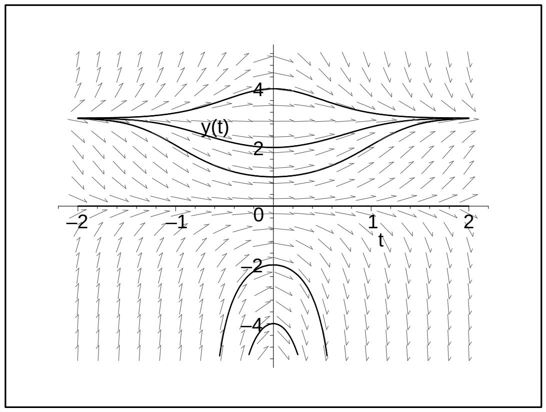
13. The equation is separable, $ydy = -4tdt$. Integrating both sides, we conclude that $y^2/2 = -2t^2 + y_0^2/2$ for $y_0 \neq 0$. The solution is defined for $y_0^2 - 4t^2 \geq 0$.

14. The equation is separable and can be written as $dy/y^2 = 2tdt$. Integrating both sides, we arrive at the solution $y = y_0/(1-y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions exist for all t.

15. The equation is separable and can be written as $dy/y^3 = -dt$. Integrating both sides, we arrive at the solution $y = y_0/(\sqrt{2ty_0^2+1})$. Solutions exist as long as $2y_0^2t + 1 > 0$.

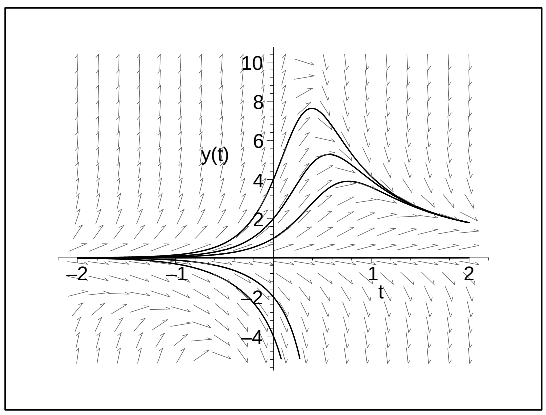
16. The equation is separable and can be written as $ydy = t^2dt/(1+t^3)$. Integrating both sides, we arrive at the solution $y = \pm(\frac{2}{3}\ln|1+t^3| + y_0^2)^{1/2}$. The sign of the solution depends on the sign of the initial data y_0. Solutions exist as long as $\frac{2}{3}\ln|1+t^3| + y_0^2 \geq 0$; that is, as long as $y_0^2 \geq -\frac{2}{3}\ln|1+t^3|$. We can rewrite this inequality as $|1+t^3| \geq e^{-3y_0^2/2}$. In order for the solution to exist, we need $t > -1$ (since the term $t^2/(1+t^3)$ has a singularity at $t = -1$. Therefore, we can conclude that our solution will exist for $[e^{-3y_0^2/2} - 1]^{1/3} < t < \infty$.

17.



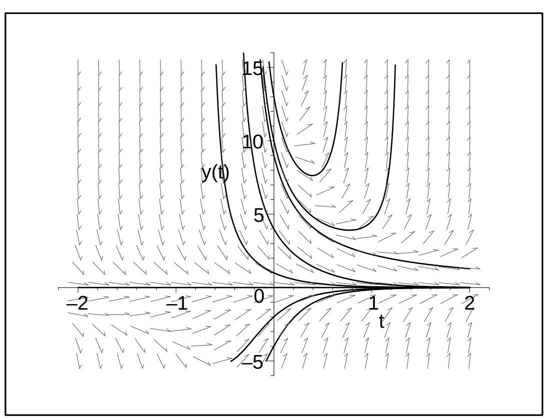
If $y_0 > 0$, then $y \to 3$. If $y_0 = 0$, then $y = 0$. If $y_0 < 0$, then $y \to -\infty$.

18.



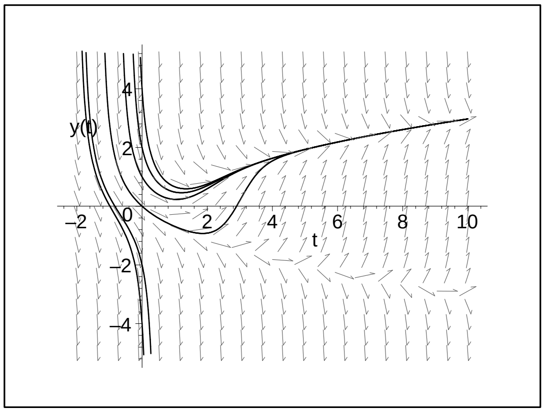
If $y_0 \geq 0$, then $y \to 0$. If $y_0 < 0$, then $y \to -\infty$.

19.



If $y_0 > 9$, then $y \to \infty$. If $y_0 < 9$, then $y \to 0$.

20.



If $y_0 < y_c \approx -0.019$, then $y \to -\infty$. Otherwise, y is asymptotic to $\sqrt{t-1}$.

21.

(a) We know that the family of solutions given by equation (19) are solutions of this initial-value problem. We want to determine if one of these passes through the point $(1,1)$. That is, we want to find $t_0 > 0$ such that if $y = [\frac{2}{3}(t-t_0)]^{3/2}$, then $(t,y) = (1,1)$. That is, we need to find $t_0 > 0$ such that $1 = \frac{2}{3}(1-t_0)$. But, the solution of this equation is $t_0 = -1/2$.

(b) From the analysis in part (a), we find a solution passing through $(2,1)$ by setting $t_0 = 1/2$.

(c) Since we need $y_0 = \pm[\frac{2}{3}(2-t_0)]^{3/2}$, we must have $|y_0| \leq [\frac{4}{3}]^{3/2}$.

22.

(a) First, it is clear that $y_1(2) = -1 = y_2(2)$. Further,

$$y_1' = -1 = \frac{-t + [(t-2)^2]^{1/2}}{2} = \frac{-t + (t^2 + 4(1-t))^{1/2}}{2}$$

and

$$y_2' = -\frac{t}{2} = \frac{-t + (t^2 - t^2)^{1/2}}{2}.$$

The function y_1 is a solution for $t \geq 2$. The function y_2 is a solution for all t.

(b) Theorem 2.4.2 requires that f and $\partial f/\partial y$ be continuous in a rectangle about the point $(t_0, y_0) = (2, -1)$. Since f is not continuous if $t < 2$ and $y < -1$, the hypothesis of Theorem 2.4.2 are not satisfied.

(c) If $y = ct + c^2$, then

$$y' = c = \frac{-t + [(t+2c)^2]^{1/2}}{2} = \frac{-t + (t^2 + 4ct + 4c^2)^{1/2}}{2}.$$

Therefore, y satisfies the equation for $t \geq -2c$.

23.

(a) $\phi(t) = e^{2t} \implies \phi' = 2e^{2t}$. Therefore, $\phi' - 2\phi = 0$. Since $(c\phi)' = c\phi'$, we see that $(c\phi)' - 2c\phi = 0$. Therefore, $c\phi$ is also a solution.

(b) $\phi(t) = 1/t \implies \phi' = -1/t^2$. Therefore, $\phi' + \phi^2 = 0$. If $y = c/t$, then $y' = -c/t^2$. Therefore, $y' + y^2 = -c/t^2 + c^2/t^2 = 0$ if and only if $c^2 - c = 0$; that is, if $c = 0$ or $c = 1$.

24. If $y = \phi$ satisfies $\phi' + p(t)\phi = 0$, then $y = c\phi$ satisfies $y' + p(t)y = c\phi' + cp(t)\phi = c(\phi' + p(t)\phi) = 0$.

25. Let $y = y_1 + y_2$, then

$$y' + p(t)y = y_1' + y_2' + p(t)(y_1 + y_2) = y_1' + p(t)y_1 + y_2' + p(t)y_2 = 0.$$

26.

(a)

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right] = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

Therefore, $y_1 = 1/\mu(t)$ and $y_2 = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds$.

(b) For $y_1 = 1/\mu(t) = e^{-\int p(t)\,dt}$, we have

$$y_1' + p(t)y_1 = -p(t)e^{-\int p(t)\,dt} + p(t)e^{-\int p(t)\,dt} = 0.$$

(c) For

$$y_2 = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds = e^{-\int p(t)\,dt}\int_{t_0}^{t} e^{\int p(s)\,ds}g(s)\,ds,$$

we have

$$\begin{aligned} y_2' + p(t)y_2 &= -p(t)e^{-\int p(t)\,dt}\int_{t_0}^{t} e^{\int p(s)\,ds}g(s)\,ds + e^{-\int p(t)\,dt}e^{\int p(t)\,dt}g(t) \\ &+ p(t)e^{-\int p(t)\,dt}\int_{t_0}^{t} e^{\int p(s)\,ds}g(s)\,ds = g(t). \end{aligned}$$

27.

(a) If $n = 0$, then $y(t) = ce^{-\int p(t)\,dt}$. If $n = 1$, then $y(t) = ce^{-\int (p(t)-q(t))\,dt}$.

(b) For $n \neq 0, 1$, let $v = y^{1-n}$. Then

$$\begin{aligned} v' &= (1-n)y^{-n}y' = (1-n)y^{-n}[-p(t)y + q(t)y^n] \\ &= (1-n)[-p(t)y^{1-n} + q(t)] = (1-n)[-p(t)v + q(t)]. \end{aligned}$$

That is, $v' + (1-n)p(t)v = (1-n)q(t)$.

28. First, rewrite as

$$y' + \frac{2}{t}y = \frac{1}{t^2}y^3.$$

Here, $n = 3$. Therefore, let $v = y^{1-3} = y^{-2}$. Making this substitution, we see that v satisfies the equation

$$v' - \frac{4}{t}v = -\frac{2}{t^2}.$$

This equation is linear with integrating factor t^{-4}. Therefore, we have

$$\left(\frac{1}{t^4}v' - \frac{4}{t^5}v\right) = -\frac{2}{t^6},$$

which can be written as $(t^{-4}v)' = -2/t^6$. The solution of this equation is given by $v = (2 + ct^5)/5t$. Then, using the fact that $y^2 = 1/v$, we conclude that $y = \pm\sqrt{5t/(2 + ct^5)}$.

29. First, rewrite as

$$y' - ry = -ky^2.$$

Here, $n = 2$. Therefore, let $v = y^{1-2} = y^{-1}$. Making this substitution, we see that v satisfies the equation

$$v' + rv = k.$$

This equation is linear with integrating factor e^{rt}. Therefore, we have

$$\left(e^{rt}v' + re^{rt}v\right) = ke^{rt},$$

which can be written as $(e^{rt}v)' = ke^{rt}$. The solution of this equation is given by $v = (k + cre^{-rt})/r$. Then, using the fact that $y = 1/v$, we conclude that $y = r/(k + cre^{-rt})$.

30. Here $n = 3$. Therefore, v satisfies

$$v' + 2\epsilon v = 2\sigma.$$

This equation is linear with integrating factor $e^{2\epsilon t}$. Its solution is given by $v = (\sigma + c\epsilon e^{-2\epsilon t})/\epsilon$. Then, using the fact that $y^2 = 1/v$, we see that $y = \pm\sqrt{\epsilon}/\sqrt{\sigma + c\epsilon e^{-2\epsilon t}}$.

31. Here $n = 3$. Therefore, v satisfies

$$v' + 2(\Gamma \cos t + T)v = 2.$$

This equation is linear with integrating factor $e^{2(\Gamma \sin t + Tt)}$. Therefore,

$$\left(e^{2(\Gamma \sin t + Tt)}v\right)' = 2e^{2(\Gamma \sin t + Tt)}$$

which implies

$$v = 2e^{-2(\Gamma \sin t + Tt)} \int e^{2(\Gamma \sin t + Tt)}\, dt + ce^{-2(\Gamma \sin t + Tt)}.$$

Then $v = y^{-2}$ implies $y = \pm\sqrt{1/v}$.

32. The solution of the initial value problem $y' + 2y = 1$ is $y = 1/2 + ce^{-2t}$. For $y(0) = 0$, we see that $c = -1/2$. Therefore, $y(t) = \frac{1}{2}(1 - e^{-2t})$ for $0 \leq t \leq 1$. Then $y(1) = \frac{1}{2}(1 - e^{-2})$.

Next, the solution of $y'+2y=0$ is given by $y=ce^{-2t}$. The initial condition $y(1)=\frac{1}{2}(1-e^{-2})$ implies $ce^{-2}=\frac{1}{2}(1-e^{-2})$. Therefore, $c=\frac{1}{2}(e^2-1)$ and we conclude that $y(t)=\frac{1}{2}(e^2-1)e^{-2t}$ for $t>1$.

33. The solution of $y'+2y=0$ with $y(0)=1$ is given by $y(t)=e^{-2t}$ for $0\le t\le 1$. Then $y(1)=e^{-2}$. Then, for $t>1$, the solution of the equation $y'+y=0$ is $y=ce^{-t}$. Since we want $y(1)=e^{-2}$, we need $ce^{-1}=e^{-2}$. Therefore, $c=e^{-1}$. Therefore, $y(t)=e^{-1}e^{-t}=e^{-1-t}$ for $t>1$.

34.

(a) Multiplying the equation by $e^{\int_{t_0}^t p(s)\,ds}$, we have

$$\left(e^{\int_{t_0}^t p(s)\,ds}y\right)'=e^{\int_{t_0}^t p(s)\,ds}g(t).$$

Integrating we have

$$e^{\int_{t_0}^t p(s)\,ds}y(t)=y_0+\int_{t_0}^t e^{\int_{t_0}^s p(r)\,dr}g(s)\,ds,$$

which implies

$$y(t)=y_0e^{-\int_{t_0}^t p(s)\,ds}+\int_{t_0}^t e^{-\int_s^t p(r)\,dr}g(s)\,ds.$$

(b) Assume $p(t)\ge p_0>0$ for all $t\ge t_0$ and $|g(t)|\le M$ for all $t\ge t_0$. Therefore,

$$\int_{t_0}^t p(s)\,ds\ge\int_{t_0}^t p_0\,ds=p_0(t-t_0)$$

which implies

$$e^{-\int_{t_0}^t p(s)\,ds}\le e^{-\int_{t_0}^t p_0\,ds}=e^{-p_0(t-t_0)}\le 1 \text{ for } t\ge t_0.$$

Also,

$$\begin{aligned}\int_{t_0}^t e^{-\int_s^t p(r)\,dr}g(s)\,ds &\le \int_{t_0}^t e^{-\int_s^t p(r)\,dr}|g(s)|\,ds\\ &\le \int_{t_0}^t e^{-p_0(t-s)}M\,ds\\ &\le M\left.\frac{e^{-p_0(t-s)}}{p_0}\right|_{t_0}^t\\ &= M\left[\frac{1}{p_0}-\frac{e^{-p_0(t-t_0)}}{p_0}\right]\\ &\le \frac{M}{p_0}\end{aligned}$$

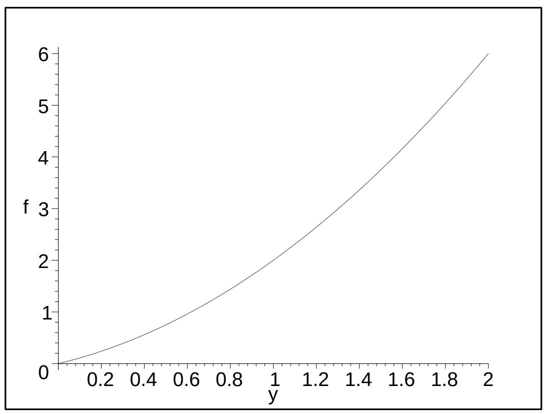
(c) Let $p(t) = 2t + 1 \geq 1$ for all $t \geq 0$ and let $g(t) = e^{-t^2}$. Therefore, $|g(t)| \leq 1$ for all $t \geq 0$. By the answer to part (a),

$$\begin{aligned}
y(t) &= e^{-\int_0^t (2s+1)\,ds} + \int_0^t e^{-\int_s^t (2r+1)\,dr} e^{-s^2}\,ds \\
&= e^{-(t^2+t)} + e^{-t^2-t} \int_0^t e^s\,ds \\
&= e^{-t^2}.
\end{aligned}$$

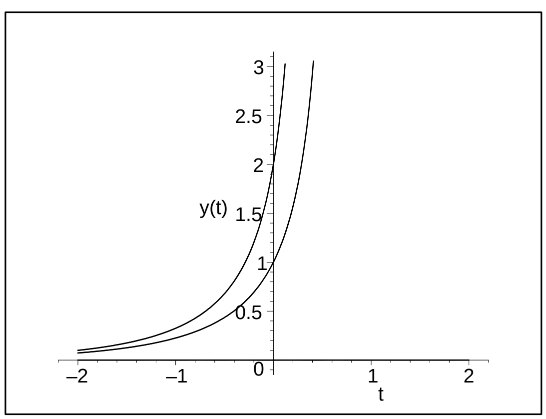
We see that y satisfies the property that y is bounded for all time $t \geq 0$.

