Instructor's Solutions Manual

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Advanced Topics in Finite Element Analysis of Structures

with Computations Using *Mathematica* and Matlab

John Wiley & Sons, Inc. 2006

CHAPTER ONE

Essential Background

1.1

An engineering analysis problem is formulated in terms of the following second order boundary value problem

$$-u\,x^4 - u' + u'' = x; \quad 0 < x < 1$$

$$u(0) = 4 \text{ and } u'(1) = 1$$

Derive a suitable weak form for use with the Galerkin method. Clearly indicate how the boundary conditions will be handled.

With $u(x)$ as an assumed solution the residual is

$$e(x) = -u(x)\,x^4 - x - u'(x) + u''(x)$$

Multiplying by $w(x)$ and writing integral over the given limits, the Galerkin weighted residual is

$$\int_0^1 (-u\,w\,x^4 - w\,x - w\,u' + w\,u'')\,dx = 0$$

Using integration by parts, the order of derivative in $w\,u''$ can be reduced to 1 as follows.

$$\int_0^1 (w\,u'')\,dx = w(1)\,u'(1) - w(0)\,u'(0) + \int_0^1 (-u'\,w')\,dx$$

Combining all terms, the weighted residual now is as follows.

$$w(1)\,u'(1) - w(0)\,u'(0) + \int_0^1 (-w\left(u\,x^4 + x + u'\right) - u'\,w')\,dx = 0$$

Consider the boundary terms

$$w(1)\,u'(1) - w(0)\,u'(0)$$

Each one of these terms gives rise to two possibilities

$-w(0)\,u'(0)$	Either $-u'(0)$ is known or $w(0) = 0$
$w(1)\,u'(1)$	Either $u'(1)$ is known or $w(1) = 0$

From these requirements the possible boundary conditions are as follows:

	NBC		EBC
1	$-u'(0)$ is given	or	$w(0) = 0 \Longrightarrow$ Must satisfy $u(0)$ boundary condition
2	$u'(1)$ is given	or	$w(1) = 0 \Longrightarrow$ Must satisfy $u(1)$ boundary condition

Given NBC for the problem:

$$u'(1) - 1 = 0$$

Rearranging: $(u'(1) \to 1)$

Given EBC for the problem:

$$u(0) - 4 = 0$$

therefore with admissible solutions (those satisfying EBC): $(w(0) \to 0)$

Thus the boundary terms in the weak form reduce to:

$$w(1)$$

Assuming admissible solutions the final weak form is as follows.

$$w(1) + \int_0^1 (-w\left(u\,x^4 + x + u'\right) - u'\,w')\,dx = 0$$

1.2

An engineering analysis problem is formulated in terms of the following ordinary differential equation

$$\frac{d^2 u}{dx^2} - x\,\frac{du}{dx} = u; \quad 0 < x < 1$$

$$u(0) = \frac{du(0)}{dx} - 2; \quad \frac{du(1)}{dx} = 1$$

Obtain a suitable weak form for the problem. What is the order of the differential equation? Is the boundary condition at $x = 0$ a natural or an essential boundary condition? Is the boundary condition at $x = 1$ a natural or an essential boundary condition?

(i) Second-order

(ii) Natural

(iii) Natural

(iv)

With $u(x)$ as an assumed solution the residual is

$$\mathrm{e}(x) = -u(x) - x\,u'(x) + u''(x)$$

Multiplying by $w_i(x)$ and writing integral over the given limits, the Galerkin weighted residual is

$$\int_0^1 (-u\,w_i - x\,u'\,w_i + u''\,w_i)\,\mathrm{d}x = 0$$

Using integration by parts, the order of derivative in $w_i\,u''$ can be reduced to 1 as follows.

$$\int_0^1 (w_i\,u'')\,\mathrm{d}x = w_i(1)\,u'(1) - w_i(0)\,u'(0) + \int_0^1 (-u'\,w_i')\,\mathrm{d}x$$

Combining all terms, the weighted residual now is as follows.

$$w_i(1)\,u'(1) - w_i(0)\,u'(0) + \int_0^1 (-u\,w_i - u'\,(x\,w_i + w_i'))\,\mathrm{d}x = 0$$

Consider the boundary terms

$$w_i(1)\,u'(1) - w_i(0)\,u'(0)$$

Each one of these terms gives rise to two possibilities

$-w_i(0)\,u'(0)$	Either $-u'(0)$ is known or $w_i(0) = 0$
$w_i(1)\,u'(1)$	Either $u'(1)$ is known or $w_i(1) = 0$

From these requirements the possible boundary conditions are as follows:

	NBC		EBC
1	$-u'(0)$ is given	or	$w_i(0) = 0 \Longrightarrow$ Must satisfy $u(0)$ boundary condition
2	$u'(1)$ is given	or	$w_i(1) = 0 \Longrightarrow$ Must satisfy $u(1)$ boundary condition

Given NBC for the problem:

$$u(0) - u'(0) + 2 = 0$$
$$u'(1) - 1 = 0$$

Rearranging: $\begin{pmatrix} u'(0) \to u(0) + 2 \\ u'(1) \to 1 \end{pmatrix}$

Thus the boundary terms in the weak form reduce to:

$$w_i(1) - (u(0) + 2)\,w_i(0)$$

Assuming admissible solutions the final weak form is as follows.

$$w_i(1) - (u(0) + 2)\,w_i(0) + \int_0^1 (-u\,w_i - u'\,(x\,w_i + w_i'))\,\mathrm{d}x = 0$$

(v)

Linear solution

Starting assumed solution: $u(x) = \mathrm{a}_0 + x\,\mathrm{a}_1$

Weighting functions $\to \{1, x\}$

Substitute into the weak form and perform integrations to get:

Weight	Equation
1	$-2\,\mathrm{a}_0 - \mathrm{a}_1 - 1 = 0$
x	$-\frac{\mathrm{a}_0}{2} - \frac{5\,\mathrm{a}_1}{3} + 1 = 0$

Solving these equations we get

$$\left\{\mathrm{a}_0 \to -\frac{16}{17},\ \mathrm{a}_1 \to \frac{15}{17}\right\}$$

Substituting into the admissible solution we get the following solution of the problem.

$$u(x) = \frac{1}{17}(15\,x - 16)$$

1.3

Steady state heat flow through long hollow circular cylinders can be described by the following ordinary differential equation.

$$\frac{d}{\mathrm{dr}}\left(k\,A\,\frac{\mathrm{dT}(r)}{\mathrm{dr}}\right) + A\,Q = 0; \quad r_i < r < r_o$$

$$T(r_i) = T_i; \quad T(r_o) = T_0$$

where r is the radial coordinate, $T(r)$ is the temperature, k is the thermal conductivity, Q is the heat generation per unit area, $A = 2\pi\,r\,L$ the surface area, L is the length of the cylinder, r_i is the inner radius, and r_o is the outer radius. The boundary conditions specify the temperature on the inside and outside of the cylinder respectively. Derive finite element equations for a typical two node linear element for the problem with nodes at $r_1$and r_2. Assume k and Q are constant over the element. Note that A is a function of r and is not constant over the element.

Derivation of element equations

Element nodes: $\{r_1, r_2\}$

Interpolation functions, $\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} \frac{r_2 - r}{r_2 - r_1} & \frac{r_1 - r}{r_1 - r_2} \end{pmatrix}$

$$\boldsymbol{B}^{\mathrm{T}} = \mathrm{d}\boldsymbol{N}^{\mathrm{T}}/\mathrm{dx} = \begin{pmatrix} \frac{1}{r_1-r_2} & \frac{1}{r_2-r_1} \end{pmatrix}$$

$$\mathrm{k}(r) = 2\,k\,L\,\pi\,r \qquad \mathrm{p}(r) = 0 \qquad \mathrm{q}(r) = 2\,L\,\pi\,Q\,r$$

$$\boldsymbol{k}_{\mathrm{k}} = \int_{r_1}^{r_2} (2\,k\,L\,\pi\,r\,\boldsymbol{B}\,\boldsymbol{B}^{\mathrm{T}})\,dr = \begin{pmatrix} \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)^2} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)(r_2-r_1)} \\ \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)(r_2-r_1)} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_2-r_1)^2} \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}}^{\mathrm{T}} = \int_{r_1}^{r_2} (2\,L\,\pi\,Q\,r\,\boldsymbol{N})\,dr = \left\{-\frac{1}{3}\,L\,\pi\,Q\,(r_1-r_2)\,(2\,r_1+r_2),\, -\frac{1}{3}\,L\,\pi\,Q\,(r_1-r_2)\,(r_1+2\,r_2)\right\}$$

The complete element equations are as follows.

$$\begin{pmatrix} \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)^2} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)(r_2-r_1)} \\ \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_1-r_2)(r_2-r_1)} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2}-\frac{r_1^2}{2}\right)}{(r_2-r_1)^2} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\,L\,\pi\,Q\,(r_1-r_2)\,(2\,r_1+r_2) \\ -\frac{1}{3}\,L\,\pi\,Q\,(r_1-r_2)\,(r_1+2\,r_2) \end{pmatrix}$$