Section 2.5

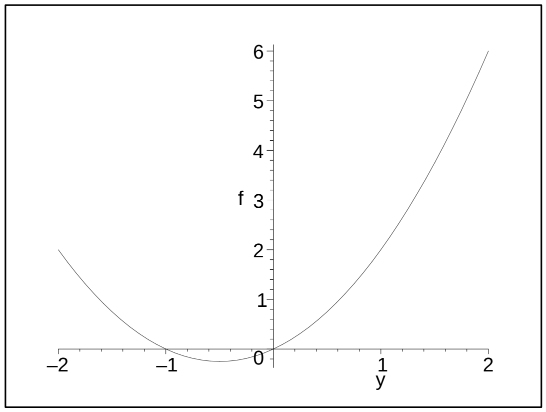
1.



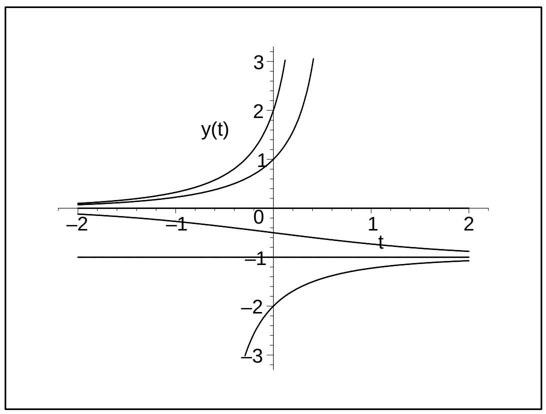
The only equilibrium point is $y^* = 0$. Since $f'(0) = a > 0$, the equilibrium point is unstable.



2.



The equilibrium points are $y^* = 0, -a/b$. $y^* = 0$ is unstable and $y^* = -a/b$ is asymptotically stable since $f'(-a/b) < 0$.



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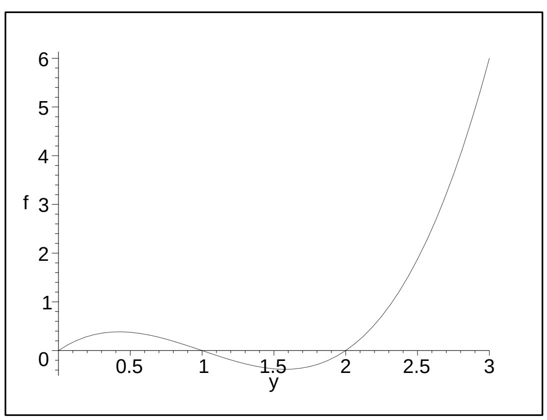
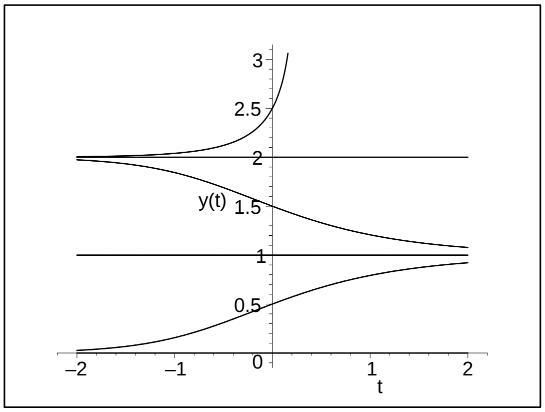


The equilibrium points are $y^* = 0, 1, 2$. Since $f'(0), f'(2) > 0$, those equilibrium point are unstable. Since $f'(1) < 0$, $y^* = 1$ is asymptotically stable.



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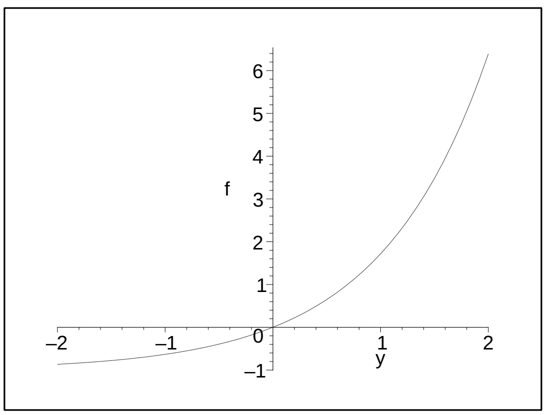
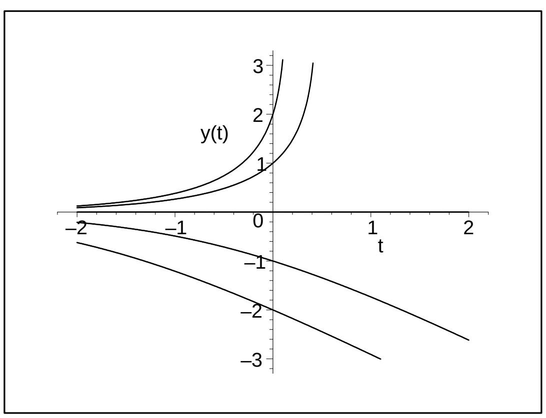


The only equilibrium point is $y^* = 0$. Since $f'(0) > 0$, the equilibrium point is unstable.



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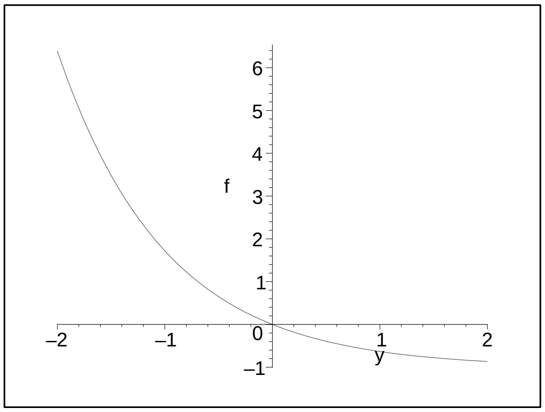
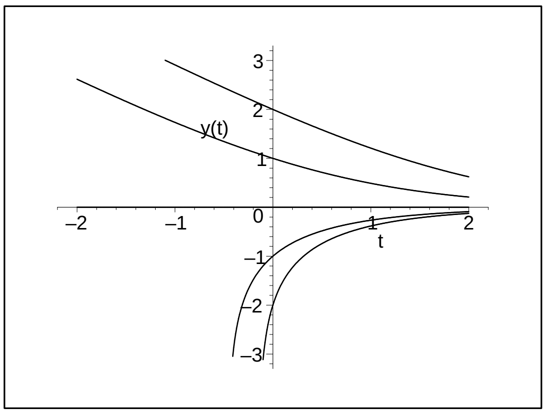


The only equilibrium point is $y^* = 0$. Since $f'(0) < 0$, the equilibrium point is asymptotically stable.



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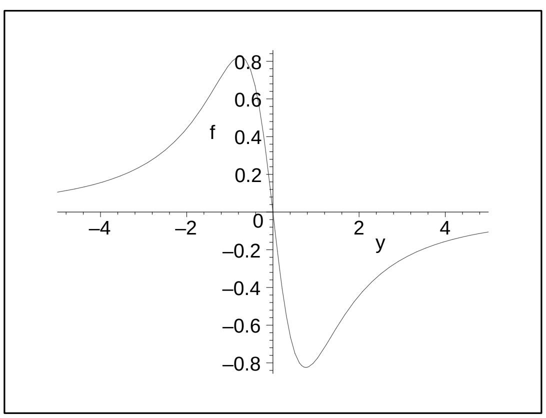
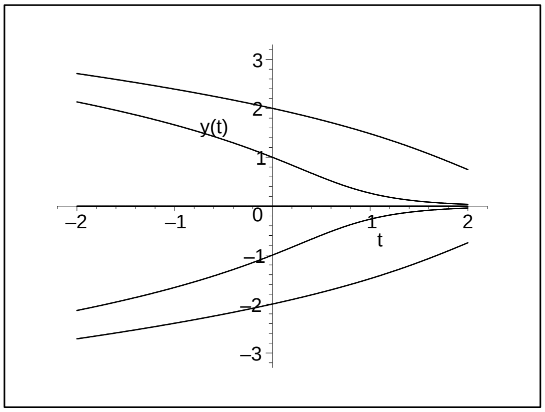


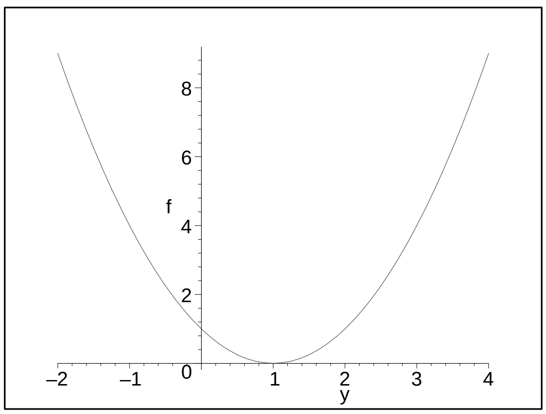
The only equilibrium point is $y^* = 0$. Since $f'(0) < 0$, the equilibrium point is asymptotically stable.



7.

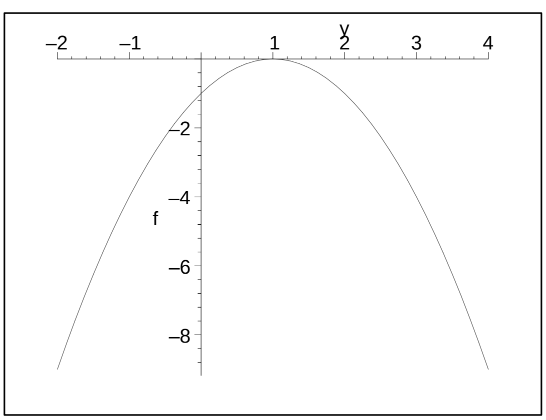
(a) The function $f(y) = k(1-y)^2 = 0 \implies y = 1$. Therefore, $y^* = 1$ is the only critical point.

(b)

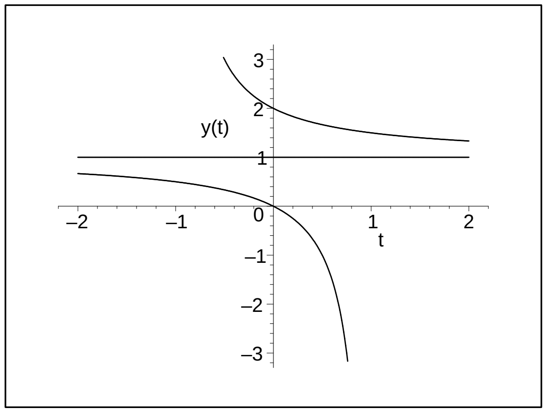


(c) This is a separable equation with solution $y(t) = [y_0 + (1-y_0)kt]/[1+(1-y_0)kt]$. If $y_0 < 1$, then $y \to 1$ as $t \to \infty$. If $y_0 > 1$, then the denominator will go to zero at some finite time $T = 1/(y_0 - 1)$. Therefore, the solution will go towards at infinity.

8.



The only equilibrium point is $y^* = 0$. Since $f'(0) < 0$. The equilibrium point is semistable.



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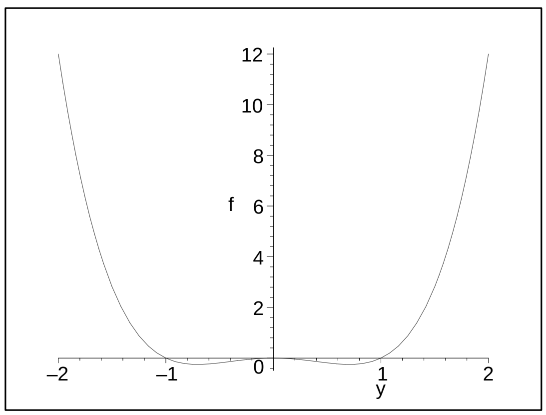
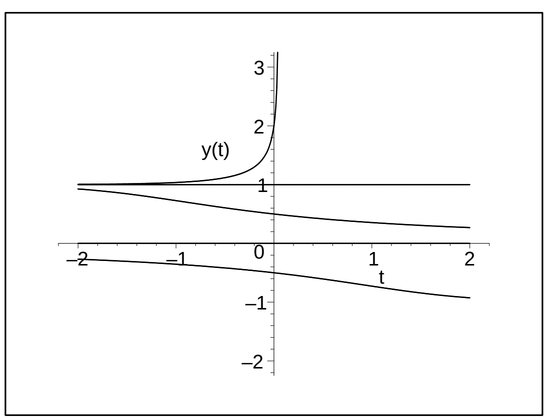


The equilibrium points are $y^* = 0, 1, -1$. Since $f'(-1) < 0$, $y = -1$ is asymptotically stable. Since $f'(1) > 0$, $y = 1$ is unstable. The equilibrium point $y = 0$ is semistable.



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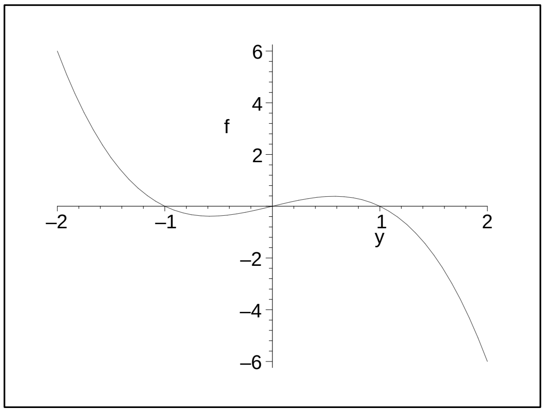
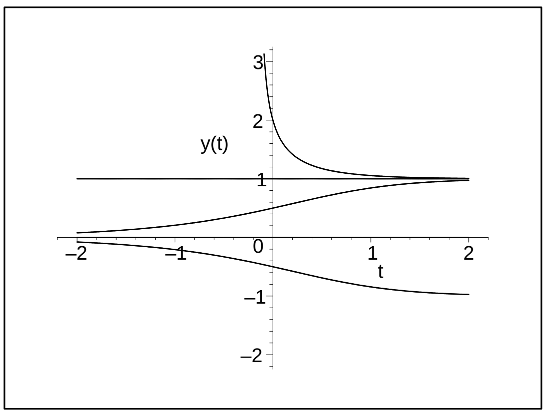


The equilibrium points are $y^* = 0, 1, -1$. Since $f'(-1), f'(1) < 0$, the equilibrium points $y = 1, -1$ are asymptotically stable. Since $f'(0) > 0$, the equilibrium point $y = 0$ is unstable.