1.4

Consider solution of the following second order boundary value problem using two node linear elements.

$$\frac{d^2u}{\mathrm{dx}^2} = \frac{\mathrm{du}}{\mathrm{dx}}; \quad 0 < x < 100$$

$$u(0) = 50; \quad u(100) = 10$$

(a) Show that the following is an appropriate weak form for a typical linear element with nodes at arbitrary locations x_1 and x_2

$$\int_{x_1}^{x_2} (u'\,(w_i + w_i'))\,\mathrm{d}x = 0$$

where $w_i(x)$ are suitable weighting functions.

(b) Using the weak form given in (a), and the assumed solution written in terms of following interpolation functions

$$u(x) = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \quad u'(x) = \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

show that the element equations for a two node linear element for this problem are as follows.

$$\begin{pmatrix} \int_{x_1}^{x_2} (N_1 + N_1')\, N_1' \,\mathrm{dx} & \int_{x_1}^{x_2} (N_1 + N_1')\, N_2' \,\mathrm{dx} \\ \int_{x_1}^{x_2} (N_2 + N_2')\, N_1' \,\mathrm{dx} & \int_{x_1}^{x_2} (N_2 + N_2')\, N_2' \,\mathrm{dx} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(c) Carrying out integrations, the element equations in (b) can be expressed as follows.

$$\frac{1}{2L} \begin{pmatrix} -L + 2 & L - 2 \\ -L - 2 & L + 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $L = x_2 - x_1$, the element length. Using three of these elements, with nodes located at 0, 60, 90, and 100 determine an approximate solution of the problem.

With $u(x)$ as an assumed solution the residual is

$$e(x) = u'(x) - u''(x)$$

Multiplying by $w_i(x)$ and writing integral over the given limits, the Galerkin weighted residual is

$$\int_{x_1}^{x_2} (w_i\, u' - w_i\, u'')\, dx = 0$$

Using integration by parts, the order of derivative in $-w_i\, u''$ can be reduced to 1 as follows.

$$\int_{x_1}^{x_2} (-w_i\, u'')\, dx = w_i(x_1)\, u'(x_1) - w_i(x_2)\, u'(x_2) + \int_{x_1}^{x_2} (u'\, w_i')\, dx$$

Combining all terms, the weighted residual now is as follows.

$$w_i(x_1)\, u'(x_1) - w_i(x_2)\, u'(x_2) + \int_{x_1}^{x_2} (u'\, (w_i + w_i'))\, dx = 0$$

Consider the boundary terms

$$w_i(x_1)\, u'(x_1) - w_i(x_2)\, u'(x_2)$$

Each one of these terms gives rise to two possibilities

$w_i(x_1)\, u'(x_1)$	Either $u'(x_1)$ is known or $w_i(x_1) = 0$
$-w_i(x_2)\, u'(x_2)$	Either $-u'(x_2)$ is known or $w_i(x_2) = 0$

From these requirements the possible boundary conditions are as follows:

	NBC		EBC
1	$u'(x_1)$ is given	or	$w_i(x_1) = 0 \Longrightarrow$ Must satisfy $u(x_1)$ boundary condition
2	$-u'(x_2)$ is given	or	$w_i(x_2) = 0 \Longrightarrow$ Must satisfy $u(x_2)$ boundary condition

Given EBC for the problem:

$$u(x_1) - u_1 = 0$$
$$u(x_2) - u_2 = 0$$

therefore with admissible solutions (those satisfying EBC): $\begin{pmatrix} w_i\,(x_1) \to 0 \\ w_i\,(x_2) \to 0 \end{pmatrix}$

All boundary terms vanish.

Assuming admissible solutions the final weak form is as follows.

$$\int_{x_1}^{x_2} (u'\, (w_i + w_i'))\, dx = 0$$

Assumed solution

$$u(x) = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{N}^T \boldsymbol{d}$$

$$u'(x) = \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{B}^T \boldsymbol{d}$$

Weighting functions $w_i = N_i$

Weak form

$$\int_{x_1}^{x_2} (N_i + N_i')\, u' \,\mathrm{dx} = 0$$

Two equations

$$\int_{x_1}^{x_2} (N_1 + N_1')\, u' \,\mathrm{dx} = 0; \quad \int_{x_1}^{x_2} (N_2 + N_2')\, u' \,\mathrm{dx} = 0$$

Writing together in a matrix form

$$\int_{x_1}^{x_2} \begin{pmatrix} N_1 + N_1' \\ N_2 + N_2' \end{pmatrix} \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mathrm{dx} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \int_{x_1}^{x_2} (N_1 + N_1')\, N_1' \,\mathrm{dx} & \int_{x_1}^{x_2} (N_1 + N_1')\, N_2' \,\mathrm{dx} \\ \int_{x_1}^{x_2} (N_2 + N_2')\, N_1' \,\mathrm{dx} & \int_{x_1}^{x_2} (N_2 + N_2')\, N_2' \,\mathrm{dx} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linear assumed solution

$$u(x) = \begin{pmatrix} \frac{x - x_2}{x_1 - x_2} & \frac{x - x_1}{x_2 - x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

```
{n1, n2} = {(x – x2) / (x1 – x2), (x – x1) / (x2 – x1)} /. x2 → (x1 + L); {b1, b2} = {D[n1, x], D[n2, x]};
```

```
Integrate[(n1 + b1) b1, {x, x1, x1 + L}] // Together
```

$$\frac{2 - L}{2\, L}$$

```
Integrate[(n1 + b1) b2, {x, x1, x1 + L}] // Together
```

$$\frac{L - 2}{2\, L}$$

```
Integrate[(n2 + b2) b1, {x, x1, x1 + L}] // Together
```

$$\frac{-L-2}{2L}$$

```
Integrate[(n2 + b2) b2, {x, x1, x1 + L}] // Together
```

$$\frac{L+2}{2L}$$

$$\frac{1}{2L}\begin{pmatrix} -L+2 & L-2 \\ -L-2 & L+2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

```
k[L_] := 1/(2 L) {{-L + 2, L - 2}, {-L - 2, L + 2}};
```

Element 1: L= 60

```
k1 = k[60]
```

$$\begin{pmatrix} -\frac{29}{60} & \frac{29}{60} \\ -\frac{31}{60} & \frac{31}{60} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{29}{60} & \frac{29}{60} \\ -\frac{31}{60} & \frac{31}{60} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 2: L= 30

```
k2 = k[30]
```

$$\begin{pmatrix} -\frac{7}{15} & \frac{7}{15} \\ -\frac{8}{15} & \frac{8}{15} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{7}{15} & \frac{7}{15} \\ -\frac{8}{15} & \frac{8}{15} \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3: L= 10

```
k3 = k[10]
```

$$\begin{pmatrix} -\frac{2}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{3}{5} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{2}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations

```
K = Table[0, {4}, {4}];
K[[{1, 2}, {1, 2}]] += k1;
K[[{2, 3}, {2, 3}]] += k2;
K[[{3, 4}, {3, 4}]] += k3; K
```

$$\begin{pmatrix} -\frac{29}{60} & \frac{29}{60} & 0 & 0 \\ -\frac{31}{60} & \frac{1}{20} & \frac{7}{15} & 0 \\ 0 & -\frac{8}{15} & \frac{2}{15} & \frac{2}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{29}{60} & \frac{29}{60} & 0 & 0 \\ -\frac{31}{60} & \frac{1}{20} & \frac{7}{15} & 0 \\ 0 & -\frac{8}{15} & \frac{2}{15} & \frac{2}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions

$$u_1 = 50; \quad u_4 = 10$$

Introducing known values and removing the first and the last equations

$$\begin{pmatrix} -\frac{31}{60} & \frac{1}{20} & \frac{7}{15} & 0 \\ 0 & -\frac{8}{15} & \frac{2}{15} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 50 \\ u_2 \\ u_3 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rearranging equations

$$\begin{pmatrix} \frac{1}{20} & \frac{7}{15} \\ -\frac{8}{15} & \frac{2}{15} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 50 \begin{pmatrix} -\frac{31}{60} \\ 0 \end{pmatrix} - 10 \begin{pmatrix} 0 \\ \frac{2}{5} \end{pmatrix}$$

Thus the final system of equations is as follows.

$$\begin{pmatrix} \frac{1}{20} & \frac{7}{15} \\ -\frac{8}{15} & \frac{2}{15} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{155}{6} \\ -4 \end{pmatrix}$$