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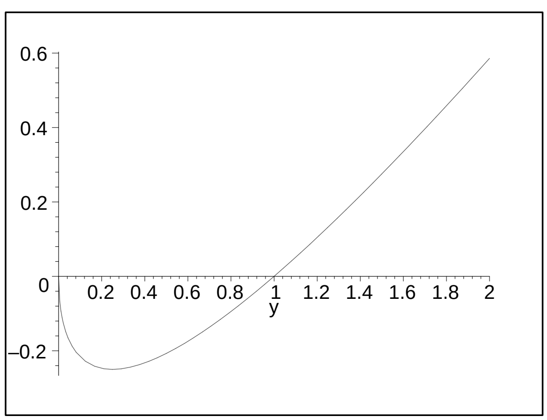
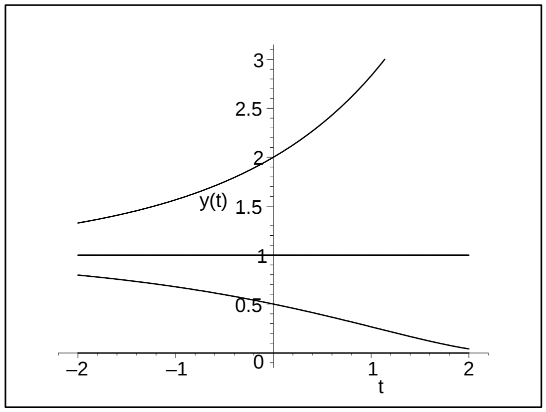


The equilibrium points are $y^* = 0, b^2/a^2$. Since $f'(0) < 0$, the equilibrium point $y = 0$ is asymptotically stable. Since $f'(b^2/a^2) > 0$, the equilibrium point $y = b^2/a^2$ is unstable.



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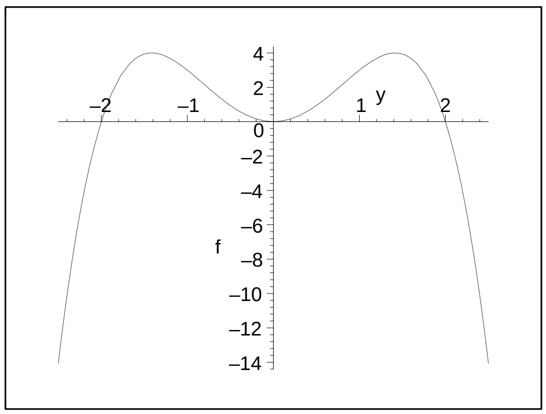
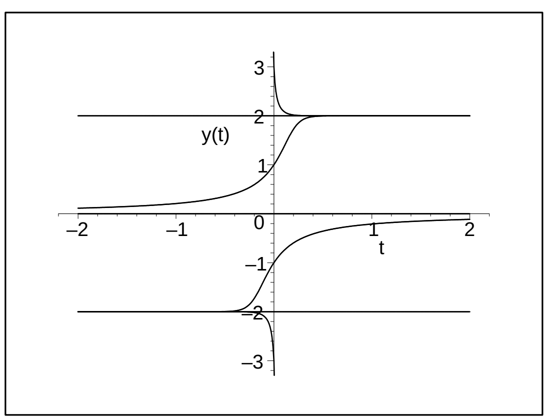


The equilibrium points are $y^* = 0, 2, -2$. The equilibrium point $y = 0$ is semistable. Since $f'(-2) > 0$, the equilibrium point $y = -2$ is unstable. Since $f'(2) < 0$, the equilibrium point $y = 2$ is asymptotically stable.



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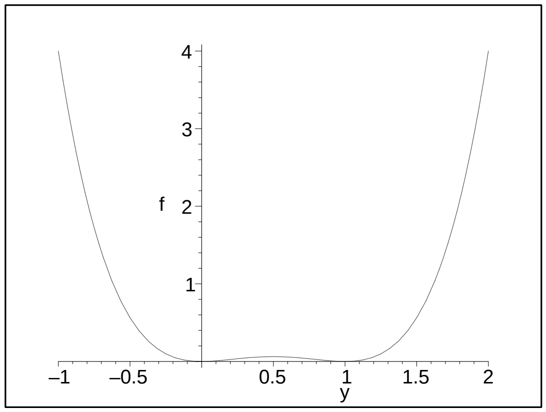
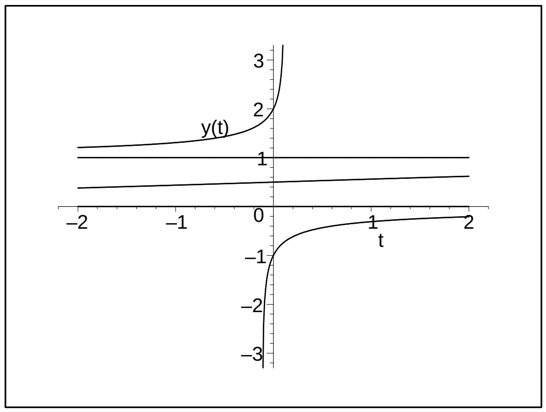


The equilibrium points are $y^* = 0, 1$. They are both semistable.



14.

(a) The equation is separable. Using partial fractions, it can be written as

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right) dy = r dt.$$

Integrating both sides and using the initial condition $y_0 = K/3$, we know the solution y satisfies

$$\ln\left|\frac{y}{1 - y/K}\right| = rt + \ln\left|\frac{K}{2}\right|.$$

To find the time τ such that $y = 2y_0 = 2K/3$, we substitute $y = 2K/3$ and $t = \tau$ into the equation above. Using the properties of logarithm functions, we conclude that $\tau = (\ln 4)/r$. If $r = 0.025$, then $\tau \approx 55.452$ years.

(b) Using the analysis from part (a), we know the general solution satisfies

$$\ln\left|\frac{y}{1 - y/K}\right| = rt + c.$$

The initial condition $y_0 = \alpha K$ implies $c = \ln|\alpha K/(1 - \alpha)|$. Therefore,

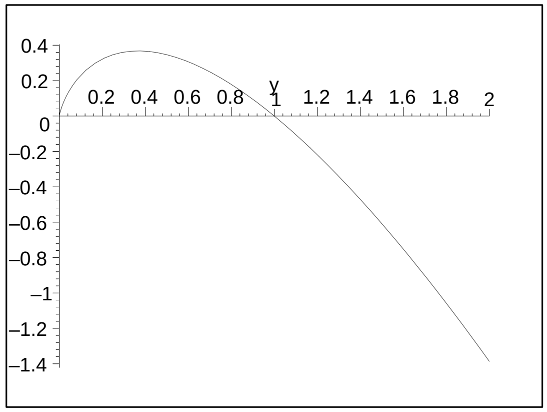
$$\ln\left|\frac{y}{1 - y/K}\right| = rt + \ln\left|\frac{\alpha K}{1 - \alpha}\right|.$$

In order to find the time T at which $y(T) = \beta K$, we use the equation above. We conclude that

$$T = (1/r)\ln|\beta(1-\alpha)/\alpha(1-\beta)|.$$

15.

(a) Below we sketch the graph of f for $r = 1 = K$.



The critical points occur at $y^* = 0, K$. Since $f'(0) > 0$, $y^* = 0$ is unstable. Since $f'(K) < 0$, $y^* = K$ is asymptotically stable.

(b) We calculate y''. Using the chain rule, we see that

$$y'' = ry'\left[\ln\left(\frac{K}{y}\right) - 1\right].$$

We see that $y'' = 0$ when $y' = 0$ (meaning $y = 0, K$) or when $\ln(K/y) - 1 = 0$, meaning $y = K/e$. Looking at the sign of y'' in the intervals $(0, K/e)$ and $(K/e, K)$, we conclude that y is concave up in the interval $(0, K/e)$ and concave down in the interval $(K/e, K)$.

16.

(a) Using the substitution $u = \ln(y/K)$ and differentiating both sides with respect to t, we conclude that $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$. The solution of this equation is $u = u_0 e^{-rt}$. Therefore,

$$\frac{y}{K} = \exp[\ln(y_0/K)e^{-rt}].$$

(b) For $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$, we conclude that $y(2) = 57.58 \times 10^6$.

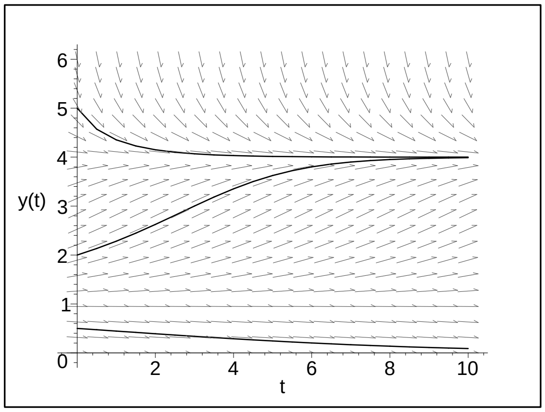
(c) Solving the equation in part (a) for t, we see that

$$t = -\frac{1}{r}\ln\left[\frac{\ln(y/K)}{\ln(y_0/K)}\right].$$

Plugging in the given values, we conclude that $\tau = 2.21$ years.

17.

(a)



(b) Consider $f(y) = -0.25(1-y)[1-(y/4)]y$. We need to differentiate $f(y)$ with respect to y. We see that

$$f'(y) = -0.25\left(\frac{3}{4}y^2 - \frac{5}{2}y + 1\right).$$

Therefore, $f'(y) = 0$ implies $3y^2 - 10y + 4 = 0$ or $y = \dfrac{5 \pm \sqrt{13}}{3}$.

(c) Since this is a separable equation, we can integrate the equation as follows:

$$\int \frac{dy}{(1-y)(1-(y/4))y} = \int -0.25\, dt.$$

Using partial fractions, we can rewrite the left-hand side as

$$\frac{1}{(1-y)(1-(y/4))y} = \frac{4/3}{1-y} + \frac{-1/12}{1-(y/4)} + \frac{1}{y}.$$

Therefore,

$$\int \frac{dy}{(1-y)(1-(y/4))y} = -\frac{4}{3}\ln|1-y| + \frac{1}{3}\ln|1-(y/4)| + \ln|y|.$$

If $y(0) = 2$, then $y(t) \to 4$ as $t \to \infty$ and moreover, $1 < y(t) < 4$ for all t. Therefore, for $1 < y_0 < 4$,

$$\begin{aligned}-\frac{4}{3}\ln|1-y| + \frac{1}{3}\ln|1-(y/4)| + \ln|y| &= -\frac{4}{3}\ln(y-1) + \frac{1}{3}\ln(1-(y/4)) + \ln(y) \\ &= \ln\left(\frac{y(1-(y/4))^{1/3}}{(y-1)^{4/3}}\right).\end{aligned}$$

We conclude that

$$\frac{y(1-(y/4))^{1/3}}{(y-1)^{4/3}} = Ce^{-0.25t}.$$

If $y(0) = 2$, then $C = 2^{2/3}$. Then if $t = 5$, we conclude that $y \approx 3.625$. Similarly, for $y_0 > 4$, we conclude that

$$\frac{y((y/4)-1)^{1/3}}{(y-1)^{4/3}} = Ce^{-0.25t}.$$

(d) Consider the equation

$$\frac{y(1-(y/4))^{1/3}}{(y-1)^{4/3}} = Ce^{-0.25t}$$

found in part (c). If $y_0 = 2$, then $C = 2^{2/3}$. Letting $y(t) = 3.95$ and solving for t, we see that $t \approx 7.97$. Similarly, using the equation found in part (c) for $y_0 > 4$, we see that if $y_0 = 6$, then $y \leq 4.05$ for $t < 7.97$. For all initial data $2 < y_0 < 6$, the conclusion also holds.

18.

(a) The surface area of the cone is given by

$$\begin{aligned} S &= \pi a\sqrt{h^2+a^2} + \pi a^2 = \pi a^2(\sqrt{(h/a)^2+1}+1) \\ &= \frac{\pi a^2 h}{3}\cdot\frac{3}{h}\left(\sqrt{(h/a)^2+1}+1\right) \\ &= c\pi\left(\frac{\pi a^2 h}{3}\right)^{2/3}\cdot\left(\frac{3a}{\pi h}\right)^{2/3} \\ &= c\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}. \end{aligned}$$

Therefore, if the rate of evaporation is proportional to the surface area, then rate out $= \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$. Therefore,

$$\begin{aligned} \frac{dV}{dt} &= \text{rate in} - \text{rate out} \\ &= k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3}\left(\frac{\pi}{3}a^2 h\right)^{2/3} \\ &= k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}. \end{aligned}$$

(b) The equilibrium volume can be found by setting $dV/dt = 0$. We see that the equilibrium volume is

$$V = \left(\frac{k}{\alpha\pi}\right)^{3/2}\left(\frac{\pi h}{3a}\right).$$

To find the equilibrium height, we use the fact that the height and radius of the conical pond maintain a constant ratio. Therefore, if h_e, a_e represent the equilibrium values for the h and a, we must have $h_e/a_e = h/a$. Further, we notice that the equilibrium volume can be written as

$$V = \frac{\pi}{3}\left(\frac{k}{\alpha\pi}\right)\left(\frac{k}{\alpha\pi}\right)^{1/2}\left(\frac{h}{a}\right) = \frac{\pi}{3}a_e^2 h_e^2,$$

where $h_e = (k/\alpha\pi)^{1/2}(h/a)$ and $a_e = (k\alpha\pi)^{1/2}$. Since $f'(V) = -\frac{2}{3}\alpha\pi(3a/\pi h)^{2/3}V^{-1/3} < 0$, the equilibrium is asymptotically stable.

(c) In order to guarantee that the pond does not overflow, we need the rate of water in to be less than or equal to the rate of water out. Therefore, we need $k - \alpha\pi a^2 \leq 0$.

19.

(a) The rate of increase of the volume is given by

$$\frac{dV}{dt} = k - \alpha a\sqrt{2gh}.$$

Since the cross-section is constant, $dV/dt = Adh/dt$. Therefore,

$$\frac{dh}{dt} = (k - \alpha a\sqrt{2gh})/A.$$

(b) Setting $dh/dt = 0$, we conclude that the equilibrium height of water is $h_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2$. Since $f'(h_e) < 0$, the equilibrium height is stable.

20.

(a) The equilibrium points are $y^* = 0, 1$. since $f'(0) = \alpha > 0$, the equilibrium solution $y^* = 0$ is unstable. Since $f'(1) = -\alpha < 0$, the equilibrium solution $y^* = 1$ is asymptotically stable.

(b) The equation is separable. The solution is given by

$$y(t) = \frac{y_0 e^{\alpha t}}{1 - y_0 + y_0 e^{\alpha t}}.$$

We see that $\lim_{t\to\infty} y(t) = 1$.

21.

(a) The solution of the separable equation is $y(t) = y_0 e^{-\beta t}$.

(b) Using the result from part (a), we see that $dx/dt = -\alpha x y_0 e^{-\beta t}$. This equation is separable with solution $x(t) = x_0 exp[-\alpha y_0(1 - e^{-\beta t})/\beta]$.

(c) As $t \to \infty$, $y \to 0$ and $x \to x_0 \exp(-\alpha y_0/\beta)$.

22.

(a) Letting $' = d/dt$, we have

$$\begin{aligned} z' &= \frac{nx' - xn'}{n^2} \\ &= \frac{-\beta nx - \mu nx + \nu\beta x^2 + \mu nx}{n^2} \\ &= -\beta\frac{x}{n} + \nu\beta\left(\frac{x}{n}\right)^2 \\ &= -\beta z + \nu\beta z^2 = -\beta z(1 - \nu z). \end{aligned}$$

(b) First, we rewrite the equation as

$$z' + \beta z = \beta \nu z^2.$$

This is a Bernoulli equation with $n = 2$. Let $w = z^{1-n} = z^{-1}$. Then, our equation can be written as

$$w' - \beta w = -\beta \nu.$$

This is a linear equation with solution $w = \nu + ce^{\beta t}$. Then, using the fact that $z = 1/w$, we see that $z = 1/(\nu + ce^{\beta t})$. Finally, the initial condition $z(0) = 1$ implies $c = 1 - \nu$. Therefore, $z(t) = 1/(\nu + (1-\nu)e^{\beta t})$.