Nodal solution

```
Kf = K[[{2, 3}, {2, 3}]]
```

$$\begin{pmatrix} \frac{1}{20} & \frac{7}{15} \\ -\frac{8}{15} & \frac{2}{15} \end{pmatrix}$$

```
Rf = -50 {K[[2, 1]], K[[3, 1]]} - 10 {K[[2, 4]], K[[3, 4]]}
```

$$\left\{\frac{155}{6}, -4\right\}$$

```
sol = LinearSolve[Kf, Rf]
```

$$\left\{\frac{478}{23}, \frac{1222}{23}\right\}$$

```
d = {50, sol[[1]], sol[[2]], 10}
```

$$\left\{50, \frac{478}{23}, \frac{1222}{23}, 10\right\}$$

Element solution

```
{n1, n2} = {(x - x2) / (x1 - x2), (x - x1) / (x2 - x1)};
ux1 = {n1, n2}.d[[{1, 2}]] /. {x1 → 0, x2 → 60} // Expand
```

$$50 - \frac{56\,x}{115}$$

```
ux2 = {n1, n2}.d[[{2, 3}]] /. {x1 → 60, x2 → 90} // Expand
```

$$\frac{124\,x}{115} - \frac{1010}{23}$$

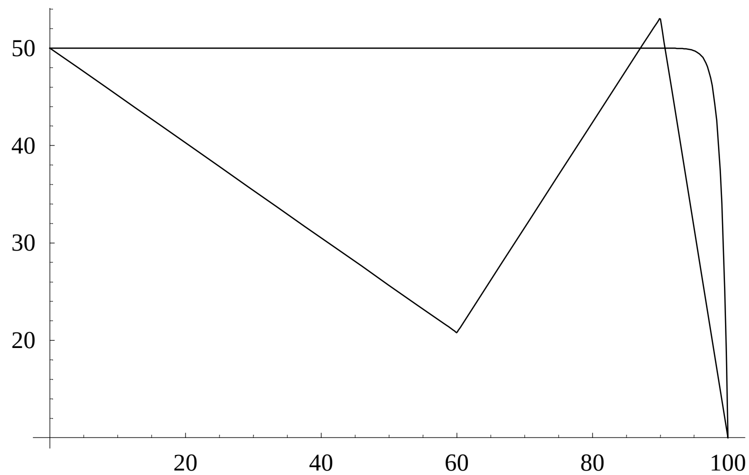
```
ux3 = {n1, n2}.d[[{3, 4}]] /. {x1 → 90, x2 → 100} // Expand
```

$$\frac{10150}{23} - \frac{496\,x}{115}$$

```
Clear[u]; exactSol = DSolve[{u''[x] == u'[x], u[0] == 50, u[100] == 10}, u[x], x]
```

$$\left\{\left\{u(x) \to -\frac{10\,(1 - 5\,e^{100} + 4\,e^{x})}{-1 + e^{100}}\right\}\right\}$$

Plot[{exactSol[[1, 1, 2]], Which[x ≤ 60, ux1, x ≤ 90, ux2, x > 90, ux3]}, {x, 0, 100}, PlotRange → All];



D[exactSol[[1, 1, 2]], {x, 2}]

$$-\frac{40\,e^{x}}{-1 + e^{100}}$$

D[exactSol[[1, 1, 2]], x]

$$-\frac{40\,e^{x}}{-1 + e^{100}}$$

1.5

Consider the following boundary value problem

$$\frac{d}{dx}\left(x\,\frac{du}{dx}\right) = \frac{2}{x^2} \quad 1 < x < 2$$

$u(1) = 2$ and $\frac{du}{dx}(2) = -\frac{1}{4}$

Compare solution and its first derivative obtained by using the following two models.

(a) Use two equal length linear finite elements.

(b) Use one quadratic finite element.

Derivation of element equations

Element nodes: $\{x_1, x_2\}$

Interpolation functions, $\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} \frac{x-x_2}{x_1-x_2} & \frac{x_1-x}{x_1-x_2} \end{pmatrix}$

$\boldsymbol{B}^{\mathrm{T}} = \mathrm{d}\boldsymbol{N}^{\mathrm{T}}/\mathrm{dx} = \begin{pmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_2-x_1} \end{pmatrix}$

$\mathrm{k}(x) = x \qquad \mathrm{p}(x) = 0 \qquad \mathrm{q}(x) = -\frac{2}{x^2}$

$$\boldsymbol{k}_{\mathrm{k}} = \int_{x_1}^{x_2} (x\,\boldsymbol{B}\,\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}x = \begin{pmatrix} -\frac{x_1+x_2}{2x_1-2x_2} & \frac{x_1+x_2}{2x_1-2x_2} \\ \frac{x_1+x_2}{2x_1-2x_2} & -\frac{x_1+x_2}{2x_1-2x_2} \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{q}}^{\mathrm{T}} = \int_{x_1}^{x_2} \left(-\frac{2}{x^2}\,\boldsymbol{N}\right) dx = \left\{ \frac{2\,((\log(x_1) - \log(x_2) - 1)\,x_1 + x_2)}{x_1\,(x_1 - x_2)},\ \frac{2\,(x_1 + (-\log(x_1) + \log(x_2) - 1)\,x_2)}{(x_1 - x_2)\,x_2} \right\}$$

The complete element equations are as follows.

$$\begin{pmatrix} \frac{\frac{x_2^2}{2} - \frac{x_1^2}{2}}{(x_1-x_2)^2} & \frac{\frac{x_2^2}{2} - \frac{x_1^2}{2}}{(x_1-x_2)(x_2-x_1)} \\ \frac{\frac{x_2^2}{2} - \frac{x_1^2}{2}}{(x_1-x_2)(x_2-x_1)} & \frac{\frac{x_2^2}{2} - \frac{x_1^2}{2}}{(x_2-x_1)^2} \end{pmatrix} \begin{pmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{pmatrix} = \begin{pmatrix} \frac{2\,((\log(x_1)-\log(x_2)-1)\,x_1+x_2)}{x_1\,(x_1-x_2)} \\ \frac{2\,(x_1+(-\log(x_1)+\log(x_2)-1)\,x_2)}{(x_1-x_2)\,x_2} \end{pmatrix}$$

2 element solution

Nodal locations: $\{1, 1.5, 2\}$

Element 1

Element nodes: $\{x_1 \to 1, x_2 \to 1.5\}$

$$\begin{pmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{pmatrix} \begin{pmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{pmatrix} = \begin{pmatrix} -0.37814 \\ -0.288527 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 2.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \mathrm{u}_3 \end{pmatrix} = \begin{pmatrix} -0.37814 \\ -0.288527 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \to 1.5, x_3 \to 2\}$

$$\begin{pmatrix} 3.5 & -3.5 \\ -3.5 & 3.5 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -0.182605 \\ -0.150728 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6. & -3.5 \\ 0 & -3.5 & 3.5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -0.37814 \\ -0.471132 \\ -0.150728 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6. & -3.5 \\ 0 & -3.5 & 3.5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -0.37814 \\ -0.471132 \\ -0.150728 \end{pmatrix}$$

Natural boundary conditions

DOF	α	β
u_3	0	$-\frac{1}{4}$

DOF	k(x)	$-\text{k(x)}\,\alpha$	$\text{k(x)}\,\beta$
u_3	2	0	$-\frac{1}{2}$

Global equations after incorporating NBC

$$\begin{pmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6. & -3.5 \\ 0 & -3.5 & 3.5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -0.37814 \\ -0.471132 \\ -0.650728 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
u_1	2

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 6. & -3.5 \\ -3.5 & 3.5 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 4.52887 \\ -0.650728 \end{pmatrix}$$

Solution for nodal unknowns

DOF	x	Solution
u_1	1	2
u_2	1.5	1.55126
u_3	2	1.36533

Solution over elements

Element 1

Nodes: $\{x_1 \to 1,\ x_2 \to 1.5\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{3. - 2.\,x,\ 2.\,x - 2.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{2,\ 1.55126\}$

Solution: $\mathrm{u}(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 2.89749 - 0.897488\,x$

Element 2

Nodes: $\{x_1 \to 1.5,\ x_2 \to 2\}$

Interpolation functions: $\boldsymbol{N}^{\mathrm{T}} = \{4. - 2.\,x,\ 2.\,x - 3.\}$

Nodal values: $\boldsymbol{d}^{\mathrm{T}} = \{1.55126,\ 1.36533\}$

Solution: $\mathrm{u}(x) = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{d} = 2.10902 - 0.371845\,x$

Solution summary

	Range	Solution
1	$1 \le x \le 1.5$	$2.89749 - 0.897488\,x$
2	$1.5 \le x \le 2$	$2.10902 - 0.371845\,x$

Derivation of element equations

Element nodes: $\left\{x_1,\ \frac{1}{2}(x_1 + x_3),\ x_3\right\}$

Interpolation functions, $\boldsymbol{N}^{\mathrm{T}} = \begin{pmatrix} \frac{(x_3-x)(-2x+x_1+x_3)}{(x_1-x_3)^2} & \frac{4(x_1-x)(x-x_3)}{(x_1-x_3)^2} & \frac{(x_1-x)(-2x+x_1+x_3)}{(x_1-x_3)^2} \end{pmatrix}$