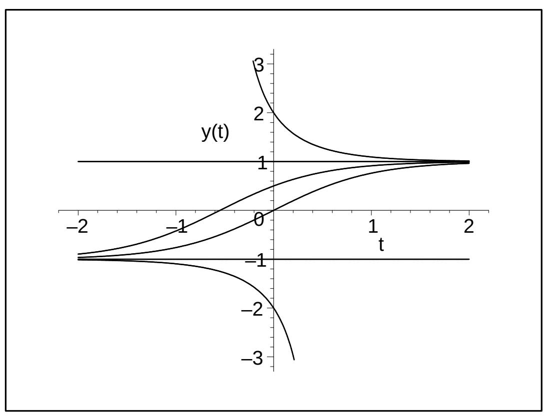
(c) Evaluating $z(20)$ for $\nu = \beta = 1/8$, we conclude that $z(20) = 0.0927$.

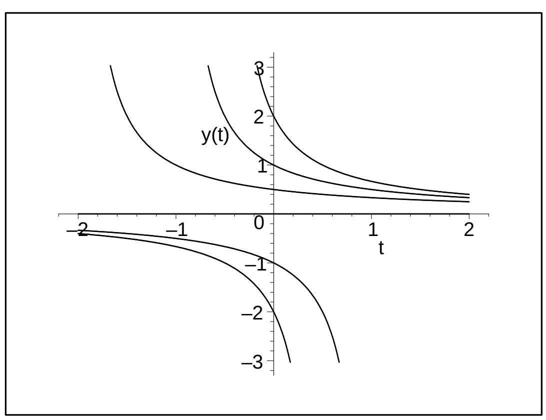
23.

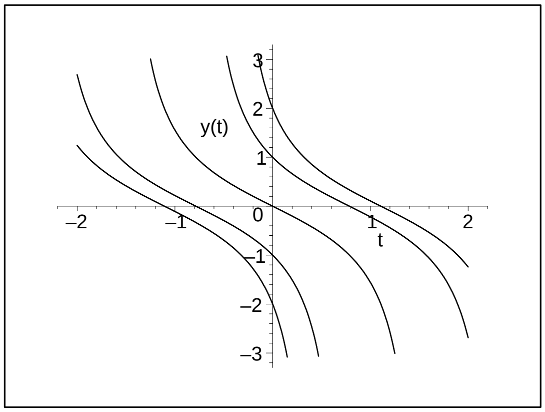
(a) The critical points occur when $a - y^2 = 0$. If $a < 0$, there are no critical points. If $a = 0$, then $y^* = 0$ is the only critical point. If $a > 0$, then $y^* = \pm\sqrt{a}$ are the two critical points.

(b) We note that $f'(y) = -2y$. Therefore, $f'(\sqrt{a}) < 0$ which implies that $\sqrt{a}$ is asymptotically stable; $f'(-\sqrt{a}) > 0$ which implies $-\sqrt{a}$ is unstable; the behavior of f' around $y^* = 0$ implies that $y^* = 0$ is semistable.

(c) Below, we graph solutions in the case $a = 1$, $a = 0$ and $a = -1$ respectively.



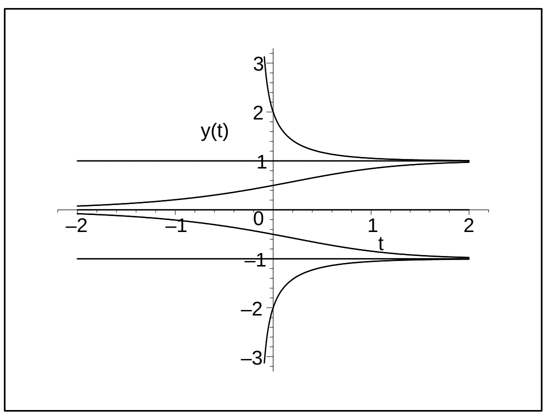


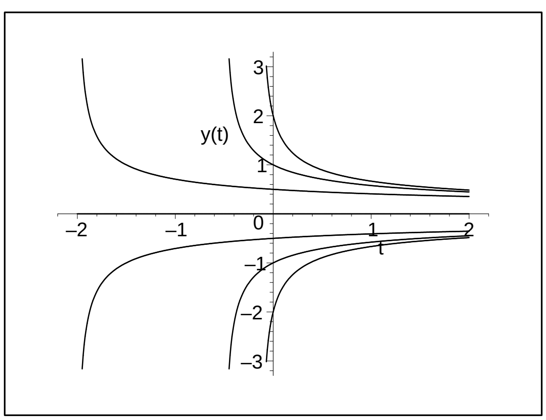


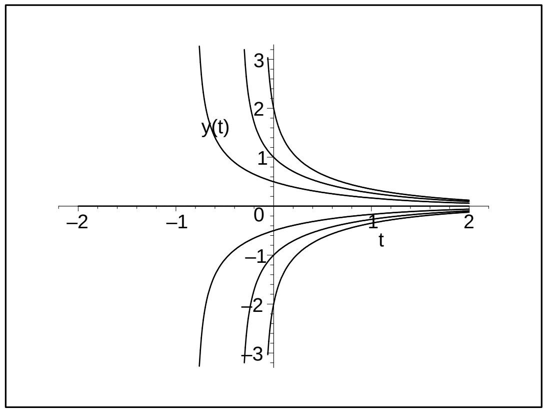
24.

(a) First, for $a < 0$, the only critical point is $y^* = 0$. Second, for $a = 0$, the only critical point is $y^* = 0$. Third, for $a > 0$, the critical points are at $y^* = 0, \pm\sqrt{a}$. Here, $f'(y) = a - 3y^2$. If $a < 0$,then $f'(y) < 0$ for all y, and, therefore, $y^* = 0$ will be asymptotically stable. If $a = 0$, then $f'(0) = 0$. From the behavior on either side of $y^* = 0$, we see that $y^* = 0$ will be asymptotically stable. If $a > 0$, then $f'(0) = a > 0$ which implies that $y^* = 0$ is unstable for $a > 0$. Further, $f'(\pm\sqrt{a}) = -2a < 0$. Therefore, $y^* = \pm\sqrt{a}$ are asymptotically stable for $a > 0$.

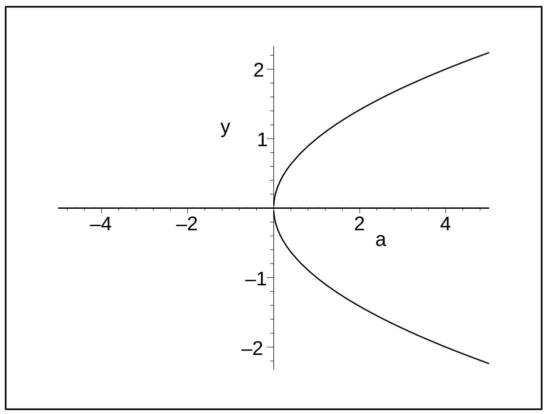
(b) Below we sketch solution curves for $a = 1, 0, -1$, respectively.







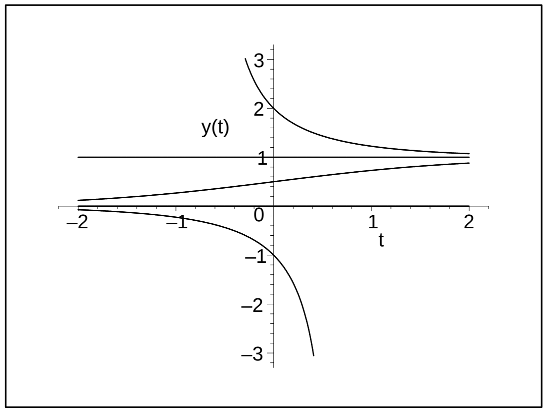
(c)

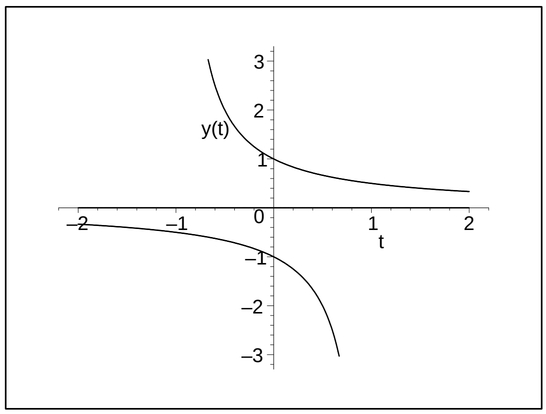


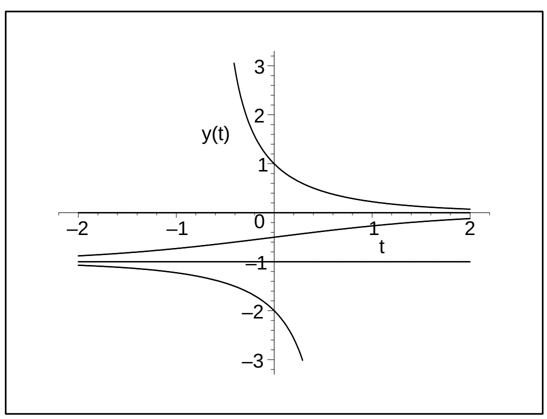
25.

(a) For $a < 0$, the critical points are $y^* = 0, a$. Since $f'(y) = a - 2y$, $f'(0) = a < 0$ and $f'(a) = -a > 0$. Therefore, $y^* = 0$ is asymptotically stable and $y^* = a$ is unstable for $a < 0$. For $a = 0$, the only critical point is $y^* = 0$. which is semistable since $f(y) = -y^2$ is concave down. For $a > 0$, the critical points are $y^* = 0, a$. Since $f'(0) = a > 0$ and $f'(a) = -a < 0$, the critical point $y^* = 0$ is unstable while the critical point $y^* = a$ is asymptotically stable for $a > 0$.

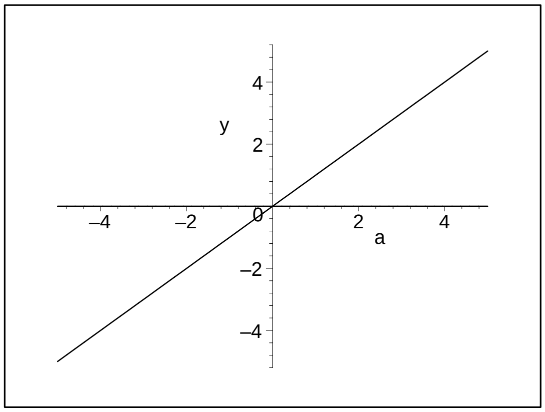
(b) Below we sketch solution curves for $a = 1, 0, -1$, respectively.







(c)



26.

(a) Since the critical points are $x^* = p, q$, we will look at their stability. Since $f'(x) = -\alpha q - \alpha p + 2\alpha x^2$, we see that $f'(p) = \alpha(2p^2 - q - p)$ and $f'(q) = \alpha(2q^2 - q - p)$. Now if $p > q$, then $-p < -q$, and, therefore, $f'(q) = \alpha(2q^2 - q - p) < \alpha(2q^2 - 2q) = 2\alpha q(q-1) < 0$ since $0 < q < 1$. Therefore, if $p > q$, $f'(q) < 0$, and, therefore, $x^* = q$ is asymptotically stable. Similarly, if $p < q$, then $x^* = p$ is asymptotically stable, and therefore, we can conclude that $x(t) \to \min\{p, q\}$ as $t \to \infty$.

The equation is separable. It can be solved by using partial fractions as follows. We can rewrite the equation as

$$\left(\frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x}\right) dx = \alpha dt,$$

which implies

$$\ln\left|\frac{p-x}{q-x}\right| = (p-q)\alpha t + C.$$

The initial condition $x_0 = 0$ implies $C = \ln|p/q|$, and, therefore,

$$\ln\left|\frac{q(p-x)}{p(q-x)}\right| = (p-q)\alpha t.$$

Applying the exponential function and simplifying, we conclude that

$$x(t) = \frac{pq(e^{(p-q)\alpha t} - 1)}{pe^{(p-q)\alpha t} - q}.$$

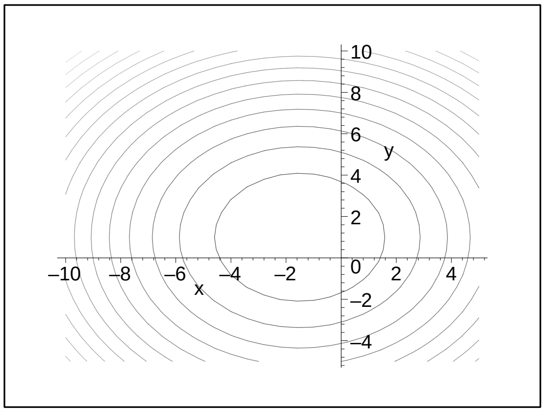
(b) In this case, $x^* = p$ is the only critical point. Since $f(x) = \alpha(p-x)^2$ is concave up, we conclude that $x^* = p$ is semistable. Further, if $x_0 = 0$, we can conclude that $x \to p$ as $t \to \infty$.

This equation is separable. Its solution is given by

$$x(t) = \frac{p^2\alpha t}{p\alpha t + 1}.$$

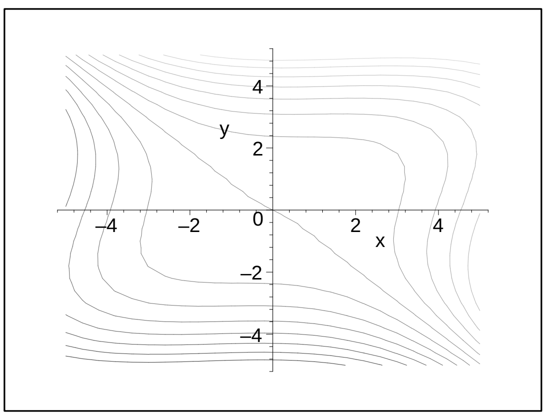
Section 2.6

1. Here $M(x, y) = 2x + 3$ and $N(x, y) = 2y - 2$. Since $M_y = N_x = 0$, the equation is exact. Since $\psi_x = M = 2x + 3$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^2 + 3x + h(y)$. Then $\psi_y = h'(y) = N = 2y - 2$ implies $h(y) = y^2 - 2y$. Therefore, $\psi(x, y) = x^2 + 3x + y^2 - 2y = c$.

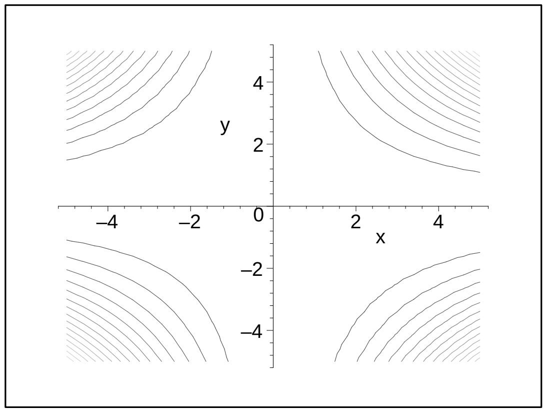


2. Here $M(x, y) = 2x + 4y$ and $N(x, y) = 2x - 2y$. Since $M_y \neq N_x$, the equation is not exact.

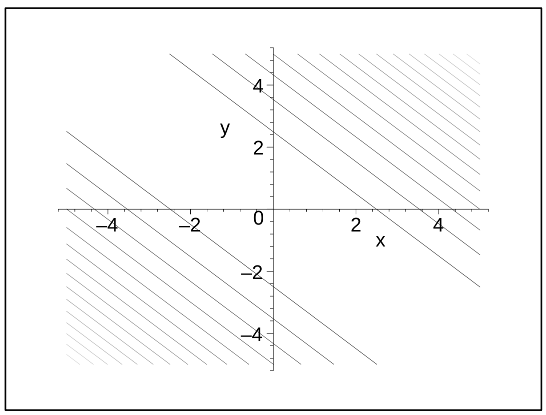
3. Here $M(x, y) = 3x^2 - 2xy + 2$ and $N(x, y) = 6y^2 - x^2 + 3$. Since $M_y = -2x = N_x$, the equation is exact. Since $\psi_x = M = 3x^2 - 2xy + 2$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^3 - x^2y + 2x + h(y)$. Then $\psi_y = -x^2 + h'(y) = N = 6y^2 - x^2 + 3$ implies $h'(y) = 6y^2 + 3$. Therefore, $h(y) = 2y^3 + 3y$ and $\psi(x, y) = x^3 - x^2y + 2x + 2y^3 + 3y = c$.