$\boldsymbol{B}^{\mathrm{T}} = \mathrm{d}\boldsymbol{N}^{\mathrm{T}}/\mathrm{dx} = \begin{pmatrix} -\frac{-4x+x_1+3x_3}{(x_1-x_3)^2} & \frac{4(-2x+x_1+x_3)}{(x_1-x_3)^2} & -\frac{-4x+3x_1+x_3}{(x_1-x_3)^2} \end{pmatrix}$

$\mathrm{k}(x) = x \qquad \mathrm{p}(x) = 0 \qquad \mathrm{q}(x) = -\frac{2}{x^2}$

$$\boldsymbol{k}_{\mathrm{k}} = \int_{x_1}^{x_3} (x\,\boldsymbol{B}\,\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}x = \begin{pmatrix} -\frac{11x_1+3x_3}{6x_1-6x_3} & \frac{2(3x_1+x_3)}{3(x_1-x_3)} & -\frac{x_1+x_3}{6x_1-6x_3} \\ \frac{2(3x_1+x_3)}{3(x_1-x_3)} & -\frac{8(x_1+x_3)}{3(x_1-x_3)} & \frac{2(x_1+3x_3)}{3(x_1-x_3)} \\ -\frac{x_1+x_3}{6x_1-6x_3} & \frac{2(x_1+3x_3)}{3(x_1-x_3)} & -\frac{3x_1+11x_3}{6x_1-6x_3} \end{pmatrix}$$

$$\boldsymbol{r}_{\text{q}}^{\text{T}} = \int_{x_1}^{x_3}(-\frac{2}{x^2}\ \boldsymbol{N})\,dx = \{-\frac{2\,((\log(x_1)-\log(x_3)-3)\,x_1^2+(3\log(x_1)-3\log(x_3)+2)\,x_3\,x_1+x_3^2)}{x_1\,(x_1-x_3)^2},$$
$$\frac{8\,((\log(x_1)-\log(x_3)-2)\,x_1+(\log(x_1)-\log(x_3)+2)\,x_3)}{(x_1-x_3)^2},$$
$$\frac{2\,(x_1^2+(-3\log(x_1)+3\log(x_3)+2)\,x_3\,x_1+(-\log(x_1)+\log(x_3)-3)\,x_3^2)}{(x_1-x_3)^2\,x_3}\}$$

The complete element equations are as follows.

$$\begin{pmatrix} -\frac{11x_1+3x_3}{6x_1-6x_3} & \frac{2(3x_1+x_3)}{3(x_1-x_3)} & -\frac{x_1+x_3}{6x_1-6x_3} \\ \frac{2(3x_1+x_3)}{3(x_1-x_3)} & -\frac{8(x_1+x_3)}{3(x_1-x_3)} & \frac{2(x_1+3x_3)}{3(x_1-x_3)} \\ -\frac{x_1+x_3}{6x_1-6x_3} & \frac{2(x_1+3x_3)}{3(x_1-x_3)} & -\frac{3x_1+11x_3}{6x_1-6x_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\frac{2((\log(x_1)-\log(x_3)-3)x_1^2+(3\log(x_1)-3\log(x_3)+2)x_3x_1+x_3^2)}{x_1(x_1-x_3)^2} \\ \frac{8((\log(x_1)-\log(x_3)-2)x_1+(\log(x_1)-\log(x_3)+2)x_3)}{(x_1-x_3)^2} \\ \frac{2(x_1^2+(-3\log(x_1)+3\log(x_3)+2)x_3x_1+(-\log(x_1)+\log(x_3)-3)x_3^2)}{(x_1-x_3)^2x_3} \end{pmatrix}$$

1 element solution

Nodal locations: {1, 1.5, 2}

Element 1

Element nodes: $\{x_1 \to 1,\ x_2 \to 1.5,\ x_3 \to 2\}$

$$\begin{pmatrix} \frac{17}{6} & -\frac{10}{3} & \frac{1}{2} \\ -\frac{10}{3} & 8 & -\frac{14}{3} \\ \frac{1}{2} & -\frac{14}{3} & \frac{25}{6} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2(5-7\log(2)) \\ 8(2-3\log(2)) \\ -7+10\log(2) \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} \frac{17}{6} & -\frac{10}{3} & \frac{1}{2} \\ -\frac{10}{3} & 8 & -\frac{14}{3} \\ \frac{1}{2} & -\frac{14}{3} & \frac{25}{6} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2(5-7\log(2)) \\ 8(2-3\log(2)) \\ -7+10\log(2) \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} \frac{17}{6} & -\frac{10}{3} & \frac{1}{2} \\ -\frac{10}{3} & 8 & -\frac{14}{3} \\ \frac{1}{2} & -\frac{14}{3} & \frac{25}{6} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2(5-7\log(2)) \\ 8(2-3\log(2)) \\ -7+10\log(2) \end{pmatrix}$$