4. Here $M(x, y) = 2xy^2 + 2y$ and $N(x, y) = 2x^2y + 2x$. Since $M_y = 4xy + 2 = N_x$, the equation is exact. Since $\psi_x = M = 2xy^2 + 2y$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^2y^2 + 2xy + h(y)$. Then $\psi_y = 2x^2y + 2x + h'(y) = N = 2x^2y + 2x$ implies $h'(y) = 0$. Therefore, $h(y) = C$ and $\psi(x, y) = x^2y^2 + 2xy = c$.

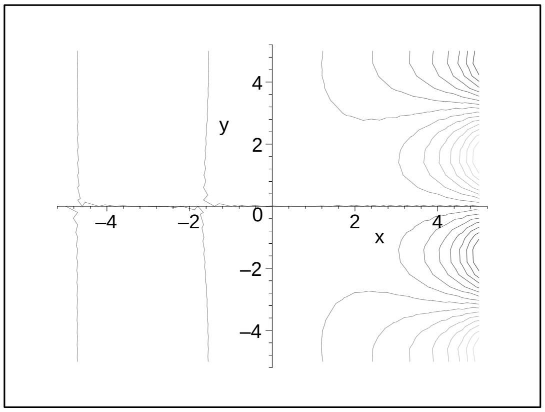


5. Here $M(x,y) = ax+by$ and $N(x,y) = bx+cy$. Since $M_y = b = N_x$, the equation is exact. Since $\psi_x = M = ax+by$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = ax^2/2+bxy+h(y)$. Then $\psi_y = bx+h'(y) = N = bx+cy$ implies $h'(y) = cy$. Therefore, $h(y) = cy^2/2$ and $\psi(x,y) = ax^2/2+bxy+cy^2/2 = c$.



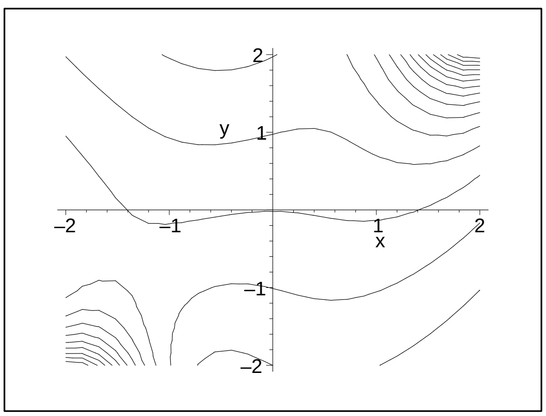
6. Here $M = ax - by$ and $N = bx - cy$. Since $M_y = -b$ and $N_x = b$, the equation is not exact.

7. Here $M(x,y) = e^x \sin y - 2y \sin x$ and $N(x,y) = e^x \cos y + 2\cos x$. Since $M_y = e^x \cos y - \sin x = N_x$, the equation is exact. Since $\psi_x = M = e^x \sin y - 2y \sin x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = e^x \sin y + 2y \cos x + h(y)$. Then $\psi_y = e^x \cos y + 2\cos x + h'(y) = N = e^x \cos y + 2\cos x$ implies $h'(y) = 0$. Therefore, $h(y) = C$ and $\psi(x,y) = e^x \sin y + 2y \cos x = c$.

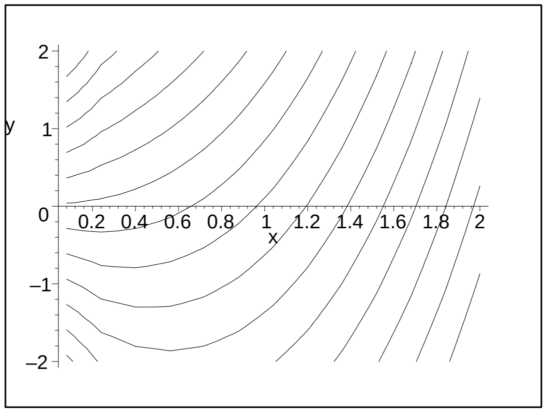


8. Here $M = e^x \sin y + 3y$ and $N = -3x + e^x \sin y$. Therefore, $M_y = e^x \cos y + 3$ and $N_x = -3 + e^x \sin y$. Since $M_y \neq N_x$, therefore, the equation is not exact.

9. Here $M(x, y) = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x$ and $N(x, y) = xe^{xy}\cos(2x) - 3$. Since $M_y = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x) = N_x$, the equation is exact. Since $\psi_x = M = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = e^{xy}\cos(2x) + x^2 + h(y)$. Then $\psi_y = xe^{xy}\cos(2x) + h'(y) = N = xe^{xy}\cos(2x) - 3$ implies $h'(y) = -3$. Therefore, $h(y) = -3y$ and $\psi(x, y) = e^{xy}\cos(2x) + x^2 - 3y = c$.

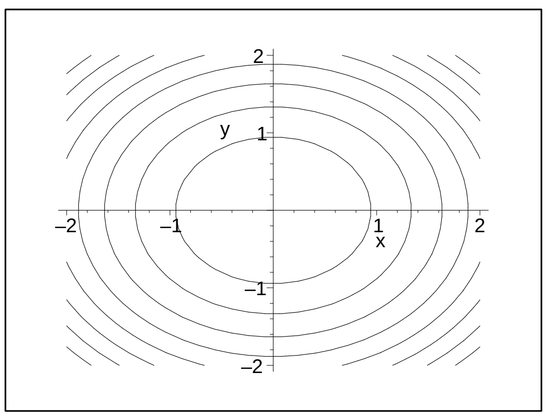


10. Here $M(x, y) = y/x + 6x$ and $N(x, y) = \ln(x) - 2$. Since $M_y = 1/x = N_x$, the equation is exact. Since $\psi_x = M = y/x + 6x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = y\ln(x) + 3x^2 + h(y)$. Then $\psi_y = \ln(x) + h'(y) = N = \ln(x) - 2$ implies $h'(y) = -2$. Therefore, $h(y) = -2y$ and $\psi(x, y) = y\ln(x) + 3x^2 - 2y = c$.



11. Here $M(x,y) = x\ln(y) + xy$ and $N(x,y) = y\ln(x) + xy$. Since $M_y = x/y + x$ and $N_x = y/x + y$, we conclude that the equation is not exact.

12. Here $M(x,y) = x/(x^2+y^2)^{3/2}$ and $N(x,y) = y/(x^2+y^2)^{3/2}$. Since $M_y = N_x$, the equation is exact. Since $\psi_x = M = x/(x^2+y^2)^{3/2}$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = -1/(x^2+y^2)^{1/2} + h(y)$. Then $\psi_y = y/(x^2+y^2)^{3/2} + h'(y) = N = y/(x^2+y^2)^{3/2}$ implies $h'(y) = 0$. Therefore, $h(y) = 0$ and $\psi(x,y) = -1/(x^2+y^2)^{1/2} = c$ which implies that $\psi(x,y) = (x^2+y^2) = c$.



13. Here $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Therefore, $M_y = N_x = -1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = x^2 - xy + h(y)$. Then $\psi_y = -x + h'(y) = N = 2y - x$ implies $h'(y) = 2y$. Therefore, $h(y) = y^2$ and we conclude that $\psi = x^2 - xy + y^2 = C$. The initial condition $y(1) = 3$ implies $c = 7$. Therefore, $x^2 - xy + y^2 = 7$. Solving for y, we conclude that $y = \dfrac{1}{2}\left[x + \sqrt{28 - 3x^2}\right]$. Therefore, the solution is valid for $3x^2 \leq 28$.

14. Here $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = -4y + x$. Therefore, $M_y = N_x = 1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = 3x^3 + xy - x + h(y)$. Then $\psi_y = x + h'(y) = N = -4y + x$ implies $h'(y) = -4y$. Therefore, $h(y) = -2y^2$ and we conclude that $\psi = 3x^3 + xy - x - 2y^2 = C$. The initial condition $y(1) = 0$ implies $c = 2$. Therefore, $3x^3 + xy - x - 2y^2 = 2$. Solving for y, we conclude that $y = [x - (24x^3 + x^2 - 8x - 16)^{1/2}]/4$. The solution is valid for $x > 0.9846$.

15. Here $M(x,y) = xy^2 + bx^2y$ and $N(x,y) = x^3 + x^2y$. Therefore, $M_y = 2xy + bx^2$ and $N_x = 3x^2 + 2xy$. In order for the equation to be exact, we need $b = c$. Taking this value for b, we integrating M with respect to x. We conclude that $\psi = x^2y^2/2 + x^3y + h(y)$. Then $\psi_y = x^2y + x^3 + h'(y) = N = x^3 + x^2y$ implies $h'(y) = 0$. Therefore, $h(y) = C$ and $\psi(x,y) = x^2y^2/2 + x^3y = C$. That is, the solution is given implicitly as $x^2y^2/2 + x^3y = c$.

16. Here $M(x,y) = ye^{2xy} + x$ and $N(x,y) = bxe^{2xy}$. Then $M_y = e^{2xy} + 2xye^{2xy}$ and $N_x = be^{2xy} + 2bxye^{2xy}$. The equation will be exact as long as $b = 1$. Integrating M with respect to x, we conclude that $\psi = e^{2xy}/2 + x^2/2 + h(y)$. Then $\psi_y = xe^{2xy} + h'(y) = N = xe^{2xy}$ implies $h'(y) = 0$. Therefore, $h(y) = 0$ and we conclude that the solution is given implicitly by the equation $e^{2xy} + x^2 = C$.

17. We notice that $\psi(x,y) = f(x) + g(y)$. Therefore, $\psi_x = f'(x)$ and $\psi_y = g'(y)$. That is,

$$\psi_x = M(x, y_0) \qquad \psi_y = N(x_0, y).$$

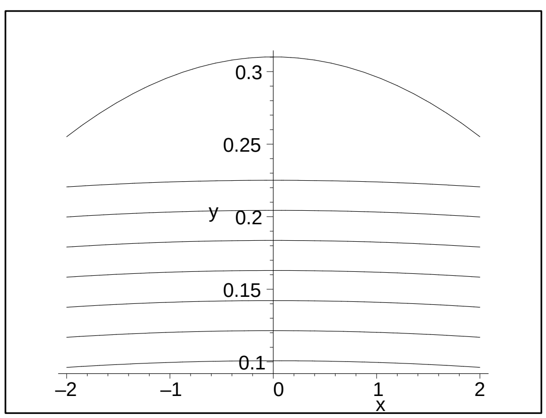
Furthermore, $\psi_{xy} = M_y$ and $\psi_{yx} = N_x$. Based on the hypothesis, $\psi_{xy} = \psi_{yx}$ and $M_y = N_x$.

18. We notice that $(M(x))_y = 0 = (N(y))_x$. Therefore, the equation is exact.

19. Here $M(x,y) = x^2y^3$ and $N(x,y) = x + xy^2$. Therefore, $M_y = 3x^2y^2$ and $N_x = 1 + y^2$. We see that the equation is not exact. Now, multiplying the equation by $\mu(x,y) = 1/xy^3$, the equation becomes

$$xdx + (1 + y^2)/y^3 dy = 0.$$

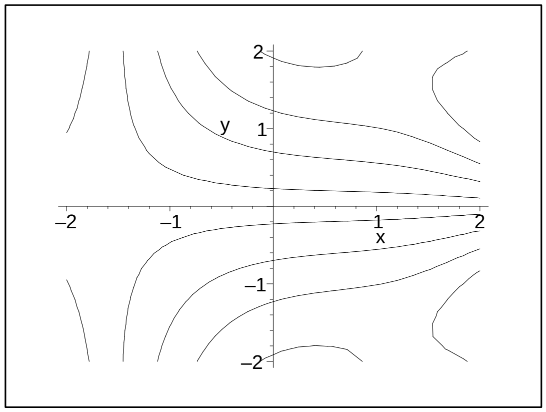
Now we see that for this equation $M = x$ and $N = (1 + y^2)/y^3$. Therefore, $M_y = 0 = N_x$. Integrating M with respect to x, we see that $\psi = x^2/2 + h(y)$. Further, $\psi_y = h'(y) = N = (1+y^2)/y^3 = 1/y^3 + 1/y$. Therefore, $h(y) = -1/2y^2 + \ln(y)$ and we conclude that the solution of the equation is given implicitly by $x^2 - 1/y^2 + 2\ln(y) = C$.



20. Multiplying the equation by $\mu(x,y) = ye^x$, the equation becomes

$$(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0.$$

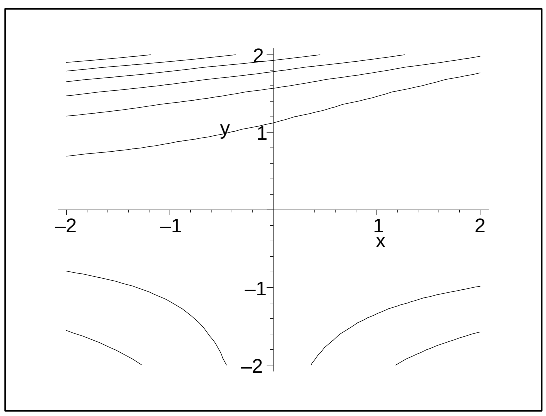
Now we see that for this equation $M = e^x \sin y - 2y \sin x$ and $N = e^x \cos y + 2\cos x$. Therefore, $M_y = e^x \cos y - 2\sin x = N_x$. Integrating M with respect to x, we see that $\psi = e^x \sin y + 2y\cos x + h(y)$. Further, $\psi_y = e^x \cos y + 2\cos x + h'(y) = N = e^x \cos y + 2\cos x$. Therefore, $h(y) = 0$ and we conclude that the solution of the equation is given implicitly by $e^x \sin y + 2y\cos x = C$.



21. Multiplying the equation by $\mu(x,y) = y$, the equation becomes

$$y^2dx + (2xy - y^2e^y)dy = 0.$$

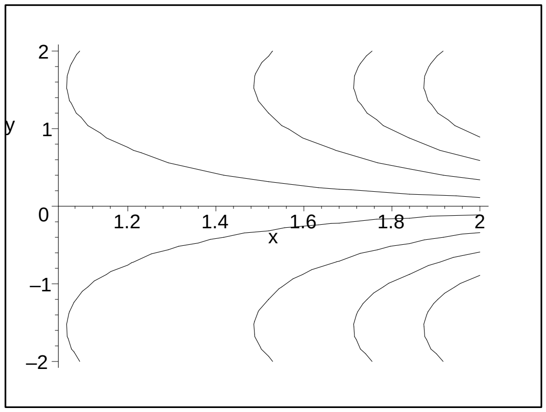
Now we see that for this equation $M = y^2$ and $N = 2xy - y^2e^y$. Therefore, $M_y = 2y = N_x$. Integrating M with respect to x, we see that $\psi = xy^2 + h(y)$. Further, $\psi_y = 2xy + h'(y) = N = 2xy - y^2e^y$. Therefore, $h'(y) = -y^2e^y$ which implies that $h(y) = -e^y(y^2 - 2y + 2)$, and we conclude that the solution of the equation is given implicitly by $xy^2 - e^y(y^2 - 2y + 2) = C$.



22. Multiplying the equation by $\mu(x,y) = xe^x$, the equation becomes

$$(x^2 + 2x)e^x \sin y dx + x^2e^x \cos y dy = 0.$$

Now we see that for this equation $M_y = (x^2 + 2x)e^x \cos y = N_x$. Integrating M with respect to x, we see that $\psi = x^2e^x \sin y + h(y)$. Further, $\psi_y = x^2e^x \cos y + h'(y) = N = x^2e^x \cos y$. Therefore, $h'(y) = 0$ which implies that the solution of the equation is given implicitly by $x^2e^x \sin y = C$.



23. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have

$$\mu M dx + \mu N dy = 0.$$

Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. Suppose μ does not depend on x. Then $\mu_x = 0$. Therefore, $\mu(N_x - M_y) = \mu_y M$. Using the assumption that $(N_x - M_y)/M = Q(y)$, we can find an integrating factor μ by choosing μ which satisfies $\mu_y/\mu = Q$. We conclude that $\mu(y) = \exp \int Q(y)\, dy$ is an integrating factor of the differential equation.

24. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have

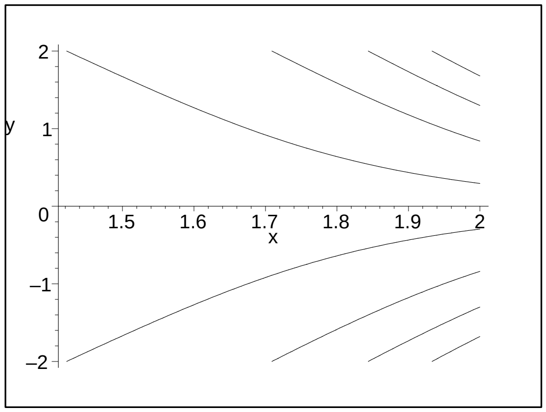
$$\mu M dx + \mu N dy = 0.$$

Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. By the given assumption, we need μ to satisfy $\mu R(xM - yN) = \mu_y M - \mu_x N$. This equation is satisfied if $\mu_y = (\mu x)R$ and $\mu_x = (\mu y)R$. Consider $\mu = \mu(xy)$. Then $\mu_x = \mu' y$ and $\mu_y = \mu' x$ where $' = d/dz$ for $z = xy$. Therefore, we need to choose μ to satisfy $\mu' = \mu R$. This equation is separable with solution $\mu = \exp(\int R(z)\, dz)$.

25. Since $(M_y - N_x)/N = 3$ is a function of x only, we know that $\mu = e^{3x}$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0.$$

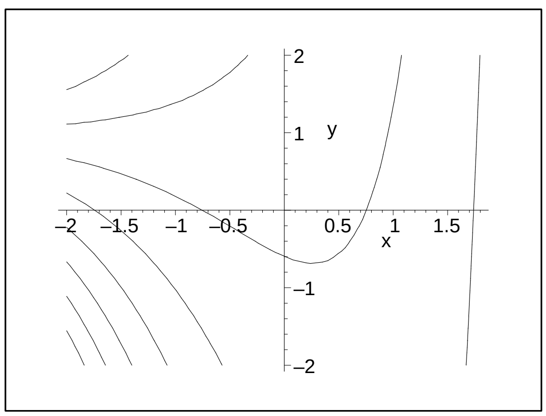
Then $M_y = e^{3x}(3x^2 + 2x + 3y^2) = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = (x^2y + y^3/3)e^{3x} + h(y)$. Then $\psi_y = (x^2 + y^2)e^{3x} + h'(y) = N = e^{3x}(x^2 + y^2)$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $(3x^2y + y^3)e^{3x} = c$.



26. Since $(M_y - N_x)/N = -1$ is a function of x only, we know that $\mu = e^{-x}$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(e^{-x} - e^{x} - ye^{-x})dx + e^{-x}dy = 0.$$

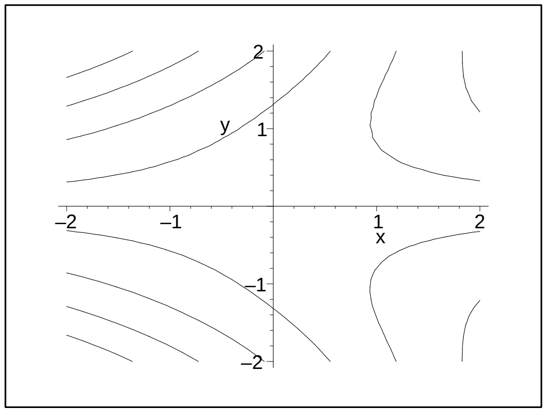
Then $M_y = -e^{-x} = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = -e^{-x} - e^{x} + ye^{-x} + h(y)$. Then $\psi_y = e^{-x} + h'(y) = N = e^{-x}$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $-e^{-x} - e^{x} + ye^{-x} = c$.



27. Since $(N_x - M_y)/M = 1/y$ is a function of y only, we know that $\mu(y) = e^{\int 1/y\, dy} = y$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$ydx + (x - y\sin y)dy = 0.$$

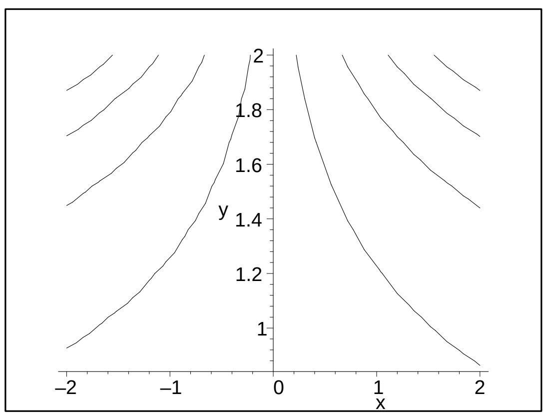
Then for this equation, $M_y = 1 = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xy + h(y)$. Then $\psi_y = x + h'(y) = N = x - y\sin y$. Therefore, $h'(y) = -y\sin y$ which implies that $h(y) = -siny + y\cos y$, and we conclude that the solution is given implicitly by $xy - \sin y + y\cos y = C$.



28. Since $(N_x - M_y)/M = (2y-1)/y$ is a function of y only, we know that $\mu(y) = e^{\int 2-1/y\,dy} = e^{2y}/y$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^{2y}dx + (2xe^{2y} - 1/y)dy = 0.$$

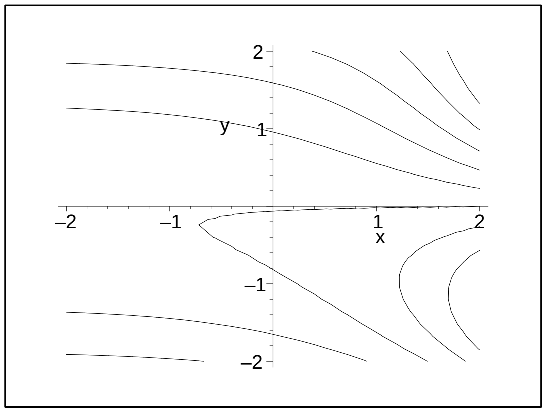
Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xe^{2y} + h(y)$. Then $\psi_y = 2xe^{2y} + h'(y) = N = 2xe^{2y} - 1/y$. Therefore, $h'(y) = -1/y$ which implies that $h(y) = -ln(y)$, and we conclude that the solution is given implicitly by $xe^{2y} - \ln(y) = C$.



29. Since $(N_x - M_y)/M = \cot(y)$ is a function of y only, we know that $\mu(y) = e^{\int \cot(y)\,dy} = \sin(y)$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^x \sin y dx + (e^x \cos y + 2y)dy = 0.$$

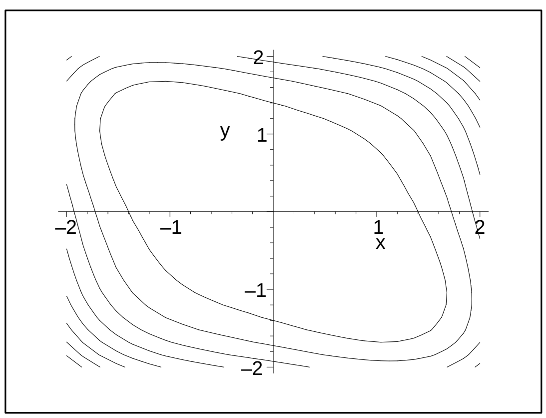
Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = e^x \sin y + h(y)$. Then $\psi_y = e^x \cos y + h'(y) = N = e^x \cos y + 2y$. Therefore, $h'(y) = 2y$ which implies that $h(y) = y^2$, and we conclude that the solution is given implicitly by $e^x siny + y^2 = C$.



30. Since $(N_x - M_y)/M = 2/y$ is a function of y only, we know that $\mu(y) = e^{\int 2/y\,dy} = y^2$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(4x^3 + 3y)dx + (3x + 4y^3)dy = 0.$$

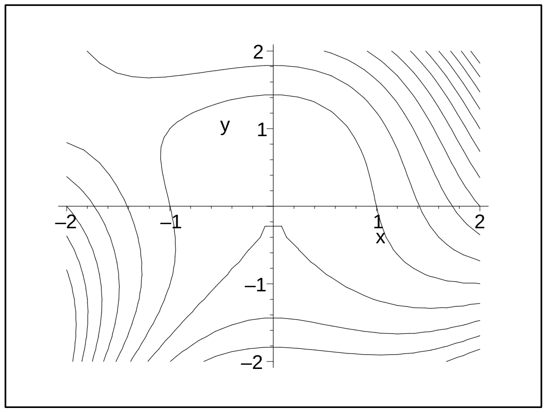
Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = x^4 + 3xy + h(y)$. Then $\psi_y = 3x + h'(y) = N = 3x + 4y^3$. Therefore, $h'(y) = 4y^3$ which implies that $h(y) = y^4$, and we conclude that the solution is given implicitly by $x^4 + 3xy + y^4 = C$.



31. Since $(N_x - M_y)/(xM - yN) = 1/xy$ is a function of xy only, we know that $\mu(xy) = e^{\int 1/xy\,dy} = xy$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0.$$

Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = x^3y + 3x^2 + h(y)$. Then $\psi_y = x^3 + h'(y) = N = x^3 + 3y^2$. Therefore, $h'(y) = 3y^2$ which implies that $h(y) = y^3$, and we conclude that the solution is given implicitly by $x^3y + 3x^2 + y^3 = C$.



32. Using the integrating factor $\mu = [xy(2x+y)]^{-1}$, this equation can be rewritten as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right] dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right] dy = 0.$$

Integrating M with respect to x, we see that $\psi = 2\ln|x| + \ln|2x+y| + h(y)$. Then $\psi_y = (2x+y)^{-1} + h'(y) = N = (2x+y)^{-1} + 1/y$. Therefore, $h'(y) = 1/y$ which implies that $h(y) = \ln|y|$. Therefore, $\psi = 2\ln|x| + \ln|2x+y| + \ln|y| = C$. Applying the exponential function, we conclude that the solution is given implicitly be $2x^3y + x^2y^2 = C$.

Section 2.7

1. The Euler formula is $y_{n+1} = y_n + h(3 + t_n - y_n)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n(1-h) + 3h + nh^2$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$1.1975, \;\; 1.38549, \;\; 1.56491, \;\; 1.73658$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$1.19631, \;\; 1.38335, \;\; 1.56200, \;\; 1.73308$$

2. The Euler formula is $y_{n+1} = y_n + h(5t_n - 3\sqrt{y_n})$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}$ with $y_0 = 2$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$1.59980, \;\; 1.29288, \;\; 1.07242, \;\; 0.930175.$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$1.61124, \;\; 1.31361, \;\; 1.10012, \;\; 0.962552$$

3. The Euler formula is $y_{n+1} = y_n + h(2y_n - 3t_n)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n(1+2h) - 3nh^2$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$1.2025, \;\; 1.41603, \;\; 1.64289, \;\; 1.88590$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$1.20388, \;\; 1.41936, \;\; 1.64896, \;\; 1.89572$$

4. The Euler formula is $y_{n+1} = y_n + h(2t_n + e^{-t_n y_n})$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + 2nh^2 + he{-}nhy_n$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$1.10244, \;\; 1.21426, \;\; 1.33484, \;\; 1.46399$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$1.10365, \;\; 1.21656, \;\; 1.33817, \;\; 1.46832$$

5. The Euler formula is $y_{n+1} = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(y_n^2 + 2nhy_n)/(3 + n^2h^2)$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$0.509239, \;\; 0.522187, \;\; 0.539023, \;\; 0.559936$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$0.509701, \;\; 0.523155, \;\; 0.540550, \;\; 0.562089$$

6. The Euler formula is $y_{n+1} = y_n + h(t_n^2 - y_n^2)\sin(y_n)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(n^2h^2 - y_n^2)\sin(y_n)$.

(a) For $h = 0.05$, the Euler approximations for y_n at $n = 2, 4, 6, 8$ are given by

$$-0.920498, \;\; -0.857538, \;\; -0.808030, \;\; -0.770038$$

(b) For $h = 0.025$, the Euler approximations for y_n at $n = 4, 8, 12, 16$ are given by

$$-0.922575, \;\; -0.860923, \;\; -0.812300, \;\; -0.774965$$

7. The Euler formula is $y_{n+1} = y_n + h(0.5 - t_n + 2y_n)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(0.5 - nh + 2y_n)$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$2.90330, \;\; 7.53999, \;\; 19.4292, \;\; 50.5614$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$2.93506, \;\; 7.70957, \;\; 20.1081, \;\; 52.9779$$

8. The Euler formula is $y_{n+1} = y_n + h(5t_n - 3\sqrt{y_n})$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(5nh - 3\sqrt{y_n})$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$0.891830, \;\; 1.25225, \;\; 2.37818, \;\; 4.07257$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$0.908902, \;\; 1.26872, \;\; 2.39336, \;\; 4.08799$$

9. The Euler formula is $y_{n+1} = y_n + h\sqrt{t_n + y_n}$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h\sqrt{nh + y_n}$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$3.95713, \;\; 5.09853, \;\; 6.41548, \;\; 7.90174$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$3.95965, \;\; 5.10371, \;\; 6.42343, \;\; 7.91255$$

10. The Euler formula is $y_{n+1} = y_n + h(2t_n + e^{-t_n y_n})$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(2nh + e^{-nhy_n})$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$1.60729, \;\; 2.46830, \;\; 3.72167, \;\; 5.45963$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$1.60996, \;\; 2.47460, \;\; 3.73356, \;\; 5.47774$$

11. The Euler formula is $y_{n+1} = y_n + h(4 - t_n y_n)/(1 + y_n^2)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(4 - nhy_n)/(1 + y_n^2)$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$-1.45865, \;\; -0.217545, \;\; 1.05715, \;\; 1.41487$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$-1.45322, \;\; -0.180813, \;\; 1.05903, \;\; 1.41244$$

12. The Euler formula is $y_{n+1} = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$ in which $t_n = t_0 + nh$. Since $t_0 = 0$, we have $y_{n+1} = y_n + h(y_n^2 + 2nhy_n)/(3 + n^2h^2)$.

(a) For $h = 0.025$, the Euler approximations for y_n at $n = 20, 40, 60, 80$ are given by

$$0.587987, \ \ 0.791589, \ \ 1.14743, \ \ 1.70973$$

(b) For $h = 0.0125$, the Euler approximations for y_n at $n = 40, 80, 120, 160$ are given by

$$0.589440, \ \ 0.795758, \ \ 1.15693, \ \ 1.72955$$

13. The Euler formula is

$$y_{n+1} = y_n + h(1 - t_n + 4y_n)$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can write

$$y_{n+1} = y_n + h - nh^2 + 4hy_n$$

with $y_0 = 1$. With $h = 0.01$, a total of 200 iterations is necessary to reach $\bar{t} = 2$. With $h = 0.001$, a total of 2000 iterations is necessary.

14. We will use the first three terms in equation (12),

$$\phi(t_{n+1}) = \phi(t_n) + f[t_n, \phi(t_n)]h + \phi''(t_n)\frac{h^2}{2}.$$

Letting $h = 0.1$, then the approximation is given by

$$y_{n+1} = y_n + h(1 - t_n + 4y_n) + \frac{h^2}{2}(3 - 4t_n + 16y_n).$$

Therefore,

$$y_1 = 1 + 0.1(1 - 0 + 4(1)) + \frac{0.1^2}{2}(3 - 4(0) + 16(1)) = 1 + 0.5 + .095 = 1.595.$$

Then, repeating this argument for y_2, we conclude that $y_2 = 2.4636$.

Solving this linear equation, we conclude that the exact solution is given by

$$y(t) = -\frac{3}{16} + \frac{1}{4}t + \frac{19}{16}e^{4t}.$$

Therefore, $y(0.1) = 1.609$ and $y(0.2) = 2.505$.

15. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = 2\phi(t) - 1.$$

Therefore,

$$\phi''(t) = 2\phi'(t) = 2(2\phi(t) - 1) = 4\phi(t) - 2.$$

Therefore,

$$e_{n+1} = (2\phi(\bar{t}_n) - 1)h^2.$$

Therefore,

$$|e_{n+1}| \leq |2M + 1|h^2$$

where $M = \max_{0 \leq t \leq 1} |\phi(t)|$.

The exact solution of this linear equation is $y(t) = 1/2 + 1/2e^{2t}$. Then, using the fact that the local truncation error is given by $e_{n+1} = \frac{1}{2}\phi(\bar{t}_n)h^2$ and $\phi(t) = 1/2 + 1/2e^{2t}$, we can conclude that

$$e_{n+1} = e^{2\bar{t}_n}h^2.$$

Therefore, $|e_1| \leq e^{0.2}(0.1)^2 \approx 0.012$. Similarly, $|e_4| \leq e^{0.8}(0.1)^2 \approx 0.022$.

16. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = \frac{1}{2} - t + 2\phi(t).$$

Therefore,

$$\phi''(t) = -1 + 2\phi'(t) = -1 + 2\left(\frac{1}{2} - t + 2\phi(t)\right) = -2t + 4\phi(t).$$

Therefore,

$$e_{n+1} = (-t_n + 2\phi(\bar{t}_n))h^2.$$

Therefore,

$$|e_{n+1}| \leq |2M + 1|h^2$$

where $M = \max_{0 \leq t \leq 1} |\phi(t)|$.

The exact solution of this linear equation is $y(t) = \frac{1}{2}t + e^{2t}$. Then, using the fact that the local truncation error is given by $e_{n+1} = \frac{1}{2}\phi(\bar{t}_n)h^2$ and $\phi(t) = \frac{1}{2}t + e^{2t}$, we can conclude that

$$e_{n+1} = 2e^{2\bar{t}_n}h^2.$$

Therefore, $|e_1| \leq 2e^{0.2}(0.1)^2 \approx 0.024$. Similarly, $|e_4| \leq 2e^{0.8}(0.1)^2 \approx 0.045$.

17. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = t^2 + (\phi(t))^2.$$

Therefore,

$$\phi''(t) = 2t + 2\phi(t)\phi'(t) = 2t + 2t^2\phi(t) + 2(\phi(t))^3$$

Therefore,

$$e_{n+1} = (\bar{t}_n + \bar{t}_n^2\phi(\bar{t}_n) + (\phi(\bar{t}_n))^3)h^2.$$

Therefore,

$$|e_{n+1}| \leq |t_{n+1} + t_{n+1}^2 M_{n+1} + M_{n+1}^3|h^2$$

where $M_{n+1} = \max_{t_n \leq t \leq t_{n_1}} |\phi(t)|$.

18. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = 5t - 3\sqrt{\phi(t)}.$$

Therefore,

$$\phi''(t) = 5 - \frac{3}{2}(\phi(t))^{-1/2}\phi'(t) = 5 - \frac{3}{2}\frac{5t - 3\sqrt{\phi(t)}}{\sqrt{\phi(t)}}.$$

Therefore,

$$\begin{aligned} e_{n+1} &= \frac{1}{2}\left(5 - \frac{3}{2}\frac{(5\bar{t}_n - 3\sqrt{\phi(\bar{t}_n)})}{\sqrt{\phi(\bar{t}_n)}}\right)h^2 \\ &= \frac{1}{4}\left(19 - 15\frac{\bar{t}_n}{\sqrt{\phi(\bar{t}_n)}}\right)h^2. \end{aligned}$$

19. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = \sqrt{t + \phi(t)}.$$

Therefore,

$$\phi''(t) = \frac{1 + \phi'(t)}{2\sqrt{t + \phi(t)}} = \frac{1}{2\sqrt{t + \phi(t)}} + \frac{1}{2}.$$

Therefore,

$$e_{n+1} = \frac{1}{4}\left[1 + \frac{1}{\sqrt{\bar{t}_n + \phi(\bar{t}_n)}}\right]h^2.$$

20. We know that $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$ where $t_n < \bar{t}_n < t_{n+1}$. Here

$$\phi'(t) = 2t + e^{-t\phi(t)}.$$

Therefore,

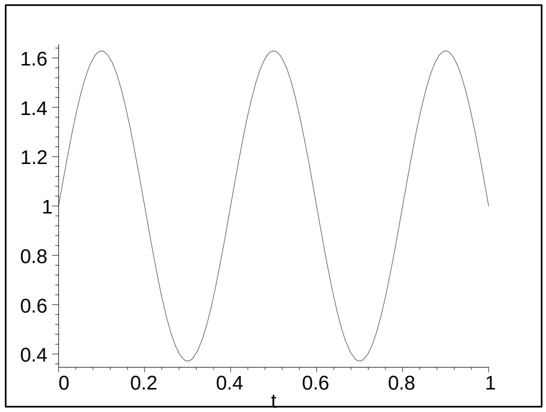
$$\phi''(t) = 2 + (-\phi(t) - t\phi'(t))e^{-t\phi(t)} = 2 + (-\phi(t) - t(2t + e^{-t\phi(t)}))e^{-t\phi(t)}.$$

Therefore,

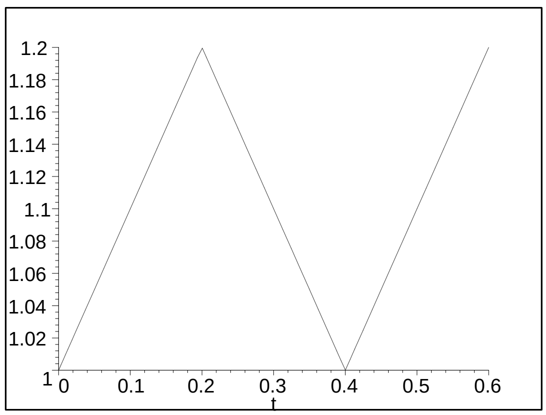
$$e_{n+1} = \frac{1}{2}\left[2 + (-\phi(\bar{t}_n) - 2\bar{t}_n^2 - \bar{t}_n e^{-\bar{t}_n\phi(\bar{t}_n)})e^{-\bar{t}_n\phi(\bar{t}_n)}\right]h^2.$$

21.

(a) The solution is given by $\phi(t) = \frac{1}{5\pi}\sin(5\pi t) + 1$.



(b) Approximate values at $t = 0.2, 0.4, 0.6$ are given by $1.2, 1.0, 1.2$, respectively.



(c) Approximate values at $t = 0.2, 0.4, 0.6$ are given by $1.1, 1.0, 1.1$, respectively.

(d) Since $\phi''(t) = -5\pi \sin(5\pi t)$, the local truncation error for the Euler method is given by

$$e_{n+1} = -\frac{5\pi h^2}{2} \sin(5\pi \bar{t}_n).$$

In order to guarantee that $|e_{n+1}| < 0.05$, we need

$$\frac{5\pi h^2}{2} < 0.05.$$

Solving this inequality, we conclude that we would need $h < 1/\sqrt{50\pi} approx 0.08$.

22.

1. The Euler formula is $y_{n+1} = y_n + h(1 - t_n + 4y_n)$. The approximate values for the solution at $t = 0.1, 0.2, 0.3, 0.4$ are given by

$$1.55, \;\; 2.34, \;\; 3.46, \;\; 5.07.$$

2. The Euler formula is $y_{n+1} = y_n + h(3 + t_n - y_n)$. The approximate values for the solution at $t = 0.1, 0.2, 0.3, 0.4$ are given by

$$1.20, \quad 1.39, \quad 1.57, \quad 1.74.$$

3. The Euler formula is $y_{n+1} = y_n + h(2y_n - 3t_n)$. The approximate values for the solution at $t = 0.1, 0.2, 0.3, 0.4$ are given by

$$1.20, \quad 1.42, \quad 1.65, \quad 1.90.$$

23.

(a)

$$1000 \cdot \begin{vmatrix} 6.0 & 18 \\ 2.0 & 6.0 \end{vmatrix} = 1000 \cdot 0 = 0.$$

(b)

$$1000 \cdot \begin{vmatrix} 6.01 & 18.0 \\ 2.00 & 6.00 \end{vmatrix} = 1000(0.06) = 60.$$

(c)

$$1000 \cdot \begin{vmatrix} 6.010 & 18.04 \\ 2.004 & 6.000 \end{vmatrix} = 1000(-0.09216) = -92.16.$$

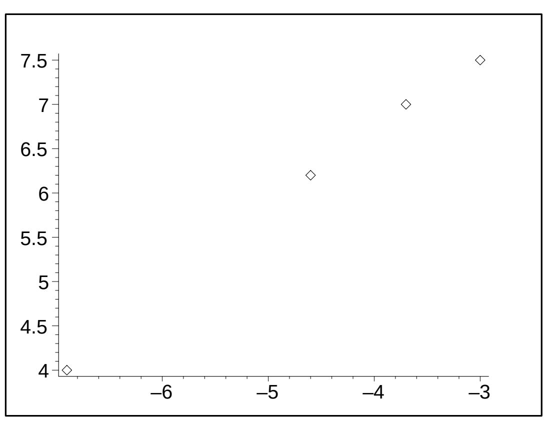
24. Rounding to three digits, $a(b-c) \approx 0.224$. Similarly, rounding to three digits, $ab \approx 0.702$ and $ac \approx 0.477$. Therefore, $ab - ac \approx 0.225$.

25.

(a) The maximum errors occur at $t = 2$. For $h = 0.001, 0.01, 0.025, 0.05$, they are given by

$$56.0393, \quad 510.8722, \quad 1107.4123, \quad 1794.5339.$$

(b)



(c) Yes.

(d) Using a curve-fitting routine, the slope of the least squares line is $\approx .909$.

Section 2.8

1.

(a) The improved Euler formula is

$$y_{n+1} = y_n + h\left(3 + \frac{1}{2}t_n + \frac{1}{2}t_{n+1} - y_n\right) - \frac{h^2}{2}(3 + t_n - y_n).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + h(3 - y_n) + \frac{h^2}{2}(y_n - 2 + 2n) - \frac{nh^3}{2}$$

with $y_0 = 1$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.19512, \quad 1.38120, \quad 1.55909, \quad 1.72956$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.19515, \quad 1.38125, \quad 1.55916, \quad 1.72965.$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.19516, \quad 1.38126, \quad 1.55918, \quad 1.72967.$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.19516, \quad 1.38127, \quad 1.55918, \quad 1.72968$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.19516, \quad 1.38127, \quad 1.55918, \quad 1.72968$$

2.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(5t_n - 3\sqrt{y_n}\right)) + \frac{h}{2}(5t_{n+1} - 3\sqrt{K_n})$$

where $K_n = y_n + h(5t_n - 3\sqrt{y_n})$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}(5nh - 3\sqrt{y_n}) + \frac{h}{2}\left[5(n+1)h - 3\sqrt{K_n}\right]$$

with $y_0 = 2$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.62283, \quad 1.33460, \quad 1.12820, \quad 0.995445$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.62243, \quad 1.33386, \quad 1.12718, \quad 0.994215$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.62234, \quad 1.33368, \quad 1.12693, \quad 0.993921$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.62231, \quad 1.33362, \quad 1.12686, \quad 0.993839$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.62230, \quad 1.33362, \quad 1.12685, \quad 0.993826$$

3.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(4y_n - 3t_n - 3t_{n+1}\right) + h^2(2y_n - 3t_n).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + 2hy_n + \frac{h^2}{2}(4y_n - 3 - 6n) - 3nh^3$$

with $y_0 = 1$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.20526, \quad 1.42273, \quad 1.65511, \quad 1.90570$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.20533, \quad 1.42290, \quad 1.65542, \quad 1.90621$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.20534, \quad 1.42294, \quad 1.65550, \quad 1.90634$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.20535, \quad 1.42295, \quad 1.65553, \quad 1.90638$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.20535, \quad 1.42296, \quad 1.65553, \quad 1.90638$$

4.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(2t_n + e^{-t_n y_n} + 2(t_n + h) + e^{-(t_n+h)K_n}\right)$$

where $K_n = y_n + h(2t_n + e^{-t_n y_n})$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\left(2nh + e^{-nhy_n} + 2h(n+1) + e^{-h(n+1)K_n}\right)$$

with $y_0 = 1$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.10483, \quad 1.21882, \quad 1.34146, \quad 1.47263$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.10484, \quad 1.21884, \quad 1.34147, \quad 1.47262$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.10484, \quad 1.21884, \quad 1.34147, \quad 1.47262$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.10484, \quad 1.21884, \quad 1.34147, \quad 1.47262$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$1.10484, \quad 1.21884, \quad 1.34147, \quad 1.47262$$

5.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{y_n^2 + 2t_n y_n}{3 + t_n^2}\right) + \frac{h}{2}\left(\frac{K_n^2 + 2t_{n+1}K_n}{3 + t_{n+1}^2}\right)$$

where $K_n = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{y_n^2 + 2nhy_n}{3 + n^2h^2}\right) + frach2\left(\frac{K_n^2 + 2(n+1)hK_n}{3 + (n+1)^2h^2}\right)$$

with $y_0 = 0.5$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$0.510164, \quad 0.524126, \quad 0.542083, \quad 0.564251$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$0.510168, \quad 0.524135, \quad 0.542100, \quad 0.564277$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$0.510169, \quad 0.524137, \quad 0.542104, \quad 0.564284$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$0.510170, \quad 0.524138, \quad 0.542105, \quad 0.564286$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$0.520169, \quad 0.524138, \quad 0.542105, \quad 0.564286$$

6.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(t_n^2 - y_n^2)\sin y_n + \frac{h}{2}(t_{n+1}^2 - K_n^2)\sin K_n$$

where $K_n = y_n + h(t_n^2 - y_n^2)\sin y_n$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}(n^2h^2 - y_n^2)\sin y_n + \frac{h}{2}\left[(n+1)^2h^2 - K_n^2\right]\sin K_n$$

with $y_0 = -1$. With $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$-0.924650, \quad -0.864338, \quad -0.816642, \quad -0.780008$$

(b) Using $h = 0.025$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$-0.924550, \quad -0.864177, \quad -0.816442, \quad -0.779781$$

(c) Using $h = 0.0125$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$-0.924525, \quad -0.864138, \quad -0.816393, \quad -0.779725$$

(d) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$-0.924517, \quad -0.864125, \quad -0.816377, \quad -0.779706$$

(e) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$ are

$$-0.924517, \quad -0.864125, \quad -0.816377, \quad -0.779706$$

7.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(0.5 - t_n + 2y_n) + \frac{h}{2}(0.5 - t_{n+1} + 2(y_n + h(0.5 - t_n + 2y_n))).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + h(2y_n + 0.5) + h^2(2y_n - n) - nh^3$$

with $y_0 = 1$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$2.96719, \quad 7.88313, \quad 20.8114, \quad 55.5106$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$2.96800, \quad 7.88755, \quad 20.8294, \quad 55.5758$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$2.96825, \quad 7.88889, \quad 20.8349, \quad 55.5957$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$2.96828, \quad 7.88904, \quad 20.8355, \quad 55.5980$$

8.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(5t_n - 3\sqrt{y_n}) + \frac{h}{2}(5t_{n+1} - 3\sqrt{K_n})$$

where $K_n = y_n + h(5t_n - 3\sqrt{y_n})$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}(5nh - 3\sqrt{y_n}) + \frac{h}{2}\left[5(n+1)h - 3\sqrt{K_n}\right]$$

with $y_0 = 2$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.926139, \quad 1.28558, \quad 2.40898, \quad 4.10386$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.925815, \quad 1.28525, \quad 2.40869, \quad 4.10359$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.925725, \quad 1.28516, \quad 2.40860, \quad 4.10350$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.925711, \quad 1.28515, \quad 2.40860, \quad 4.10350$$

9.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{t_n + y_n} + \frac{h}{2}\sqrt{t_{n+1} + K_n}$$

where $K_n = y_n + h\sqrt{t_n + y_n}$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{nh + y_n} + \frac{h}{2}\sqrt{(n+1)h + K_n}$$

with $y_0 = 3$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$3.96217, \quad 5.10887, \quad 6.43134, \quad 7.92332$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$3.96218, \quad 5.10889, \quad 6.43138, \quad 7.92337$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$3.96219, \;\; 5.10890, \;\; 6.43139, \;\; 7.92338$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$3.96219, \;\; 5.10890, \;\; 6.43139, \;\; 7.92338$$

10.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(2t_n + e^{-t_n y_n} + 2(t_n + h) + e^{-(t_n+h)K_n}\right)$$

where $K_n = y_n + h(2t_n + e^{-t_n y_n})$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\left(2nh + e^{-nhy_n} + 2h(n+1) + e^{-h(n+1)K_n}\right)$$

with $y_0 = 1$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$1.61263, \;\; 2.48097, \;\; 3.74556, \;\; 5.49595$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$1.61263, \;\; 2.48092, \;\; 3.74550, \;\; 5.49589$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$1.61262, \;\; 2.48091, \;\; 3.74548, \;\; 5.49587$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$1.61262, \;\; 2.48091, \;\; 3.74548, \;\; 5.49587$$

11.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{4 - t_n y_n}{1 + y_n^2}\right) + \frac{h}{2}\left(\frac{4 - t_{n+1}K_n}{1 + K_n^2}\right)$$

where $K_n = y_n + h(4 - t_n y_n)/(1 + y_n^2)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{4 - nhy_n}{1 + y_n^2}\right) + \frac{h}{2}\left(\frac{4 - h(n+1)K_n}{1 + K_n^2}\right)$$

with $y_0 = -2$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$-1.44768, \quad -0.144478, \quad 1.06004, \quad 1.40960$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$-1.44765, \quad -0.143690, \quad 1.06072, \quad 1.40999$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$-1.44764, \quad -0.143543, \quad 1.06089, \quad 1.41008$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$-1.44764, \quad -0.143427, \quad 1.06095, \quad 1.41011$$

12.