Natural boundary conditions

DOF	α	β
u_3	0	-0.25

DOF	k(x)	$-k(x)\,\alpha$	$k(x)\,\beta$
u_3	2	0	-0.5

Global equations after incorporating NBC

$$\begin{pmatrix} \frac{17}{6} & -\frac{10}{3} & \frac{1}{2} \\ -\frac{10}{3} & 8 & -\frac{14}{3} \\ \frac{1}{2} & -\frac{14}{3} & \frac{25}{6} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2\,(5 - 7\log(2)) \\ 8\,(2 - 3\log(2)) \\ -0.568528 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
u_1	2

Incorporating EBC the final system of equations is

$$\begin{pmatrix} 8 & -\frac{14}{3} \\ -\frac{14}{3} & \frac{25}{6} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} + 8\,(2 - 3\log(2)) \\ -1.56853 \end{pmatrix}$$

Solution for nodal unknowns

DOF	x	Solution
u_1	1	2
u_2	1.5	1.54124
u_3	2	1.34975

Solution over elements

Element 1

Nodes: $\{x_1 \to 1,\ x_2 \to 1.5,\ x_3 \to 2\}$

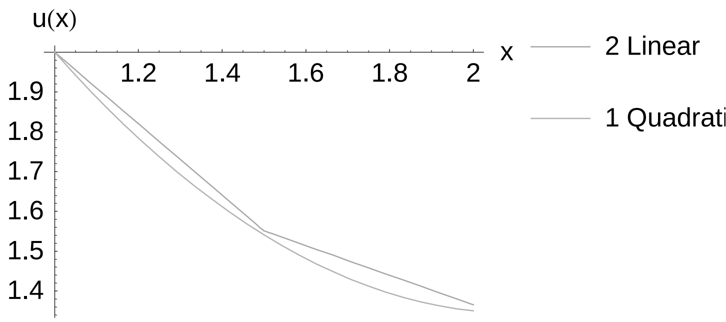
Interpolation functions: $\boldsymbol{N}^T = \{2x^2 - 7x + 6,\ -4x^2 + 12x - 8,\ 2x^2 - 5x + 3\}$

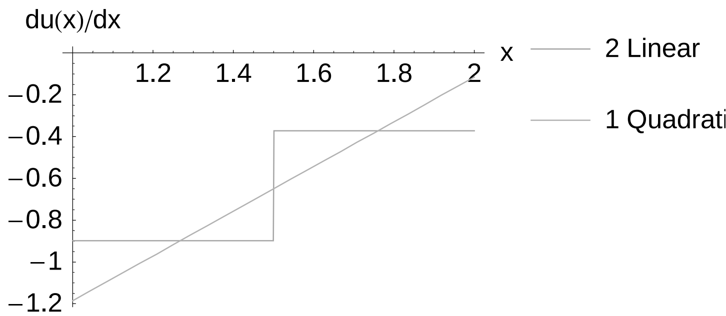
Nodal values: $\boldsymbol{d}^T = \{2,\ 1.54124,\ 1.34975\}$

Solution: $u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.534517x^2 - 2.25381x + 3.71929$

Solution summary

	Range	Solution
1	$1 \le x \le 2$	$0.534517x^2 - 2.25381x + 3.71929$





1.6

A four node quadrilateral element is shown in Figure 1.12.

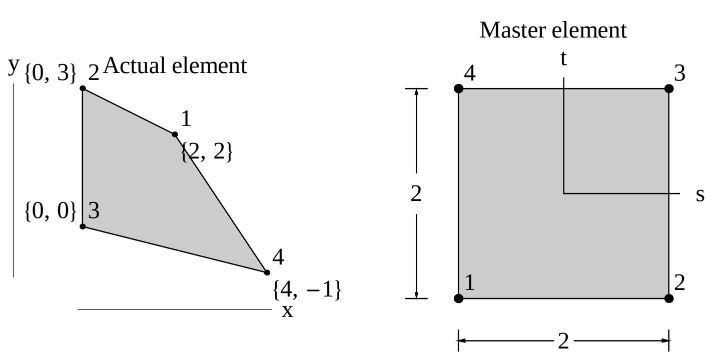


Figure 1.12.