(a) The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{y_n^2 + 2t_n y_n}{3 + t_n^2}\right) + \frac{h}{2}\left(\frac{K_n^2 + 2t_{n+1}K_n}{3 + t_{n+1}^2}\right)$$

where $K_n = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, this formula can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{y_n^2 + 2nhy_n}{3 + n^2h^2}\right) + \frac{h}{2}\left(\frac{K_n^2 + 2(n+1)hK_n}{3 + (n+1)^2h^2}\right)$$

with $y_0 = 0.5$. With $h = 0.025$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.590897, \quad 0.799950, \quad 1.16653, \quad 1.74969$$

(b) Using $h = 0.0125$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.590906, \quad 0.799988, \quad 1.16663, \quad 1.74992$$

(c) Using the Runge-Kutta method with $h = 0.1$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.590909, \;\; 0.800000, \;\; 1.166667, \;\; 1.75000$$

(d) Using the Runge-Kutta method with $h = 0.05$, the approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$ are

$$0.590909, \;\; 0.800000, \;\; 1.166667, \;\; 1.75000$$

13. The improved Euler method is

$$y_{n+1} = y_n + \frac{h}{2}(1 - t_n + 4y_n) + \frac{h}{2}[1 - (t_n + h) + 4K_n]$$

where $K_n = y_n + h(1 - t_n + 4y_n)$. Since $t_n = nh + t_0$ and $t_0 = 0$, this equation can be simplified to

$$y_{n+1} = y_n + \frac{h}{2}(1 - nh + 4y_n) + \frac{h}{2}[1 - h(n+1) + 4K_n]$$

with $y_0 = 1$.

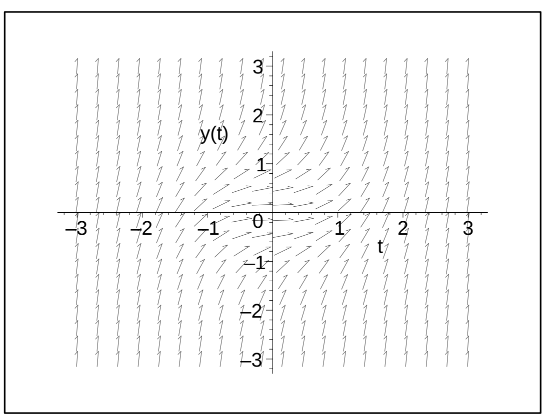
14. The differential equation is linear. Its exact solution is given by $y(t) = \frac{19}{16}e^{4t} + \frac{1}{4}t - \frac{3}{16}$. The improved Euler method is

$$y_{n+1} = y_n + \frac{h}{2}(1 - t_n + 4y_n) + \frac{h}{2}[1 - (t_n + h) + 4K_n]$$

where $K_n = y_n + h(1 - t_n + 4y_n)$.

15.

(a)

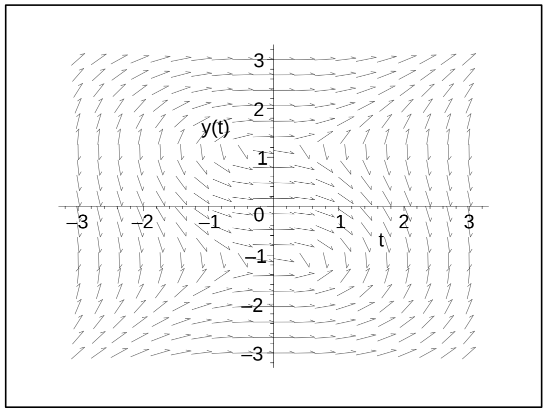


(b) The following are the approximate values of the solution at $t = 0.8, 0.9, 0.95$ using the Runge-Kutta method with $h = 0.01$:

$$5.848616, 14.304785, 50.436365.$$

16.

(a)



(b) For the integral curve staring at $(0, 0)$, the slope becomes infinite near $t_M \approx 1.5$. We note that the exact solution is given implicitly as $y^3 - 4y = t^3$.

(c) Based on the direction field, the solution of the initial value problem with initial condition $y(0) = 0$ should decrease monotonically to the limiting value $y = -2/\sqrt{3}$. Using the Runge-Kutta method, we calculate the approximate value of t_M by looking at the approximate time in the iteration process that the calculated values begin to increase. For $h = 0.1, 0.05, 0.025, 0.01$, the respective times are given by $t_M \approx 1.9, 1.65, 1.55, 1.455$.

(d) These values are not associated with the integral curve starting at $(0, 0)$. These values are approximations to nearby integral curves.

(e) Suppose now that $y(0) = 1$. The exact solution is given by $y^3 - 4y = t^3 - 3$. For the integral curve starting at $(0, 1)$, the slope becomes infinite near $t_M \approx 2.0$. Using the Runge-Kutta method, we calculate the following approximate values for t_M. For $h = 0.1, 0.05, 0.025, 0.01$, the respective times are given by $t_M \approx 1.85, 1.85, 1.86, 1.835$.

17.

(a) First we notice that

$$\begin{aligned}\phi'(t_n)h - \frac{f(t_n, y_n)h}{2} &= \phi'(t_n)h - \frac{y_n' h}{2} \\ &= \phi'(t_n)h - \frac{\phi'(t_n)h}{2} = \frac{\phi'(t_n)h}{2}.\end{aligned}$$

Using this fact, it follows that $\phi(t_{n+1}) - y_{n+1}$ satisfies the given equation.

(b) First, using the Taylor approximation, we see that

$$\begin{aligned}f[t_n + h, y_n + hf(t_n, y_n)] - f(t_n, y_n) &= f_t(t_n, y_n)h + f_y(t_n, y_n)hf(t_n, y_n) \\ &\quad + \frac{1}{2!}(h^2 f_{tt} + 2hkf_{ty} + k^2 f_{yy})\Big|_{x=\xi, y=\eta}\end{aligned}$$

Next, we see that

$$\begin{aligned}\phi''(t_n)h &= f_t(t_n, \phi(t_n))h + f_y(t_n, \phi(t_n))\phi'(t_n)h \\ &= f_t(t_n, y_n)h + f_y(t_n, y_n)hf(t_n, y_n).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\frac{1}{2!}\left[\phi''(t_n)h - \{f[t_n + h, y_n + hf(t_n, y_n)] - f(t_n, y_n)\}\right]h \\ &\quad = h\left(\frac{1}{2!}(h^2 f_{tt} + 2hk f_{ty} + k^2 f_{yy})\Big|_{x=\xi, y=\eta}\right)\end{aligned}$$

is proportional to h^3.

(c) If f is linear in t and y, then $f_{tt} = f_{ty} = f_{yy} = 0$. Therefore, the terms from part (b) above are all zero.

18. The exact solution is given by $\phi(t) = -3/16 + t/4 + (19/16)e^{4t}$. Then, by the results of Problem 17(c), the error will be given by

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)h^3}{3!}.$$

Here $\phi' = 1/4 + (19/4)e^{4t}$, $\phi'' = 19e^{4t}$, $\phi''' = 76e^{4t}$. Therefore,

$$e_{n+1} = \frac{38e^{4\bar{t}_n}h^3}{3}.$$

Therefore, on the interval $0 \le t \le 2$, we conclude that

$$|e_{n+1}| \le \frac{38e^8 h^3}{3} = 37,758.8h^3.$$

Then for $h = 0.05$, we conclude that

$$|e_1| \le \frac{38e^{4(0.05)}(0.05)^3}{3} = 0.00193389.$$

19. The exact solution of the initial value problem is $\phi(t) = \frac{1}{2} + \frac{1}{2}e^{2t}$. Based on the result from problem 17(c), the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3.$$

Here $\phi' = e^{2t}$, $\phi'' = 2e^{2t}$, $\phi''' = 4e^{2t}$. Therefore,

$$e_{n+1} = \frac{2}{3}e^{2\bar{t}_n}h^3.$$

Further, on the interval $0 \le t \le 1$,

$$|e_{n+1}| \le \frac{2}{3} e^2 h^3 = 4.92604 h^3.$$

Letting $h = 0.1$,

$$|e_1| \le \frac{2}{3} e^{2(0.1)} (0.1)^3 = 0.000814269.$$

Using the improved Euler method, with $h = 0.1$, we have $y_1 \approx 1.11000$. The exact value of the solution is $\phi(0.1) = 1.1107014$.

20. The exact solution of the initial value problem is $\phi(t) = \frac{1}{2}t + e^{2t}$. Based on the result from problem 17(c), the local truncation error for a linear differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n) h^3.$$

Here $\phi' = \frac{1}{2} + 2e^{2t}$, $\phi'' = 4e^{2t}$, $\phi''' = 8e^{2t}$. Therefore,

$$e_{n+1} = \frac{4}{3} e^{2\bar{t}_n} h^3.$$

Further, on the interval $0 \le t \le 1$,

$$|e_{n+1}| \le \frac{4}{3} e^2 h^3 = 9.85207 h^3.$$

Letting $h = 0.1$,

$$|e_1| \le \frac{4}{3} e^{2(0.1)} (0.1)^3 = 0.00162854.$$

21. The Euler formula is

$$y_{n+1} = y_n + h(0.5 - t_n + 2y_n).$$

Since $t_0 = 0$, $y_0 = 1$ and $h = 0.1$, we have

$$y_1 = 1 + 0.1(0.5 - 0 + 2) = 1.25.$$

For $t_0 = 0$, the improved Euler formula is

$$y_{n+1} = y_n + h(2y_n + 0.5) + h^2(2y_n - n) - nh^3.$$

Therefore, for $y_0 = 1$ and $h = 0.1$,

$$y_1 = 1 + 0.1(2 + 0.5) + (0.1)^2(2 - 0) - 0(0.1)^3 = 1.27.$$

Therefore, the estimated error of the Euler method is $e_{n+1}^{\text{ext}} = 1.27 - 1.25 = .02$. If we want the error of the Euler method to be less than 0.0025, we need to multiply the original step size of 0.1 by the factor $\sqrt{0.0025/0.02} \approx 0.35$. Therefore, the required step size is estimated to be $h \approx (0.1)(0.35) = 0.035$.

22. For $t_0 = 0$, the Euler formula is

$$y_{n+1} = y_n + h(5nh - 3\sqrt{y_n}).$$

Therefore, for $y_0 = 2$ and $h = 0.1$, we have

$$y_1 = 2 + 0.1(0 - 3\sqrt{2}) = 1.575736.$$

For $t_0 = 0$, the improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}(5nh - 3\sqrt{y_n}) + \frac{h}{2}\left[5(n+1)h - 3\sqrt{K_n}\right]$$

where $K_n = y_n + h(5t_n - 3\sqrt{y_n})$. Therefore, for $y_0 = 2$ and $h = 0.1$,

$$y_1 = 2 + 0.05(0 - 3\sqrt{2}) + 0.05[5(0.1) - 3\sqrt{2 + 0.1(0 - 3\sqrt{2})}] = 1.624575.$$

Therefore, the estimated error of the Euler method is $e_{n+1}^{\text{ext}} = 1.624575 - 1.575736 = 0.048839$. If we want the error of the Euler method to be less than 0.0025, we need to multiply the original step size of 0.1 by the factor $\sqrt{0.0025/0.048839} \approx 0.226$. Therefore, the required step size is estimated to be $h \approx (0.1)(0.226) = 0.0226$.

23. For $t_0 = 0$, the Euler formula is

$$y_{n+1} = y_n + h\sqrt{nh + y_n}.$$

Therefore, for $y_0 = 3$ and $h = 0.1$, we have

$$y_1 = 3 + 0.1\sqrt{0 + 3} = 3.173205.$$

For $t_0 = 0$, the improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{nh + y_n} + \frac{h}{2}\sqrt{(n+1)h + K_n}$$

where $K_n = y_n + h\sqrt{t_n + y_n}$. Therefore, for $y_0 = 3$ and $h = 0.1$,

$$y_1 = 3 + 0.05\sqrt{0 + 3} + 0.05\sqrt{0.1 + (3 + 0.1\sqrt{0 + 3})} = 3.177063.$$

Therefore, the estimated error of the Euler method is $e_{n+1}^{\text{ext}} = 3.177063 - 3.173205 = 0.003858$. If we want the error of the Euler method to be less than 0.0025, we need to multiply the original step size of 0.1 by the factor $\sqrt{0.0025/0.003858} \approx 0.805$. Therefore, the required step size is estimated to be $h \approx (0.1)(0.226) = 0.0805$.

24. For $t_0 = 0$, the Euler formula is

$$y_{n+1} = y_n + h(y_n^2 + 2nhy_n)/(3 + n^2h^2).$$

Therefore, for $y_0 = 0.5$ and $h = 0.1$, we have

$$y_1 = 0.5 + 0.1(0.5^2 + 0)/(3 + 0) = 0.508334.$$

For $t_0 = 0$, the improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{y_n^2 + 2nhy_n}{3 + n^2h^2}\right) + \frac{h}{2}\left(\frac{K_n^2 + 2(n+1)hK_n}{3 + (n+1)^2h^2}\right)$$

where $K_n = y_n + h(y_n^2 + 2t_ny_n)/(3 + t_n^2)$. Therefore, for $y_0 = 0.5$ and $h = 0.1$,

$$y_1 = 0.5 + 0.05\frac{0.5^2 + 0}{3 + 0} + 0.05\frac{(0.5 + 0.1(0.5^2/3))^2 + 2(0.1)(0.5 + 0.1(0.5^2/3))}{3 + 0.1^2} = 0.510148.$$

Therefore, the estimated error of the Euler method is $e_{n+1}^{\text{ext}} = 0.510148 - 0.598334 = 0.0018$. The local truncation error is less than the given tolerance. Therefore, if we allow an error tolerance of 0.0025, we can multiply the original step size of 0.1 by the factor $\sqrt{0.0025/0.0018} \approx 1.1785$. Therefore, the required step size is estimated to be $h \approx (0.1)(1.1785) = 0.11785$.